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Algebra and quantum geometry of multifrequency resonance

M. V. Karasev and E. M. Novikova

Abstract. The algebra of symmetries of a quantum resonance oscillator in the case of three or more frequencies is described using a finite (minimal) basis of generators and polynomial relations. For this algebra, we construct quantum leaves with a complex structure (an analogue of classical symplectic leaves) and a quantum Kähler 2-form, a reproducing measure, and also the corresponding irreducible representations and coherent states.

Keywords: frequency resonance, algebra of symmetries, non-linear commutation relations, quantum Kähler forms, coherent states.

§ 1. Introduction

In wave and quantum mechanics of multidimensional systems, a fundamental role is played by states localized near a stable equilibrium [1]–[3]. The harmonic part of such systems, namely, the oscillator, determines the principal component of motion, while the anharmonic part is a perturbation. After quantum averaging, this perturbation will commute with the harmonic part, that is, it specifies an element in the commutant (the algebra of symmetries). When the frequencies of the harmonic part are in resonance, the algebra of symmetries is non-commutative. This non-commutativity gives rise to a non-trivial dynamics of the averaged system and produces several interesting effects.

In classical mechanics, the symmetries of resonance oscillators have been studied for a long time (see, for example, [4], [5] and the references in the fundamental survey [6]). The Poisson structure on the space of symmetries has been studied in several particular cases of resonance, for example, in [7]–[9].

It was shown in [10] that the algebra of symmetries of a general resonance oscillator is an algebra with finitely many generators and polynomial relations. It is called a *resonance algebra*.

The harmonic part of the original system is a Casimir element (the centre) of the resonance algebra. The averaged anharmonic part determines another Hamiltonian on the resonance algebra. In the stable case, when all the resonance frequencies are positive, this Hamiltonian corresponds to a quasi-particle arising from the resonance. We call this quasi-particle a *gyron*. It turned out that the description of the resonance algebra itself and the analysis of the gyron states (precessions) gave rise to the appearance of unusual quantum geometry [11]–[14]. The new algebraic and geometric objects discovered here enabled one to solve the old problem of finding

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the asymptotic behaviour of the spectrum and wave states in nano- and microzones near resonances.

The resonance problem under consideration has two important aspects. Firstly, it covers a wide range of basic models in wave optics and quantum nanophysics. Secondly, it turned out that this problem leads to an interesting class of algebras defined by finitely many generators and (in general non-linear) polynomial relations. One can construct a complete theory of irreducible representations of such algebras, including a quantum-geometric generalization of the well-known orbit method (originally developed for Lie algebras [15]).

Systematic studies of quantum algebras with non-linear relations were initiated by the schools of Maslov and Faddeev [16]–[25] in the 1970s, although the first attempts to use such algebras were made much earlier by physicists.

For a long time, the list of physical systems, where algebras with non-linear relations play a significant role in the description of the spectrum and dynamics, was confined to infinite-dimensional field systems and spin chains (see the references in [22]). Examples of finitely generated non-Lie algebras whose properties manifest themselves in fundamental effects of quantum mechanics (the Zeeman and Zeeman–Stark effects) were discovered and studied in detail in [26]–[28]. The structure of these families is similar to that of the algebras considered in [16]: the number of their generators coincides with the dimension of the spectrum.

The algebras discovered in [26]–[28] are quantum algebras, that is, deformations of certain classical Poisson algebras (with a polynomial Poisson tensor). The resonance algebras corresponding to multifrequency quantum oscillators, which were found in [11]–[13], also belong to the class of quantum algebras, but the number of their generators generally exceeds the dimension of the spectrum.

The quantum oscillator with frequencies $f_1, \dots, f_n$ is defined by the Hamiltonian

$$\widehat{H} = -\frac{\hbar^2}{2}\Delta + \frac{1}{2}(f_1^2 q_1^2 + \dots + f_n^2 q_n^2) - \frac{\hbar}{2}(f_1 + \dots + f_n).$$

This operator acts on variables q_j in the space $L^2(\mathbb{R}^n)$. We assume that the parameter $\hbar$ is positive and, for simplicity, consider only the stable case $f_j > 0$, $j = 1, \dots, n$.

We define the annihilation operators

$$\hat{z}_j = \sqrt{\frac{f_j}{2}}q_j + \frac{\hbar}{\sqrt{2f_j}}\frac{\partial}{\partial q_j} \tag{1.1}$$

and consider their adjoint operators $\hat{z}_j^*$. The following relations hold:

$$[\hat{z}_j, \hat{z}_k] = 0, \qquad [\hat{z}_k, \hat{z}_j^*] = \hbar\delta_{j,k}. \tag{1.2}$$

By introducing the action operators $\widehat{S}_j = \hat{z}_j^*\hat{z}_j$, $j = 1, \dots, n$, one can write the oscillator Hamiltonian as the scalar product of the frequency vector and the action operator vector:

$$\widehat{H} = \langle f, \widehat{S}\rangle. \tag{1.3}$$

The action operators commute with each other, and their common spectrum forms a lattice of step $\hbar$ in n-dimensional space: $\operatorname{Spectr}(\widehat{S}) = \{\hbar k \mid k \in \mathbb{Z}_+^n\}$.

The study of the algebra of symmetries of the operator $\langle f, \widehat{S}\rangle$ can be reduced to that of simple *resonance* cases, in which the frequencies $f \in \mathbb{N}^n$ are integer and simple, that is, no pair of frequencies has a non-trivial common divisor (see, for example, [10]). We consider precisely such resonance cases.

It follows from the relations (1.2) that the algebra of symmetries of the resonance oscillator, that is, the algebra of operators in $L^2(\mathbb{R}^n)$ commuting with $\langle f, \widehat{S}\rangle$, is generated by the elements

$$(\hat{z}^*)^r \hat{z}^l, \qquad \text{where} \quad \langle f, l-r\rangle = 0, \qquad l, r \in \mathbb{Z}^n_+. \tag{1.4}$$

We note that the generators $(\hat{z}^*)^r \hat{z}^l$ are not independent and satisfy several identities.

In classical mechanics, the functions $\overline{z}^r z^l$ satisfying the condition $\langle f, l-r\rangle = 0$ are called *resonance normal forms* (see [5], [29]). They determine the algebra of functions that are in involution with the classical resonance oscillator. As shown in [10], one can choose a *finite basis* of '*minimal*' *generators* in this infinite family of generators. A minimal basis also exists in the quantum case.

Once this first non-trivial fact is established, the following problem arises: describe the relations holding in the associative algebra generated by this basis. In other words, it is required to determine the algebra of symmetries of the resonance oscillator not via its specific representation by the (minimal) normal forms (1.4) but as an abstract algebra with finitely many generators and algebraic relations. We call it a *resonance algebra*.

In the framework of classical mechanics, this problem can be reformulated as follows: find the minimal algebraic manifold with algebraic Poisson structure, and a canonical map of the standard phase space $\mathbb{R}^{2n}$ to this manifold such that this map is in involution with the resonance oscillator $\langle f, S\rangle$.

Since the Hamiltonian trajectories are circles, we come to the problem of the algebraic description of the Poisson reduction of the space $\mathbb{R}^{2n}$ by a circle action (for details of general reduction by circle actions see, for example, [30]). This problem was solved in [10] in the case of general resonance. Other approaches to some degenerate cases of resonance were developed in [7], [9].

We note that the space of frequencies is endowed with a natural notion of arithmetic equivalence under the action of the group $\mathrm{SL}(n, \mathbb{Z})$ of integer matrices with unit determinant. However, the resonance algebras and Poisson manifolds of arithmetically equivalent sets of frequencies can be essentially different. For example, although any set of coprime frequencies is arithmetically equivalent in the two-dimensional case to the set of unit frequencies 1:1, the resonance algebra for the first set is generally determined by non-linear commutation relations and cannot be reduced to a finite-dimensional Lie algebra, while the resonance algebra for the second set is the simple three-dimensional Lie algebra su(2). Thus, arithmetic equivalence does not imply algebraic equivalence. On the other hand, sets of frequencies that differ only by permutations are certainly equivalent, both arithmetically and algebraically. Furthermore, a set of frequencies having a common integer divisor is algebraically equivalent to the same set with this divisor deleted. Moreover, any set of frequencies can always be reduced in the algebraic sense to elementary sets, where the frequencies are pairwise coprime (see [10]). It is such elementary sets of frequencies that will be considered in this paper.

One cannot say that the quantum version of the problem of the resonance algebra simply duplicates the classical version. Of course, it is easy to describe the quantum resonance algebra in the two-frequency case (this was done in [12], [13] along with a description of the classical algebra). But the case of quantum multifrequency resonance for $n \geqslant 3$ is much more complicated than its classical version.

The cases of three or more frequencies differ essentially from the two-frequency case because the resonance lattice of 'creation' vectors (of transitions between points of the spectrum of the action operators) has an anomaly [10]. In the present paper, we show that one can determine maximal normal sublattices in this anomalous lattice. They play a significant role in the description of both the Poisson tensor and the quantum relations of the resonance algebra. The same sublattices are used to construct coherent transformations (in this case, the normality of the sublattices removes the problem of ordering the 'creation' operators).

Since the multifrequency case exhibits non-commuting 'creations', we must introduce [10] the notion of commutator between the vectors of an integer lattice. The sublattices of normal resonance can alternatively be defined as maximal commutative sublattices. The commutator between the vectors of a lattice determines brackets between the 'creation'–'creation' and 'creation'–'annihilation' coordinates in the resonance Poisson algebra.

In the quantum multifrequency case we must introduce several new operations on the integer resonance lattice. Describing the resonance algebra in this case requires more arithmetic. In particular, we encounter interesting 'structure polynomials' on the integer lattice, which play the role of structure functions in quantum commutation relations.

Besides the description of the resonance algebra, the problem under consideration also contains the following subproblem: that of constructing the irreducible representations of this algebra. Here it unexpectedly turns out that the classical geometric objects (Poisson tensor, symplectic leaves) cannot be used to construct quantum irreducible representations, so that there is no direct analogy with the orbit method. For example, the dimension of a quantum irreducible representation cannot be expressed in terms of the classical Liouville volume of symplectic leaves.

In the general quantization scheme, it is known that the dimension is determined by the reproducing measure on a quantum leaf, and the irreducible representations of the algebra are constructed over the same leaf [31], [32]. In the present paper, we explicitly describe quantum leaves for the multifrequency oscillator. The construction is based on the group structure of the resonance lattice. This enables us to introduce a universal family of complex manifolds determined only by the resonance frequencies and oscillator eigenvalues. On one hand, these manifolds can be embedded in the resonance Poisson manifold as the closures of symplectic leaves. Hence the classical closed Kirillov 2-form (with singularities) is defined on them. Being Kähler with respect to the complex structure, this form has no reproducing measure. On the other hand, the same manifolds admit a quantum Kähler 2-form without singularities,[1] for which a reproducing measure exists. This measure gives a formula for the dimension of an irreducible representation of the resonance algebra and also determines a Hilbert structure on the representation space. By these means, the operators of an irreducible representation can be constructed explicitly as differential operators with polynomial coefficients. The order of such an operator is generally greater than 1. This corresponds to the fact that the relations in the

[1] This form is closed but degenerate. Hence it gives an example of a vortex structure [33].

algebra are non-linear and, moreover, the complex polarization is non-invariant. Only in the case of isotropic resonance, where all the frequencies are equal to 1, the resonance algebra is a Lie algebra, the polarization is invariant, and the operators of an irreducible representation are of order 1.

The quantum 2-form and the corresponding reproducing measure introduced in this context are called the *objects of quantum geometry*. In the classical limit, outside the singular points, these objects respectively become the classical Kirillov form and the Liouville measure on symplectic leaves, but this limit cannot be realized near singularities (on the closure of the leaves). In principle, one can try to avoid the use of quantum geometry in this problem and deal only with classical leaves and general geometric quantization for non-invariant polarizations following the scheme given in [34], but then the whole construction would be much more complicated and the operators of an irreducible representation would contain singularities and be pseudo-differential rather than differential as they are in our version (see the discussion in [14], [31], [32]).

We also give a simple formula for coherent transformations intertwining the original representation of the symmetry algebra of the resonance oscillator with its irreducible representations on the space of antiholomorphic sections of a Hermitian line bundle over quantum leaves. The resulting coherent states of the resonance oscillator also enable us to realize representations of the resonance algebra on the space of functions over Lagrangian submanifolds of the quantum leaf. This is convenient for calculating the semiclassical asymptotics (see [13], [14]). In particular, this realization uses the phase function (action) generated by a quantum 2-form.

§ 2. Minimal resonance vectors

A set of vectors with integer Cartesian coordinates in $\mathbb{R}^n$ is called a *lattice* if it is a semigroup with respect to addition.

We fix a frequency vector $f \in \mathbb{N}^n$. The *resonance lattice* $\mathcal{R} = \mathcal{R}[f]$ is defined to be the set of all integer vectors in $\mathbb{R}^n$ that are orthogonal to the frequency vector. Thus, we have $\sigma \in \mathcal{R}$ if all the coordinates σ_j are integer and

$$\langle f, \sigma \rangle = 0.$$

Such vectors σ are called *resonance* vectors.

The intersections of the resonance lattice $\mathcal{R}$ with the Cartesian sectors are called *normal sublattices*. Each of the Cartesian coordinates preserves its sign on such sublattices. These sublattices can be indexed as follows:

$$\mathcal{R}^j, \mathcal{R}^{jk}, \mathcal{R}^{jkl}, \dots, \tag{2.1}$$

where the superscripts indicate the indices of those Cartesian coordinates that must be non-negative (and the others non-positive). In intersections of these sublattices, the corresponding coordinates are identically zero. Thus, in the intersection $\mathcal{R}^j \cap \mathcal{R}^{jk}$, the coordinate with index k is zero, that with index j is non-negative, and the others are non-positive.

The total number of normal resonance sublattices (2.1) is given by the formula

$$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} = 2^n - 2.$$

Together they cover the entire resonance lattice $\mathcal{R}$.

When $n = 2$, there are only two normal lattices, $\mathcal{R}^1$ and $\mathcal{R}^2$. They consist of two-dimensional integer vectors lying on the line orthogonal to the frequency vector $f \in \mathbb{N}^2$ on either side of the origin 0.

When $n = 3$, there are six normal resonance sublattices:

$$\mathcal{R}^1, \mathcal{R}^2, \mathcal{R}^3, \mathcal{R}^{12}, \mathcal{R}^{23}, \mathcal{R}^{31} \subset \mathbb{Z}^3.$$

They correspond to the six sectors into which Cartesian planes divide the resonance plane orthogonal to the frequency vector $f \in \mathbb{N}^3$.

Definition 2.1. A non-zero resonance vector is said to be *minimal* if it cannot be written as the sum of two non-zero vectors in a normal sublattice.

It is natural to consider the set $\mathcal{M}$ of all minimal vectors as a set of elementary generators of the resonance lattice. The notion of a minimal vector was introduced in somewhat different terms in [10], where the following completeness theorem was also proved: *any resonance vector can be represented as a linear combination of minimal vectors with positive integer coefficients*. This result can be improved.

A subset of the resonance lattice is said to be *normal* if it lies in one of the normal resonance sublattices.

A resonance vector is said to be *internal* if it belongs to only one normal sublattice. Otherwise it is called a *face* vector.

Theorem 2.1. *Any resonance vector σ can be decomposed into a sum of minimal vectors with non-negative integer coefficients*:

$$\sigma = \sum_{\varkappa \in \mathcal{M}_\sigma} n^\sigma_\varkappa \varkappa, \qquad n^\sigma_\varkappa \in \mathbb{Z}_+, \quad \mathcal{M}_\sigma \subset \mathcal{M}. \tag{2.2}$$

Here $\mathcal{M}_\sigma$ stands for the set of minimal vectors in the intersection of all normal sublattices containing σ. In particular, $\mathcal{M}_\sigma$ is a normal subset.

Proof. We assume that σ is not minimal. Then it can be decomposed into a sum $\sigma = \varkappa' + \varkappa''$, where $\varkappa'$ and $\varkappa''$ lie in a normal sublattice containing σ. If $\varkappa'$ and $\varkappa''$ are minimal, then the proof is complete. If at least one of them is not minimal, then we again decompose it into a sum of two vectors of the same normal lattice and so on. The procedure stops after finitely many steps because the integer components of the decomposed vector are represented at each step as sums of integers of the same sign and zeros (by the definition of a normal sublattice).

We note that the decomposition (2.2) is generally non-unique.

For example, in the three-frequency case $f_1 = 1$, $f_2 = 2$, $f_3 = 3$, the resonance vector $\sigma = (2, -4, 2)$ has two decompositions into a sum of minimal vectors:

$$\sigma = 2\varkappa, \qquad \sigma = \varkappa' + \varkappa'',$$

where $\varkappa = (1, -2, 1)$, $\varkappa' = (2, -1, 0)$, and $\varkappa'' = (0, -3, 2)$.

Furthermore, the number of terms in (2.2) does not exceed the *total number $M \overset{\text{def}}{=} \#\mathcal{M}$ of minimal resonance vectors*. It was proved in [10] that M is finite.

We give an explicit description of the set of minimal resonance vectors in the case $n = 3$.

First, we note that it suffices to consider only the case of coprime frequencies. Indeed, if two frequencies, say, f_1 and f_2, have a common divisor:

$$f_1 = mf_1', \qquad f_2 = mf_2',$$

then the situation is easily reduced to the set of frequencies f_1', f_2', f_3. This reduction is done as follows. We first reduce the situation to the case when f_3 is coprime to m. Then we use the following fact: $(\sigma_1, \sigma_2, \sigma_3)$ *is a minimal resonance vector for the set of frequencies* (f_1, f_2, f_3) *if and only if* $(\sigma_1, \sigma_2, m\sigma_3)$ *is a minimal resonance vector for the set of frequencies* (f_1', f_2', f_3).

For example, the study of the resonance $3:6:2$ can be reduced to that of the resonance[2] $1:1:1$.

For the sake of simplicity, we consider only the case of positive frequencies (one can similarly treat the cases when some frequencies are negative). Therefore we assume that the following conditions hold:

(A) the frequencies are pairwise coprime,

(B) all the frequencies are positive.

Consider the Diophantine equation

$$\mu f_1 + \nu f_2 + f_3 = 0 \tag{2.3}$$

for unknown integers μ and ν. We use the following simple assertion.

Lemma 2.1. *Equation* (2.3) *has a unique solution satisfying the condition*

$$0 \leqslant \nu \leqslant f_1 - 1. \tag{2.4}$$

In particular, if $f_1 = 1$, *then this solution is* $\mu = -f_3$, $\nu = 0$. *If* $f_1 \geqslant 2$, *then* $\nu \geqslant 1$.

If $f_1 \geqslant 2$, then for each $l = 1, 2, \dots, f_1 - 1$, we write

$$\nu^{(l)} = l\nu \,(\operatorname{mod} f_1), \qquad \mu^{(l)} = -\frac{lf_3 + \nu^{(l)} f_2}{f_1},$$

where ν is the solution of (2.3) satisfying (2.4). Both $\mu^{(l)}$ and $\nu^{(l)}$ are integers and $0 \leqslant \nu^{(l)} \leqslant f_1 - 1$. Of course, for $l = 1$ we have $\mu^{(1)} = \mu$, $\nu^{(1)} = \nu$.

Theorem 2.2. *In the three-frequency case, under conditions* (A) *and* (B), *the minimal vectors in the resonance lattice* $\mathcal{R} = \mathcal{R}^{23} \cup \mathcal{R}^{31} \cup \mathcal{R}^{12} \cup \mathcal{R}^{1} \cup \mathcal{R}^{2} \cup \mathcal{R}^{3}$ *have the following structure.*

a) *If* $f_1 = 1$, *then there are no internal minimal vectors in the sublattice* $\mathcal{R}^{23}$.

b) *If* $f_1 \geqslant 2$, *then all internal minimal vectors in* $\mathcal{R}^{23}$ *are determined by the sequence*

$$(\mu^{(l)}, \nu^{(l)}, l), \qquad l = 1, \dots, f_1 - 1, \tag{2.5}$$

and the vector with index l *is present in* (2.5) *only if*

$$\nu^{(l)} < \nu^{(j)}, \qquad j = 1, \dots, l - 1. \tag{2.6}$$

The face minimal vectors in $\mathcal{R}^{23}$ *have the form*

$$(-f_3, 0, f_1) \in \mathcal{R}^{23} \cap \mathcal{R}^{3}, \qquad (-f_2, f_1, 0) \in \mathcal{R}^{23} \cap \mathcal{R}^{2}. \tag{2.7}$$

[2]Resonances of this degenerate type, which can be reduced to the case of unit frequencies, were considered in [9].

The minimal vectors in the normal sublattices $\mathcal{R}^{31}$ *and* $\mathcal{R}^{12}$ *can be obtained from the above description of the vectors in* $\mathcal{R}^{23}$ *by cyclic permutation of the indices* $1, 2, 3$. *The minimal vectors in the normal sublattice* $\mathcal{R}^{j}$ *have the form* $(-\sigma)$, *where* σ *is a minimal vector in the sublattice* $\mathcal{R}^{kl}$ *and* k, l *are the indices complementing* j *to the triple* $1, 2, 3$.

A proof is given in [35].

It follows from condition (2.6) that all the numbers $\nu^{(l)}$ in the sequence (2.5) must be less than ν. Since f_1 and f_3 are coprime, the sequence (2.5) cannot contain two vectors with the same second coordinate. Therefore the number of terms in (2.5) does not exceed $\nu \leqslant f_1 - 1$. This gives an upper bound for the number of internal minimal vectors in the sublattice $\mathcal{R}^{23}$. Similar estimates hold for the sublattices $\mathcal{R}^{31}$ and $\mathcal{R}^{12}$ (with f_1 replaced by f_2 or f_3). Thus the total number of internal minimal vectors does not exceed $2[(f_1-1)+(f_2-1)+(f_3-1)] = 2(f_1+f_2+f_3)-6$ and there are exactly six face minimal vectors (see (2.7)).

Corollary 2.1. *In the three-frequency case* $n = 3$, *the number of minimal resonance vectors has the following upper bound*:

$$M \leqslant 2(f_1 + f_2 + f_3).$$

§ 3. The Poisson algebra of symmetries of a resonance oscillator

Following [10], we describe the resonance algebra of the classical oscillator

$$H = \frac{1}{2}(p_1^2 + \dots + p_n^2 + f_1^2 q_1^2 + \dots + f_n^2 q_n^2) \tag{3.1}$$

in the elliptic case (see condition (B) in § 2).

We introduce the following operations on the lattice $\mathbb{Z}^n$ (the index j runs through the values $1, \dots, n$):

$$\begin{aligned}
&\alpha \to |\alpha|, && |\alpha| \overset{\text{def}}{=} |\alpha_1| + \dots + |\alpha_n|, \\
&\alpha \to \theta(\alpha), && \theta(\alpha)_j \overset{\text{def}}{=} \begin{cases} 1 & \text{for } \alpha_j > 0, \\ 0 & \text{for } \alpha_j \leqslant 0, \end{cases} \\
&\alpha, \beta \to \alpha \cdot \beta, && (\alpha \cdot \beta)_j \overset{\text{def}}{=} \alpha_j \beta_j, \\
&\alpha \to \alpha_\pm, && \alpha_\pm \overset{\text{def}}{=} \pm\alpha \cdot \theta(\pm\alpha), \\
&\alpha, \beta \to \alpha|\beta, && (\alpha|\beta)_j \overset{\text{def}}{=} \min\{(\alpha_-)_j, (\beta_+)_j\}, \\
&\alpha, \beta \to [\alpha|\beta], && [\alpha|\beta] \overset{\text{def}}{=} \alpha|\beta - \beta|\alpha, \\
&\alpha, \beta \to \alpha \overset{\circ}{+} \beta, && \alpha \overset{\circ}{+} \beta \overset{\text{def}}{=} \alpha|\beta + \beta|\alpha, \\
&\alpha \to \overset{\circ}{\alpha}, && \overset{\circ}{\alpha} \overset{\text{def}}{=} \frac{1}{2}(\alpha \overset{\circ}{+} (-\alpha)) \equiv \frac{1}{2}(\alpha_+ + \alpha_-), \\
&\alpha, \beta \to \alpha \circ \beta, && \alpha \circ \beta = \alpha \cdot \overset{\circ}{\beta}, \\
&\alpha, \beta \to [\alpha, \beta], && [\alpha, \beta] \overset{\text{def}}{=} \alpha_+ \cdot \beta_- - \alpha_- \cdot \beta_+.
\end{aligned} \tag{3.2}$$

Another useful operation will be introduced in (3.18) below. Various properties of the operations in (3.2) were listed in [10]. The last operation in (3.2) was called a *commutator* in [10]. The operation $\overset{\circ}{+}$ was called an *anomaly*.

We note that the operation $\circ$ (introduced in (3.2)) is associative, and the commutator (3.2) can be represented as

$$[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha. \tag{3.3}$$

But the operation $\circ$ is not distributive in the second argument with respect to addition:

$$\alpha \circ (\beta + \gamma) = \alpha \circ \beta + \alpha \circ \gamma - \alpha \cdot (\beta \overset{\circ}{+} \gamma),$$

where $\overset{\circ}{+}$ denotes an anomaly. Hence the 'Jacobi identity' for the commutator (3.3) has a non-standard form (see [10]).

We shall say that two vectors in a lattice commute if their commutator is zero. A simple calculation shows that

$$\alpha \overset{\circ}{+} \beta = 0 \quad \Longleftrightarrow \quad [\alpha, \beta] = 0 \quad \Longleftrightarrow \quad [\alpha|\beta] = 0. \tag{3.4}$$

Lemma 3.1. *Two vectors in $\mathbb{Z}^n$ commute if and only if they belong to the same normal sublattice. A normal sublattice can be defined as a maximal commutative subset of a given lattice.*

Proof. We first note that

$$[\alpha, \beta] = 0 \quad \Longleftrightarrow \quad \alpha_+ \cdot \beta_- = \alpha_- \cdot \beta_+ = 0.$$

The last (double) equality means that corresponding components of α and β cannot have opposite signs. Thus these vectors belong to the same normal sublattice.

Let $\mathcal{R} = \mathcal{R}[f]$ be the resonance lattice in $\mathbb{Z}^n$ generated by the set of frequencies $f \in \mathbb{N}^n$ of the oscillator (3.1). We recall that $\mathcal{M}$ is the subset of minimal resonance vectors in $\mathcal{R}$ and $\mathcal{M}_\sigma$ are the normal subsets of $\mathcal{M}$ used in decomposition (2.2) of a resonance vector σ. Lemma 3.1 yields the following assertion.

Proposition 3.1. *Each subset $\mathcal{M}_\sigma$ is commutative. All elements $\varkappa \in \mathcal{M}_\sigma$ in the decomposition* (2.2) *commute.*

We consider the space

$$\mathcal{N}^{\#} = \mathbb{C}^M \times \mathbb{R}^n \tag{3.5}$$

and define constraints of three different types on this space.

We recall that M is the number of elements in $\mathcal{M}$. The complex coordinates in the first factor $\mathbb{C}^M$ in (3.5) are indexed by elements $\sigma \in \mathcal{M}$. We denote these coordinates by A_σ. The real coordinates in the second factor $\mathbb{R}^n$ in (3.5) are denoted by S_j, $j = 1, \dots, n$. For every $\alpha \in \mathbb{Z}_+^n$ we put $S^\alpha = S_1^{\alpha_1} \cdots S_n^{\alpha_n}$.

Constraints of Hermitian type. For any minimal vector σ, the opposite vector $(-\sigma)$ corresponds to conjugation of the complex coordinate:

$$\overline{A}_\sigma = A_{-\sigma}. \tag{3.6}$$

Constraints of commutative type. If two families $\{\rho\}$ and $\{\sigma\}$ of commuting minimal vectors determine the same linear combinations

$$\sum_\rho k_\rho\, \rho = \sum_\sigma m_\sigma\, \sigma, \qquad k_\rho, m_\sigma \in \mathbb{N}, \tag{3.7}$$

then

$$\prod_\rho (A_\rho)^{k_\rho} = \prod_\sigma (A_\sigma)^{m_\sigma}. \tag{3.8}$$

Remark 3.1. A relation of the form (3.7) is said to be *reducible* if the coefficients k_ρ and m_σ can be written as $k_\rho = k'_\rho + k''_\rho$ and $m_\sigma = m'_\sigma + m''_\sigma$ with $\sum_\rho k'_\rho \rho = \sum_\sigma m'_\sigma \sigma$, where $k'_\rho, k''_\rho, m'_\sigma, m''_\sigma \in \mathbb{Z}_+$, $\big(\sum_\rho k'_\rho\big)\big(\sum_\rho k''_\rho\big) \neq 0$. Otherwise (3.7) is said to be *irreducible*. It is easy to show that the number of irreducible relations (3.7), and hence the number of independent constraints (3.8) of commutative type, is finite.

Constraints of non-commutative type. If minimal vectors ρ and σ do not commute and $\rho \neq -\sigma$, then

$$A_\rho A_\sigma = S^{\rho \overset{\circ}{+} \sigma} \prod_{\varkappa \in \mathcal{M}_{\rho+\sigma}} A_\varkappa^{n_\varkappa^{\rho+\sigma}}, \tag{3.9}$$

where $\overset{\circ}{+}$ is the anomaly introduced in (3.2) and $n_\varkappa^{\rho+\sigma}$ are the coefficients in the decomposition (2.2) of the vector $\rho + \sigma$ into minimal vectors in $\mathcal{M}_{\rho+\sigma}$. We note that if ρ and σ commute, then the relation (3.9) follows from constraints (3.8) of commutative type.

Let $\mathcal{N} \subset \mathcal{N}^\#$ be the submanifold in the space (3.5) simultaneously determined by all the constraint equations (3.6), (3.8), (3.9).

Given any non-minimal resonance vector σ, we define a function

$$A_\sigma = \begin{cases} 1 & \text{for} \quad \sigma = 0, \\ \displaystyle\prod_{\varkappa \in \mathcal{M}_\sigma} A_\varkappa^{n_\varkappa^\sigma} & \text{for} \quad \sigma \neq 0 \end{cases} \tag{3.10}$$

on the submanifold $\mathcal{N}$. Here the subsets $\mathcal{M}_\sigma$ and the numbers $n_\varkappa^\sigma$ are defined by (2.2). By the constraints of commutative type, this function is independent of the choice of a decomposition (2.2) of the vector σ into minimal vectors.

We now define brackets on $\mathcal{N}$ following [10]:

$$\{S_j, S_k\} = 0, \qquad \{S_j, A_\rho\} = i\rho_j A_\rho, \qquad \{A_\rho, A_\sigma\} = -if_{\rho,\sigma}(S) A_{\rho+\sigma}, \tag{3.11}$$

where $j, k = 1, \dots, n$, the vectors ρ and σ belong to $\mathcal{M}$, and the polynomials $f_{\rho,\sigma}$ on $\mathbb{R}^n$ are given by

$$f_{\rho,\sigma}(s) \overset{\text{def}}{=} \frac{[\rho, \sigma]}{s}\, s^{\rho \overset{\circ}{+} \sigma}, \qquad s \in \mathbb{R}^n. \tag{3.12}$$

In (3.12), the commutator $[\rho, \sigma]$ and the anomaly $\rho \overset{\circ}{+} \sigma$ are given by the formulae (3.2) and fractions of the form $\frac{\gamma}{s}$ are used as a shorthand notation for the sums

$$\frac{\gamma}{s} \equiv \sum_{j=1}^n \frac{\gamma_j}{s_j}. \tag{3.13}$$

We see from (3.4) that the functions (3.12) have no singularities at $s = 0$.

We recall that a function on a Poisson manifold is called a *Casimir function* if it is in involution with all functions. We shall say that a function is *quasi-Casimir* if it is in involution with all the functions on its zero level surface. The zero level surface of a quasi-Casimir function is a Poisson submanifold of the original Poisson manifold.

Theorem 3.1. *The formulae* (3.11) *determine Poisson brackets on* $\mathcal{N}$. *The function*

$$C \overset{\text{def}}{=} \langle f, S\rangle \tag{3.14}$$

is a Casimir function on $\mathcal{N}$. *For each minimal vector* ρ, *the function*

$$C_\rho \overset{\text{def}}{=} A_\rho A_{-\rho} - S^{2\overset{\circ}{\rho}} \tag{3.15}$$

is a quasi-Casimir function on $\mathcal{N}$. *More precisely, we have*

$$\{C_\rho, S_j\} = \{C_\rho, A_\rho\} = \{C_\rho, A_{-\rho}\} = 0,$$
$$\{C_\rho, A_\sigma\} = 2i\,\frac{\sigma\circ\rho}{S}\,A_\sigma C_\rho, \qquad \sigma\ne\pm\rho.$$

Here $\overset{\circ}{\rho}$ *and* $\sigma\circ\rho$ *are defined in* (3.2). *In particular, when* $n=2$, *the quasi-Casimir function* (3.15) *is a Casimir function for the bracket* (3.11).

Definition 3.1. We define a submanifold $\mathcal{N}_0\subset\mathcal{N}$ as the common zero level surface of the quasi-Casimir functions:

$$\mathcal{N}_0 = \mathcal{N}\big|_{\{C_\rho=0\,|\,\rho\in\mathcal{M}\}},$$

and call it the *resonance Poisson manifold*.

Conjecture 3.1. The resonance Poisson manifold $\mathcal{N}_0$ coincides with $\mathcal{N}$ for $n\geqslant 3$.

We note that the following relations hold on the resonance manifold by the constraints of commutative and non-commutative types:

$$A_\rho A_\sigma = S^{\rho\overset{\circ}{+}\sigma} A_{\rho+\sigma} \tag{3.16}$$

for any minimal vectors ρ and σ.

Proposition 3.2. *The relations* (3.16) *hold on the resonance manifold* $\mathcal{N}_0$ *for all* (*not necessarily minimal*) *resonance vectors* ρ *and* σ.

The proof is similar to that of Proposition 11.1, which will be given in § 11, where the quantum case is considered.

Corollary 3.1. *If the resonance vector* $\varkappa$ *can be represented as a combination of resonance vectors* $\varkappa^{(j)}$, *that is*,

$$\varkappa = k_1\varkappa^{(1)}+\dots+k_l\varkappa^{(l)}, \qquad k_j\in\mathbb{N},$$

then the following relation holds on the resonance manifold $\mathcal{N}_0$:

$$S^{k_1\varkappa^{(1)}\overset{\circ}{+}\cdots\overset{\circ}{+}k_l\varkappa^{(l)}} A_\varkappa = \prod_{j=1}^{l} (A_{\varkappa^{(j)}})^{k_j}. \tag{3.17}$$

Here we use the l-position anomaly, which is defined as follows:

$$\alpha_1 \overset{\circ}{+} \cdots \overset{\circ}{+} \alpha_l \overset{\text{def}}{=} \sum_{m=1}^{l-1} \big((\alpha_1 + \cdots + \alpha_m) \overset{\circ}{+} \alpha_{m+1}\big) = \sum_{m=1}^{l-1} \big((\alpha_l + \cdots + \alpha_{m+1}) \overset{\circ}{+} \alpha_m\big). \tag{3.18}$$

The right-hand side of (3.18) *contains the* 2*-position anomaly* $\overset{\circ}{+}$ *introduced in* (3.2).

We now construct a map $\mathbb{R}^{2n} \to \mathcal{N}^{\#}$ generated by the symmetries (1.4) of the resonance oscillator:

$$A_\sigma(q,p) = \overline{z}^{\sigma_+} z^{\sigma_-}, \qquad \sigma \in \mathcal{M}, \tag{3.19}$$

$$S_j(q,p) = \overline{z}_j z_j, \qquad j = 1, \dots, n. \tag{3.20}$$

Here z_j are complex coordinates on $\mathbb{R}^{2n}$ given by the formulae

$$z_j \overset{\text{def}}{=} \sqrt{\frac{f_j}{2}} q_j + \frac{i}{\sqrt{2f_j}} p_j, \tag{3.21}$$

where q, p are Cartesian coordinates in $\mathbb{R}^{2n} = \mathbb{R}^n_q \times \mathbb{R}^n_p$.

Lemma 3.2. *The image of the symmetry map* (3.19), (3.20), $(q,p) \to (A,S)$, *coincides with the part* $\mathcal{N}_0^+$ *of the resonance manifold* $\mathcal{N}_0$ *on which*

$$S_1 \geqslant 0, \ \dots, \ S_n \geqslant 0. \tag{3.22}$$

The dimension of the resonance manifold $\mathcal{N}_0$ *is equal to* $2n - 1$.

Proof. We claim that the functions (3.19) and (3.20) satisfy the constraint equations. Indeed, the constraints of Hermitian type follow from the obvious property

$$(-\rho)_\pm = \rho_\mp \qquad \forall\, \rho \in \mathbb{Z}^n.$$

Those of non-commutative type follow from the relation

$$(\rho + \sigma)_\pm = \rho_\pm + \sigma_\pm - (\rho \overset{\circ}{+} \sigma),$$

which holds for any $\rho, \sigma \in \mathbb{Z}^n$. In particular, if ρ and σ commute, then $\rho \overset{\circ}{+} \sigma = 0$ and we see that

$$(\rho + \sigma)_\pm = \rho_\pm + \sigma_\pm.$$

This yields the constraints of commutative type for the functions (3.19).

Furthermore, any function A_σ (see (3.19)) can be expressed using the relations (3.17) in terms of the independent functions $A_{\rho^{(1)}}, \dots, A_{\rho^{(n-1)}}$, $S_1, \dots, S_n$ outside the points where $S_j = 0$ (see Remark 4.1 below). Therefore $\dim \mathcal{N}_0 = 2n-1$.

Proposition 3.3. *The submanifold* $\mathcal{N}_0^+ \subset \mathcal{N}_0$ *determined by the inequalities* (3.22) *is a Poisson manifold. The symmetry map* (3.19), (3.20) *from* $\mathbb{R}^{2n}$ *to* $\mathcal{N}_0^+$ *is a Poisson map, that is, it takes the canonical bracket on* $\mathbb{R}^{2n} = \mathbb{R}^n_q \times \mathbb{R}^n_p$ *to the bracket* (3.11). *Under this map, the Casimir function* C (*see* (3.14)) *corresponds to the oscillator Hamiltonian* H (*see* (3.1)).

The proof repeats that of Theorem 2.5 in [10].

§ 4. The partial complex structure on the resonance manifold

We shall show how to endow the resonance manifold $\mathcal{N}_0$ with a partial complex structure induced by the symmetry map (3.19), (3.20) and consistent with the Poisson brackets (3.11).

Each minimal vector $\varkappa \in \mathcal{M}$ and a vector $z = (z_1, \dots, z_n)$, (3.21), of complex coordinates determine a function $z^\varkappa$ on $\mathbb{R}^{2n}$. It follows from (3.19), (3.20) that

$$z^\varkappa = \frac{A_{-\varkappa}(q,p)}{S(q,p)^{\varkappa_-}}.$$

Thus, under the symmetry map (3.19), (3.20), the function $z^\varkappa$ on $\mathbb{R}^{2n}$ corresponds to the function

$$W_\varkappa \overset{\text{def}}{=} \frac{A_{-\varkappa}}{S^{\varkappa_-}} \tag{4.1}$$

on the resonance manifold $\mathcal{N}_0$. The function (4.1) is smooth everywhere except for a certain subset of the boundary $\partial\mathcal{N}_0^+$, where singularities are possible because the coordinates S_j can be zero.

We can decompose an arbitrary resonance vector σ into minimal vectors according to (2.2) and thus define

$$W_\sigma \overset{\text{def}}{=} \prod_{\varkappa \in \mathcal{M}_\sigma} (W_\varkappa)^{n^\sigma_\varkappa}, \tag{4.2}$$

where the function $W_\varkappa$ for minimal $\varkappa$ is given in (4.1). In particular, $W_0 \equiv 1$.

Lemma 4.1. *The function W_σ, (4.2), on the resonance manifold $\mathcal{N}_0$ is independent of the choice of representation (2.2) of the vector $\sigma \in \mathcal{R}$. We have*

$$W_{\rho+\sigma} = W_\rho W_\sigma$$

for any $\rho, \sigma \in \mathcal{R}$.

A proof follows from the relations (3.8) and (3.16).

We now show how to construct local complex coordinates on $\mathcal{N}_0$.

Definition 4.1. A set of linearly independent vectors $\rho^{(1)}, \dots, \rho^{(n-1)} \in \mathcal{M}$ is called a *resonance basis* if the coefficients of a decomposition of any resonance vector $\sigma \in \mathcal{R}$ into vectors $\rho^{(k)}$ are integers:

$$\sigma = \sum_{k=1}^{n-1} N^{(k)}_\sigma \rho^{(k)}, \qquad N^{(k)}_\sigma \in \mathbb{Z}. \tag{4.3}$$

Lemma 4.2. *There is a resonance basis.*

(Explicit examples of resonance bases are given in Examples 4.1, 5.2 and Remark 6.4 below.)

Remark 4.1. If σ is a minimal vector, then formula (4.3) and constraints of the form (3.17) yield the identity

$$S^{N^{(1)}_\sigma \rho^{(1)} \overset{\circ}{+} \dots \overset{\circ}{+} N^{(n-1)}_\sigma \rho^{(n-1)}} A_\sigma = \prod_{j=1}^{n-1} (A_{\rho^{(j)}})^{N^{(j)}_\sigma}.$$

With every resonance basis we associate a set of complex coordinates w_k defined by (4.1):

$$w_k \stackrel{\text{def}}{=} W_{\rho^{(k)}}, \qquad k = 1, \dots, n-1. \tag{4.4}$$

Remark 4.1 shows that the functions (4.2) are meromorphically expressed in terms of these coordinates:

$$W_\sigma = w^{N_\sigma} \qquad \Bigl(W_\sigma = \prod_{k=1}^{n-1} (w_k)^{N_\sigma^{(k)}} \Bigr). \tag{4.5}$$

When we pass from the resonance basis $\{\rho^{(j)}\}$ to another resonance basis $\{\widetilde{\rho}^{(j)}\}$, the transformation $w \to \widetilde{w}$ of coordinates (4.4) is described by a power law:

$$\widetilde{w} = w^{\mathfrak{N}} \qquad \Bigl(\widetilde{w}_j = \prod_{k=1}^{n-1} (w_k)^{\mathfrak{N}_k^j} \Bigr). \tag{4.6}$$

Here the matrix $\mathfrak{N}$ with integer entries is determined by the decomposition of the basis vectors $\widetilde{\rho}^{(j)}$ with respect to the basis vectors $\rho^{(k)}$:

$$\widetilde{\rho}^{(j)} = \sum_{k=1}^{n-1} \mathfrak{N}_k^j \rho^{(k)}. \tag{4.7}$$

We note that the inverse matrix $\mathfrak{N}^{-1}$ also has integer entries.

We now cover the whole manifold $\mathcal{N}_0$ except for the point 0 (at which $S = 0$ and $A_\sigma = 0$ for any σ) by charts and assign a resonance basis to each chart in such a way that the coordinate functions $w_1, \dots, w_{n-1}$, (4.4), corresponding to this basis have no singularities in the given chart.

Example 4.1. Consider the following three resonance bases in the case of a three-frequency resonance $f_1 : f_2 : f_3$.

The first basis consists of two (commuting) vectors of the resonance lattice:

$$(-f_2, f_1, 0), \qquad (\mu, \nu, 1), \tag{4.8}$$

where μ, ν is the solution of the Diophantine equation (2.3) satisfying condition (2.4). Only the first component in this pair of vectors is negative. Therefore, the complex coordinates (defined by (4.4) and (4.1)) corresponding to this resonance basis have no singularities for $S_1 \neq 0$.

The second basis is obtained from (4.8) by the cyclic permutation $f_1 : f_2 : f_3 \to f_2 : f_3 : f_1$ of the frequencies and the permutation $(1, 2, 3) \to (3, 1, 2)$ of the components of the resonance vectors. The complex coordinates corresponding to this basis have no singularities for $S_2 \neq 0$.

The third basis is obtained by the other cyclic permutation. Here the complex coordinates have no singularities for $S_3 \neq 0$.

These three bases determine local charts (and complex coordinates in them) which cover the whole resonance manifold except for the point 0.

In the case of general multifrequency resonance, we obtain the following result.

Theorem 4.1. (a) *All possible resonance bases $\{\rho^{(j)}\}$ generate an atlas of charts on the resonance manifold $\mathcal{N}_0 \setminus \{0\}$ and determine a partial complex structure of* *type*[3] $(n-1,1)$. *The common real coordinate of all the charts is the Casimir function* C, (3.14), *and the local complex coordinates are given by formula* (4.4). *When we pass from one chart to another, the complex coordinates are holomorphically transformed by formula* (4.6).

(b) *The formulae* (4.1) *and* (4.2) *determine a representation* $\sigma \to W_\sigma$ *of the resonance lattice* $\mathcal{R}$ *in the space of meromorphic functions on the resonance manifold* $\mathcal{N}_0$. *Under complex conjugation, the functions* W_σ *are transformed as follows*:

$$\overline{W_\sigma} = S^\sigma W_{-\sigma}, \qquad |W_\sigma|^2 = S^\sigma. \tag{4.9}$$

(c) *The partial complex structure on* $\mathcal{N}_0 \setminus \{0\}$ *is consistent with the Poisson structure* (3.11) *in the sense of* [31]. *The meromorphic functions* (4.1) *and* (4.2) *are in involution with each other*:

$$\{W_\rho, W_\sigma\} = 0. \tag{4.10}$$

Moreover,

$$\{W_\rho, \overline{W_\sigma}\} = i\frac{\rho\cdot\sigma}{S} W_\rho \overline{W_\sigma}, \tag{4.11}$$

in the notation of (3.13).

Proof. Only the last two formulae must be proved. Formula (4.10) follows from (3.11), (3.16), and the following property of the commutator on the lattice $\mathbb{Z}^n$:

$$[-\rho, -\sigma] = -[\rho, \sigma].$$

The relations (3.11) and (3.16) and the identity

$$[-\rho, \sigma] + \rho_- \cdot \sigma + \rho \cdot \sigma_- = \rho \cdot \sigma$$

on $\mathbb{Z}^n$ similarly prove (4.11). The proof of the theorem is complete.

§ 5. Symplectic leaves and resonance Darboux coordinates

We introduce *angular coordinates* Φ_σ on $\mathcal{N}_0$ by the formula

$$A_\sigma = S^{\overset{\circ}{\sigma}} \exp\{i\Phi_\sigma\} \qquad \text{or} \qquad W_\sigma = |W_\sigma| \exp\{-i\Phi_\sigma\}. \tag{5.1}$$

Here we use the notation $\overset{\circ}{\sigma}$ introduced in (3.2) and take the constraint equations (3.6) and (3.16) into account:

$$|A_\sigma| = S^{\overset{\circ}{\sigma}}.$$

Lemma 5.1. *The angular coordinates are in involution with each other*:

$$\{\Phi_\sigma, \Phi_\rho\} = 0. \tag{5.2}$$

Moreover,

$$\{A_\sigma, \Phi_\rho\} = \frac{\rho\circ\sigma}{S} A_\sigma, \qquad \{S_j, \Phi_\rho\} = \rho_j, \tag{5.3}$$

where the operation $\rho\circ\sigma$ *is defined in* (3.2).

[3]The charts are modelled on the space $\mathbb{C}^{n-1}\times\mathbb{R}^1$ with $(n-1)$ complex coordinates and one real coordinate (see [31]).

Proof. It follows from (3.11), (3.12), and (3.3) that

$$\{e^{i\Phi_\sigma}, e^{i\Phi_\rho}\} = 0, \qquad \{A_\sigma, e^{i\Phi_\rho}\} = i\,\frac{\rho\circ\sigma}{S}\,A_\sigma e^{i\Phi_\rho}, \qquad \{S_j, e^{i\Phi_\rho}\} = i\rho_j e^{i\Phi_\rho}.$$

These relations yield (5.2) and (5.3). The proof of the lemma is complete.

Let $\{S = s(C)\}$ be a curve in $\mathbb{R}^n$ parametrized by the Casimir coordinate (3.14). We represent the coordinates S on the resonance manifold as

$$S = s(C) + \sum_{k=1}^{n-1} x^{(k)}\rho^{(k)}, \tag{5.4}$$

where $\{\rho^{(k)}\}$ is a certain resonance basis and $x^{(k)} \in \mathbb{R}$. Thus, $x^{(1)}, \dots, x^{(n-1)}$ are new coordinate functions on the resonance manifold with respect to the family of 'initial points' $s(C)$ and the basis $\{\rho^{(k)}\}$.

These functions are clearly in involution and the relations (5.3) imply that their brackets with the angular coordinates are equal to constants:

$$\{x^{(j)}, x^{(k)}\} = 0, \qquad \{x^{(k)}, \Phi_\sigma\} = N_\sigma^{(k)} \tag{5.5}$$

(the numbers $N_\sigma^{(k)}$ are given in (4.3)). In particular, introducing the special angular coordinates

$$\varphi_k \stackrel{\text{def}}{=} \Phi_{\rho^{(k)}}, \qquad k = 1, \dots, n-1, \tag{5.6}$$

corresponding to this resonance basis, we obtain the following canonical brackets from (5.2) and (5.3):

$$\{x^{(j)}, x^{(k)}\} = 0, \qquad \{\varphi_j, \varphi_k\} = 0, \qquad \{x^{(j)}, \varphi_k\} = \delta_{j,k}.$$

Thus the functions $x^{(j)}$, φ_j, $j = 1, \dots, n-1$, and the Casimir function C determine *Darboux coordinates* everywhere on the resonance manifold $\mathcal{N}_0$ except for the singularity submanifold $\partial\mathcal{N}_0^+$.

We now consider symplectic leaves in $\mathcal{N}_0$ (that is, the surfaces on which the bracket (3.11) is non-degenerate). All the leaves lie on the submanifolds $\{C=\text{const}\}$. The Poisson bracket on the leaves is induced by the bracket (3.11).

Lemma 5.2. *The symplectic leaves in the resonance manifold $\mathcal{N}_0$ have maximal dimension $2n-2$ (we recall that $\dim\mathcal{N}_0 = 2n-1$). The closure of all leaves in the domain $\mathcal{N}_0^+$ is compact, but when at least one of the frequencies f_j is greater than* 1, *it may happen that the leaf does not coincide with its closure. All degenerate leaves lie on the boundary $\partial\mathcal{N}_0^+$.*

Example 5.1. In the three-frequency case ($n = 3$), the resonance manifold has dimension $\dim\mathcal{N}_0 = 5$. If all the frequencies f_j, $j = 1, 2, 3$, are equal to 1, then there is only one zero-dimensional leaf, the point 0. The other symplectic leaves are four-dimensional and coincide with their closure. If some frequency f_j is greater than 1, then all the points of the form

$$\{A_\rho = 0,\ S_k = S_l = 0\} \in \mathcal{N}_0 \tag{5.7}$$

(where j, k, l is a permutation of $1, 2, 3$) are zero-dimensional leaves. In this case, the leaves Ω of maximal dimension 4 are obtained from their closures $\overline{\Omega}$ by removing points of the form (5.7) corresponding to frequencies other than 1.

In the general multifrequency case, all the functions defined on $\mathcal{N}_0$ (including the complex coordinates w_j and the canonical coordinates $x^{(j)}$, φ_j) can be restricted to symplectic leaves of maximal dimension.

Theorem 5.1. *The symplectic leaves $\Omega \subset \mathcal{N}_0^+ \cap \{C = \text{const}\}$ of maximal dimension in the resonance manifold $\mathcal{N}_0$ are Kähler manifolds with respect to the complex structure induced from $\mathcal{N}_0$. The symplectic (Kähler) form on the leaves can be written as*

$$\omega = dx \wedge d\varphi, \tag{5.8}$$

where x and φ are the coordinates (5.4) *and* (5.6) *determined by some resonance basis $\{\rho^{(k)}\} \subset \mathcal{R}$ and the initial point $s(C)$. The form* (5.8) *has singularities on the set $\overline{\Omega} \setminus \Omega$ (which lies in the boundary $\partial\mathcal{N}_0^+$).*

Example 5.2. In the three-frequency case, consider the resonance basis

$$\rho^{(1)} = (f_2, -f_1, 0), \qquad \rho^{(2)} = (\mu, \nu, 1), \tag{5.9}$$

where μ, ν is the solution of Diophantine equation (2.3) satisfying (2.4). In contrast to (4.8), this basis is non-commutative: $[\rho^{(1)}, \rho^{(2)}] \neq 0$. We introduce the corresponding complex coordinates by formulae (4.4) and (4.1):

$$w_1 = \frac{A_{-\rho^{(1)}}}{S_2^{f_1}}, \qquad w_2 = \frac{A_{-\rho^{(2)}}}{S_1^{|\mu|}}. \tag{5.10}$$

They have no singularities in the domain

$$S_1 \neq 0, \qquad S_2 \neq 0. \tag{5.11}$$

The vertex $v = (0, C/f_2, 0)$ of the classical simplex $\{S \in \mathbb{R}^3_+ \mid \langle f, S\rangle = C\}$ is associated with zero values $w_1 = w_2 = 0$ of the complex variables and lies on the boundary of this domain.

The Kähler form on the leaves Ω is given in the coordinates (5.10) by the formula $\omega = i\sum_{k,l=1,2}(\partial^2 F/\partial\,\overline{w}_k\partial w_l)\,d\,\overline{w}_k \wedge dw_l$ in terms of the Kählerian potential[4]

$$F = S_1 + S_2 + S_3 - \frac{C}{\omega_2}\ln S_2 + \text{const}\,. \tag{5.12}$$

Here the functions S_j are expressed in terms of $|w_1|$ and $|w_2|$ by formulae (4.9):

$$|w_1|^2 = \frac{S_1^{f_2}}{S_2^{f_1}}, \qquad |w_2|^2 = \frac{S_3 S_2^{\nu}}{S_1^{|\mu|}}, \tag{5.13}$$

with (3.14) taken into account. In particular, as $|w_1| \to 0$, we obtain

$$S_1 \sim \left(\frac{C}{f_2}\right)^{f_1/f_2} |w_1|^{2/f_2},$$

$$S_2 \sim \frac{C}{f_2} - \frac{f_1}{f_2}\left(\frac{C}{f_2}\right)^{f_1/f_2} |w_1|^{2/f_2} - \frac{f_3}{f_2}\left(\frac{C}{f_2}\right)^{f_3/f_2} |w_1|^{2|\mu|/(f_1 f_2)}|w_2|^2,$$

$$S_3 \sim \left(\frac{C}{f_2}\right)^{f_3/f_2} |w_1|^{2|\mu|/f_2}|w_2|^2.$$

[4]The general formula for the potential in the n-frequency case has the form $F = |S| - \ln S^v +$ const, where v is a vertex of the classical simplex (see (6.1)). Such formulae can be derived from the general definition of the Kähler form $\omega = \sum_{k,l}((\{w_k, \overline{w}_l\}))^{-1} dw_l \wedge d\,\overline{w}_k$ using (4.11).

Although the complex structure is also defined at the vertex v of the classical simplex (v lies in the closure of the chart (5.11)), the Kählerian potential (5.12) near this vertex has the form

$$F \sim \mathrm{const} + \left(\frac{C}{f_2}\right)^{f_1/f_2} |w_1|^{2/f_2}, \qquad |w_1| \to 0, \quad |w_2| \to 0.$$

When $f_2 \neq 1$, the symplectic form corresponding to this potential,

$$\omega \sim \left(\frac{C}{f_2}\right)^{f_1/f_2} \frac{i\, d\overline{w}_1 \wedge dw_1}{|w_1|^{2(f_2-1)/f_2}}, \qquad |w_1| \to 0, \quad |w_2| \to 0, \tag{5.14}$$

has a (weak) singularity at the point corresponding to $w = 0$ in the closure $\overline{\Omega}$ of the symplectic leaf. (Similar calculations in the two-frequency case are given in [14].)

The Liouville measure on the leaf Ω has the form

$$\frac{1}{2}|\omega \wedge \omega| = \frac{S_1 S_2 S_3}{(f_1)^2 S_1 + (f_2)^2 S_2 + (f_3)^2 S_3} \frac{|d\,\overline{w} \wedge dw|}{|w_1|^2 |w_2|^2},$$

and the Liouville volume of the whole leaf Ω is given by the formula

$$\int_\Omega \frac{|\omega \wedge \omega|}{2} = \frac{2\pi^2 C^2}{f_1 f_2 f_3}.$$

In the n-frequency case, we have

$$\int_\Omega |\underbrace{\omega \wedge \cdots \wedge \omega}_{n-1}| = \frac{(2\pi)^{n-1} C^{n-1}}{f_1 \cdots f_n}.$$

The formula for calculating the Liouville volume follows, for example, from the representation (5.8) and reduces to calculating the volume of the simplex in the x-space given by the inequalities $S_j \geqslant 0$ (see (5.4)).

§ 6. Resonance simplices

It follows from formulae (1.1) that the common spectrum of the quantum action operators $\widehat{S}_j = \hat{z}_j^* \hat{z}_j$ in the space $L^2(\mathbb{R}^n)$ coincides with the lattice $\hbar\mathbb{Z}_+^n = \{S = \hbar m \mid m \in \mathbb{Z}_+^n\}$. This spectral lattice may be regarded as a subset of the classical space $\mathbb{R}_+^n$, which is the range of the coordinate S on the resonance manifold $\mathcal{N}_0^+$.

The following subsets of $\mathbb{R}_+^n$ are called *classical simplices*:

$$\blacktriangle[c] \stackrel{\text{def}}{=} \{S \mid \langle f, S\rangle = c\}, \qquad c \geqslant 0. \tag{6.1}$$

We define *quantum resonance simplices* as the non-empty intersections of classical simplices with the spectral lattice $\hbar\mathbb{Z}_+^n$.

To each number $M \in \mathbb{Z}_+$ we assign its *Diophantine skeleton*:

$$\boldsymbol{\Delta}[M] \stackrel{\text{def}}{=} \{k \in \mathbb{Z}_+^n \mid \langle f, k\rangle = M\}. \tag{6.2}$$

Here $f = (f_1, \dots, f_n)$ is a resonance set of frequencies satisfying conditions (A) and (B).

Let $d[M]$ be the number of points in the Diophantine skeleton $\boldsymbol{\Delta}[M]$. This number is called the *multiplicity of the skeleton*.

Lemma 6.1. *All the quantum simplices are of the form* $\hbar\boldsymbol{\Delta}[M]$ *for* $d[M] \geqslant 1$, *and*

$$\hbar\boldsymbol{\Delta}[M] \subset \blacktriangle[\hbar M].$$

The spectrum of the oscillator $\widehat{H} = \langle f, \widehat{S}\rangle$, (1.3), *consists of those points* $\hbar M$ *for which* $d[M] \geqslant 1$. *The skeleton multiplicity* $d[M]$ *coincides with the multiplicity of the eigenvalue* $\hbar M$.

Thus, a quantum simplex is a subset of the spectrum of actions which corresponds to a given point of the spectrum of the resonance oscillator under the map $S \to \langle f, S\rangle$.

Lemma 6.2. *If at least one of the frequencies* f_j *is equal to* 1, *then* $d[M] \geqslant 1$ *for all* $M \geqslant 0$. *Hence quantum resonance simplices correspond to all numbers* $M \in \mathbb{Z}_+$, *and all skeletons* $\boldsymbol{\Delta}[M]$ *are non-empty.*

In the general case it is rather difficult to describe those M for which the Diophantine skeleton is non-empty: $\boldsymbol{\Delta}[M] \neq \varnothing$.

We first consider the two-frequency case ($n = 2$). We use the following consequence of Lemma 2.1.

Lemma 6.3. *Any number* $M \in \mathbb{Z}_+$ *can be uniquely represented as*

$$M = M_{12} f_1 f_2 + m_{21} f_1 + m_{12} f_2, \tag{6.3}$$

where M_{12}, m_{12}, *and* m_{21} *are integers and*

$$M_{12} \geqslant -1, \qquad 0 \leqslant m_{12} \leqslant f_1 - 1, \qquad 0 \leqslant m_{21} \leqslant f_2 - 1. \tag{6.4}$$

Knowing the number M_{12} in the representation (6.3), we can calculate the multiplicity $d[M]$ by the method used in [13].

Lemma 6.4. *The multiplicity of the eigenvalue* $\hbar M$ *of the quantum oscillator* $\widehat{H} = f_1\widehat{S}_1 + f_2\widehat{S}_2$ *is given by the formula*

$$d[M] = M_{12} + 1.$$

The Diophantine skeleton $\boldsymbol{\Delta}[M]$ *is empty if and only if* $M_{12} = -1$ *in the representation* (6.3) *of the number* M. *The point* $\hbar M$ *belongs to the spectrum of* $\widehat{H}$ *if and only if* $M_{12} \geqslant 0$ *in the representation* (6.3). *All the eigenvalues of the form* $\hbar(m_{21} f_1 + m_{12} f_2)$, *where* m_{21} *and* m_{12} *satisfy* (6.4), *and only these eigenvalues, have multiplicity* 1.

Example 6.1. Let $f_1 = 3$ and $f_2 = 2$. Then

$$d[1] = 0, \qquad d[0] = d[2] = d[3] = d[4] = d[5] = d[7] = 1, \qquad d[6] = 2.$$

In the general n-frequency case, for each pair of indices $j, k \in (1, \dots, n)$, we consider the following representation of the number M in a form similar to (6.3):

$$M = M_{jk} f_j f_k + m_{kj} f_j + m_{jk} f_k, \qquad \text{where} \quad 0 \leqslant m_{kj} \leqslant f_k - 1. \tag{6.5}$$

Theorem 6.1. *We have*

$$d[M] \geqslant M_{jk} + 1.$$

Thus, if $M_{jk} \geqslant 0$ *for at least one pair of indices* j, k, *then* $d[M] \geqslant 1$, *that is, the Diophantine skeleton* $\mathbf{\Delta}[M]$ *is non-empty and* $\hbar M$ *belongs to the spectrum of the oscillator* $\widehat{H}$, (1.3). *If, in addition,* $M_{jk} \geqslant 1$, *then* $d[M] \geqslant 2$, *that is,* $\hbar M$ *is a multiple eigenvalue.*

Proof. It follows from (6.5) that we can represent M as $M = \langle f, k\rangle$, $k \in \mathbb{Z}_+^n$, choosing

$$k_j = lf_k + m_{kj}, \qquad k_k = (M_{jk} - l)f_j + m_{jk}, \qquad k_m = 0, \quad m \neq j, k,$$

where l is any fixed integer in the interval $0 \leqslant l \leqslant M_{jk}$. There are $M_{jk} + 1$ ways to choose l. Hence the multiplicity $d[M]$ is at least $M_{jk} + 1$. The proof of the theorem is complete.

Example 6.2. If $M \geqslant (f_j - 1)(f_k - 1)$ for some pair of indices j, k, then $d[M] \geqslant 1$. This follows from Theorem 6.1 and the fact that $M_{jk} \geqslant 0$ (since if $M_{jk} = -1$, we have $M = -f_j f_k + m_{kj} f_j + m_{jk} f_k < (f_j - 1)(f_k - 1)$).

Example 6.3. If none of the f_j exceed 4 in the three-frequency case ($n = 3$), then $d[M] \geqslant 1$ for any $M \in \mathbb{Z}_+$. Indeed, since the f_j are coprime, it follows that at least one of them is equal to 1, and we can apply Lemma 6.2.

We now take $f_1 = 5$, $f_2 = 3$ and $f_3 = 2$. Then $d[1] = 0$. In this case, the number $\hbar$ does not belong to the spectrum of the oscillator $5\widehat{S}_1 + 3\widehat{S}_2 + 2\widehat{S}_3$ while all the other numbers $\hbar M$ ($M \geqslant 0$, $M \neq 1$) do belong to this spectrum.

Among all non-empty Diophantine skeletons, we distinguish those consisting of exactly one point, that is, $d[M] = 1$. This case is trivial.

In what follows we assume that $d[M] \geqslant 2$.

A point $r \in \mathbf{\Delta}[M]$ is called a *vertex* of the Diophantine skeleton if there is a resonance basis such that, for all $l \in \mathbf{\Delta}[M]$, the vector $l - r$ is co-directional with all vectors in the basis (that is, all the coefficients in the decomposition of $l - r$ in terms of basis vectors are non-negative).

Remark 6.1. We note the following fact: *to each resonance basis there corresponds at most one vertex.* Indeed, if there were two vertices r' and r'' for a given basis $\{\rho\}$, then both $r' - r''$ and $r'' - r'$ would be co-directional with the basis $\{\rho\}$. This is possible only if $r' - r'' = 0$.

Remark 6.2. We note the following fact: *the same vertex may correspond to distinct bases.* For example, take $f = (1, 1, f_3)$ and consider two resonance bases:

$$\rho^{(1)} = (1, -1, 0), \quad \rho^{(2)} = (-f_3, 0, 1) \quad \text{and} \quad \widetilde{\rho}^{(1)} = (1, -1, 0), \quad \widetilde{\rho}^{(2)} = (0, -f_3, 1).$$

The first is the non-commutative basis in Example 5.2, and the second is the commutative basis in Example 4.1. For every $M \in \mathbb{Z}_+$ the point $r = (0, M, 0)$ is a vertex of the skeleton

$$\mathbf{\Delta}[M] = \{l \in \mathbb{Z}_+^3 \mid l_1 + l_2 + f_3 l_3 = M\}$$

with respect to these two bases. Indeed, every point $l \in \mathbf{\Delta}[M]$ admits two decompositions:

$$l = r + N^{(1)}\rho^{(1)} + N^{(2)}\rho^{(2)} = r + \widetilde{N}^{(1)}\widetilde{\rho}^{(1)} + \widetilde{N}^{(2)}\widetilde{\rho}^{(2)}$$

with non-negative coefficients

$$N^{(1)} = l_1 + f_3 l_3, \quad N^{(2)} = l_3, \qquad \widetilde{N}^{(1)} = l_1, \quad \widetilde{N}^{(2)} = l_3.$$

Remark 6.3. Consider the three-frequency case. We note the following fact: *if the Diophantine skeleton* $\mathbf{\Delta}[M]$ *is non-empty, then it contains a vertex* (*to be denoted by* $r^{(2)}$) *corresponding to the basis* (5.9), *and the coordinates of this vertex are given by the following formulae*:[5]

$$r_3^{(2)} = l_3^* \stackrel{\text{def}}{=} \min_{l\in\mathbf{\Delta}[M]} l_3, \qquad r_1^{(2)} = \min_{\substack{l\in\mathbf{\Delta}[M]\\ l_3=l_3^*}} l_1, \qquad r_2^{(2)} = \max_{\substack{l\in\mathbf{\Delta}[M]\\ l_3=l_3^*}} l_2. \tag{6.6}$$

To prove this fact, it suffices to consider the decomposition of the vectors $l - r^{(2)}$ (for $l \in \mathbf{\Delta}[M]$) with respect to the basis (5.9):

$$l_1 - r_1^{(2)} = N^{(1)} f_2 + N^{(2)}\mu, \qquad l_2 - r_2^{(2)} = -N^{(1)} f_1 + N^{(2)}\nu,$$
$$l_3 - r_3^{(2)} = N^{(2)},$$

and show that the coefficients $N^{(1)}$ and $N^{(2)}$ are non-negative for all $l \in \mathbf{\Delta}[M]$. The bound $N^{(2)} \geqslant 0$ follows from the third relation and formula for $r_3^{(2)}$, and the bound $N^{(1)} \geqslant 0$ follows from the first relation if we take into account that $\mu < 0$ and $r_1^{(2)} < f_2$ in this relation. Indeed, assuming that $N^{(1)} < 0$, we have

$$l_1 = (r_1^{(2)} + N^{(1)} f_2) + N^{(2)}\mu < 0,$$

which contradicts the condition $l_1 \geqslant 0$.

Now let (j,k,s) be a cyclic permutation of $(1,2,3)$. We consider the resonance basis obtained by an appropriate cyclic permutation of the basis (5.9). Then the following point $r^{(k)}$ is a vertex of the skeleton $\mathbf{\Delta}[M]$ with respect to this basis:

$$r_s^{(k)} = l_s^* \stackrel{\text{def}}{=} \min_{l\in\mathbf{\Delta}[M]} l_s, \qquad r_j^{(k)} = \min_{\substack{l\in\mathbf{\Delta}[M]\\ l_s=l_s^*}} l_j, \qquad r_k^{(k)} = \max_{\substack{l\in\mathbf{\Delta}[M]\\ l_s=l_s^*}} l_k. \tag{6.6a}$$

Moreover, if M_{jk} is non-negative in (6.5), then the formulae (6.6a) become

$$r_s^{(k)} = 0, \qquad r_j^{(k)} = m_{kj}, \qquad r_k^{(k)} = M_{jk} f_j + m_{jk}. \tag{6.6b}$$

Remark 6.4. Formula (6.6b) can easily be generalized to the n-frequency case. We assume that M_{12} is non-negative in the representation (6.3). Then the point

$$r^{(2)} = \Big(m_{21}, M_{12} f_1 + m_{12}, \underbrace{0,\dots,0}_{n-2}\Big) \tag{6.7}$$

is a vertex of the Diophantine skeleton $\mathbf{\Delta}[M]$. The corresponding resonance basis may be chosen to have the form

$$\begin{aligned}
\rho^{(1)} &= (f_2, -f_1, 0, 0, \dots, 0, 0),\\
\rho^{(2)} &= (\mu_3, \nu_3, 1, 0, \dots, 0, 0),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\rho^{(n-1)} &= (\mu_n, \nu_n, 0, 0, \dots, 0, 1).
\end{aligned}$$

[5] The equation $f_1 l_1 + f_2 l_2 = M - f_3 l_3^*$ implies that the minimum of l_1 and the maximum of l_2 over the set $\{l \in \mathbf{\Delta}[M] \mid l_3 = l_3^*\}$ are attained at the same vector l.

This is an n-dimensional analogue of the basis (5.9). In these formulae, μ_l and ν_l give a solution of the Diophantine equation

$$\mu_l f_1 + \nu_l f_2 + f_l = 0$$

satisfying the condition $0 \leqslant \nu_l \leqslant f_1 - 1$ (here $l = 3, \dots, n$).

By taking different pairs of indices j and k for which $M_{jk} \geqslant 0$ in the representation (6.5), we can similarly obtain other vertices $r^{(k)}$ of the skeleton $\mathbf{\Delta}[M]$.

We note that to each vertex $r \in \mathbf{\Delta}[M]$ there corresponds a point $\hbar r$ in the quantum simplex. It is called a *vertex of the quantum simplex*.

Lemma 6.5. *When $\hbar \to 0$ and $M \sim \hbar^{-1}$, the vertices of the quantum simplex $\hbar\mathbf{\Delta}[M]$ are close to those of the classical simplex $\blacktriangle[\hbar M]$.*

Proof. For large M, all the M_{jk} in the representation (6.5) are also large (and necessarily positive). The vertex of the quantum simplex corresponding to each pair of indices j, k via a formula of the form (6.7), where $j = 1$ and $k = 2$, differs from the vertex $(0, \frac{\hbar M}{f_2}, 0, \dots, 0)$ of the classical simplex by the vector

$$\biggl(\hbar m_{21}, -\frac{\hbar m_{21} f_1}{f_2}, 0, \dots, 0\biggr),$$

which tends to zero as $\hbar \to 0$. Hence Lemma 6.5 holds for vertices of the form (6.7). The other vertices of the quantum simplex $\hbar\mathbf{\Delta}[M]$ with $M \sim \hbar^{-1} \to \infty$ lie closer to the vertices of the classical simplex $\blacktriangle[\hbar M]$ than those of the form (6.7). Thus the lemma also holds for them.

§ 7. Quantum leaves and spaces of holomorphic sections

A *frame* of a Diophantine skeleton $\mathbf{\Delta}[M]$ is a pair $R = (r, \{\rho\})$ consisting of a vertex r and a resonance basis $\{\rho\}$ corresponding to it.

To each frame we assign the model space $\mathbb{C}^{n-1}$ of a local chart. Consider two charts indexed by frames $R = (r, \{\rho\})$ and $\widetilde{R} = (\widetilde{r}, \{\widetilde{\rho}\})$. Their complex coordinates w and $\widetilde{w}$ are determined from the resonance bases $\{\rho\}$ and $\{\widetilde{\rho}\}$ by formula (4.4). The charts are glued together by formula (4.6), where $\mathfrak{N}$ is the transition matrix (4.7) (with integer entries) from the basis $\{\rho\}$ to the basis $\{\widetilde{\rho}\}$. As a result, we obtain a complex manifold. It is called a *quantum leaf* and is denoted by $\Omega_\hbar[M]$.

We recall that $\Omega = \Omega[c]$ stands for the symplectic leaves of maximal dimension lying in $\mathcal{N}_0^+ \subset \mathcal{N}_0$ on the level surfaces of the Casimir function $\{C = c\}$, (3.14). The construction of the complex structure on $\mathcal{N}_0$ (see formulae (4.1) and (4.9)) leads to the following assertion.

Proposition 7.1. *The classical symplectic leaf $\Omega = \Omega[\hbar M]$ is densely embedded in the quantum leaf $\Omega_\hbar = \Omega_\hbar[M]$, that is,*

$$\Omega \subset \overline{\Omega} \approx \Omega_\hbar.$$

This embedding preserves the complex structure.

Corollary 7.1. *The quantum leaves $\Omega_\hbar$ are compact.*

We note that the classical Kähler form ω cannot be extended from classical leaves to quantum leaves if there are frequencies other than 1 (for example, this follows from (5.14)). We show how to introduce a quantum Kähler form on quantum leaves using the group structure of the resonance lattice and following the general ideology developed in [31], [32]. But first we introduce the space of holomorphic sections over a quantum leaf.

We begin by constructing a sheaf of germs of holomorphic functions over a quantum leaf and define the Hilbert space of sections of that sheaf.

We define a family of functions U_R^t in the chart with index R on the quantum leaf $\Omega_\hbar = \Omega_\hbar[M]$ by the formula

$$U_R^t \stackrel{\text{def}}{=} \sqrt{\frac{\hbar^{|r|}\, r!}{\hbar^{|t|}\, t!}}\, W_{t-r}, \qquad t \in \mathbf{\Delta}[M]. \tag{7.1}$$

Here r stands for a vertex of the frame R, and the meromorphic functions W_σ are defined by formula (4.5) in terms of the complex coordinates in the chart with index R. These functions are monomials in these coordinates because r is a vertex of the skeleton $\mathbf{\Delta}[M]$.

If the point t in (7.1) coincides with a vertex of the skeleton $\mathbf{\Delta}[M]$, then we also use another notation:

$$V_R^T \stackrel{\text{def}}{=} U_R^t, \tag{7.2}$$

where T is any frame with vertex t. Thus, the function V_R^T is the same for distinct frames T with the same vertex t. Clearly, $V_R^R \equiv 1$.

The group property of the map $\rho \to W_\rho$ (Lemma 4.1) implies the following relation on the intersection of the charts with indices R and Q for any $t \in \mathbf{\Delta}[M]$:

$$U_Q^t = V_Q^R U_R^t. \tag{7.3}$$

In particular, for any frames R, L, and Q we have the following identity on the intersection of the corresponding three charts:

$$V_Q^L = V_Q^R V_R^L. \tag{7.4}$$

The property (7.4) means that the set $\{V_Q^R\}$ of holomorphic functions on the intersections of charts of the quantum leaf $\Omega_\hbar$ can be treated as *matching functions for the sheaf* $\Pi(\Omega_\hbar)$ *of germs of holomorphic functions on* $\Omega_\hbar$.

Moreover, formula (7.3) shows that the set of functions $U^t = \{U_R^t\}$ determines a section of the sheaf $\Pi(\Omega_\hbar)$ for any fixed $t \in \mathbf{\Delta}[M]$. Thus we obtain the following result.

Lemma 7.1. *The points t of the Diophantine skeleton $\mathbf{\Delta}[M]$ are associated with sections U^t of the sheaf $\Pi(\Omega_\hbar)$ of germs of holomorphic functions on the quantum leaf $\Omega_\hbar$. These sections are linearly independent.*

We now introduce a Hilbert structure in the space of sections of the sheaf $\Pi(\Omega_\hbar)$ in such a way that the sections $\{U^t \mid t \in \mathbf{\Delta}[M]\}$ be orthonormal:

$$(U^k, U^t) = \delta_{k,t}. \tag{7.5}$$

Thus we have constructed a certain canonical Hilbert space $\mathcal{L}(\Omega_\hbar)$ of holomorphic sections over the quantum leaf $\Omega_\hbar$. This space is induced by the group structure of the resonance lattice according to (7.1) and Lemma 4.1.

We shall say that the numerical coefficients in (7.1) are of *Fock* form, which imitates the form of the coefficients in an ordinary normalized Fock basis in the space of holomorphic functions on Euclidean space [36].

Let $\mathcal{K}$ be the *reproducing kernel* of the Hilbert space $\mathcal{L}(\Omega_\hbar)$. This is the section of the sheaf $(\Pi^* \times \Pi)(\Omega_\hbar)$ given by the following formula in the chart with index R:

$$\mathcal{K}_R \stackrel{\text{def}}{=} \sum_t |\psi^t_R|^2, \tag{7.6}$$

where the sum is taken with respect to an arbitrary orthonormal basis of sections $\{\psi^t\}$ of the sheaf $\Pi(\Omega_\hbar)$. It is well known [37] that the section (7.6) is independent of the choice of basis. Taking the set of sections $\{U^t\}$ for a basis, we thus obtain the following formula for the reproducing kernel in the chart with index R:

$$\mathcal{K}_R = \sum_{t\in\mathbf{\Delta}[M]} |U^t_R|^2. \tag{7.7}$$

It remains to substitute the explicit expressions (7.1) and (4.9) into this formula.

Theorem 7.1. *Suppose that* $M \in \mathbb{Z}_+$ *and* $d[M] \geqslant 1$, *so that the Diophantine skeleton* $\mathbf{\Delta}[M]$ *is non-empty. Define a polynomial* $\mathcal{P}^{[M]}$ *on* $\mathbb{R}^{n-1}$ *by the formula*

$$\mathcal{P}^{[M]}(s) \stackrel{\text{def}}{=} \sum_{\substack{\langle t,f\rangle=M\\ t\in\mathbb{Z}^n_+}} \frac{s^t}{\hbar^{|t|}\, t!}. \tag{7.8}$$

The Hilbert space $\mathcal{L}(\Omega_\hbar)$ *of holomorphic sections over the quantum leaf* $\Omega_\hbar = \Omega_\hbar[M]$ *has dimension* $d[M]$. *The reproducing kernel of this space in the chart with index* R *is calculated in terms of the polynomial* (7.8) *as follows*:

$$\mathcal{K}_R = \frac{r!\,\hbar^{|r|}}{S^r}\mathcal{P}^{[M]}(S). \tag{7.9}$$

Here r *is the vertex of the frame* R, *and the coordinates* S *are transferred to* $\Omega_\hbar$ *from the closure of the symplectic leaf* $\Omega[\hbar M]$ *by formulae* (4.9) *and* (3.14):

$$S^\rho = |W_\rho|^2, \qquad \langle f, S\rangle = \hbar M,$$

where $\{\rho\}$ *is the basis corresponding to the frame* R.

The section (7.9) *is expressed in the local complex coordinates* w *of the chart with index* R *as a polynomial*:

$$\mathcal{K}_R = \sum_{\sigma\in\mathcal{R}_r} \frac{r!}{\hbar^{|\sigma_+|-|\sigma_-|}\,(r+\sigma)!}\overline{w}^{N_\sigma} w^{N_\sigma}. \tag{7.10}$$

Here the subset $\mathcal{R}_r \subset \mathcal{R}$ *is given by the condition*

$$\sigma \in \mathcal{R}_r \iff r+\sigma \in \mathbf{\Delta}[M], \tag{7.11}$$

and the vectors $N_\sigma \in \mathbb{Z}^{n-1}_+$ *are determined by the decomposition* (4.3) *of the vector* σ *with respect to the resonance basis of the frame* R *with vertex* r.

§ 8. The quantum reproducing measure and quantum Kählerian structure

Following [31], [32], we can now try to write the inner product (7.5) in the Hilbert space $\mathcal{L}(\Omega_\hbar)$ as an integral over a measure:

$$(\varphi,\psi)=\frac{1}{(2\pi\hbar)^{n-1}}\int_{\Omega_\hbar}\rho_{\psi|\varphi}\,dm_\hbar. \tag{8.1}$$

Here φ and ψ are arbitrary sections in $\mathcal{L}(\Omega_\hbar)$ and their mutual *density function* $\rho_{\psi|\varphi}$ on $\Omega_\hbar$ is given by the formula

$$\rho_{\psi|\varphi}\overset{\text{def}}{=}\frac{\overline{\psi}\varphi}{\mathcal{K}}, \tag{8.2}$$

while the *reproducing measure* $dm_\hbar$ is determined in the local coordinates w in the chart with index R by the relation

$$dm_\hbar=\mathcal{K}_R\mathcal{J}_R\,d\overline{w}\,dw. \tag{8.3}$$

This representation of the measure uses the reproducing kernel $\mathcal{K}$, (7.9), and a section $\mathcal{J}$ of the sheaf $(\Pi^{-1*}\times\Pi^{-1})(\Omega_\hbar)$.

Theorem 8.1. *The inner product in the space $\mathcal{L}(\Omega_\hbar)$ of holomorphic sections over the quantum leaf $\Omega_\hbar$ is calculated from formulae* (8.1)–(8.3) *when the section $\mathcal{J}$ in* (8.3) *has the following form in the chart with index R*:

$$\mathcal{J}_R=\frac{S^{r-\Sigma\rho}}{r!\,\hbar^{|r|}}Q^{[M]}(S). \tag{8.4}$$

Here r is the vertex of the frame R, $\{\rho\}$ is the corresponding resonance basis, $\sum\rho=\sum_{j=1}^{n-1}\rho^{(j)}$, $M=\langle f,r\rangle$, and the function $Q^{[M]}$ is given by the formula

$$Q^{[M]}(s)\overset{\text{def}}{=}\frac{s_1\cdots s_n}{\hbar}\int_0^\infty y^{M+|f|-1}\exp\biggl\{-\frac{1}{\hbar}\sum_{j=1}^{n-1}s_jy^{f_j}\biggr\}\,dy. \tag{8.5}$$

Proof. Proving this theorem is equivalent to verifying the 'reproducing property' for the polynomial $\mathcal{K}_R$, (7.6)–(7.10):

$$\frac{1}{(2\pi\hbar)^{n-1}}\int\mathcal{K}_R(w'\overline{w})\mathcal{K}_R(w\overline{w}'')\mathcal{J}_R\,dw\,d\overline{w}=\mathcal{K}_R(w'\overline{w}''),$$

where the arguments of the polynomial $\mathcal{K}_R$ are defined via componentwise multiplication of the points of $\mathbb{C}^{n-1}$. We note that if the coordinates w are related to the coordinates z by the symmetry map

$$w_j=z^{\rho^{(j)}},\qquad j=1,\dots,n-1,$$

then by the residue theorem, the polynomial $\mathcal{K}_R$ can be written as an integral:

$$\mathcal{K}_R(w'\overline{w}'')=\frac{\hbar^{|r|}\,r!}{2\pi(z'\overline{z}'')^r}\int_0^{2\pi}\exp\biggl\{i\langle f,r\rangle t+\frac{1}{\hbar}\sum_{j=1}^{n}z'_j\overline{z}''_je^{-if_jt}\biggr\}\,dt.$$

We now consider the well-known Fock formula

$$\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{C}^n}\exp\Big\{\frac{\langle z',\overline{z}\rangle+\langle\overline{z}'',z\rangle-\langle\overline{z},z\rangle}{\hbar}\Big\}\,dz\,d\overline{z}=\exp\Big\{\frac{\langle z',\overline{z}''\rangle}{\hbar}\Big\},$$

replace z'_j by $z'_j e^{-if_jt'}$ and z''_j by $z''_j e^{-if_jt''}$ and integrate with respect to t' and t'' from 0 to 2π. Then the above integral representation for $\mathcal{K}_R$ yields the identity

$$\frac{1}{(2\pi\hbar)^{n-1}}\int_{\mathbb{C}^n}\mathcal{K}_R(w'\overline{w})\mathcal{K}_R(w\overline{w}'')\frac{|z^r|^2}{2\pi\hbar^{|r|+1}\,r!}e^{-|z|^2/\hbar}\,dz\,d\overline{z}=\mathcal{K}_R(w'\overline{w}'').$$

We make the change of variables $w_1=z^{\rho^{(1)}},\ \dots,\ w_{n-1}=z^{\rho^{(n-1)}},\ w_n=z^f$ in the integrand of this identity; its Jacobian is equal to[6]

$$\Big(\langle f,f\rangle\frac{|w_1\cdots w_n|}{|z_1\cdots z_n|}\Big)^2.$$

We then obtain the desired 'reproducing property' for $\mathcal{K}_R$, where the density $\mathcal{J}_R$ is given by the formula

$$\mathcal{J}_R(w\overline{w})=\frac{1}{2\pi\hbar^{|r|+1}\,r!\,\langle f,f\rangle^2|w_1\cdots w_{n-1}|^2}\int_{\mathbb{C}}\frac{|z^r|^2|z_1\cdots z_n|^2}{|w_n|^2}e^{-\frac{|z|^2}{\hbar}}\,dw_n\,d\overline{w}_n.$$

Here the coordinates z_j in the integrand are expressed in terms of $w_1,\dots,w_{n-1},w_n$ by the formulae

$$z_j=w_1^{p_j^{(1)}}\cdots w_n^{p_j^{(n)}},$$

where $p^{(1)},\dots,p^{(n)}$ are the vectors comprising the inverse matrix of the matrix composed of the vectors $\rho^{(1)},\dots,\ \rho^{(n-1)},\ \rho^{(n)}\equiv f$, that is,

$$\sum_{l=1}^n p_j^{(l)}\rho_k^{(l)}=\delta_{j,k},\qquad j,k=1,\dots,n.$$

Moreover, $p_j^{(n)}=f_j/\langle f,f\rangle$, and all the numbers $p_j^{(l)}$, $1\leqslant l\leqslant n-1$, are integers.

In the last integral, we pass to the coordinates x, φ by the formula

$$w_n=\sqrt{x^{\langle f,f\rangle}}e^{i\varphi}$$

and take into account that the polar angle runs through the interval $[0,2\pi\langle f,f\rangle]$. This yields another formula

$$\mathcal{J}_R=\frac{1}{\hbar^{|r|+1}\,r!}\prod_{k=1}^{n-1}|w_k|^{2(\langle r,p^{(k)}\rangle+|p^{(k)}|-1)}$$
$$\times\int_0^\infty x^{\langle f,r\rangle+|f|-1}\exp\Big\{-\frac{1}{\hbar}\sum_{j=1}^n x^{f_j}\prod_{k=1}^{n-1}|w_k|^{2p_j^{(k)}}\Big\}\,dx.$$

It remains to note that it is more convenient to use the coordinates $S_1,\dots,S_n$ (see (4.9)) instead of the $|w_j|$. The change $x=S^{f/\langle f,f\rangle}y$ in the integrand of the last integral finally yields (8.4) and (8.5).

[6]Here we have used the following identity between the determinant of the matrix composed of the resonance basis vectors and the frequency vector: $|\det(\rho^{(1)}\cdots\rho^{(n-1)}f)|=\langle f,f\rangle$.

In particular, Theorem 8.1 implies that the following equations hold by (7.6) for any orthonormal basis $\{\psi^t\}$ of the space $\mathcal{L}(\Omega_\hbar)$:

$$\dim \mathcal{L}(\Omega_\hbar) = \sum_t \|\psi^t\|^2 = \sum_t \frac{1}{(2\pi\hbar)^{n-1}} \int_{\Omega_\hbar} \frac{|\psi^t|^2}{\mathcal{K}}\, dm_\hbar = \frac{1}{(2\pi\hbar)^{n-1}} \int_{\Omega_\hbar} dm_\hbar.$$

Therefore Lemma 6.1 and Theorem 7.1 yield the following result.

Corollary 8.1. *The multiplicity of an eigenvalue $\hbar M$ of the resonance oscillator* (1.3) *is calculated from the formula*

$$d[M] = \frac{1}{(2\pi\hbar)^{n-1}} \int_{\Omega_\hbar} dm_\hbar. \tag{8.6}$$

Here the quantum reproducing measure has the form

$$dm_\hbar = (\mathcal{P}^{[M]} Q^{[M]})(S)\, \frac{d\,\overline{w}\, dw}{S^{\Sigma\rho}}, \tag{8.7}$$

where $\mathcal{P}^{[M]}$ and $Q^{[M]}$ are given by (7.8) *and* (8.5), *w stands for the complex coordinates in the local chart with index R on the quantum leaf $\Omega_\hbar[M]$, and ρ is the resonance basis in the frame R.*

We now construct a closed 2-form $\omega_\hbar$ on the quantum leaf $\Omega_\hbar$. This 2-form is generated by the reproducing kernel of the space $\mathcal{L}(\Omega_\hbar)$ and given by the following formula in the chart with index R and complex coordinates w:

$$\omega_\hbar = i\hbar \sum_{l,j=1}^{n-1} \frac{\partial^2}{\partial \overline{w}_l\, \partial w_j} (\ln \mathcal{K}_R)\, d\,\overline{w}_l \wedge dw_j. \tag{8.8}$$

This form is independent of the choice of the chart. Following [31], [32], we call it the *quantum Kähler form* on $\Omega_\hbar$.

We now find the asymptotic behaviour of the quantum Kähler form and the reproducing measure as $\hbar \to 0$.

Lemma 8.1. *Outside a neighbourhood of the points where at least one coordinate S_j is zero, the reproducing kernel $\mathcal{K}_R$,* (7.9), *and the density $\mathcal{J}_R$,* (8.4), *have the following asymptotic behaviour as $\hbar \to 0$ and $M \sim 1/\hbar$:*

$$\mathcal{K}_R = c_r \frac{\exp\{|S|/\hbar\}}{S^r} \big(\varkappa(S) + O(\hbar)\big),$$

$$\mathcal{J}_R = \hbar^{-|r|-1/2} \frac{\sqrt{2\pi}}{r!}\, S^r \exp\biggl\{-\frac{|S|}{\hbar}\biggr\} \biggl(\frac{S_1 \cdots S_n}{S^{\rho^{(1)}+\dots+\rho^{(n-1)}}} \varkappa(S) + O(\hbar)\biggr).$$

Here ρ and r are the resonance basis and the vertex of the frame R, and the function $\varkappa$ and the constant c_r are given by the formulae

$$\varkappa(S) = (f_1^2 S_1 + \dots + f_n^2 S_n)^{-1/2},$$

$$c_r = \hbar^{|r|+1/2}\, r!\, e^{-|r|} \sqrt{2\pi} (f_1^2 r_1 + \dots + f_n^2 r_n)^{1/2} \sum_{\substack{\langle t-r, f\rangle = 0\\ t \in \mathbb{Z}_+^n}} \frac{r^t}{t!}.$$

Proof. This follows from the system of differential equations which are satisfied by the functions $\mathcal{K}_R$ and $\mathcal{J}_R$. These equations are given in Remark 13.1. The asymptotic formula for solutions of these equations as $\hbar \to 0$ has the WKB-form

$$\mathcal{K}_R = \exp\left\{\frac{F}{\hbar}\right\}(\varphi + O(\hbar)), \qquad \mathcal{J}_R = \exp\left\{-\frac{F}{\hbar}\right\}(\psi + O(\hbar)),$$

where F is the Kählerian potential of the classical symplectic form ω (see formula (5.12)). We obtain simple equations for the amplitudes φ and ψ, and their solutions lead to the formulae given in the lemma.

We recall (Proposition 7.1) that the quantum leaf $\Omega_\hbar = \Omega_\hbar[M]$ is identified with the closure $\overline{\Omega}$ of the classical symplectic leaf $\Omega = \Omega[\hbar M]$ of the resonance manifold. The following result is obtained from Lemma 8.1 and formulae (8.7) and (8.8).

Theorem 8.2. *At those points of the quantum leaf $\Omega_\hbar$ that correspond to points of the classical leaf Ω, the quantum Kähler form* (8.8) *can be approximated by the classical symplectic form* (5.8), *and the reproducing measure* (8.3) *is approximated by the classical Liouville measure $dm = \frac{1}{(n-1)!}|\underbrace{\omega \wedge \dots \wedge \omega}_{n-1}|$ as $\hbar \to 0$, $M \sim 1/\hbar$:*

$$\omega_\hbar = \omega + O(\hbar), \qquad dm_\hbar = dm + O(\hbar). \tag{8.9}$$

The asymptotic formulae (8.9) *do not hold at those points of the quantum leaf $\Omega_\hbar$ that correspond to the boundary points $\overline{\Omega} \setminus \Omega$ of the classical leaf. The classical symplectic form ω and the measure dm are singular at these points.*

Remark 8.1. It follows from (8.9) that the quantum Kähler form $\omega_\hbar$ with sufficiently small $\hbar$ and $M \sim 1/\hbar$ is non-degenerate at the points of the classical symplectic leaf, but can degenerate on its boundary while still remaining smooth. The form $\omega_\hbar$ is non-degenerate at the boundary point corresponding to a vertex $r \in \Delta[M]$ if and only if all the points obtained from r by means of the basis vectors belong to the quantum simplex:

$$(r + \rho^{(j)}) \in \Delta[M], \qquad j = 1, \dots, n-1. \tag{8.10}$$

This follows from the explicit formula for $\omega_\hbar$ in the complex coordinates w on the quantum leaf, where the vertex r corresponds to $w = 0$:

$$\omega_\hbar|_{w=0} = i\hbar \sum_{(r+\rho^{(j)})\in\Delta[M]} c^r_{\rho^{(j)}}\, d\overline{w}_j \wedge dw_j, \qquad c^r_{\rho^{(j)}} = \frac{r!}{\hbar^{|\rho^{(j)}_+|-|\rho^{(j)}_-|}\,(r+\rho^{(j)})!}.$$

An example where the quantum form is degenerate is given by the resonance $1 : 2 : 3$ for $M = 2$ at the vertex $r = (2, 0, 0)$ with basis (4.8).

We note that a result similar to Theorem 8.2 holds for several other quantum systems whose algebra of symmetries is not a Lie algebra [13], [14], [31], [32]: the quantum geometric objects $\omega_\hbar$ and $dm_\hbar$ are globally defined and smooth on $\Omega_\hbar$, in contrast to the classical geometric objects.

Remark 8.2. The asymptotic expansions (8.9) can be extended to terms of arbitrary order in $\hbar$:

$$\omega_\hbar \simeq \omega + \sum_{k=1}^{\infty} \hbar^k \lambda_k, \qquad dm_\hbar = dm\left(1 + \sum_{k=1}^{\infty} \hbar^k d_k\right). \tag{8.11}$$

Here the coefficients λ_k are closed 2-forms and the coefficients d_k are smooth functions on Ω. Each d_k is explicitly expressible [32] in terms of the forms $\omega, \lambda_1, \dots, \lambda_k$.

Substituting (8.11) into (8.6), we obtain the following formula for the multiplicity $d[M]$, which contains only the first $n-1$ of the functions d_k, and the parameter $\hbar$ can be set equal to 1 (see [32]):

$$d[M] = \frac{1}{(2\pi)^{n-1}} \int_{\Omega[M]} \biggl(1 + \sum_{k=1}^{n-1} d_k\biggr) \frac{1}{(n-1)!} |\omega \wedge \dots \wedge \omega|.$$

§ 9. Cohomology of quantum leaves

Since the quantum leaf $\Omega_\hbar = \Omega_\hbar[M]$ is compact, all the cohomology groups $H^{2k}(\Omega_\hbar)$ with $k = 1, \dots, n-1$ are non-trivial. Moreover, the cohomology class of the quantum Kähler form $\frac{1}{2\pi\hbar}\omega_\hbar$, defined in (8.8) must be integer-valued [38].

For the sake of simplicity, we consider the case $n = 3$.

Theorem 9.1. *The non-contractible* 2*-cycles in the quantum leaf* $\Omega_\hbar[M]$ *correspond to the edges of the classical resonance simplex* $\blacktriangle[\hbar M]$. *Namely, if* (j, k, s) *is a cyclic permutation of* $(1, 2, 3)$, *then the edge connecting the* j*th and* k*th vertices is associated with the* (*positively oriented*) *cycle* $\Sigma_{jk} = \{f_j S_j + f_k S_k = \hbar M\}$. *The following formulae hold*:

$$\frac{1}{2\pi\hbar} \int_{\Sigma_{jk}} \omega_\hbar = \biggl[\frac{r_k^{(k)}}{f_j}\biggr]. \tag{9.1}$$

Here $r^{(k)}$ *is the vertex* (6.6a) *of the Diophantine skeleton* $\boldsymbol{\Delta}[M]$, *and the square brackets stand for the integer part of a number. If* $M_{jk} \geqslant 0$ *in the representation* (6.5), *then formula* (9.1) *becomes*

$$\frac{1}{2\pi\hbar} \int_{\Sigma_{jk}} \omega_\hbar = M_{jk}. \tag{9.1a}$$

As $\hbar \to 0$, $M \sim 1/\hbar$, *the integer* (9.1) *can be approximated by*

$$\frac{1}{2\pi\hbar} \int_{\Sigma_{jk}} \omega = \frac{M}{f_j f_k}, \tag{9.2}$$

where ω *is the classical symplectic form on the leaf.*

Proof. To be definite, we set $j = 1$ and $k = 2$. We introduce the real coordinates X_j, Φ_j (see (5.1)) instead of the complex coordinates w_j, (5.10), corresponding to the resonance basis $\rho^{(1)}$, $\rho^{(2)}$, (5.9):

$$w_j = \sqrt{X_j} \exp\{-i\Phi_j\}, \qquad 0 < X_j < \infty, \quad 0 \leqslant \Phi_j < 2\pi.$$

We have $w_2 = 0$ on the edge $\{S_3 = 0\}$ of the classical simplex $\blacktriangle[\hbar M]$ connecting the first and second vertices (see (5.13)). Hence

$$\omega_\hbar\big|_{\Sigma_{12}} = \hbar \frac{\partial}{\partial X_1} \biggl(X_1 \frac{\partial}{\partial X_1} \ln \mathcal{K}_R(X)\biggr)\bigg|_{X_2 \to 0} dX_1 \wedge d\Phi_1.$$

If a point moves along the edge $\{S_3 = 0\}$ from the first vertex (where $S_1 = \hbar M/f_1$ and $S_2 = 0$) to the second vertex (where $S_2 = \hbar M/f_2$ and $S_1 = 0$), then X_1 varies from ∞ to 0 (see (5.13)). Therefore,

$$\frac{1}{2\pi\hbar}\int_{\Sigma_{12}} \omega_\hbar = \left(X_1 \frac{\partial}{\partial X_1} \ln \mathcal{K}_R(X)\right)\Big|_{\substack{X_1\to\infty\\ X_2\to 0}}.$$

We recall that the polynomial $\mathcal{K}_R$ is given by formulae (7.10) and (7.11):

$$\mathcal{K}_R(X) = \sum_{\sigma\in\mathcal{R}_r} c_\sigma^r X_1^{N_\sigma^{(1)}} X_2^{N_\sigma^{(2)}}, \qquad c_\sigma^r \stackrel{\text{def}}{=} \frac{r!}{\hbar^{|\sigma_+|-|\sigma_-|}\,(r+\sigma)!},$$

where the numbers $N_\sigma^{(j)} \in \mathbb{Z}_+$ are determined by the decomposition (4.3):

$$\sigma = N_\sigma^{(1)}\rho^{(1)} + N_\sigma^{(2)}\rho^{(2)}.$$

When $X_2 = 0$, the sum defining the polynomial $\mathcal{K}_R$ contains only the terms with $N_\sigma^{(2)} = 0$. It follows that

$$\left(X_1 \frac{\partial}{\partial X_1} \ln \mathcal{K}_R(X)\right)\Big|_{\substack{X_1\to\infty\\ X_2\to 0}} = \max_{N_\sigma^{(1)}\rho^{(1)}\in\mathcal{R}_r} N_\sigma^{(1)}.$$

The inclusion $N_\sigma^{(1)}\rho^{(1)} \in \mathcal{R}_r$ means that the vector $r + N_\sigma^{(1)}\rho^{(1)}$ is contained in the quantum simplex $\mathbf{\Delta}[M]$, where $M = \langle f, r\rangle$. This is equivalent to the inequalities

$$r_l + N_\sigma^{(1)}\rho_l^{(1)} \geqslant 0, \qquad l = 1, 2, 3.$$

Since we are using the basis (5.9) that corresponds to the vertex $r \equiv r^{(2)}$, (6.6), we obtain the system of inequalities

$$r_1^{(2)} + N_\sigma^{(1)} f_2 \geqslant 0, \qquad r_2^{(2)} - N_\sigma^{(1)} f_1 \geqslant 0, \qquad r_3^{(2)} \geqslant 0.$$

The second inequality yields that the maximal value of $N_\sigma^{(1)}$ for which this system of inequalities holds is equal to $[\frac{r_2^{(2)}}{f_1}]$. Thus we have

$$\max_{N_\sigma^{(1)}\rho^{(1)}\in\mathcal{R}_r} N_\sigma^{(1)} = \left[\frac{r_2^{(2)}}{f_1}\right].$$

This proves formula (9.1).

If $M_{12} \geqslant 0$ in the representation (6.5), then the formulae (6.6b) determining the vertex yield that

$$r_2^{(2)} = M_{12} f_1 + m_{12}.$$

Here $0 \leqslant m_{12} \leqslant f_1 - 1$. Therefore, in this case, $[\frac{r_2^{(2)}}{f_1}] = M_{12}$, and we obtain formula (9.1a).

Furthermore, Theorem 8.2 yields that the integral (9.1) is approximated by $\frac{1}{2\pi\hbar}\int_{\Sigma_{jk}} \omega$ as $\hbar \to 0$, where ω is the classical symplectic form on the leaf. It is now convenient to write it in the form (5.8) in the Darboux coordinates x_1, x_2, (5.4), φ_1, φ_2, (5.6). We choose a family $s(\hbar M) = \hbar r^{(2)}$ of initial points in (5.4). Then

$$S_1 = \hbar r_1^{(2)} + f_2 x_1 + \mu x_2, \qquad S_2 = \hbar r_2^{(2)} - f_1 x_1 + \nu x_2, \qquad S_3 = \hbar r_3^{(2)} + x_2,$$

so that the coordinate x_2 remains constant ($x_2 = -\hbar r_3^{(2)}$) along the edge $\{S_3 = 0\}$ connecting the first and second vertices. Hence

$$\omega\big|_{\Sigma_{12}} = dx_1 \wedge d\varphi_1.$$

Here φ_1 varies from 0 to 2π and the coordinate x_1 varies from $\hbar(r_2^{(2)} - r_3^{(2)}\nu)/f_1$ to $\hbar(\mu r_3^{(2)} - r_1^{(2)})/f_2$ as we move along the edge from the first vertex to the second. Therefore we have formula (9.2):

$$\frac{1}{2\pi\hbar}\int_{\Sigma_{12}} \omega = \frac{1}{f_1 f_2}\big(f_1 r_1^{(2)} + f_2 r_2^{(2)} - (f_1\mu + f_2\nu) r_3^{(2)}\big) = \frac{M}{f_1 f_2}.$$

(Here we have used the Diophantine equation (2.3) and the formula $\langle f, r^{(2)}\rangle = M$.)

The integers M_{jk}, (9.1a), are called the *principal quantum numbers* of the leaf $\Omega_\hbar[M]$. We note that these numbers are not independent: they are all uniquely determined by the number M (see also Corollary 9.2).

The coefficients m_{jk} in the representation (6.5) are called the *adjoint numbers* of the leaf $\Omega_\hbar[M]$. The following assertion is obtained from formulae (9.2) and (6.5).

Corollary 9.1. *The cohomology class of the classical symplectic form on the leaf is given by the formula*

$$\frac{1}{2\pi\hbar}\int_{\Sigma_{jk}} \omega = M_{jk} + \frac{m_{jk}}{f_j} + \frac{m_{kj}}{f_k}, \tag{9.3}$$

where M_{jk} are the principal numbers and m_{jk} are the adjoint numbers of the leaf, and

$$0 \leqslant \frac{m_{jk}}{f_j} + \frac{m_{kj}}{f_k} \leqslant 2 - \left(\frac{1}{f_j} + \frac{1}{f_k}\right) < 2.$$

Remark 9.1. It follows from (9.3) that if at least one of the frequencies f_j is not equal to 1, then the cohomology class of the classical symplectic form $\omega/(2\pi\hbar)$ is not integer-valued on leaves with non-zero adjoint numbers. This explains the inconvenience of using the form ω in quantum geometric constructions.

We now calculate the class of the quantum 4-form $\omega_\hbar \wedge \omega_\hbar$. Since the leaves $\Omega[\hbar M] \subset \Omega_\hbar[M]$ are four-dimensional when $n = 3$, this 4-form determines their quantum volume.

Theorem 9.2. *In the three-frequency case* ($n = 3$), *the quantum volume is calculated as follows*:

$$\frac{1}{(2\pi\hbar)^2}\int_{\Omega} \frac{\omega_\hbar \wedge \omega_\hbar}{2} = \frac{1}{2}\underset{\substack{(j,k,s)\\=(1,2,3)}}{\mathfrak{S}} \big(\widetilde{r}_j - r_j^{(s)}\big)\left[\frac{r_s^{(s)}}{f_k}\right], \tag{9.4}$$

where the vertices $r^{(1)}$, $r^{(2)}$, and $r^{(3)}$ are defined as in Remark 6.3, *$\widetilde{r}$ is one of these vertices, and the sum $\mathfrak{S}$ is taken over all cyclic permutations of* $(1,2,3)$. *For example, when $\widetilde{r} = r^{(2)}$, the right-hand side of* (9.4) *has the form*

$$\frac{1}{2}\left((r_1^{(2)} - r_1^{(3)})\left[\frac{r_3^{(3)}}{f_2}\right] + (r_2^{(2)} - r_2^{(1)})\left[\frac{r_1^{(1)}}{f_3}\right]\right).$$

Here and in (9.4), *the square brackets stand for the integer part of a number. If* $M_{12} \geqslant 0$, $M_{23} \geqslant 0$, *and* $M_{31} \geqslant 0$ *in the representation* (6.5), *then* (9.4) *becomes*

$$\frac{1}{(2\pi\hbar)^2}\int_\Omega \frac{\omega_\hbar \wedge \omega_\hbar}{2} = \frac{1}{2}\big(M_{ks}(M_{sj}f_s + m_{sj}) + M_{jk}m_{js}\big). \tag{9.4a}$$

Here (j,k,s) *is an arbitrary cyclic permutation of* $(1,2,3)$.

Theorem 9.2 will be proved in the Appendix.

We note that the quantum vortex volume (9.4) can be approximated as $\hbar \to 0$, $M \sim 1/\hbar$, by the classical volume

$$\frac{1}{(2\pi\hbar)^2}\int_\Omega \frac{\omega \wedge \omega}{2} = \frac{M^2}{2f_1f_2f_3}. \tag{9.5}$$

In this case, the numbers (9.4a) and (9.5) approximate the dimension $d[M]$, (8.6), but do not coincide with it.

Moreover, Theorem 9.2 yields interesting identities for the principal and adjoint quantum numbers.

Corollary 9.2. *Suppose that* $n = 3$ *and all the numbers* M_{12}, M_{23}, M_{31} *in the representation* (6.5) *are non-negative. Then the following identities hold*:

$$\begin{aligned} M_{31}M_{12}f_1 + M_{31}m_{12} + M_{23}m_{21} &= M_{12}M_{23}f_2 + M_{12}m_{23} + M_{31}m_{32} \\ &= M_{23}M_{31}f_3 + M_{23}m_{31} + M_{12}m_{13}. \end{aligned}$$

The proof follows from formula (9.4a): each of the three numbers coincides with twice the quantum volume.

§ 10. The quantum resonance algebra

For any $a \in \mathbb{R}$ and $m \in \mathbb{Z}$, we define

$$(a)_m \overset{\text{def}}{=} \begin{cases} (a+\hbar)\cdots(a+m\hbar) & \text{for} \quad m \geqslant 1, \\ 1 & \text{for} \quad m = 0, \\ a(a-\hbar)\cdots(a-\hbar(|m|-1)) & \text{for} \quad m \leqslant -1. \end{cases} \tag{10.1}$$

Given any vectors $s \in \mathbb{R}^n$ and $\rho \in \mathbb{Z}^n$, we set

$$(s)_\rho \overset{\text{def}}{=} (s_1)_{\rho_1}\cdots(s_n)_{\rho_n}, \tag{10.1a}$$

where all the factors are determined by (10.1).

The operations (10.1), (10.1a) generalize the well-known ‘Pochhammer symbols’ to the case of negative subscripts. With our modified definition of these symbols, they have the following important property.

Lemma 10.1. *The operation* $\rho \to (s)_\rho$ *has an analogue of the group property*:

$$(s)_{\rho+\sigma} = \frac{(s+\hbar\sigma)_\rho (s)_\sigma}{(s+\hbar\sigma)^2_{[\sigma|\rho]}},$$

where the vector $[\sigma|\rho]$ *is defined by* (3.2). *In particular, if* $[\rho,\sigma] = 0$ (*see* (3.2)), *then* $(s)_{\rho+\sigma} = (s+\hbar\sigma)_\rho(s)_\sigma$.

Given any pair of vectors $\rho, \sigma \in \mathbb{Z}^n$, we define the *structure polynomial* $g_{\rho,\sigma}$ on $\mathbb{R}^n$ by the formula

$$g_{\rho,\sigma}(s) \stackrel{\text{def}}{=} (s - \hbar\rho)_{[\sigma|\rho]}, \qquad s \in \mathbb{R}^n. \tag{10.2}$$

The structure polynomials $g_{\rho,\sigma}$ have many interesting properties. Here we mention only those identities that will be used later.

Lemma 10.2. *The following identities hold.*

1) *If* $[\rho, \sigma] = 0$ (*see* (3.2)), *then* $g_{\rho,\sigma}(s) \equiv 1$.

2) $g_{\rho,\sigma}(s) g_{\rho+\sigma,\varkappa}(s) = g_{\rho,\sigma+\varkappa}(s) g_{\sigma,\varkappa}(s - \hbar\rho)$.

3) $g_{\rho,\sigma}(s + \hbar\rho + \hbar\sigma) = g_{-\sigma,-\rho}(s)$.

They have the following corollaries.

4) $g_{\rho,-\rho}(s) = g_{-\rho,\rho}(s - \hbar\rho)$.

5) *If* $[\rho, \sigma] = 0$, *then* $g_{-\rho,\sigma}(s) g_{-\rho,\rho}(s - \hbar\sigma) = g_{\sigma,-\rho}(s) g_{-\rho,\rho}(s)$.

6) *If* $[\rho', \rho''] = [\sigma', \sigma'']$, *then* $g_{\rho',\sigma'}(s - \hbar\rho'') g_{\rho'+\sigma',\sigma''}(s - \hbar\rho'') g_{\rho'',\rho'+\sigma'+\sigma''}(s) = g_{\rho'+\rho'',\sigma'+\sigma''}(s)$.

We now introduce the quantum resonance algebra. The generators of this algebra are denoted by $\mathbf{A}_\sigma$ and $\mathbf{S}_j$ (analogous to the classical coordinates A_σ and S_j), where $j = 1, \dots, n$ and σ runs through the set $\mathcal{M}$ of minimal resonance vectors.

We replace the constraints (3.6), (3.8), (3.9) and the Poisson brackets (3.11) by quantum constraints and commutation relations between the generators $\mathbf{A}_\sigma$ and $\mathbf{S}$.

The *quantum constraints of Hermitian type* are defined as

$$\mathbf{S}_j^* = \mathbf{S}_j, \qquad \mathbf{A}_\sigma^* = \mathbf{A}_{-\sigma} \tag{10.3}$$

for any $j = 1, \dots, n$ and any $\sigma \in \mathcal{M}$.

The *quantum constraints of commutative type* are defined as

$$\prod_\rho (\mathbf{A}_\rho)^{k_\rho} = \prod_\sigma (\mathbf{A}_\sigma)^{m_\sigma} \tag{10.4}$$

for any families of commuting vectors $\rho, \sigma \in \mathcal{M}$ and numbers $k_\rho, m_\sigma \in \mathbb{N}$ with

$$\sum_\rho k_\rho \rho = \sum_\sigma m_\sigma \sigma.$$

The *quantum constraints of non-commutative type* are defined as follows. If two minimal vectors ρ and σ do not commute and $\rho \neq -\sigma$, then

$$\mathbf{A}_\rho \mathbf{A}_\sigma = g_{\rho,\sigma}(\mathbf{S}) \prod_{\varkappa \in \mathcal{M}_{\rho+\sigma}} (\mathbf{A}_\varkappa)^{n_\varkappa^{\rho+\sigma}}, \tag{10.5}$$

where $g_{\rho,\sigma}$ is the structure polynomial (10.2) and the $n_\varkappa^{\rho+\sigma}$ are the coefficients in the decomposition (2.2) of the vector $\rho + \sigma$ into minimal vectors in $\mathcal{M}_{\rho+\sigma}$.

The *commutation relations* are defined as

$$[\mathbf{S}_j, \mathbf{S}_k] = 0, \qquad [\mathbf{S}_j, \mathbf{A}_\rho] = \hbar\rho_j \mathbf{A}_\rho, \qquad [\mathbf{A}_{-\rho}, \mathbf{A}_\rho] = \hbar F_{-\rho,\rho}(\mathbf{S}) \tag{10.6}$$

for any $j, k = 1, \dots, n$ and $\rho \in \mathcal{M}$, where the polynomials $F_{\rho,\sigma}$ are given by the formula

$$F_{\rho,\sigma} \stackrel{\text{def}}{=} \frac{1}{\hbar}(g_{\rho,\sigma} - g_{\sigma,\rho}). \tag{10.7}$$

Remark 10.1. The set of constraints (10.5) of non-commutative type consists of the commutation relations

$$[\mathbf{A}_\rho, \mathbf{A}_\sigma] = \hbar F_{\rho,\sigma}(\mathbf{S}) \prod_{\varkappa\in\mathcal{M}_{\rho+\sigma}} (\mathbf{A}_\varkappa)^{n_\varkappa^{\rho+\sigma}}$$

and anti-commutation relations

$$[\mathbf{A}_\rho, \mathbf{A}_\sigma]_+ = \big(g_{\rho,\sigma}(\mathbf{S}) + g_{\sigma,\rho}(\mathbf{S})\big) \prod_{\varkappa\in\mathcal{M}_{\rho+\sigma}} (\mathbf{A}_\varkappa)^{n_\varkappa^{\rho+\sigma}}. \qquad (10.5\text{a})$$

The relations (10.5a) are called *actual constraints of non-commutative type*.

The set (10.4) of constraints of commutative type also contains commutation relations. These are the constraints (10.4) corresponding to the equalities $\rho+\sigma = \sigma+\rho$ for vectors $\rho,\sigma\in\mathcal{M}$ in a normal sublattice. Removing the commutation relations from the set of constraints (10.4), we obtain *actual constraints of commutative type*.

The actual constraints thus defined are not independent. The number of independent actual constraints is equal to $M-n+1$, where M is the number of minimal vectors.

Definition 10.1. The *resonance algebra* $\mathcal{A}$ is an algebra with involution having generators $\mathbf{A}_\sigma,\ \sigma\in\mathcal{M},\ \mathbf{S}_j,\ j=1,\dots,n$, and relations (10.3)–(10.6).

As in (3.10), we associate each non-minimal resonance vector σ with the following element $\mathbf{A}_\sigma$ of the resonance algebra:

$$\mathbf{A}_\sigma = \begin{cases} \mathbf{I} & \text{if} \quad \sigma = 0, \\ \prod_{\varkappa\in\mathcal{M}_\sigma} \mathbf{A}_\varkappa^{n_\varkappa^\sigma} & \text{if} \quad \sigma \neq 0. \end{cases} \qquad (10.8)$$

Here the subsets $\mathcal{M}_\sigma$ and numbers $n_\varkappa^\sigma$ are defined by (2.2). The constraints (10.4) of commutative type show that this notation is well defined, that is, independent of the decomposition of σ into minimal vectors.

We use this notation to write (10.5) in the form

$$\mathbf{A}_\rho\mathbf{A}_\sigma = g_{\rho,\sigma}(\mathbf{S})\mathbf{A}_{\rho+\sigma}$$

(here $\rho,\sigma\in\mathcal{M}$ and $\rho\neq-\sigma$).

For any minimal vectors ρ and σ, the last relation and (10.6) imply the following commutation relation:

$$[\mathbf{A}_\rho, \mathbf{A}_\sigma] = \hbar F_{\rho,\sigma}(\mathbf{S})\mathbf{A}_{\rho+\sigma}, \qquad (10.9)$$

where $F_{\rho,\sigma}$ are polynomials (10.7).

We now write down some useful permutation relations that follow from the constraints (10.4), (10.5) and the permutation relations (10.6).

Corollary 10.1. *The generators of the resonance algebra* $\mathcal{A}$ *satisfy the following permutation relations.*

(a) *If* $\rho,\sigma\in\mathcal{M}$ *and* $\rho\neq-\sigma$, *then* $g_{\sigma,\rho}(\mathbf{S})\mathbf{A}_\rho\mathbf{A}_\sigma = g_{\rho,\sigma}(\mathbf{S})\mathbf{A}_\sigma\mathbf{A}_\rho$.

(b) *If* $\rho\in\mathcal{M}$, P *is a polynomial and* $k\in\mathbb{N}$, *then* $(\mathbf{A}_\rho)^k P(\mathbf{S}) = P(\mathbf{S}-\hbar k\rho)(\mathbf{A}_\rho)^k$.

(c) *If* $\rho\in\mathcal{M}$ *and* $k\in\mathbb{N}$, *then* $[\mathbf{A}_{-\rho}, (\mathbf{A}_\rho)^k] = g_{-\rho,\rho}(\mathbf{S})(\mathbf{A}_\rho)^{k-1} - (\mathbf{A}_\rho)^{k-1} g_{\rho,-\rho}(\mathbf{S})$.

To prove the relation (c), we use the fact that the structure polynomials satisfy the identity 4) in Lemma 10.2.

For any $s \in \mathbb{R}^n$ and any $\rho, \sigma, \sigma_j \in \mathbb{Z}^n$, $j = 1, \dots, l$, $l \geqslant 2$, we have the following formulae as $\hbar \to 0$:

$$(s)_{[\rho|\sigma]} = s^{\rho \overset{\circ}{+} \sigma} + O(\hbar), \qquad g_{\rho,\sigma}(s) = s^{\rho \overset{\circ}{+} \sigma}\biggl(1 + \hbar \frac{\rho \& \sigma}{s} + O(\hbar^2)\biggr),$$

where

$$\rho \& \sigma \overset{\text{def}}{=} \frac{1}{2}(\sigma_- - \rho_+) \cdot (\sigma|\rho) + \frac{1}{2}(\rho_- - \sigma_+) \cdot (\rho|\sigma) + \frac{1}{2}(\rho \overset{\circ}{+} \sigma) - \frac{1}{2}[\rho, \sigma].$$

It follows that

$$F_{\rho,\sigma}(s) = -s^{\rho \overset{\circ}{+} \sigma} \frac{[\rho, \sigma]}{s} + O(\hbar).$$

Corollary 10.2. *As $\hbar \to 0$, the quantum resonance algebra $\mathcal{A}$ corresponds to the Poisson algebra of functions on the resonance manifold $\mathcal{N}$.*

Proof. It suffices to compare the quantum and classical commutation relations (10.6), (10.9), (3.11) with the quantum and classical constraint equations (10.3)–(10.5), (3.6), (3.8), (3.9) and use the asymptotic formulae above. For example, the asymptotic formula

$$F_{\rho,\sigma}(s) = -f_{\rho,\sigma}(s) + O(\hbar)$$

shows that the commutation relation (10.9) becomes the third relation in (3.11) as $\hbar \to 0$.

The following theorem gives two elementary properties of the resonance algebra.

Theorem 10.1. *The centre of the resonance algebra $\mathcal{A}$ contains the element*

$$\mathbf{C} = f_1 \mathbf{S}_1 + \dots + f_n \mathbf{S}_n. \tag{10.10}$$

The resonance algebra has a representation

$$\mathbf{A}_\sigma \to \widehat{A}_\sigma, \qquad \mathbf{S}_j \to \widehat{S}_j \tag{10.11}$$

in the space $L^2(\mathbb{R}^n)$ via the symmetry operators

$$\widehat{S}_j = \hat{z}_j^* \hat{z}_j, \qquad \widehat{A}_\sigma = (\hat{z}^*)^{\sigma_+} \hat{z}^{\sigma_-} \tag{10.12}$$

of the resonance oscillator. In this representation, the element (10.10) *coincides with the oscillator Hamiltonian* (1.3).

Proof. The fact that the operators (10.12) satisfy all the constraint equations and commutation relations (10.3)–(10.6) follows from the simple permutation formulae

$$\hat{z}\widehat{S}_j = (\widehat{S}_j + \hbar)\hat{z}_j, \qquad \hat{z}_j^m (\hat{z}_j^*)^m = (\widehat{S}_j)_m, \qquad (\hat{z}_j^*)^m \hat{z}_j^m = (\widehat{S}_j)_{-m}, \quad m \in \mathbb{Z}_+.$$

The proof of the theorem is complete.

§ 11. Vacuum vectors and irreducible representations

We now consider an abstract representation of the resonance algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}$. The representation operators are denoted by the same letters $\mathbf{S}_j$, $\mathbf{A}_\sigma$, $j=1,\dots,n$, $\sigma\in\mathcal{M}$, as the generators of $\mathcal{A}$. For a non-minimal resonance vector σ, let $\mathbf{A}_\sigma$ be the operator defined by formula (10.8).

We choose a number $M\in\mathbb{Z}_+$ such that $d[M]\neq 0$ (that is, the Diophantine skeleton $\mathbf{\Delta}[M]$, (6.2), is non-empty). Let $R=(r,\{\rho\})$ be a frame of the Diophantine skeleton $\mathbf{\Delta}[M]$ and write $\mathcal{R}_R^-$ for the set of vectors σ in the resonance lattice $\mathcal{R}$ for which $r+\sigma\notin\mathbf{\Delta}[M]$.

We assume that for at least one frame R there is a normalized vector $\mathfrak{p}_R$ in $\mathcal{H}$ such that

$$\begin{aligned}\mathbf{A}_\rho\mathfrak{p}_R&=0, &\qquad \rho&\in\mathcal{R}_R^-,\\ \mathbf{S}_j\mathfrak{p}_R&=\hbar r_j\mathfrak{p}_R, &\qquad j&=1,\dots,n,\end{aligned}\tag{11.1}$$

where r is the vertex of the frame R. The vector $\mathfrak{p}_R$ is called a *vacuum* vector.

Applying all possible representation operators $\mathbf{A}_\sigma$ to the vacuum vector, we construct a vector subspace $\mathcal{H}_M\subseteq\mathcal{H}$:

$$\mathcal{H}_M=\overline{\operatorname{span}}\{\mathbf{A}_\sigma\mathfrak{p}_R\mid\sigma\in\mathcal{R}\}.\tag{11.2}$$

We note that $\mathcal{H}_M$ is independent of the choice of the frame R in (11.1). This follows from Lemma 12.1, which will be proved later.

Proposition 11.1. *The following relation holds on $\mathcal{H}_M$ for any resonance vectors ρ and σ:*

$$\mathbf{A}_\rho\mathbf{A}_\sigma=g_{\rho,\sigma}(\mathbf{S})\mathbf{A}_{\rho+\sigma}.\tag{11.3}$$

Remark 11.1. If ρ and σ are minimal resonance vectors with $\rho\neq-\sigma$, then (11.3) holds on the whole space $\mathcal{H}$ because of the constraints (10.5) of non-commutative type.

To prove Proposition 11.1, we need the following three lemmas.

Lemma 11.1. *Let ρ and σ be resonance vectors.*

(a) *If $\rho\notin\mathcal{R}_R^-$, then either $\rho=0$ or $-\rho\in\mathcal{R}_R^-$.*

(b) *If $\rho\in\mathcal{R}_R^-$ and $[\rho,\sigma]=0$, then $(\rho+\sigma)\in\mathcal{R}_R^-$.*

Lemma 11.2. *Let ρ, σ and θ be resonance vectors.*

(a) *If $(\rho+\theta)\in\mathcal{R}_R^-$ and $\theta\notin\mathcal{R}_R^-$, then $g_{-\rho,\rho}(\hbar r+\hbar\theta)=0$.*

(b) *If $\sigma\notin\mathcal{R}_R^-$ and $\theta\notin\mathcal{R}_R^-$, then $g_{\theta-\sigma,\rho}(\hbar r+\hbar\theta)>0$ for any $\rho\in\mathcal{R}$.*

Lemma 11.3. *If $\rho\in\mathcal{M}$, $\varkappa\in\mathcal{R}$ and $[\rho,\varkappa]=0$, then*

$$\mathbf{A}_{-\rho}\mathbf{A}_\rho\mathbf{A}_\varkappa\mathfrak{p}_R=g_{-\rho,\rho}(\mathbf{S})\mathbf{A}_\varkappa\mathfrak{p}_R.\tag{11.4}$$

Proof. By Corollary 10.1, (b) and the properties (11.1) of the vacuum vector, it suffices to prove the following equivalent formula:

$$\mathbf{A}_{-\rho}\mathbf{A}_\rho\mathbf{A}_\varkappa\mathfrak{p}_R=g_{-\rho,\rho}(\hbar r+\hbar\varkappa)\mathbf{A}_\varkappa\mathfrak{p}_R.\tag{11.4a}$$

If $(\rho+\varkappa)\in\mathcal{R}_R^-$, then the left-hand side of (11.4a) is equal to zero because $\mathbf{A}_\rho\mathbf{A}_\varkappa\mathfrak{p}_R=\mathbf{A}_{\rho+\varkappa}\mathfrak{p}_R=0$ by the conditions (11.1). The right-hand side of (11.4a) is also equal to zero in this case. Indeed, if $\varkappa\in\mathcal{R}_R^-$, this follows from the equality $\mathbf{A}_\varkappa\mathfrak{p}_R=0$. If $\varkappa\notin\mathcal{R}_R^-$, it follows from the fact that the coefficient is zero:

$g_{-\rho,\rho}(\hbar r+\hbar\varkappa)=0$ (using Lemma 11.2 (a)). This completes the proof of (11.4a) in the case when $(\rho+\varkappa)\in\mathcal{R}_R^-$.

Suppose that $(\rho+\varkappa)\notin\mathcal{R}_R^-$. We consider in turn the following possible cases of decomposition of the resonance vector $\varkappa$ into a sum of commuting minimal vectors:

$\langle 1\rangle$ $\varkappa=k\rho$;

$\langle 2\rangle$ $\varkappa=k\rho+\sigma$, where $[\rho,\sigma]=0$, $\rho\neq\sigma$;

$\langle 3\rangle$ $\varkappa=k\rho+\sigma+\sigma'$, where $[\rho,\sigma]=[\rho,\sigma']=[\sigma,\sigma']=0$, $\rho\neq\sigma$, $\rho\neq\sigma'$;

. . . .

Here $\sigma,\sigma',\dots\in\mathcal{M}$ and $k\in\mathbb{Z}_+$.

In case $\langle 1\rangle$ we have $(k+1)\rho\notin\mathcal{R}_R^-$. Therefore $\rho\notin\mathcal{R}_R^-$ by Lemma 11.1, and then $-\rho\in\mathcal{R}_R^-$. Hence, first, $\mathbf{A}_{-\rho}\mathfrak{p}_R$=0 and, second, $g_{\rho,-\rho}(\hbar r)=0$. Thus Corollary 10.1 (b), (c) yields that (11.4a) holds for $\varkappa=k\rho$:

$$\begin{aligned}\mathbf{A}_{-\rho}\mathbf{A}_{\rho}\mathbf{A}_{k\rho}\mathfrak{p}_R&=\mathbf{A}_{-\rho}(A_\rho)^{k+1}\mathfrak{p}_R=[\mathbf{A}_{-\rho},(A_\rho)^{k+1}]\mathfrak{p}_R\\&=\bigl(g_{-\rho,\rho}(\mathbf{S})(\mathbf{A}_\rho)^k-(\mathbf{A}_\rho)^k g_{\rho,-\rho}(\mathbf{S})\bigr)\mathfrak{p}_R=g_{-\rho,\rho}(\mathbf{S})\mathbf{A}_{k\rho}\mathfrak{p}_R.\end{aligned}$$

In case $\langle 2\rangle$ we can apply Corollary 10.1(a) since $\rho\neq\sigma$:

$$g_{\sigma,-\rho}(\mathbf{S})\mathbf{A}_{-\rho}\mathbf{A}_\sigma=g_{-\rho,\sigma}(\mathbf{S})\mathbf{A}_\sigma\mathbf{A}_{-\rho}.$$

Using the equation proved in case $\langle 1\rangle$, and then Corollary 10.1 (b) and part 5) of Lemma 10.2, we obtain

$$\begin{aligned}&g_{\sigma,-\rho}(\mathbf{S})\mathbf{A}_{-\rho}\mathbf{A}_\rho\mathbf{A}_{k\rho+\sigma}\mathfrak{p}_R=g_{\sigma,-\rho}(\mathbf{S})\mathbf{A}_{-\rho}\mathbf{A}_\sigma\mathbf{A}_\rho\mathbf{A}_{k\rho}\mathfrak{p}_R\\&\quad=g_{-\rho,\sigma}(\mathbf{S})\mathbf{A}_\sigma\mathbf{A}_{-\rho}\mathbf{A}_\rho\mathbf{A}_{k\rho}\mathfrak{p}_R=g_{-\rho,\sigma}(\mathbf{S})\mathbf{A}_\sigma g_{-\rho,\rho}(\mathbf{S})\mathbf{A}_{k\rho}\mathfrak{p}_R\\&\quad=g_{-\rho,\sigma}(\mathbf{S})g_{-\rho,\rho}(\mathbf{S}-\hbar\sigma)\mathbf{A}_\sigma\mathbf{A}_{k\rho}\mathfrak{p}_R=g_{\sigma,-\rho}(\mathbf{S})g_{-\rho,\rho}(\mathbf{S})\mathbf{A}_{k\rho+\sigma}\mathfrak{p}_R\end{aligned}\tag{11.5}$$

(here it is important that $[\rho,\sigma]=0$). By property (11.1) of the vacuum vector we can replace the operator-valued factor $g_{\sigma,-\rho}(\mathbf{S})$ (contained in the initial and final parts of the formula) by the number $g_{\sigma,-\rho}(\hbar r+\hbar k\rho+\hbar\sigma)$. This number is different from zero by Lemmas 11.1 (b) and 11.2 (b). Dividing (11.5) by this number, we obtain the identity (11.4a) in case $\langle 2\rangle$.

In case $\langle 3\rangle$ we argue as in $\langle 2\rangle$ and use the result of $\langle 2\rangle$ to arrive at (11.4a). Proceeding by induction, we see that (11.4a) holds for any $\varkappa$ commuting with ρ.

Proof of Proposition 11.1. We first note that it suffices to prove (11.3) for the vacuum vector $\mathfrak{p}_R$:

$$\mathbf{A}_\rho\mathbf{A}_\sigma\mathfrak{p}_R=g_{\rho,\sigma}(\mathbf{S})\mathbf{A}_{\rho+\sigma}\mathfrak{p}_R.\tag{11.3a}$$

Indeed, assuming that (11.3) is true for $\mathfrak{p}_R$, we see that (11.3) also holds for all vectors of the form $\mathbf{A}_\varkappa\mathfrak{p}_R$, where $\varkappa\in\mathcal{R}$:

$$\begin{aligned}\mathbf{A}_\rho\mathbf{A}_\sigma\mathbf{A}_\varkappa\mathfrak{p}_R&=\mathbf{A}_\rho g_{\sigma,\varkappa}(\mathbf{S})\mathbf{A}_{\sigma+\varkappa}\mathfrak{p}_R=g_{\sigma,\varkappa}(\mathbf{S}-\hbar\rho)\mathbf{A}_\rho\mathbf{A}_{\sigma+\varkappa}\mathfrak{p}_R\\&=g_{\sigma,\varkappa}(\mathbf{S}-\hbar\rho)g_{\rho,\sigma+\varkappa}(\mathbf{S})\mathbf{A}_{\rho+\sigma+\varkappa}\mathfrak{p}_R\\&=g_{\rho,\sigma}(\mathbf{S})g_{\rho+\sigma,\varkappa}(\mathbf{S})\mathbf{A}_{\rho+\sigma+\varkappa}\mathfrak{p}_R=g_{\rho,\sigma}(\mathbf{S})\mathbf{A}_{\rho+\sigma}\mathbf{A}_\varkappa\mathfrak{p}_R.\end{aligned}$$

Here we have used Corollary 10.1(b) and Lemma 10.2, 2).

Let $\rho,\sigma\in\mathbb{Z}^n$. The *non-commutativity index* $m(\rho,\sigma)$ is the number of j for which $[\rho,\sigma]_j\neq 0$. We have $m(\rho,\sigma)\in\{0,1,\dots,n\}$.

It is easy to verify the following lemma.

Lemma 11.4. *Suppose that* $\rho = \rho' + \rho''$, $\sigma = \sigma' + \sigma''$, *and let* $[\rho', \rho''] = [\sigma', \sigma''] = 0$. *Then the following assertions hold.*

(a) $m(\rho' + \sigma', \sigma'') \leqslant m(\rho, \sigma)$.

(b) $m(\rho'', \rho' + \sigma) \leqslant m(\rho, \sigma)$.

(c) *If there is* j *such that* $[\rho, \sigma]_j \neq 0$, $\rho'_j + \sigma'_j = 0$, *then* $m(\rho' + \sigma', \sigma'') \leqslant m(\rho, \sigma)$.

(d) *If there is* j *such that* $[\rho, \sigma]_j \neq 0$, $\rho''_j = 0$, *then* $m(\rho'', \rho' + \sigma) \leqslant m(\rho, \sigma)$.

We prove (11.3a) by induction on the value of $m(\rho, \sigma)$.

If $m(\rho, \sigma) = 0$, that is, $[\rho, \sigma] = 0$, then (11.3) follows from the constraints (10.4) of commutative type and part 1) of Lemma 10.2.

The induction hypothesis is formulated as follows.

(H1) Formula (11.3a) holds for all $\rho, \sigma \in \mathcal{R}$ with $m(\rho, \sigma) \leqslant m$.

Under this assumption, we shall prove (11.3a) for all $\rho, \sigma \in \mathcal{R}$ with $m(\rho, \sigma) = m + 1$.

Since $m(\rho, \sigma) \geqslant 1$, it follows that at least one component of the commutator $[\rho, \sigma]$ is different from zero, that is,

$$\exists\, j: \quad [\rho, \sigma]_j \neq 0.$$

Therefore the vectors ρ and σ can be represented as

$$\begin{aligned} &\rho = \rho' + \rho'', && \rho' \in \mathcal{M}, && \rho'' \in \mathcal{R}, && [\rho', \rho''] = 0, && \rho'_j \neq 0, \\ &\sigma = \sigma' + \sigma'', && \sigma' \in \mathcal{M}, && \sigma'' \in \mathcal{R}, && [\sigma', \sigma''] = 0, && \sigma'_j \neq 0. \end{aligned} \tag{11.6}$$

Hence we have

$$\mathbf{A}_\rho \mathbf{A}_\sigma \mathfrak{p}_R = \mathbf{A}_{\rho''} \mathbf{A}_{\rho'} \mathbf{A}_{\sigma'} \mathbf{A}_{\sigma''} \mathfrak{p}_R.$$

The product $\mathbf{A}_{\rho'} \mathbf{A}_{\sigma'}$ in the right-hand side can be replaced by $g_{\rho', \sigma'}(\mathbf{S}) \mathbf{A}_{\rho' + \sigma'}$. Indeed, if $\rho' \neq -\sigma'$, this follows from the constraints (10.6) of non-commutative type. If $\rho' = -\sigma'$, it follows from Lemma 11.3 (see also the notation (10.8)). Using Corollary 10.1 (b), we thus obtain the relation

$$\mathbf{A}_\rho \mathbf{A}_\sigma \mathfrak{p}_R = g_{\rho', \sigma'}(\mathbf{S} - \hbar \rho'') \mathbf{A}_{\rho''} \mathbf{A}_{\rho' + \sigma'} \mathbf{A}_{\sigma''} \mathfrak{p}_R. \tag{11.7}$$

We now proceed by induction on $M_j(\rho, \sigma) \stackrel{\text{def}}{=} |[\rho, \sigma]_j|$.

When $M_j(\rho, \sigma) = 1$, we obviously have $\rho'_j = -\sigma'_j$, $\rho''_j = 0$. Parts (c) and (d) of Lemma 11.4 imply that

$$m(\rho' + \sigma', \sigma'') < m + 1, \qquad m(\rho'', \rho' + \sigma) < m + 1.$$

Hence, by the induction hypothesis (H1), we have

$$\begin{aligned} \mathbf{A}_{\rho' + \sigma'} \mathbf{A}_{\sigma''} \mathfrak{p}_R &= g_{\rho' + \sigma', \sigma''}(\mathbf{S}) \mathbf{A}_{\rho' + \sigma' + \sigma''} \mathfrak{p}_R, \\ \mathbf{A}_{\rho''} \mathbf{A}_{\rho' + \sigma} \mathfrak{p}_R &= g_{\rho'', \, \rho' + \sigma}(\mathbf{S}) \mathbf{A}_{\rho'' + \rho' + \sigma} \mathfrak{p}_R. \end{aligned} \tag{11.8}$$

Substituting these relations in (11.7) and using Corollary 10.1 (b) and the identity 6) in Lemma 10.2, we obtain the desired relation (11.3a) in the case when $m(\rho, \sigma) = m + 1$, $M_j(\rho, \sigma) = 1$:

$$\begin{aligned} \mathbf{A}_\rho \mathbf{A}_\sigma \mathfrak{p}_R &= g_{\rho', \sigma'}(\mathbf{S} - \hbar \rho'') \mathbf{A}_{\rho''} g_{\rho' + \sigma', \sigma''}(\mathbf{S}) \mathbf{A}_{\rho' + \sigma} \mathfrak{p}_R \\ &= g_{\rho', \sigma'}(\mathbf{S} - \hbar \rho'') g_{\rho' + \sigma', \sigma''}(\mathbf{S} - \hbar \rho'') g_{\rho'', \rho' + \sigma}(\mathbf{S}) \mathbf{A}_{\rho + \sigma} \mathfrak{p}_R \\ &= g_{\rho' + \rho'', \, \sigma' + \sigma''}(\mathbf{S}) \mathbf{A}_{\rho + \sigma} \mathfrak{p}_R = g_{\rho, \sigma}(\mathbf{S}) \mathbf{A}_{\rho + \sigma} \mathfrak{p}_R. \end{aligned}$$

The induction hypothesis is formulated as follows.

(H2) Formula (11.3a) holds for all $\rho,\sigma\in\mathcal{R}$ with
$m(\rho,\sigma)=m+1$ and $M_j(\rho,\sigma)\leqslant M_j$.

We also consider $\rho,\sigma\in\mathcal{R}$ such that $m(\rho,\sigma)=m+1$ and $M_j(\rho,\sigma)=M_j+1$. Then, first, parts (a) and (b) of Lemma 11.4 imply the inequalities

$$m(\rho'+\sigma',\sigma'')\leqslant m+1,\qquad m(\rho'',\rho'+\sigma)\leqslant m+1$$

and, second, the relations (11.6) imply the inequalities

$$M_j(\rho'+\sigma',\sigma'')\leqslant M_j,\qquad M_j(\rho'',\rho'+\sigma)\leqslant M_j.$$

This means that one of the induction hypotheses (H1) or (H2) holds. In any case, the formulae (11.8) hold. Substituting them in (11.7), we obtain formula (11.3a) in the case when $m(\rho,\sigma)=m+1,\ M_j(\rho,\sigma)=M_j+1$.

Thus we have proved by induction that formula (11.3a) holds for all values of $m(\rho,\sigma)\in\{0,1,\dots,n\},\ n\in\mathbb{N}$.

Corollary 11.1. *The subspace* $\mathcal{H}_M$, (11.2), *is invariant and minimal under the representation of the resonance algebra* (*that is,* $\mathcal{H}_M$ *is the space of an irreducible representation of* $\mathcal{A}$). *The representation operators act on vectors in* $\mathcal{H}_M$ *as follows*:

$$\mathbf{S}_j\mathbf{A}_\sigma\mathfrak{p}_R=\hbar(r_j+\sigma_j)\mathbf{A}_\sigma\mathfrak{p}_R,$$
$$\mathbf{A}_\rho\mathbf{A}_\sigma\mathfrak{p}_R=g_{\rho,\sigma}(\hbar r+\hbar\rho+\hbar\sigma)\mathbf{A}_{\rho+\sigma}\mathfrak{p}_R.$$

Combining this with (11.1) and Lemma 10.2, 4), we get the following assertion.

Corollary 11.2. *Suppose that* $\sigma,\theta\in\mathcal{R}$.

(a) *If* $(r+\sigma)\notin\mathbf{\Delta}[M]$, *then* $\mathbf{A}_\sigma\mathfrak{p}_R=0$.

(b) *If* $(r+\sigma)\in\mathbf{\Delta}[M]$, *then* $\|\mathbf{A}_\sigma\mathfrak{p}_R\|=(\hbar r)_{-\sigma}^{-1/2}$.

(c) *If* $\sigma\neq\theta$, *then* $(\mathbf{A}_\sigma\mathfrak{p}_R,\mathbf{A}_\theta\mathfrak{p}_R)=0$.

§ 12. Coherent states

We consider irreducible representations of the resonance algebra in the space $\mathcal{H}_M\subset\mathcal{H}$, (11.2), generated by the vacuum vector. We associate each point of the skeleton $t\in\mathbf{\Delta}[M]$ with a vector in $\mathcal{H}_M$ by the formula

$$\mathfrak{p}^t\overset{\text{def}}{=}(\hbar r)_{t-r}^{-1/2}\mathbf{A}_{t-r}\mathfrak{p}_R,\tag{12.1}$$

where $\mathfrak{p}_R$ is the vacuum vector and r is the vertex of the frame R.

Proposition 12.1. (1) *The vectors* $\mathfrak{p}^t$, (12.1), *form an orthonormal basis in* $\mathcal{H}_M$, (11.2).

(2) *The representation operators of the algebra* $\mathcal{A}$ *in the basis* $\{\mathfrak{p}^t\mid t\in\mathbf{\Delta}[M]\}$ *have the form*

$$\mathbf{S}_j\mathfrak{p}^t=\hbar t_j\mathfrak{p}^t,\qquad j=1,\dots,n,\tag{12.2}$$
$$\mathbf{A}_\rho\mathfrak{p}^t=(\hbar t)_\rho^{1/2}\mathfrak{p}^{\rho+t},\qquad \rho\in\mathcal{R}.\tag{12.3}$$

Proof. Only formula (12.3) must be explained. Suppose that $(\rho + t) \in \mathbf{\Delta}[M]$. By Corollary 11.1 and definition (12.1) we have

$$\mathbf{A}_\rho \mathfrak{p}^t = \sqrt{\frac{(\hbar r)_{\rho+t-r}}{(\hbar r)_{t-r}}}\, g_{\rho,t-r}(\hbar\rho + \hbar t)\mathfrak{p}^{\rho+t}.$$

By Lemma 10.1 and definition (10.2) we have the identity

$$(g_{\rho,\sigma}(s))^2 = \frac{(s-\hbar\rho)_\rho (s-\hbar\rho-\hbar\sigma)_\sigma}{(s-\hbar\rho-\hbar\sigma)_{\rho+\sigma}}, \qquad s \in \mathbb{R}^n, \quad \rho,\sigma \in \mathbb{Z}^n.$$

This and Lemma 11.2 imply that

$$g_{\rho,t-r}(\hbar\rho + \hbar t) = \sqrt{\frac{(\hbar t)_\rho (\hbar r)_{t-r}}{(\hbar r)_{\rho+t-r}}},$$

and we obtain formula (12.3).

Suppose that $(\rho + t) \notin \mathbf{\Delta}[M]$. Then the left-hand side of (12.3) is zero by Corollary 11.2, and the right-hand side of (12.3) is zero because of the coefficient: $(\hbar t)_\rho = 0$. Thus formula (12.3) also holds in this case.

Lemma 12.1. *Suppose that the vacuum vector* $\mathfrak{p}_R$, (11.1), *exists for some frame* R *of the resonance skeleton* $\mathbf{\Delta}[M]$. *Then there is a vacuum vector* $\mathfrak{p}_L$ *for any other frame* L *of* $\mathbf{\Delta}[M]$, *namely,*

$$\mathfrak{p}_L = \mathfrak{p}^l,$$

where l *is the vertex of the resonance frame* L *and the vector* $\mathfrak{p}^l$ *is given by* (12.1).

Proof. We recall that $\sigma \in \mathcal{R}_L^- \iff (l+\sigma) \notin \mathbf{\Delta}[M]$. It follows by Proposition 12.1 that the vector $\mathfrak{p}_L = \mathfrak{p}^l$ satisfies conditions (11.1) for the vacuum vector:

$$\mathbf{A}_\sigma \mathfrak{p}_L = 0, \qquad 0 \in \mathcal{R}_L^-,$$
$$\mathbf{S}_j \mathfrak{p}_L = \hbar l \mathfrak{p}_L, \qquad j = 1, \dots, n.$$

Moreover, Proposition 12.1, 1) yields that the vector $\mathfrak{p}_L$ is normalized: $\|\mathfrak{p}_L\| = 1$. Hence $\mathfrak{p}_L$ is the vacuum vector corresponding to the frame L.

Lemma 12.2. *The vector* $\mathfrak{p}^t$, (12.1), *is independent of the choice of the frame* R.

Proof. By formula (12.3), we have

$$\mathbf{A}_{t-l}\mathfrak{p}^l = (\hbar l)_{t-l}^{1/2}\mathfrak{p}^t.$$

Hence the vector $\mathfrak{p}^t$ can be expressed by the formula

$$\mathfrak{p}^t = (\hbar l)_{t-l}^{-1/2}\mathbf{A}_{t-l}\mathfrak{p}_L$$

in terms of the vacuum vector $\mathfrak{p}_L = \mathfrak{p}^l$ corresponding to an arbitrary frame L. Comparing the resulting expression with (12.1), we conclude that $\mathfrak{p}^t$ is independent of the choice of the frame.

Example 12.1. If $\mathcal{H} = L^2(\mathbb{R}^n)$ and the representation of $\mathcal{A}$ is given by (10.12), then the basis vectors $\mathfrak{p}^t$ are functions on $\mathbb{R}^n$:

$$\mathfrak{p}^t(q) = \frac{1}{\sqrt{2^{|t|}\, t!}} \prod_{j=1}^{n} \sqrt[4]{\frac{f_j}{\pi\hbar}}\, H_{t_j}\Big(\sqrt{\frac{f_j}{\hbar}} q_j\Big) \exp\Big\{-\frac{1}{2\hbar}\sum_{j=1}^{n} f_j q_j^2\Big\}, \qquad q \in \mathbb{R}^n.$$

The H_k here are the standard Hermite polynomials

$$H_k(\xi) \stackrel{\text{def}}{=} (-1)^k e^{\xi^2} \frac{d^k}{d\xi^k}(e^{-\xi^2}).$$

Indeed, in this case, the vacuum vector is given by the formula

$$\mathfrak{p}_R = c(r)(\hat{z}^*)^r \chi_0,$$

where

$$c(r) \stackrel{\text{def}}{=} \sqrt[4]{\frac{f_1 \cdots f_n}{(\pi\hbar)^n}} \, \frac{1}{\sqrt{\hbar^{|r|}\, r!}}, \qquad \chi_0(q) = \exp\biggl\{-\frac{1}{2\hbar}\sum_{j=1}^n f_j q_j^2\biggr\}.$$

It follows from definition (12.1) that

$$\begin{aligned}
\mathfrak{p}^t &= c(r)(\hbar r)_{t-r}^{-1/2}\, (\hat{z}^*)^{(t-r)_+}\, \hat{z}^{(t-r)_-}\, (\hat{z}^*)^r \chi_0 \\
&= c(t)\, (\hat{z}^*)^t \chi_0 = c(t)\, \frac{\hbar^{|t|/2}}{2^{|t|/2}} \prod_{j=1}^n H_{t_j}\biggl(\sqrt{\frac{f_j}{\hbar}}\, q_j\biggr)\chi_0.
\end{aligned}$$

This yields the above formula for $\mathfrak{p}^t(q)$.

We recall from § 7 that the set of all vertices $t \in \boldsymbol{\Delta}[M]$ is associated with an orthonormal basis $\{U^t\}$ in the space $\mathcal{L}(\Omega_\hbar)$ of holomorphic sections of the sheaf $\Pi(\Omega_\hbar)$.

Let $\Pi(\Omega_\hbar, \mathcal{H}) = \Pi(\Omega_\hbar) \otimes \mathcal{H}$ be the sheaf of $\mathcal{H}$-valued holomorphic functions on $\Omega_\hbar$ with scalar matching functions (7.2) as above, and let $\mathcal{L}(\Omega_\hbar, \mathcal{H})$ be the space of its sections $\Psi = \{\Psi_R\}$ equipped with the natural Hilbert norm

$$\|\Psi\| = \biggl(\frac{1}{(2\pi\hbar)^{n-1}} \int_{\Omega_\hbar} \frac{\|\Psi_R\|_{\mathcal{H}}^2}{\mathcal{K}_R}\, dm_\hbar\biggr)^{1/2}.$$

We clearly have an embedding $\mathcal{L}(\Omega_\hbar, \mathcal{H}_M) \subset \mathcal{L}(\Omega_\hbar, \mathcal{H})$ for the subspace $\mathcal{H}_M \subset \mathcal{H}$.

Definition 12.1. The vectors

$$\mathfrak{P} \stackrel{\text{def}}{=} \sum_{t \in \boldsymbol{\Delta}[M]} U^t \mathfrak{p}^t \tag{12.4}$$

are called the *coherent states* of the algebra $\mathcal{A}$ corresponding to the quantum leaf $\Omega_\hbar = \Omega_\hbar[M]$.

Formula (12.4) determines an element of the space $\mathcal{L}(\Omega_\hbar, \mathcal{H}_M)$. By (7.1), the vectors $\mathfrak{P} = \mathfrak{P}_R(w)$ are holomorphic functions of the complex coordinates w in the local chart with index $R = (r, \{\rho\})$:

$$\mathfrak{P}_R(w) = \sum_{t \in \boldsymbol{\Delta}[M]} U_R^t(w)\mathfrak{p}^t = \sum_{t \in \boldsymbol{\Delta}[M]} \sqrt{\frac{\hbar^{|r|}\, r!}{\hbar^{|t|}\, t!}} \prod_{k=1}^{n-1} w^{N_{t-r}^{(k)}} \mathfrak{p}^t. \tag{12.5}$$

Here the non-negative exponents $N_{t-r}^{(k)}$ are determined by the decomposition (4.3) of the resonance vector $t - r$ with respect to the basis $\{\rho\}$ at the vertex r.

Example 12.2. Suppose that $n = 3$ and $f_1 = 1$. Then, for each $M \in \mathbb{Z}_+$, the Diophantine skeleton $\mathbf{\Delta}[M]$ consists of the points

$$(M - f_2 j - f_3 k, j, k), \qquad \text{where} \quad j \in \mathbb{Z}_+, \quad k \in \mathbb{Z}_+, \qquad f_2 j + f_3 k \leqslant M.$$

We take the vertex $r = (M, 0, 0)$ and the basis (4.8): $\rho^{(1)} = (-f_2, 1, 0)$, $\rho^{(2)} = (-f_3, 0, 1)$ for the resonance frame $R = (r, \{\rho\})$. We then obtain the following formula for coherent states (12.5) in the local chart with index R:

$$\mathfrak{P}_R(w) = \sum_{\substack{j \in \mathbb{Z}_+,\, k \in \mathbb{Z}_+ \\ f_2 j + f_3 k \leqslant M}} \frac{1}{j!\, k!} \left(\frac{w_1 \mathbf{A}_{\rho^{(1)}}}{\hbar} \right)^j \left(\frac{w_2 \mathbf{A}_{\rho^{(2)}}}{\hbar} \right)^k \mathfrak{p}^r. \tag{12.6}$$

Here $\mathfrak{p}^r$ is the vacuum vector satisfying equations (11.1):

$$\mathbf{S}_1 \mathfrak{p}^r = \hbar M \mathfrak{p}^r, \qquad \mathbf{S}_2 \mathfrak{p}^r = \mathbf{S}_3 \mathfrak{p}^r = 0$$

and the normalization condition $\|\mathfrak{p}^r\| = 1$, while w_1 and w_2 are the complex coordinates related to the resonance basis $\rho^{(1)}$, $\rho^{(2)}$ by the formulae (4.4). We note that the vectors $\rho^{(1)}$ and $\rho^{(2)}$ commute in the sense of (3.2): $[\rho^{(1)}, \rho^{(2)}] = 0$. Hence the operators $\mathbf{A}_{\rho^{(1)}}$ and $\mathbf{A}_{\rho^{(2)}}$ also commute:

$$[\mathbf{A}_{\rho^{(1)}}, \mathbf{A}_{\rho^{(2)}}] = 0.$$

We also note that $(\mathbf{A}_{\rho^{(1)}})^j (\mathbf{A}_{\rho^{(2)}})^k \mathfrak{p}^r = 0$ for all $j, k \in \mathbb{Z}_+$, $f_2 j + f_3 k > M$. Hence the summation over j and k in formula (12.6) can be extended to $+\infty$. As a result, we obtain the following expression for coherent states:

$$\mathfrak{P}_R(w) = \exp\left\{ \frac{1}{\hbar} \big(w_1 \mathbf{A}_{\rho^{(1)}} + w_2 \mathbf{A}_{\rho^{(2)}} \big) \right\} \mathfrak{p}^r.$$

We also mention general properties of coherent states [39], [40].

Lemma 12.3. (a) *The scalar product* $\mathcal{K} = (\mathfrak{P}, \mathfrak{P})_{\mathcal{H}}$ *of coherent states coincides with the reproducing kernel* (7.7), (7.9) *of the space* $\mathcal{L}(\Omega_\hbar)$.

(b) *The sum of the projection operators* $\boldsymbol{\pi}_\hbar$ *to the one-dimensional subspaces in* $\mathcal{H}_M$ *generated by the coherent states is equal to the projection operator* $\mathbf{\Pi}_\hbar[M]$ *whose image is* $\mathcal{H}_M$ (*that is, a whole irreducible component of the representation of* $\mathcal{A}$ *in* $\mathcal{H}$):

$$\frac{1}{(2\pi\hbar)^{n-1}} \int_{\Omega_\hbar[M]} \boldsymbol{\pi}_\hbar \, dm_\hbar = \mathbf{\Pi}_\hbar[M].$$

§ 13. Realization of irreducible representations over quantum leaves

We now exhibit a universal realization of irreducible representations of the quantum resonance algebra $\mathcal{A}$ in the spaces of antiholomorphic sections over the quantum leaves $\Omega_\hbar[M]$.

We fix a local chart with index $R = (r, \{\rho\})$ in $\Omega_\hbar[M]$. Here r is a vertex of the Diophantine skeleton $\mathbf{\Delta}$ and $\{\rho\}$ is a resonance basis. We denote the complex coordinates in this chart by w.

We define vector functions

$$s(m) \stackrel{\text{def}}{=} \hbar r + \hbar \sum_{k=1}^{n-1} m_k \rho^{(k)}, \qquad m \in \mathbb{Z}^n.$$

Theorem 13.1. *The differential operators*

$$\begin{aligned} \overset{\circ}{S}_j &\stackrel{\text{def}}{=} s_j\biggl(\overline{w}\,\frac{\partial}{\partial \overline{w}}\biggr), \qquad j = 1, \dots, n, \\ \overset{\circ}{A}_\sigma &\stackrel{\text{def}}{=} (\overset{\circ}{S})_{\sigma_-} \overline{W_\sigma}, \qquad \sigma \in \mathcal{M}, \end{aligned} \tag{13.1}$$

in local charts on $\Omega_\hbar = \Omega_\hbar[M]$ are consistent on intersections of charts and determine an irreducible representation of the quantum resonance algebra $\mathcal{A}$ in the Hilbert space $\mathcal{L}^*(\Omega_\hbar)$ of (antiholomorphic) sections of the sheaf $\Pi^*(\Omega_\hbar)$. For this representation, the vacuum vector corresponding to the frame R is the section of $\Pi^*(\Omega_\hbar)$ determined by the identity function in the chart with index R.

The second formula in (13.1) uses the notation (10.1a) and (4.5).

Proof of Theorem 13.1. The verification of consistency on the intersections of charts is a routine matter. The fact that the constraint equations and commutation relations (10.4)–(10.6) hold follows from the permutation formulae

$$\overline{w}_k \frac{\partial}{\partial \overline{w}_k} \circ \overline{W_\sigma} = \overline{W_\sigma} \circ \biggl(\overline{w}_k \frac{\partial}{\partial \overline{w}_k} + N_\sigma^{(k)}\biggr), \qquad \overset{\circ}{S}_j \circ \overline{W_\sigma} = \overline{W_\sigma} \circ (\overset{\circ}{S}_j + \hbar\sigma_j),$$

where the numbers $N_\sigma^{(k)}$ are defined by (4.3). Consider the conditions (10.3) saying that the operators (13.1) are Hermitian. It suffices to verify them on the vectors $\overline{U^t}$ of the orthonormal basis (7.1), (7.5). These vectors are eigenvectors of the operators $\overset{\circ}{S}_j$:

$$\overset{\circ}{S}_j \overline{U^t} = \hbar t_j \overline{U^t}. \tag{13.2}$$

The operators $\overset{\circ}{A}_\sigma$ act as follows:

$$\overset{\circ}{A}_\sigma \overline{U^t} = \sqrt{(\hbar t)_\sigma}\; \overline{U^{t+\sigma}}. \tag{13.3}$$

Hence, using the property $(s - \hbar\sigma)_\sigma = (s)_{-\sigma}$ of the symbols (10.1a), we obtain

$$\begin{aligned} (\overset{\circ}{A}_\sigma \overline{U^k}, \overline{U^t}) &= \sqrt{(\hbar k)_\sigma}\; (\overline{U^{k+\sigma}}, \overline{U^t}) = \sqrt{(\hbar k)_\sigma}\; \delta_{k+\sigma,t} \\ &= \sqrt{(\hbar t)_{-\sigma}}\; \delta_{k,t-\sigma} = (\overline{U^k}, \overset{\circ}{A}_{-\sigma} \overline{U^t}), \end{aligned}$$

that is, $\overset{\circ}{A}{}_\sigma^* = \overset{\circ}{A}_{-\sigma}$.

For $t = r$, since $(\hbar r)_\sigma = 0$ for any $\sigma \in \mathcal{R}_R^-$, it follows from (13.2) and (13.3) that

$$\overset{\circ}{S}_j \overline{U^r} = \hbar r_j \overline{U^r}, \quad \overset{\circ}{A}_\sigma \overline{U^r} = 0, \qquad j = 1, \dots, n, \quad \sigma \in \mathcal{R}_R^-,$$

that is, the section $\overline{U^r}$ is a vacuum vector for the operators (13.1). This section is identically equal to 1 in the chart with index R: $\overline{U_R^r} = 1$ (see (7.1)).

Remark 13.1. Since the scalar product in the space $\mathcal{L}^*(\Omega_\hbar)$ is represented by the integral (8.1), one can rewrite the condition that representation (13.1) is Hermitian as several equations for the density $\mathcal{J}$ and the reproducing kernel $\mathcal{K}$ (see [25] for this well-known procedure). This system of equations on $\mathcal{J}$ and $\mathcal{K}$ has the form

$$(\overset{\circ}{S}_i)^T\mathcal{J} = \overline{(\overset{\circ}{S}_i)}^T\mathcal{J}, \qquad (\overset{\circ}{A}_\sigma)^T\mathcal{J} = \overline{(\overset{\circ}{A}_{-\sigma})}^T\mathcal{J},$$
$$\overset{\circ}{S}_i\mathcal{K} = \overline{\overset{\circ}{S}_i}\mathcal{K}, \qquad \overset{\circ}{A}_\sigma\mathcal{K} = \overline{\overset{\circ}{A}_{-\sigma}}\mathcal{K}.$$

Here we write $(\dots)^T$ for the operation of transposing a differential operator with respect to the standard measure dw (or $d\overline{w}$) in the local complex coordinates. The bar means the operator of complex conjugation.

The left-hand column of equations of this system (where $i = 1, \dots, n$) ensures that the density $\mathcal{J}$ and the kernel $\mathcal{K}$ depend only on the moduli of the complex coordinates w, that is, on the real variables $X_j = |w_j|^2$. It suffices to write the right-hand column of equations (where $\sigma \in \mathcal{M}$) only for the vectors in the basis $\rho^{(j)}$. It has the form

$$\big[X_j(\hbar\widetilde{r} - \hbar D)_{\rho_-^{(j)}} - (\hbar\widetilde{r} - \hbar D)_{\rho_+^{(j)}}\big]\mathcal{J}_R = 0,$$
$$\big[X_j(\hbar r + \hbar D)_{-\rho_-^{(j)}} - (\hbar r + \hbar D)_{-\rho_+^{(j)}}\big]\mathcal{K}_R = 0, \qquad j = 1, \dots, n-1.$$

Here the frame R indicates a chart on the quantum leaf, r is the vertex of R and $\{\rho^{(j)}\}$ is the resonance basis of R, $\widetilde{r} = r - \rho^{(1)} - \dots - \rho^{(n-1)}$, and $D = \sum_{j=1}^{n-1} \rho^{(j)} X_j \frac{\partial}{\partial X_j}$. This system of equations for $\mathcal{J}_R$ and $\mathcal{K}_R$ was used in Lemma 8.1 to find the asymptotic behaviour as $\hbar \to 0$.

To conclude, we show how to intertwine any abstract representation of $\mathcal{A}$ in $\mathcal{H}$ which has a vacuum vector (such as the representation (10.11), (10.12)) with the universal irreducible representations (13.1).

We consider coherent states $\mathfrak{P}$, (12.4), in $\mathcal{H}$. For each section ψ of the sheaf $\Pi^*(\Omega_\hbar)$ we define a vector $\mathcal{P}[\psi] \in \mathcal{H}$ by the formula

$$\mathcal{P}[\psi] \overset{\text{def}}{=} \frac{1}{(2\pi\hbar)^{n-1}} \int_{\Omega_\hbar} \frac{\psi\mathfrak{P}}{\mathcal{K}}\, dm_\hbar.$$

The map

$$\psi \to \mathcal{P}[\psi] \tag{13.4}$$

is called a *coherent transformation*.

Theorem 13.2. *The coherent transformation* (13.4) *intertwines the representation of the resonance algebra $\mathcal{A}$ in the Hilbert space $\mathcal{H}$ with the irreducible representation in the space $\mathcal{L}^*(\Omega_\hbar)$ of antiholomorphic sections over the quantum leaf $\Omega_\hbar = \Omega_\hbar[M]$:*

$$\mathbf{A}_\sigma\mathcal{P}[\psi] = \mathcal{P}[\overset{\circ}{A}_\sigma\psi], \qquad \mathbf{S}_j\mathcal{P}[\psi] = \mathcal{P}[\overset{\circ}{S}_j\psi].$$

Here the differential operators $\overset{\circ}{A}_\sigma$ and $\overset{\circ}{S}_j$ are given by the formulae (13.1).

Proof. It follows from (12.4), (8.1), (8.2), (7.5) that $\mathcal{P}$ takes the vectors $\overline{U^t}$, (7.1), of an orthonormal basis in $\mathcal{L}^*(\Omega_\hbar)$ to vectors $\mathfrak{p}^t \in \mathcal{H}$, (13.1):

$$\mathcal{P}[\overline{U^t}] = \frac{1}{(2\pi\hbar)^{n-1}} \int_{\Omega_\hbar} \frac{\overline{U^t}}{\mathcal{K}} \sum_{t'\in\mathbf{\Delta}[M]} \overline{U^{t'}}\mathfrak{p}^{t'}\, dm_\hbar = \sum_{t'\in\mathbf{\Delta}[M]} (\overline{U^{t'}}, \overline{U^t})\mathfrak{p}^{t'} = \mathfrak{p}^t.$$

Therefore it suffices to prove that

$$\mathbf{A}_\sigma \mathfrak{p}^t = \mathcal{P}[\mathring{A}_\sigma \overline{U^t}], \qquad \mathbf{S}_j \mathfrak{p}^t = \mathcal{P}[\mathring{S}_j \overline{U^t}]. \tag{13.5}$$

We use (12.3) to obtain

$$\mathbf{A}_\sigma \mathfrak{p}^t = \sqrt{(\hbar t)_\sigma}\, \mathfrak{p}^{t+\sigma}.$$

Comparing this formula with (13.3) and taking into account that $P[\overline{U^t}] = \mathfrak{p}^t$, we obtain the first equality in (13.5).

Using the formulae (13.2) and (12.2), we similarly obtain the second.

Appendix. Proof of Theorem 9.2

Precisely as in the proof of Theorem 9.1, we use the 'polar' coordinates X, Φ and represent the quantum Kähler form (8.8) as

$$\omega_\hbar = \hbar\Big(\frac{\partial g_1}{\partial X_1}\,dX_1\wedge d\Phi_1 + \frac{\partial g_2}{\partial X_1}\,dX_1\wedge d\Phi_2 + \frac{\partial g_1}{\partial X_2}\,dX_2\wedge d\Phi_1 + \frac{\partial g_2}{\partial X_2}\,dX_2\wedge d\Phi_2\Big),$$

where

$$g_j \overset{\text{def}}{=} X_j \frac{\partial}{\partial X_j} \ln \mathcal{K}_R.$$

Calculating the integral of the quantum volume form over the leaf Ω reduces to calculating the integral of the primitive of this 4-form over the boundary $\partial\Omega = \overline{\Omega} \cap \partial\mathcal{N}_0^+$:

$$\int_\Omega \frac{\omega_\hbar \wedge \omega_\hbar}{2} = \hbar^2 \int_{\partial\Omega} \theta \wedge d\Phi_1 \wedge d\Phi_2, \qquad \theta \overset{\text{def}}{=} G_1\,dX_1 + G_2\,dX_2.$$

Here

$$G_1(X_1, X_2) \overset{\text{def}}{=} g_2 \frac{\partial g_1}{\partial X_1} - g_1 \frac{\partial g_2}{\partial X_1}, \qquad G_2(X_1, X_2) \overset{\text{def}}{=} g_2 \frac{\partial g_1}{\partial X_2} - g_1 \frac{\partial g_2}{\partial X_2}.$$

On $\partial\Omega$, the angular variables Φ_j vary from 0 to 2π, and the values of the radial variables X_j lie on the boundary (the union of the three edges) of the classical simplex (triangle) $\blacktriangle[\hbar M]$. Since this boundary can occur in the boundary of the coordinate chart, we allow the radial variables to take the values zero and infinity. By (5.13) we have $X_2 = 0$ (and hence $\theta|_{\Sigma_{12}} = 0$) on the edge joining the first and second vertices. On the edge joining the second and third vertices, we have

$$X_2 = \infty, \qquad X_1 = \tau X_2^{f_2/\mu}$$

and hence

$$\theta\big|_{\Sigma_{23}} = \big(X_2^{f_2/\mu} G_1(\tau X_2^{f_2/\mu}, X_2)\big)\big|_{X_2=\infty}\, d\tau,$$

where τ goes from 0 to ∞ as we move from the second vertex to the third. On the edge joining the third and first vertices, we have

$$X_1 = \infty, \qquad X_2 = t X_1^{-\nu/f_1}$$

and hence

$$\theta\big|_{\Sigma_{31}} = \big(X_1^{-\nu/f_1} G_2(X_1, t X_1^{-\nu/f_1})\big)\big|_{X_1=\infty}\, dt,$$

where t goes from ∞ to 0 as we move from the third vertex to the first. Thus,

$$\frac{1}{(2\pi\hbar)^2}\int_\Omega \frac{\omega_\hbar\wedge\omega_\hbar}{2} = \frac{1}{2}\bigg\{\int_0^\infty \big(X_2^{f_2/\mu}G_1(\tau X_2^{f_2/\mu}, X_2)\big)\big|_{X_2=\infty}\,d\tau - \int_0^\infty \big(X_1^{-\nu/f_1}G_2(X_1, tX_1^{-\nu/f_1})\big)\big|_{X_1=\infty}\,dt\bigg\}. \tag{A.1}$$

We calculate the first integral in the right-hand side. Using the explicit formula for $\mathcal{K}_R(X_1,X_2)$ as in the proof of Theorem 9.1, we obtain

$$\mathcal{K}_R(\tau X_2^{f_2/\mu}, X_2) = \sum_{\sigma\in\mathcal{R}_r} c_\sigma^r X_2^{f_2 N_\sigma^{(1)}/\mu + N_\sigma^{(2)}}\tau^{N_\sigma^{(1)}}.$$

It follows that

$$\mathcal{K}_R(\tau X_2^{f_2/\mu}, X_2) = \bigg(\sum_{\sigma\in\mathcal{R}_r^\lambda} c_\sigma^r \tau^{N_\sigma^{(1)}}\bigg) X_2^\lambda + O(X_2^{\lambda-1}) \tag{A.2}$$

as $X_2\to\infty$. Here we write

$$\lambda \stackrel{\text{def}}{=} \max_{\sigma\in\mathcal{R}_r}\bigg(\frac{f_2}{\mu}N_\sigma^{(1)} + N_\sigma^{(2)}\bigg), \qquad \mathcal{R}_r^\lambda \stackrel{\text{def}}{=} \bigg\{\sigma\in\mathcal{R}_r \;\bigg|\; \frac{f_2}{\mu}N_\sigma^{(1)} + N_\sigma^{(2)} = \lambda\bigg\}. \tag{A.3}$$

We also introduce a shorthand notation for sums over $\mathcal{R}_r^\lambda$. Namely, we put

$$\Sigma[\gamma(N_\sigma^{(1)})] \stackrel{\text{def}}{=} \sum_{\sigma\in\mathcal{R}_r^\lambda} \gamma(N_\sigma^{(1)})\, c_\sigma^r\, \tau^{N_\sigma^{(1)}}.$$

In particular, in this notation formula (A.2) has the form

$$\mathcal{K}_R(\tau X_2^{f_2/\mu}, X_2) = \Sigma[1]\, X_2^\lambda + O(X_2^{\lambda-1}).$$

We similarly obtain

$$\begin{aligned}
(D_1\mathcal{K}_R)(\tau X_2^{f_2/\mu}, X_2) &= \Sigma[N_\sigma^{(1)}]\, X_2^\lambda + O(X_2^{\lambda-1}),\\
(D_2\mathcal{K}_R)(\tau X_2^{f_2/\mu}, X_2) &= \Sigma\bigg[\lambda - \frac{f_2}{\mu}N_\sigma^{(1)}\bigg] X_2^\lambda + O(X_2^{\lambda-1}),\\
(D_1^2\mathcal{K}_R)(\tau X_2^{f_2/\mu}, X_2) &= \Sigma[(N_\sigma^{(1)})^2]\, X_2^\lambda + O(X_2^{\lambda-1}),\\
(D_1D_2\mathcal{K}_R)(\tau X_2^{f_2/\mu}, X_2) &= \Sigma\bigg[N_\sigma^{(1)}\bigg(\lambda - \frac{f_2}{\mu}N_\sigma^{(1)}\bigg)\bigg] X_2^\lambda + O(X_2^{\lambda-1}),
\end{aligned}$$

where $D_j = X_j\frac{\partial}{\partial X_j}$. Substituting these asymptotic expressions into the formula

$$G_1(X_1,X_2) = \frac{D_2\mathcal{K}_R\cdot D_1^2\mathcal{K}_R - D_1D_2\mathcal{K}_R\cdot D_1\mathcal{K}_R}{X_1(\mathcal{K}_R)^2}$$

for G_1, we obtain

$$\begin{aligned}
&\big(X_2^{f_2/\mu} G_1(\tau X_2^{f_2/\mu}, X_2)\big)\Big|_{X_2=\infty}\\
&\quad = \frac{\Sigma[\lambda - \frac{f_2}{\mu}N_\sigma^{(1)}]\Sigma[(N_\sigma^{(1)})^2] - \Sigma[N_\sigma^{(1)}(\lambda - \frac{f_2}{\mu}N_\sigma^{(1)})]\Sigma[N_\sigma^{(1)}]}{\tau(\Sigma[1])^2}\\
&\quad = \lambda \frac{\Sigma[1]\Sigma[(N_\sigma^{(1)})^2] - (\Sigma[N_\sigma^{(1)}])^2}{\tau(\Sigma[1])^2} = \lambda \frac{\partial}{\partial \tau}\bigg(\frac{\Sigma[N_\sigma^{(1)}]}{\Sigma[1]}\bigg).
\end{aligned}$$

Therefore, the desired integral has the form

$$\begin{aligned}
\int_0^\infty \big(X_2^{f_2/\mu} G_1(\tau X_2^{f_2/\mu}, X_2)\big)\Big|_{X_2=\infty} d\tau &= \lambda \frac{\Sigma[N_\sigma^{(1)}]}{\Sigma[1]}\bigg|_{\tau=0}^{\infty}\\
&= \lambda\Big(\max_{\sigma\in\mathcal{R}_r^\lambda} N_\sigma^{(1)} - \min_{\sigma\in\mathcal{R}_r^\lambda} N_\sigma^{(1)}\Big). \qquad \text{(A.4)}
\end{aligned}$$

Writing $l = r^{(2)} + \sigma$, it follows from (7.11) that

$$\sigma \in \mathcal{R}_r \iff l \in \mathbb{Z}_+^3.$$

We use the explicit formulae (5.9) for the basis $\rho^{(1)}$, $\rho^{(2)}$ and the representation (4.3):

$$\begin{aligned}
l_1 &= r_1^{(2)} + N_\sigma^{(1)} f_2 + N_\sigma^{(2)}\mu,\\
l_2 &= r_2^{(2)} - N_\sigma^{(1)} f_1 + N_\sigma^{(2)}\nu, \qquad \text{(A.5)}\\
l_3 &= r_3^{(2)} + N_\sigma^{(2)}.
\end{aligned}$$

The first relation in (A.5) and the definition (A.3) imply that

$$\lambda = \frac{1}{|\mu|}\big(r_1^{(2)} - \min_{l\in\mathbf{\Delta}[M]} l_1\big) = \frac{r_1^{(2)} - r_1^{(3)}}{|\mu|} \qquad \text{(A.6)}$$

in view of the formulae (6.6a): $\min_{l\in\mathbf{\Delta}[M]} l_1 = r_1^{(3)}$.

Furthermore, it follows from the first and third relations in (A.5) that

$$N_\sigma^{(1)} = \frac{|\mu|}{f_2}(l_3 - r_3^{(2)}) + \frac{1}{f_2}(l_1 - r_1^{(2)}),$$

whence we obtain

$$\max_{\sigma\in\mathcal{R}_r^\lambda} N_\sigma^{(1)} = \frac{|\mu|}{f_2} \max_{\substack{l\in\mathbf{\Delta}[M]\\ l_1=r_1^{(3)}}} l_3 + \frac{1}{f_2}(r_1^{(3)} - r_1^{(2)} - |\mu| r_3^{(2)}),$$

$$\min_{\sigma\in\mathcal{R}_r^\lambda} N_\sigma^{(1)} = \frac{|\mu|}{f_2} \min_{\substack{l\in\mathbf{\Delta}[M]\\ l_1=r_1^{(3)}}} l_3 + \frac{1}{f_2}(r_1^{(3)} - r_1^{(2)} - |\mu| r_3^{(2)}).$$

Lemma A.1. *Let* (j,k,s) *be a cyclic permutation of* $(1,2,3)$ *and let* l^* *be the point at which* $\min_{l\in\mathbf{\Delta}[M]} l_s$ *is attained. Then the following relations hold*:

$$\frac{1}{f_k}\bigg(\max_{\substack{l\in\mathbf{\Delta}[M]\\ l_s=l_s^*}} l_j - \min_{\substack{l\in\mathbf{\Delta}[M]\\ l_s=l_s^*}} l_j\bigg) = \left[\frac{l_j^*}{f_k}\right] + \left[\frac{l_k^*}{f_j}\right] = \left[\frac{r_k^{(k)}}{f_j}\right],$$

where $r^{(k)}$ *is the vertex* (6.6a) *of the Diophantine skeleton* $\mathbf{\Delta}[M]$.

By Lemma A.1, we have

$$\max_{\sigma\in\mathcal{R}_r^\lambda} N_\sigma^{(1)} - \min_{\sigma\in\mathcal{R}_r^\lambda} N_\sigma^{(1)} = \frac{|\mu|}{f_2}\biggl(\max_{\substack{l\in\mathbf{\Delta}[M]\\ l_1=r_1^{(3)}}} l_3 - \min_{\substack{l\in\mathbf{\Delta}[M]\\ l_1=r_1^{(3)}}} l_3\biggr) = |\mu|\left[\frac{r_3^{(3)}}{f_2}\right].$$

Using the last formula and (A.6), we see from (A.4) that

$$\int_0^\infty \bigl(X_2^{f_2/\mu} G_1(\tau X_2^{f_2/\mu}, X_2)\bigr)\Big|_{X_2=\infty}\,d\tau = (r_1^{(2)} - r_1^{(3)})\left[\frac{r_3^{(3)}}{f_2}\right].$$

We similarly evaluate the integral

$$-\int_0^\infty \bigl(X_1^{-\nu/f_1} G_2(X_1, tX_1^{-\nu/f_1})\bigr)\Big|_{X_1=\infty}\,dt = (r_2^{(2)} - r_2^{(1)})\left[\frac{r_1^{(1)}}{f_3}\right].$$

Summing these two integrals in (A.1), we obtain (9.4).

If all three of the numbers M_{12}, M_{23}, and M_{31} in the representation (6.5) are non-negative, then the coordinates of the vertices $r^{(s)}$ in (9.4) are given by the formulae (6.6b). Substituting them into (9.4), we obtain (9.4a).

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M. V. Karasev
Moscow State University of Electronics
and Mathematics (MIEM)
E-mail: karasev@miem.edu.ru

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E. M. Novikova
Moscow State University of Electronics
and Mathematics (MIEM)
E-mail: novikova@miem.edu.ru