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# ON THE EQUIVALENCE OF SOME SOLUTION CONCEPTS IN SPATIAL VOTING THEORY

We prove that, for a spatial voting setting with convex preferences, the locally uncovered set, proposed by Schofield (1999), is closely related to the dimension-by-dimension median of Shepsle (1979). It is shown that every point in the interior of the locally uncovered set can be supported as a dimension-by-dimension median by some set of basis vectors for the space of alternatives. Moreover, for a two-dimensional policy space, the locally uncovered set and the set of dimension-by-dimension medians coincide. We also introduce a measure that allows to differentiate locally uncovered alternatives and chose the best one.

## Introduction

It is well known that the preference relationship generated by majority voting is not transitive [Plott, 1968]. Several solution concepts have been investigated in theoretical and empirical voting literature. The uncovered set [Miller, 1980; McKelvey, 1984] is a set of all alternatives that beat any other alternative either directly, or with the help of one intermediate alternative (and hence are not «covered»). It is a well-known construction but is difficult to obtain analytically or numerically [Miller, 2007; Bianco, Jeliazkov, Sened, 2004]. The locally uncovered set, or heart [Schofield, 1999; Schofield, Sened, 2006] is another set-based solution concept. A locally uncovered alternative is not covered by any alternative in some small neighborhood around it. The locally uncovered set is much simpler to calculate; in this work, we relate it to a well-known point solution concept – the dimension by dimension median[1], interpreted as a structurally induced equilibrium by Shepsle (1979) and Shepsle and Weingast (1981). Its principal drawback is that it is not invariant to linear transformation of voter ideal points.

The following example illustrates the concept of the dimension-by-dimension median, and how it can depend on the voting agenda. A three-member committee

[1] There are several other solution concepts in the social choice theory, such as the Banks set, the competitive solution.

votes to determine the allocation $x = (x_1, x_2)$ of a budget to two projects. There are three committee members with Euclidian preferences:

$$u_i(x) = -|v_i - x|,$$

where $v_i$ is the best alternative of committee member $i$. Let $v_1 = (1,1)$, $v_2 = (2,1)$ and $v_3 = (1,2)$. Suppose that the committee votes on each issue separately. Fix any value of $x_2$. Then the preferences of all three members are single-peaked with respect to $x_1$. Moreover, the best value of $x_1$ for each member does not depend on the value of $x_2$. Hence, voter 3 is the median voter with respect to the first dimension, with the median position being $x_1^* = 1$. Similarly, the median with respect to the second dimension is $x_2^* = 1$.

Now suppose that the candidates first vote on the combined budget $x_1 + x_2$. The preferences with respect to combined budget will once again be single-peaked, with the best alternatives being $x_1 + x_2 = 2$ for committee member 1 and $x_1 + x_2 = 3$ for members 2 and 3. Hence we have $x_1^* + x_2^* = 3$. The second vote is on the distribution of the agreed-upon budget between the two projects. Here, committee member 1 will be the median voter, preferring $x_1^* = x_2^*$. Finally, this gives us $x_1^* = x_2^* = \frac{3}{2}$.

One can see that the selection of voting agenda can be used to strategically manipulate the outcome of the voting process. But what is the set of outcomes that can be achieved by choosing different voting agendas? Geometrically, the outcomes in the first and second case are related. One can obtain the second outcome by rotating the best alternative of each voter by 45 degrees, calculating the median position along each dimension, and then rotating the resulting dimension-by dimension median back by –45 degrees. Feld and Grofman (1988) proposed a generalized solution concept, the Schattschneider set, as the locus of all such dimension-by-dimension median points that could be obtained by some rotation of the set of voter ideal points. They also derived bounds on this set, relating it to another solution concept – the yolk[2].

It was shown that the Schattschneider set must lie within $\sqrt{K}$ yolk radii from the center of the yolk, where $K$ is the number of policy space dimensions.

Austen-Smith and Banks (2005) suggested a generalization of this concept, by considering not only rotations, but arbitrary nondegenerate linear transformations. They have proven the existence of such a solution for any linear transformation, and for any concave voter preferences. In this work we take a step further, investigating

[2] The yolk is the smallest sphere that intersects all median hyperplanes.

the properties of the generalized Schattschneider set. We find that this set contains the locally uncovered set; for the two-dimensional case, the two sets coincide.

## Results

Let $L = \Re^K$ be a finite-dimensional space of alternatives. Suppose that $I = \{1,\ldots,N\}$ is the set of voters, with $N$ an odd number. Assume that each voter $i \in I$ has a binary preference relationship defined over $L$ that described by a continuously differentiable and strictly concave utility function $u_i : L \to \Re$. Denote by $U : L \to \Re^N$ the profile of utility functions. Denote by $U$ the space of such utility function profiles on $L$. Define by $\succeq$ the majority preference relationship on $L$:

$$For\ every\ \ x, y \in L, x \succeq y \Leftarrow \#\{i \in I \mid u_i(x) \geq u_i(y)\} > \frac{n}{2}.$$

Let $\succ$ be the strict counterpart of $\succeq$.

Denote by $B = (b_1,\ldots,b_K)$ a basis of $\Re^K$. Denote by $B$ the set of all bases. Let $l(x,b_j) = \{x + \alpha b_j \mid \alpha \in \Re\}$ be the line parallel to the $j^{th}$ axis running through $x$. As the utility functions of the voters are strictly quasi concave, the preferences of all voters are single-peaked on any $l(x,b_j)$, for any $x \in L$ and $B \in B$. The following definition can now be given.

**Definition 1.** *Let* $B \in B$ *be a basis. We say that* $x^* \in L$ *is a* dimension-by-dimension median *with respect to* $B$, *if for any* $j \in K = \{1,\ldots,K\}$, *we have* $x^* \succ y$ *for all* $y \in l(x^*, b_j)$, $y \neq x^*$.

In the example given in the introduction, the voting procedure where the committee votes separately on each issue corresponds to basis vectors $b_1 = (1,0)$ and $b_2 = (0,1)$. For the second case, the basis vectors are $b_1 = (1,1)$ and $b_2 = (-1,1)$. For this special case of Euclidian preferences, for any two pairs of basis vectors, the median alternative with respect to the first coordinate does not depend on the value of the second coordinate; the dimension-by-dimension median is $(1,1)$ for the first case, and $\left(\frac{3}{2},\frac{3}{2}\right)$ for the second.

This makes the computation of the dimension-by-dimension median for Eucledian preferences very straightforward.

In the general case, the median with respect to the first coordinate does depend on the value of other coordinates; however, Austen-Smith and Banks (2005, Theorem 5.1) have shown that such median always exist[3]:

[3] Their setting was more general, as they allowed a different social choice rule for each dimension. Here, we only consider majority voting.

**Theorem 1 [Austen-Smith, Banks, 2005].** *Suppose that the strategy space for each candidate is some convex and compact subset of* $L$. *Then for all* $U \in U$ *and* $B \in B$, *there exists a dimension-by-dimension median.*

We now give the definition of local covering.

**Definition 2.** *An alternative* $x \in L$ *is locally uncovered if for every its neighborhood* $N(x)$ *there exists a neighborhood* $\widetilde{N}(x)$ *such that for every* $y \succ x$, $y \in \widetilde{N}(x)$ *there exists* $z \in N(x)$ *such that* $x \succ z \succ y$. *The set*

$$H = \{x \in L \mid x \text{ is locally uncovered}\} \tag{1}$$

is called the *heart* or the *locally uncovered set*.

If an alternative is not locally uncovered, we say that it is locally covered[4]. In terms of sequences it formally means that an alternative $x$ has a neighborhood $N(x)$ such that there exists a sequence $y_\xi \to x$, such that for every $\xi$ conditions $z_\xi \succ y_\xi \succ x$ and $x \succ z_\xi$ are true for no $z_\xi \in N(x)$.

For the two-dimensional case and Euclidian preferences, the locally uncovered set has a neat geometric interpretation that makes it possible to compute the set easily: An alternative belongs to the locally uncovered set if and only if there are two voter ideal points that satisfy two conditions. First, the two voter ideal points and the alternative are not collinear. Second, there exist two median lines, running through the alternative and each of the two voter ideal points. Thus the boundary of the set is formed by some segments of those median lines that run through at least two voter ideal points. As we shall see below for two-dimensional case and any profiles the locally uncovered set coincides with the set of dimension-by-dimension medians defined for all possible bases. Moreover, Lemmas 3 and 2 will give us a tool to construct the basis for any two-dimensional case.

Every locally uncovered point is not Pareto-dominated. Indeed, if $x$ is Pareto dominated by $y$, i.e., if $u_i(y) > u_i(x)$ for all $i$ then there can be no majority coalitions $S, T \subset I$ that there is $z$ such that $u_i(z) > u_i(y)$ for all $i \in S$ and $u_i(z) < u_i(x)$ for all $i \in T$ (otherwise for $i \in S \cap T$ one derives $u_i(y) < u_i(x)$ ).

The locally uncovered set and the uncovered set are geometrically unrelated. One can construct examples (see [Bianko, Jeliazkov, Sened, 2005] of alternatives belonging to the uncovered set, but not to the locally uncovered set, and vice versa.

We now proceed to derive the relationship between the locally uncovered set and the dimension-by-dimension medians. First, we provide several supplementary definitions.

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[4] Schofield (1999) have shown that the locally uncovered set is, in general, nonempty and closed.

Let $S \subseteq I$ be a coalition of voters. Denote by

$$V(x,S) = \{v \in \Re^K \mid \langle \nabla u_i(x), v\rangle > 0 \ \ \forall i \in S\}$$

the set of directions in which all members of $S$ would agree to move away from $x$. Let $k = \dfrac{N+1}{2}$ be the size of the smallest winning coalition. Put

$$W(x) = \bigcup_{S\subseteq I, |S|\geq k} V(x,S).$$

The structure of the set $W(x)$ is similar in shape to the set $\omega(x)$ in a small neighborhood of $x$.

Two properties of $W = W(x)$ are immediately apparent. First, the set $W$ is an open cone, i.e., for all $\lambda > 0$ we have $\lambda W \subseteq W$. In general it is not convex. Second, if $\nabla u_i(x) \neq 0$ for all $i \in I$, then we have

$$W \bigcap (-W) = \varnothing \ \ \& \ \ L = W \bigcup (-W) \bigcup \mathrm{b}d(W \cup (-W)), \tag{2}$$

where $\mathrm{b}d(C)$ denotes the boundary of $C \subset L$.

The following two lemmas present supplementary results.

**Lemma 1.** *If* $x \in H$, *then*

$$\forall v \in W(x) \ \exists w \in \mathrm{c}l(W(x)) \ \textit{such that} \ v + w \notin W(x). \tag{3}$$

Notice that in view of (2) this is not equivalent to $v + w \in -\mathrm{c}l(W(x))$. Here $\mathrm{c}l(A)$ denotes the closure of the set $A \subseteq \Re^K$.

**Lemma 2.** *An alternative* $x \in L$ *is a dimension-by-dimension median for some basis* $B \in B$ *if and only if*

$$\exists v, w \in W(x) \ \textit{such that} \ v + w \notin W(x). \tag{4}$$

Consequently $x \in L$ is not a dimension-by-dimension median for any basis if and only if $W(x)$ is an open half-space.

Now the first main result of this work follows immediately.

**Theorem 2.** *Suppose that* $x \in H$. *Then there exists a basis* $B \in B$ *such that* $x$ *is a dimension-by-dimension median with respect to* $B$.

**Proof of Theorem 2.**

Suppose that some $x \in H$ is not a dimension-by-dimension median for any basis. Then, by Lemma 2, $W(x)$ is an open halfspace. But this implies $W(x) + \mathrm{c}l(W(x)) = W(x)$, which contradicts Lemma 1. *Q.E.D.*

Finally, for the two-dimensional case, the sets of dimension-by-dimension medians and the locally uncovered set are identical. First we state a supplemental result.

**Lemma 3.** *Let $K = 2$. Then an alternative $x \in L$ is a dimension-by-dimension median for some basis $B \in B$ if and only if*

$$\forall v \in W(x)\, \exists w \in W(x) \text{ such that } v + w \notin W(x). \tag{5}$$

Notice that the only difference between (4) and (5) is in the first quantor: existential in (4) and universal in (5).

**Theorem 3.** *Let $K = 2$. Then $x \in H$ if and only if it is a dimension-by-dimension median with respect to some basis.*

For three and more dimensional policy space there may exist a basis for a dimension-by-dimension median even if the alternative is locally covered, as the following example demonstrates.

**Example 1.** Let $L = \Re^3$, $I = \{1,2,3\}$, $x = 0$ and the gradients of the utility functions be:

$$\nabla u_1(x) = (0,0,1), \nabla u_2(x) = (1,-1,1), \nabla u_3(x) = (-1,-1,1).$$

Consider the following basis:

$$b_1 = (1,2,1), b_2 = (-1,2,1), b_3 = (2,1,0).$$

The alternative $x$ is a dimension-by-dimension median according to the basis. Indeed, $\langle \nabla u_1(x), b_1 \rangle = 1$, $\langle \nabla u_2(x), b_1 \rangle = 0$, $\langle \nabla u_3(x), b_1 \rangle = -2$; therefore voter 2 is the median voter according to the basis vector $b_1$. Similarly, a median voter exists for all other basis vectors.

On the other hand, $x$ is not locally uncovered. Moreover, it is Pareto-dominated: if we take $v = (0,0,1)$, we are going to have $\langle \nabla u_1(x), v \rangle > 0$, $\langle \nabla u_2(x), v \rangle > 0$, and $\langle \nabla u_3(x), v \rangle > 0$. It follows that for some $\varepsilon > 0$, alternative $x + \varepsilon v$ is preferred by all three voters to $x$. Hence $x$ is locally covered.

## Discussion

Austen-Smith and Banks (2005) have shown that the dimension-by-dimension equilibrium exists for any set of basis vectors. Their proof, however, was non-constructive and gave no hint to where the median would actually be located. In this work, we prove that the set of dimension by-dimension medians contains the locally uncovered set. This result signifies the importance of both solution concepts in the social choice theory.

## Proofs

**Proof of Lemma 1.**

Without loss of generality put $x = 0$. Take $v \in W$. Let $v_\varepsilon = \varepsilon \dfrac{v}{\| v \|}$. For all sufficiently small $\varepsilon > 0$ we have $v_\varepsilon \succ x$, that is, $\exists R \subset I,\ | R | > \dfrac{N}{2}$, such that $u_i(v_\varepsilon) > u_i(x),\ \ i \in R$. Clearly we have also $v \in V(R)$. According to Definition 2, take $N(x)$ to be the open ball of radius $\delta > 0$, and $\widetilde{N}(x)$ to be the open ball of radius $\varepsilon > 0$. Take $y_\delta \in N(x)$ such that

$$y_\delta \succ v_\varepsilon \succ x \ and\ x \succ y_\delta .$$

From Definition 2 it follows that for any $\delta > 0$ there exists $\varepsilon > 0$ such that the pair $(v_\varepsilon, y_\delta)$ does exist. Hence one can find a sequence $(\varepsilon_\xi, \delta_\xi) \to 0$ such that for $\xi \to \infty$ the following holds:

- we have $\dfrac{y_\xi}{\| y_\xi \|} \to s$ and, as soon as $s$ and $v$ are not collinear (since $s \notin W$ ), we also have $\dfrac{\| y_\xi \|}{\varepsilon_\xi} \to \alpha > 0;$
- we have $\dfrac{y_\xi - v_\xi}{\| y_\xi - v_\xi \|} \to s',$ while $\dfrac{\| y_\xi - v_\xi \|}{\varepsilon_\xi} \to \beta > 0;$
- there exist $S, T \subseteq I,\ \ | S | = | T | = k$ such that $u_i(y_\xi) < u_i(x),\ \ \forall i \in S$ & $u_i(y_\xi) > u_i(v_\xi)$, $\forall i \in T.$

Consider $(iii)$ and the equality

$$\frac{v}{\| v \|} + \frac{\| y_\xi - v_\xi \|}{\varepsilon_\xi} \cdot \frac{y_\xi - v_\xi}{\| y_\xi - v_\xi \|} = \frac{\| y_\xi \|}{\varepsilon_\xi} \cdot \frac{y_\xi}{\| y_\xi \|}.$$

Let $\xi \to \infty$. In the limiting case we obtain

$$s \notin W(x) \;\; \& \;\; s' \in \mathrm{c}l(V(T)) \;\; \& \;\; v + \beta \| v \| s' = \alpha \| v \| s.$$

It remains to put $w = \beta \| v \| s'$. *Q.E.D.*

**Proof of Lemma 2.**

**Necessity.** Suppose that $x = 0$ and that for all $v, w \in W$, we have $v + w \in W \;\Leftarrow\; W + W \subseteq W$. Since $W$ is a cone, the condition $W + W \subseteq W$ implies the convexity of $W$. It follows that both $W$ and $-W$ are open and convex. Now by (2) it is easy to see that $\mathrm{b}d(W)$ is a hyperplane of dimensionality $K-1$. Indeed, applying the separation theorem one can find $h \in \Re^K$ strictly separates $W$ and $-W$, i.e. there exists a real number $\gamma$ such that

$$W \subseteq \{y \in \Re^K \mid \langle h, y \rangle > \gamma\} \;\; \& \;\; -W \subseteq \{y \in \Re^K \mid \langle h, y \rangle < \gamma\}.$$

Clearly, $\gamma = \langle h, x \rangle = 0$. Moreover, assuming that there exists

$$z \in \{y \in \Re^K \mid \langle h, y \rangle > 0\} \setminus W$$

and again applying the separation theorem, one can find $p \in \Re^K$ such that

$$W \subseteq \{y \in \Re^K \mid \langle p, y \rangle > \langle p, z \rangle\}.$$

Now we have constructed the set

$$\{y \in \Re^K \mid \langle p, y \rangle < \langle p, z \rangle\} \cap \{y \in \Re^K \mid \langle h, y \rangle > 0\} \neq \varnothing$$

that has no common points neither with $W$ nor with $-W$, which contradicts (2). The similar is true for $-W$, so we have proven that

$$\mathrm{b}d(W) = \mathrm{b}d(-W) = \mathrm{b}d(W \cup (-W)) = \{y \in \Re^K \mid \langle h, y \rangle = 0\}.$$

Suppose that $x$ is a dimension-by-dimension median for some basis $B$. Now if $W + W \subseteq W$ we must have $l(x, b_j) \subseteq \mathrm{b}d(W)$ for all $j \in K$, since otherwise $l(x, b_j) \cap W \neq \varnothing$ for some $j \in K$, so for all $y \in l(x, b_j) \cap W$ we will have $y \succ x$, that contradicts the choice of $x$. But $l(x, b_j) \subseteq \mathrm{b}d(W)$ is impossible for every $j \in K$ as a set of $K-1$ dimensionality cannot contain a basis of $\Re^K$. Therefore $W + W \subseteq W$ cannot be true for a dimension-by-dimension median.

**Sufficiency.** Let $v, w \in W$ with $v + w \notin W$. One can think also that each linear functional $\langle \nabla u_i(x), \cdot \rangle$ is non-zero on the space spanned by $v$ and $w$: since

$W$ is open then suitable points from the neighborhoods of $v$ and $w$ can always be selected.

Now consider the following linear interval:

$$(v, v+w) = \{\lambda v + (1-\lambda)(v+w) \mid \lambda \in (0,1)\}.$$

Remember that for the right endpoint we have $v + w \notin W$. Define $s = v + w$ if $(v, v+w) \subset W$. If it is not so then find a point $s \in (v, v+w)$ such that $(v,s) \subset W$ and $s \in \mathrm{b}d(W)$. Since $(v, v+w) \not\subset W$, it is possible to find such a point. We have $s \neq 0$, as $v$ and $v + w$ are not collinear. By construction

$$(v,s) \subset W \ \textit{and} \ s \in \mathrm{b}d(W).$$

It now follows that for some majority coalition $S$ we have $s \in \mathrm{c}l(V(x,S))$ and, without lost of generality, $v \in V(x,S)$. It also follows that for some other majority coalition $T$ we have $\langle \nabla u_j(x), s\rangle \leq 0, \ \ j \in T$. So, it follows that there are two majority coalitions $S$ and $T$, such that

$$\langle \nabla u_i(x), s\rangle \geq 0 \ \ \forall i \in S \ \textit{and} \ \langle \nabla u_j(x), s\rangle \leq 0 \ \ \forall j \in T.$$

From this we conclude that for all $y \in l(x,s), \ \ y \neq x$ we have $x \succ y$, or equivalently that $x$ is the median along $l(x,s)$. This is indeed the case, as the maxima of the utility functions for members of $S$ over $l(x,s)$ are located in the ray $l^+(x,s) = \{x + \alpha s \mid \alpha \geq 0\}$, while the maxima for the members of $T$ are located on $l^-(x,s) = \{x + \alpha s \mid \alpha \leq 0\}$.

As both $S$ and $T$ are majority coalitions, their intersection $S \cap T$ is nonempty and forms the set of median voters on $l(x,s)$. Indeed, for any $k \in S \cap T$ we must have $\langle \nabla u_k(x), s\rangle = 0$. Now take and fix some $k \in S \cap T$. It follows that $\langle \nabla u_k(x), v - s\rangle > 0$ since $k \in S$. That implies $\langle \nabla u_k(x), v\rangle > 0 > \langle \nabla u_k(x), w\rangle$ if $s \neq v + w$ and $\langle \nabla u_k(x), v\rangle = -\langle \nabla u_k(x), w\rangle > 0$ for $s = v + w$. As a result we have

$$\langle \nabla u_k(x), v\rangle\langle \nabla u_k(x), w\rangle < 0, \ \langle \nabla u_k(x), s\rangle = 0, \textit{and} \ s \neq 0. \tag{6}$$

Consider another similarly constructed interval $(v, -w)$, where for the right endpoint we have $-w \in -W$. We can carry out for this interval an exercise similar to the above one and find $s' \neq 0$ such that $x$ is the median along $l(x,s')$, and such that for a corresponding median voter $m \in I$ we have[5]

[5] Notice that in this case $s' \neq v$ and $s' \neq -w$ simultaneously.

$$\langle \nabla u_m(x), v\rangle\langle \nabla u_m(x), w\rangle > 0,\ \langle \nabla u_m(x), s'\rangle = 0,\ and\ s' \neq 0. \tag{7}$$

Comparing (6) and (7), we conclude that neither $\nabla u_k(x)$ and $\nabla u_m(x)$, nor $s$ and $s'$ are collinear. The latter statement follows from the fact that both vectors $s$ and $s'$ belong to the two-dimensional space spanned by $v$ and $w$, and they are nontrivial solutions of the equations $\langle \nabla u_k(x), s\rangle = 0$ and $\langle \nabla u_m(x), s'\rangle = 0$ where $\nabla u_k(x)$ and $\nabla u_m(x)$ are non-collinear vectors.

Now put $b_1 = s$ and $b_2 = s'$: they are the first two vectors of the basis $B$ that we are constructing. In order to construct the third vector, consider the subspace $L$ spanned by $\{b_1, b_2\}$, and any $s \in W$, $s \notin L$. Consider an open linear interval $(s, s')$ with $s' \in L \cap -W$, where one can put $s' = -v$. Choose $s$ such that all functionals $\langle \nabla u_i(x), \cdot\rangle$ are non-constant on $(s, s')$. Using the above method one can find $b_3 \in (s, s')$ (from $b_3 \notin W$ and $(s, b_3) \subset W$ ) such that $x \succ y$ for all $y \in l(x, b_3)$. As $b_3 \notin L$, the vectors $\{b_1, b_2, b_3\}$ are linearly independent. Repeating this procedure $K$ times, we construct the required basis $B$. *Q.E.D.*

**Proof of Lemma 3.**

We need to prove that, in the two-dimensional case, condition (4) implies (5). There are two possibilities:

(i) there exists a pair of vectors $v, w \in W(x)$ such that $v + w \in -W$;

(ii) for every pair of vectors $v, w \in W(x)$, such that $v + w \notin W$, we have $v + w \in \mathrm{cl}W$.

Consider (i) and put $s = -(v + w)$. By construction we have $v, w, s \in W$. Any pair of these vectors is non-collinear and forms a basis for the two-dimensional space. One can also see that the space is covered by the cones spanned by those vectors[6]:

$$L = \mathrm{c}on\{v, w\} \cup \mathrm{c}on\{v, s\} \cup \mathrm{c}on\{s, w\}. \tag{8}$$

Also we have $v + w = -s$, $v + s = -w$, $w + s = -v$, but $-v, -w, -s \in -W$. Thus the space is a union of six convex cones. One edge of each cone lies in $W$, another one in $-W$. An arbitrary chosen point $s' \in W$ must lie in one of the cones in the right hand side of (8). Let it be $con\{v, w\} \ni -s$. Then $s'$ has to belong to either $\mathrm{c}on\{v, -s\}$, or $\mathrm{c}on\{-s, w\}$. If, for instance, $s' \in \mathrm{c}on\{v, -s\}$, then $-s \in \mathrm{c}on\{s', w\}$. Hence, there are $\alpha > 0$, $\beta > 0$, such that $w + \alpha s' = -\beta s$. This can be repeated for all possible locations of $s'$ that proves (5) for (i).

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[6] By definition $z \in \mathrm{c}on\{x, y\}$ if there exist $\alpha \geq 0$, $\beta \geq 0$, such that $z = \alpha x + \beta y$.

Analyzing all possibilities via (2) one can conclude that (ii) is possible only then $W$ is open half-plane intersected with a finite number of lines going though the origin. This is also true in the general case because for (ii) we have $\mathrm{c}l(W(x))+\mathrm{c}l(W(x))=\mathrm{c}l(W(x))$. That means that the cone $\mathrm{c}l(W(x))$ is convex and, therefore, $\mathrm{c}l(W(x))$ is a closed half-space.

It follows that (5) is true. Moreover, the fact that $x \in H$ can be checked directly: for $z$, according to Definition 2, one needs take points such that $z-y$ lies on the bisectrice of the angle forming $W$.

**Proof of Theorem 3.**

The set $W$ is a union of pairwise non-intersected open convex cones $K_\xi$, $\xi=1,\ldots,m$. Index these cones so that adjoining cones have adjoining numbers (the last number adjoins with 1). From each cone take vector $a_\xi \in K_\xi$, $\xi=1,\ldots,m$. Consider the cone hulls $M_\xi = \mathrm{c}on\{a_\xi, a_{\xi+1}\}$ of adjoining vectors couples. Notice that angle between vectors $a_\xi$ and $a_{\xi+1}$ is always less than $\pi$. The family of cones constructed in this manner has the following properties:

(i) We have $\cup_1^m M_\xi = \Re^2$, i.e. this is a covering of plane.

(ii) Forevery $\xi$ $a_\xi, a_{\xi+1} \in W$, $\mathrm{c}on\{a_\xi, a_{\xi+1}\} = M_\xi \not\subset W \Rightarrow \exists\ b_\xi \in M_\xi$, $b_\xi \notin W$. One can think that $\exists\, \alpha_\xi > 0$: $a_\xi + \alpha_\xi a_{\xi+1} = b_\xi$.

(iii) For every $\xi$, for every $v \in M_\xi \cap W$ there is $\beta > 0$ such that ether $v + \beta a_{\xi+1} \in r(b_\xi)$, or $v + \beta a_\xi \in r(b_\xi)$, and $\beta \to 0$ for $\| v \| \to 0$ at that. Here $r(b_\xi)$ is a ray spanned the vector $b_\xi$.

(iv) For every $\xi$, for every $z \in r(b_\xi)$, $z \neq x$ one has $x \succ z$ by majority rule, i.e. there is $T_\xi \subseteq I$, $|T_\xi| > \dfrac{N}{2}$ such that $u_i(z) < u_i(x)$ for all $i \in T_\xi$. To see this recall that (for $x=0$ without loss of generality) $z \notin W$ and, therefore, $T_\xi = \{i \in I \mid \langle \nabla u_i(x), z\rangle \le 0\}$ is a majority coalition. However, in the view of strictly concave utilities, one has $u_i(x) > u_i(z)$ for all $i \in T_\xi$.

We can now construct the two neighborhoods in the definition of a locally uncovered alternative, see Definition 2.

We start by constructing the neighborhood $N(x)$. First specify the auxiliary neighborhood via the conditions

$$y \in \mathrm{c}on\{a_\xi, b_\xi\} \cap W \mid \langle \nabla u_i(y), a_{\xi+1}\rangle > \frac{1}{2}\langle \nabla u_i(x), a_{\xi+1}\rangle > 0 \quad \forall \xi, i \in S_{\xi+1},$$

$$y \in \mathrm{c}on\{b_\xi, a_{\xi+1}\} \cap W \mid \langle \nabla u_i(y), a_\xi\rangle > \frac{1}{2}\langle \nabla u_i(x), a_\xi\rangle > 0 \quad \forall \xi, i \in S_\xi,$$

where

$$S_\xi = \{i \in I \mid \langle \nabla u_i(x), a_\xi \rangle > 0\}, \ \forall \xi.$$

For continuously differentiable utilities this method defines an open neighborhood $M(x)$ of $x$. Let $\| y \| < 1 \ \ \forall y \in M(x)$. Now construct a circular neighborhood $N(x)$ with the radius equal to the maximal length of any vector $z = y + \beta a_\xi$ or $z = y + \beta a_{\xi+1}$ such that $y \in M(x)$ and $z$ belongs to $r(b_\xi), \ \forall \xi$. So it is defined as follows

$$N(x) = \{z \in \Re^K \mid \| z \| < \max_\xi \{d_\xi \vee c_\xi\}\}$$

where for $\xi = 1,2,\dots,m$

$$d_\xi = \sup\{\| y + \beta a_\xi \| \mid y \in con\{a_\xi, b_\xi\} \cap W \cap M(x) \ \& \ y + \beta a_{\xi+1} \in r(b_\xi)\},$$

$$c_\xi = \sup\{\| y + \beta a_{\xi+1} \| \mid y \in con\{a_\xi, b_\xi\} \cap W \cap M(x) \ \& \ y + \beta a_\xi \in r(b_\xi)\}.$$

Now construct the neighborhood $\widetilde{N}(x)$. Decompose agents' utilities $u_i(\cdot)$ in a neighborhood of $y$:

$$u_i(z) = u_i(y) + \langle \nabla u_i(y), z - y \rangle + o_i(\| z - y \|), \ i \in I.$$

For $z$, running rays $r(b_\xi)$ and $y$ from $con\{a_\xi, b_\xi\} \cap W$ (or from $con\{b_\xi, a_{\xi+1}\} \cap W$ ), where $z = y + \beta a_\xi$ (or $z = y + \beta a_{\xi+1}$ ) one will have

$$u_i(z) = u_i(y) + \beta \langle \nabla u_i(y), a_\xi \rangle + o_i(\beta), \ i \in I.$$

Here $o_i(\beta) = o_i(\beta)(y)$, i.e. it is a function of two variables, $y$ and $\beta \geq 0$, such that $\dfrac{o_i(\beta)(y)}{\beta} \to 0$ for every fixed $y$ when $\beta \to 0$. Moreover, since all utility functions are $C^1$, this function is continuous. For $y \in M(x)$ and $i \in S_\xi$ one can estimate

$$u_i(z) - u_i(y) = \beta \langle \nabla u_i(y), a_\xi \rangle + o_i(\beta) \geq \beta \frac{1}{2} \langle \nabla u_i(x), a_\xi \rangle + o_i(\beta)(y).$$

Now the neighborhood that we are looking for can be chosen from the condition

$$\frac{o_i(\beta)(y)}{\beta} < \frac{1}{2} \langle \nabla u_i(x), a_\xi \rangle, a_\xi \rangle, \ i \in S_\xi, \forall \xi.$$

This is possible because $\langle \nabla u_i(x), a_\xi \rangle > 0$ and by definition of the infinitesimal $o_i(\beta)(y) = \beta\gamma(\beta, y)$, where $\gamma(\beta, y) = 0$ for $\beta = 0$ and moreover $\gamma(\beta, y)$ is continuous in both variables (utility is continuously differentiable).

So we have constructed the neighborhoods $N(x)$ and $\widetilde{N}(x)$ such that if

$$y \in \widetilde{N}(x),\ y \succ x \Rightarrow$$

$$y - x = v \in W\ \&\ \exists\, \xi : y - x = v \in con\{a_\xi, b_\xi\} \Rightarrow$$

$$\exists\, \beta > 0 : z - x = y - x + \beta a_{\xi+1} \in r(b_\xi) \Rightarrow$$

$$x \succ z \succ y.$$

Finally, if $N'(x)$ is an arbitrary neighborhood of $x$, then, according to Definition 2, take $\delta > 0$ such that $\delta N(x) \subset N'(x)$ and take $\delta\widetilde{N}(x)$ as a neighborhood of $x$ applied for changes of $y$. $Q.E.D.$

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