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Weakly damped KdV soliton dynamics with the random force

N. Zahibo [a], E. Pelinovsky [a,b,*], A. Sergeeva [b]

[a] *Department of Physics, University of Antilles and Guyane, Pointe-a-Pitre, Guadeloupe, France*
[b] *Department of Nonlinear Geophysical Processes, Institute of Applied Physics, 46 Uljanov Street, Nizhny Novgorod 603950, Russia*

Abstract

The soliton dynamics in the random field is studied in the framework of the Korteweg–de Vries–Burgers equation. Asymptotic solution of this equation with weak dissipation is found and the average wave field is analyzed. All formulas can be given explicitly for the uniform (table-top) distribution function of the random field. Weakly damped KdV soliton on large times transforms to the "thick" soliton or KdV-like soliton depending from the statistical properties of the force. New scenario of KdV soliton transformation into the thick soliton and then again in KdV-like soliton is predicted for certain conditions.
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1. Introduction

Forced Korteweg–de Vries equation is written as

$$\frac{\partial \zeta}{\partial t} + 6\zeta \frac{\partial \zeta}{\partial x} + \frac{\partial^3 \zeta}{\partial x^3} = F(x, t, \varsigma) \tag{1}$$

and its generalizations are canonical models of the soliton interaction with external force; see, for instance, pioneer paper by Wadati [26] and also [27,14,1,2,10,3,22,7,8,11,4,12,6,21]. Some analytical solutions demonstrated the major features of the wave dynamics have been obtained [26–28,2,8,24,5,21,16]. In this note, the soliton propagation in the random weakly viscous media is studied in the framework of the forced Korteweg–de Vries–Burgers (KdVB) equation

$$\frac{\partial \zeta}{\partial t} + 6\zeta \frac{\partial \zeta}{\partial x} + \frac{\partial^3 \zeta}{\partial x^3} - v\frac{\partial^2 \zeta}{\partial x^2} = F(t), \tag{2}$$

where v is the viscosity coefficient and the force, $F(t)$ depends only on time. In the context of the water waves in shallow water, the dissipation term in (2) is associated with horizontal diffusion, not with fluid viscosity [9,19]. For the force in the form $F(t)$, Eq. (2) can be reduced to the Korteweg–de Vries–Burgers equation with constant coefficients, and the force action induces the phase fluctuation of the nonlinear waves only. The averaged soliton dynamics is investigated in details for various statistical characteristics. The existence of new scenario of KdV soliton transformation into the thick soliton and then again in KdV-like soliton is predicted.

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* Corresponding author. Address: Department of Nonlinear Geophysical Processes, Institute of Applied Physics, 46 Uljanov Street, Nizhny Novgorod 603950, Russia. Tel.: +7 8312 164839; fax: +7 8312 365976.
E-mail address: pelinovsky@hydro.appl.sci-nnov.ru (E. Pelinovsky).

2. Reduction to deterministic KdVB equation

Forced KdVB equation can be reduced to the classical Korteweg–de Vries–Burgers equation with constant coefficients. The first substitution

$$\zeta(x,t) = \eta(x,t) + Z(t), \quad Z(t) = \int_0^t F(t')\,\mathrm{d}t', \tag{3}$$

reduces Eq. (2) to the variable-coefficient KdVB equation

$$\frac{\partial \eta}{\partial t} + 6Z(t)\frac{\partial \eta}{\partial x} + 6\eta\frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} - v\frac{\partial^2 \eta}{\partial x^2} = 0. \tag{4}$$

The stochastic equation (4) with no dispersion has been studied by Gurbatov et al. [13] in the context of the shock waves. The second substitution

$$x = y + 6V(t), \quad V(t) = \int_0^t Z(t')\,\mathrm{d}t', \tag{5}$$

transforms the stochastic equation (4) to the deterministic KdVB equation with constant coefficients

$$\frac{\partial \eta}{\partial t} + 6\eta\frac{\partial \eta}{\partial y} + \frac{\partial^3 \eta}{\partial y^3} - v\frac{\partial^2 \eta}{\partial y^2} = 0. \tag{6}$$

This simple derivation confirms that the force that depends only on time (random or deterministic) allows to obtain the nonlinear evolution equation in a normal form. It is valid for quadratic nonlinear medium with advective nonlinearity. For instance, this procedure can not be applied to the forced modified Korteweg–de Vries equation (with cubic nonlinearity) which is also actively studied [25,17].

3. Averaged weakly damped soliton

The damping of the KdVB soliton in weakly viscous media in the frame of the deterministic Korteweg–de Vries equation (6) is well studied and the approximated one-soliton solution can be obtained [20,18,9]

$$\eta(x,t) \approx A(t)\mathrm{sech}^2\left[\sqrt{\frac{A(t)}{2}}\left(x - x_0 - 2\int A(t)\,\mathrm{d}t\right)\right], \tag{7}$$

and its amplitude is attenuated as

$$A(t) = \frac{A_0}{1 + 8A_0 vt/15}. \tag{8}$$

It is important to mention that solitons "forget" the initial value of wave amplitude by long-term estimation, and their amplitude is inverse proportional to time. The averaged soliton will also damp in time. The soliton width increases in time

$$L_s \sim \sqrt{\frac{2}{A(t)}} = \sqrt{\frac{2(1 + 8A_0 vt/15)}{A_0}}, \tag{9}$$

and also it "forgets" initial value of soliton amplitude long-term estimated.

The mean field is found by the statistical averaging the function ζ, contained a pedestal and solitary wave. The pedestal is uniform in space and does not influence on the averaged characteristics of nonlinear wave, and the major effect here is the random phase shifts of the soliton in a space. The mean soliton field can be presented in integral form

$$\langle \eta(x,t)\rangle = A(t)\int_{-\infty}^{+\infty}\mathrm{sech}^2\left[\sqrt{\frac{A(t)}{2}}(x - x_0 - 2\int A(t)\,\mathrm{d}t - 6V)\right]W(V,t)\,\mathrm{d}V, \tag{10}$$

with distribution function $W(V,t)$ which we assume to be uniform for simplicity

$$W(V,t) = \frac{1}{2\sigma}\begin{cases} 1 & |V| < \sigma \\ 0 & |V| > \sigma \end{cases} \quad \sigma = \sigma_0 t^\gamma \tag{11}$$

with fixed parameters σ_0 and γ. In this case the integral (10) is calculated explicitly

$$\langle \eta(X,t) \rangle = \frac{\sqrt{2A(t)}}{12\sigma} \left\{ \tanh \left[\sqrt{\frac{A(t)}{2}}(X+6\sigma) \right] - \tanh \left[\sqrt{\frac{A(t)}{2}}(X-6\sigma) \right] \right\}, \tag{12}$$

where

$$X = x - x_0 - 2 \int A(t)\mathrm{d}t \tag{13}$$

is a coordinate (trajectory of soliton motion) moved with a speed of damped soliton. In the variables (prima omitted)

$$\eta' = \frac{\langle \eta \rangle}{A(t)}, \quad x' = \sqrt{\frac{A(t)}{2}}X, \quad \sigma' = \sqrt{18A(t)}\sigma_V(t), \tag{14}$$

averaged soliton shape depends only on one parameter σ (Fig. 1)

$$\eta(x,\sigma) = \frac{1}{2\sigma}\{\tanh(x+\sigma) - \tanh(x-\sigma)\}. \tag{15}$$

If for weak fluctuations the wave is close to the classical KdV soliton shape, then for large fluctuations we obtain the mean filed the so-called "thick" or table-top soliton; this term is used in the theory of large-amplitude solitons for the two-layer fluid described by the same expression (15), see [15,23]. So, the soliton length increases when the fluctuations (phase shifts) are increased, and soliton crest is more flatted. The character of wave evolution in time is determined by the equivalent dispersion, σ', which depends on two function: $\sigma_V(t)$ и $A(t)$ according to (14), and here several scenarios are possible.

If external force, $F(t)$ is delta-correlated, the dispersion of phase shift, V is proportional to $t^{3/2}(\gamma = 3/2)$. If the correlation time is very prolonged (almost constant force), then $\gamma = 2$. In many physical applications statistical characteristics are known directly for the integral from the force, function $Z(t)$, which is for instance the fluctuations of the atmospheric pressure in the context of shallow water waves. If this function is delta-correlated also, then $\gamma = 1/2$. If the correlation time of function $Z(t)$ is prolonged, then $\gamma = 1$. Other values of γ are possible also. For $\gamma > 1/2$ equivalent dispersion increases always instead of soliton amplitude attenuation, and, therefore, the initial KdV soliton is transformed into the table-top soliton with time. Its amplitude is proportional to $A^{1/2}(t)/\sigma_V(t) \sim t^{-\gamma-1/2}$. For instance, in the case of delta-correlated external force $F(t)$ the soliton amplitude decreases as t^{-2}, and its width increases linearly in time. In case of almost constant force, the soliton amplitude decreases more rapidly, as $t^{-5/2}$, and its width grows as $t^{3/2}$. This process is displayed in Fig. 2.

If the process V is stationary (in statistical sense) with constant dispersion, the equivalent dispersion decreases always and the wave will be more deterministic in time. The asymptotic of the nonlinear wave in average is the damped KdV soliton. The external force in this case results to the initial scattering of the solitons in space, but this scattering decrease due to soliton damping. Soliton amplitude is proportional to t^{-1} on large times. In fact this scenario acts for unsteady random processes with $\gamma < 1/2$ also. It is illustrated in Fig. 3.

In fact, the new, third scenario of soliton transformation is possible if $\gamma < 1/2$. The equivalent dispersion after substitution of (9) and (11) is

$$\sigma' = \sigma_0 \sqrt{18A_0} \left(\frac{15}{8A_0 v} \right)^\gamma \frac{\tau^\gamma}{\sqrt{1+\tau}} \quad (\tau = 8A_0 vt/15). \tag{16}$$
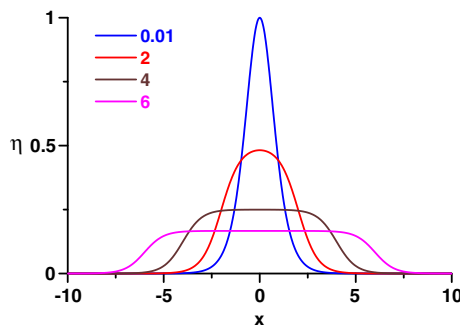

Fig. 1. Averaged shape of the solitary waves for various values of σ.
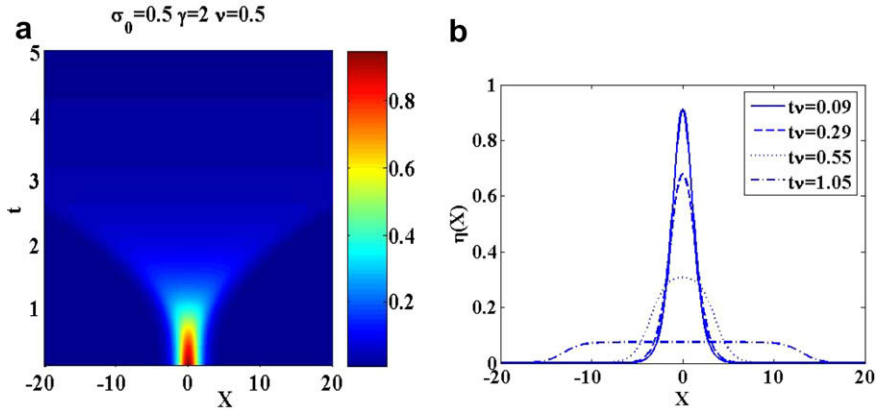
Fig. 2. Transformation of the KdV soliton into table-top soliton: (a) spatial–temporal diagram and (b) wave shaped for different times.
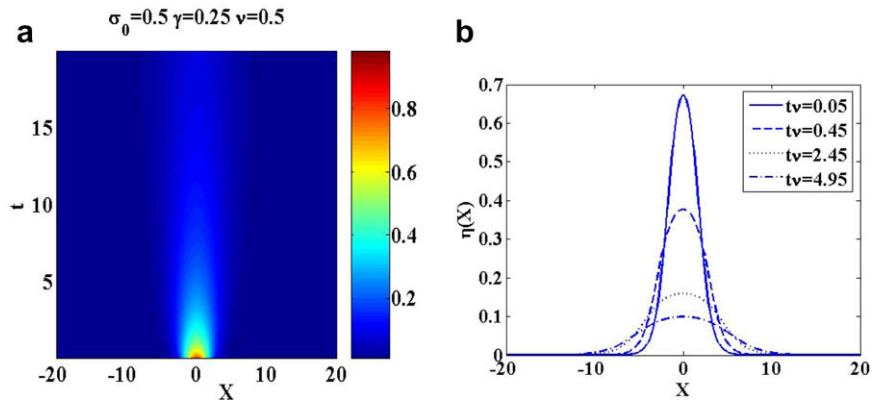

Fig. 3. Transformation of the weakly damped KdV soliton in the case $\gamma < 1/2$: (a) spatial–temporal diagram and (b) wave shape for various times.
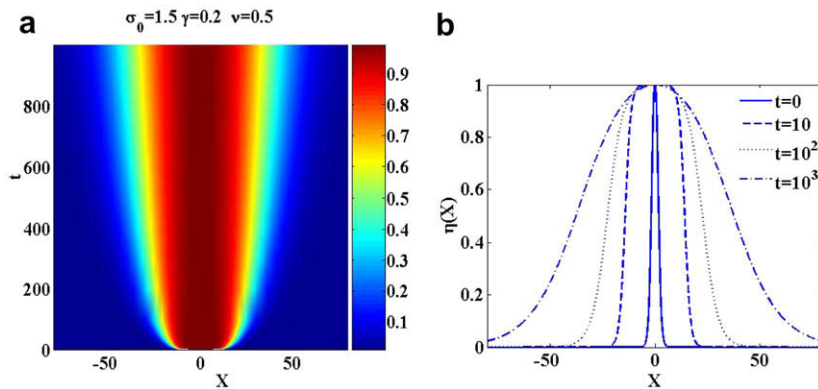

Fig. 4. Transformation of initial KdV soliton into table-top soliton and then again into KdV soliton in the case $\gamma < 1/2$ and big value of maximum of equivalent dispersion: (a) spatial–temporal diagram and (b) wave shape for various times (normalized on the wave amplitude).

Short-term estimated the equivalent dispersion is small, and process is deterministic; soliton amplitude damps due to viscosity. Long-term estimated the equivalent dispersion is small again, and therefore, the process asymptotically is deterministic again, as it was described for the second scenario. But the maximum value of dispersion on the intermediate time

$$\sigma'_{max} = \sigma_0 \sqrt{18 A_0} (2\gamma)^\gamma (1 - 2\gamma)^{1/2 - \gamma} \left(\frac{15}{8 A_0 v} \right)^\gamma \tag{17}$$

can be large. In this case the KdV soliton transforms into the table-top soliton on intermediate times, and then again in the KdV soliton. This process is displayed in Fig. 4. The soliton amplitude decreases as $t^{-\gamma + 1/2}$ on intermediate times and t^{-1} for long-term estimations. The existence of this scenario has not been pointed out in the literature.

The case $\gamma = 1/2$ is marginal, and here the equivalent dispersion is constant long-term estimated. The soliton amplitude decreases as t^{-1}, but the soliton shape will not change. We will not illustrate this marginal case.

4. Conclusion

The soliton dynamics in the random medium is studied in the framework of the forced Korteweg–de Vries–Burgers equation. This equation can be reduced to the constant-coefficient Korteweg–de Vries–Burgers equation in certain conditions. As a result, all solution of stochastic equation can be expressed through deterministic solutions with variable (random) phase. Two asymptotic regimes for weakly damped KdV solitons are predicted depending on the parameter, γ, determined the ratio of correlation length to the soliton width. If this parameter is large, the KdV soliton transforms into the table-top soliton long-term estimated. If this parameter is small, the KdV soliton remains to be the KdV soliton for all times. Both scenarios are separated by the marginal value $\gamma = 1/2$. New scenario is found when the initial KdV soliton transforms into the table-top soliton and then again in KdV-like soliton.

Acknowledgement

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