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Parastatistics and Phase Transition from a Cluster as a Fluctuation to a Cluster as a Distinguishable Object

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Abstract. A phase transition of the first kind is a jump of a function, a phase transition of the second kind is a jump of its first derivative, a phase transition of the third kind, a jump of the second derivative. A phase transition from one statistic to another is very gradual, but finally it is as considerable as the phase transition of the first kind. However, we cannot introduce a clearly defined parameter to which this transition corresponds. This is due to the fact that the fluctuations near the critical point are huge, and this violates, in the vicinity of that point, the main law of equilibrium thermodynamics, which asserts that fluctuations are relatively small.

The paper describes the transition in the supercritical fluid region of equilibrium thermodynamics from parastatistics to mixed statistics, in which the Boltzmann statistics is realized for long-living clusters. In economics this corresponds to a negative nominal credit rate. Examples of this non-standard situation are presented.

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General theories such as probability theory and mathematical statistics are interesting not only as mathematical conceptions. Their value is mainly determined by their practical applications. But the latter are not always easy to implement.

We have compared the results of our own mathematical conception [1, 2] with the data of recent experiments concerning gas and thermodynamics. Here we did not take into account the interaction of particles, and so all our arguments are of general nature. The arguments work in any "independent" system. In probability theory, one has the definition of independent events. But this definition is essentially related to the Boltzmann–Maxwell distribution. In the Kolmogorov complexity theory, a different definition is used, although it is not explicitly stated: independent systems are those systems for which entropy is additive (i.e., the numbers of variants are multiplied).

It is precisely such a general conception that we are constructing, by introducing new notions (variables): the temperature T, the number of degrees of freedom D, the chemical potential μ, or its analog in economics, the nominal credit rate R. All these quantities possess dual values: temperature – entropy, number of particles (or "money," or other objects – people, animals, plants, and so on) – chemical potential. All these quantities constitute a Lagrangian manifold Λ^2 in 4-dimensional phase space.

The main notion is the number N (the number of particles, the amount of money, etc.) which possesses the property of "density." The mean density in a large volume coincides (up to $\sqrt{N}$) with the density in a smaller subvolume. In the language of physics, this means that the fluctuations in density are not very large.

The processes that were described in the framework of this conception were called relaxation processes by the authors. Such processes, in physics, correspond to equilibrium thermodynamics. As we explained, measurements are carried out after one of the variables is changed (perturbed) only after a relatively long passage of time, when the processes return to equilibrium. Therefore, we can say that time does not exist in equilibrium thermodynamics. The tool that returns the system to equilibrium is viscosity, dissipation.

In thermodynamics, however, there is a domain in which these basic assumptions no longer hold. This is the domain of the so-called critical point. The critical point $T_c, P_c, \rho_c = N_c/V_c$ differs for different pure gases. For example, the critical temperature in different gases can differ 1000 fold.

As we have seen, the value of the dimension D at a critical point largely determines the distribution that the corresponding relaxation process obeys.

The paradox of our conception is that it is exactly at the critical point that our main axiom (concerning fluctuations), on which the distribution is based, breaks down. It is precisely in the neighborhood of the critical point that the fluctuations are so large that no longer obey that axiom: they can be considerable greater than $\sqrt{N}$. This implies that we should avoid the neighborhood of the critical point. There a kind of "phase transition" occurs. The Bose–Einstein distribution, as well as the parastatistical distribution, takes place outside some neighborhood of the critical point, and this allows us to avoid it.

In the case when interaction between particles is present, the pasting together with asymptotic formulas on two different sheets [3] or the passage through the Stokes line [4–7] makes it possible to get rid of this discrepancy in the logic of our constructions. But in the present paper, we shall try to avoid the case of interacting particles and so carry over thermodynamics to mathematical statistics.

According to the Large Encyclopedic Dictionary of the Russian Language (T.E. Efremova, Editor), fluctuations are random deviations of a physical quantity from its average value. The average observation depends on the interval of time during which the observation is made. If this interval is sufficiently large (because the observer must wait for the system to reach equilibrium), the fluctuation can "live" for a fairly long time, and for the observer, it will remain a fluctuation. Further, when the life span of the object that we regarded as a fluctuation will be of the order of the observation time, then this object will no longer be considered to be a fluctuation.

This passage from a fluctuation cluster to a cluster as a distinguishable object is the passage to the appearance of the Boltzmann statistics, or the phenomenon of dissipation, viscosity.

Let us note first of all that there are domains where fluctuations and clusters are essentially the same things. And if we consider clusters consisting of $K_c = 200$ particles or less, then from the parastatistical formula, we can find admissible fluctuations.

In the case of negative μ, we can use the following parastatistical formulas for pressure and density (where $b = 1/T$):

$$\begin{aligned} P &= T^{\gamma+2}C\Big(\operatorname{Li}_{\gamma+2}(e^{b\mu}) - \frac{1}{(K+1)^{\gamma+1}}\operatorname{Li}_{\gamma+2}(e^{b\mu(K+1)})\Big) \\ &= \frac{T^{\gamma+2}C}{\Gamma(\gamma+2)}\int_0^\infty \Big(\frac{1}{e^{x-b\mu}-1} - \frac{K+1}{e^{(K+1)(x-b\mu)}-1}\Big)x^{\gamma+1}\,dx, \end{aligned} \tag{1}$$

$$\begin{aligned} \rho &= T^{\gamma+1}C\Big(\operatorname{Li}_{\gamma+1}(e^{b\mu}) - \frac{1}{(K+1)^{\gamma}}\operatorname{Li}_{\gamma+1}(e^{b\mu(K+1)})\Big) \\ &= \frac{T^{\gamma+1}C}{\Gamma(\gamma+1)}\int_0^\infty \Big(\frac{1}{e^{x-b\mu}-1} - \frac{K+1}{e^{(K+1)(x-b\mu)}-1}\Big)x^{\gamma}\,dx. \end{aligned} \tag{2}$$

For the constant C we can take

$$C = \frac{1}{\zeta(\gamma+2)}. \tag{3}$$

For the compressibility factor we have

$$Z = \frac{P}{\rho T} = \frac{\operatorname{Li}_{\gamma+2}(e^{b\mu}) - \frac{1}{(K+1)^{\gamma+1}}\operatorname{Li}_{\gamma+2}(e^{b\mu(K+1)})}{\operatorname{Li}_{\gamma+1}(e^{b\mu}) - \frac{1}{(K+1)^{\gamma}}\operatorname{Li}_{\gamma+1}(e^{b\mu(K+1)})}. \tag{4}$$

When $\mu = 0$, we obtain

$$P = T^{\gamma+2}C\zeta(\gamma+2)\Big(1 - \frac{1}{(K+1)^{\gamma+1}}\Big) = T^{\gamma+2}\Big(1 - \frac{1}{(K+1)^{\gamma+1}}\Big), \tag{5}$$

$$\rho = T^{\gamma+1}C\zeta(\gamma+1)\Big(1 - \frac{1}{(K+1)^{\gamma}}\Big) = T^{\gamma+1}\frac{\zeta(\gamma+1)}{\zeta(\gamma+2)}\Big(1 - \frac{1}{(K+1)^{\gamma}}\Big), \tag{6}$$

$$Z = \frac{\zeta(\gamma+2)(1 - \frac{1}{(K+1)^{\gamma+1}})}{\zeta(\gamma+1)(1 - \frac{1}{(K+1)^{\gamma}})}. \tag{7}$$

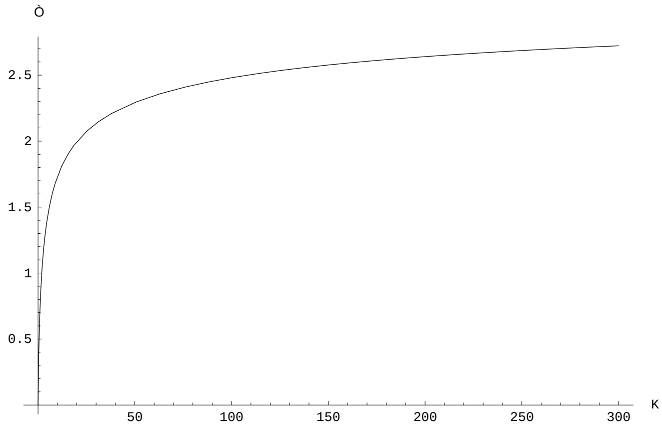

Fig. 1. Three families of isotherms for the initial values $K_c = 98.7$, $K_c = 191$, $K_c = 514$.

Note that, for the isotherm $T = 1$ for $\gamma = 0.222, K = 100$, we see that $Z(\mu = 0) = 0.45$, $P(\mu = 0) = 0.996$ (compare this with formulas (1–3) from the paper [3]).

The value of $K(T_r, \gamma)$ must satisfy the condition

$$\rho_c^{(\gamma)} = \frac{N_c^{(\gamma)}}{V} = C(\gamma)\zeta(1+\gamma) = T_r^{1+\gamma} C(\gamma) \left[1 - \left(\frac{1}{1+K}\right)^{\gamma}\right] \zeta(1+\gamma). \tag{8}$$

This condition leads to a relation for $K(T_r, \gamma)$ when $\mu = 0$ (see (9)).

In this case, we use the dependence for $K(T_r)$, which for $\mu = 0$, is given by the equation

$$T_r = (1 - (K+1)^{-\gamma})^{-1/(\gamma+1)}, \qquad \gamma = \gamma(T_r), \tag{9}$$

with the additional condition $Z|_{\mu=0} = 1$:

$$\frac{1 - (K(T_r)+1)^{-\gamma(T_r)-1}}{1 - (K(T_r)+1)^{-\gamma(T_r)}} \cdot \frac{\zeta(2+\gamma(T_r))}{\zeta(1+\gamma(T_r))} = 1. \tag{10}$$

Here, according to parastatistics, we come to a nearby point $P < P_c$, which differs from P_c by 1/100%. At the same time, the point Z is approximately equal to 0.40, and the value of T is also close to T_c.

The isochore that passes through that point coincides with the experimental isochore for methane. Thus we come to a certain constant K in thermodynamics, and this constant is related to the so-called correlation length (radius), and in the new mathematical statistics, with a certain admissible fluctuation.

We have discussed the critical point in the paper [8], regarding it as a focal inflection point, similar to the point where shock waves and Riemann waves arise, and we discussed the connection between Maxwell's area rule and Riemann's area rule. However, that model, which is adequately described by the tunnel (Wiener) canonical operator, involves only dissipation and the Lagrangian manifold. Actually, at that point, as in a shock wave, vorticities arise together with dissipation. This influences the opaliscence picture near the critical point and rules out the simple passage from nonequilibrium to equilibrium thermodynamics.

In equilibrium thermodynamics, we must go through the closed interval $0.4 \geqslant Z \geqslant 0.2$ on which small variations of one of the parameters lead to huge jumps in the values of the other thermodynamical parameters, so that this interval cannot correspond to equilibrium thermodynamics.

The same picture naturally appears in abstract mathematical statistics. Hence, the number K_c, which determines the domain of unstable oscillations, must characterize the general relaxation step process.

We avoid the interval near the critical point and join by a line segment the corresponding values of the isotherms in the (Z, P)-plane.

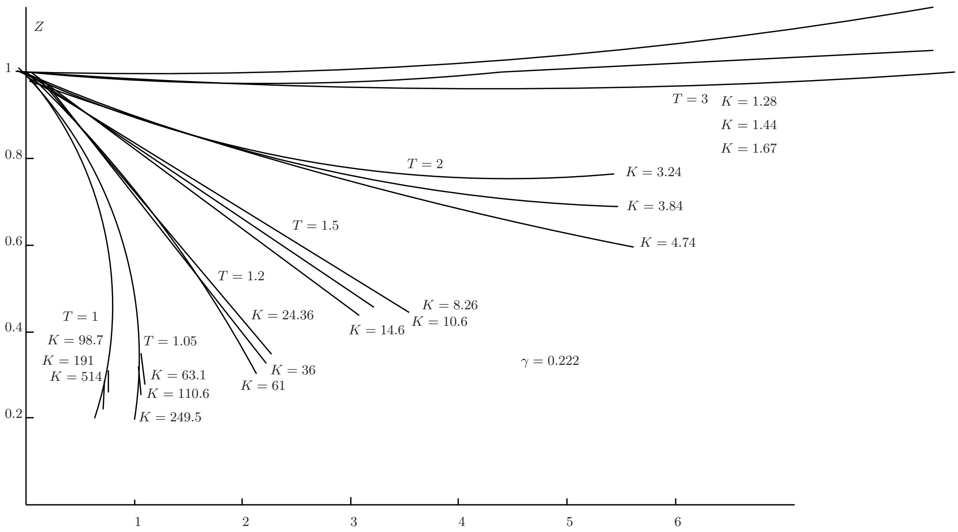

Fig. 2. Three families of isotherms for initial values $K_c = 98.7$, $K_c = 191$, $K_c = 514$ for $\mu \leqslant 0$.

Now if we consider the spectrum of the general Hamilton operator and, at the same time, wish to take into account the pairwise interaction of particles, then, sadly enough, the passage to the new mathematical statistics will not be the kind of rigorous mathematical approach to which the authors adhere, but something called "modeling" in the contemporary language. However, as previously indicated, if we do take into account the interactions, then in the supercritical state, we obtain a two-liquid model, similar to superfluid helium 4.

As is well known, if we rotate such a system of superfluid helium in a cylinder at tempertures lower than the λ point, then it turns out that the normal component (the viscous component) will finally be carried along and will rotate with the cylinder, while the superfluid component will remain in place or at least will behave differently. This means that the normal component passes through superfluidity without viscosity! And conversely – the superfluid component passes through the normal viscous liquid without friction and dissipation.

Landau guessed that this, if we put it in mathematical terms, means that there are two series separated by a barrier. Then Bogolyubov rigorously proved, essentially, that the two series exist. The minimum of the trough of the superfluid series is higher than the lowest point of the normal series. Hence, even at the bottom of their trough, the series is superfluid as compared to the normal series. This situation corresponds to the two-liquid model

As shown in [3], mathematically exactly the same situation arises for supercritical states. Normal clusters constitute a cell-like structure which continuously changes in time (until the jamming effect occurs – "glass with holes" similar to pumice-stone is formed). At the same time, monomers, which correspond to series with higher minimum, go through that cell structure ("normal liquid") without resistance.[1]

Actually, we should not be allowed to use the term "cell-like structure" while clusters are only fluctuations in equilibrium thermodynamics.

We have pointed out more than once that the type of statistics depends on our subjective approach. For example, when we are carrying out observations in equilibrium thermodynamics and await the equilibrium of the system, we can only talk of the percentage of dimers in the gas. In such measurements, we cannot follow each of the dimers. A dimer lives for only 10^{-15} seconds, but when

[1] The simplest way to establish the presence of two series (the two-liquid model) experimentally is, apparently, by determining the second sound in the supercritical fluid.

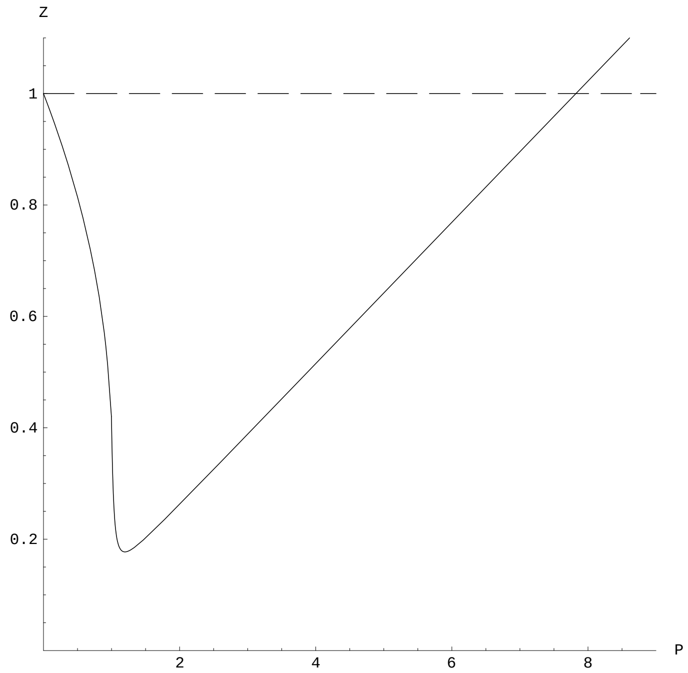

Fig. 3. Theoretical critical isotherm for methane.

we turn on the heating of our houses, we must wait for some hours until the mean temperature (approximately the same everywhere – the equilibrium temperature) is established.

In this case, the statistics of "dimer density" and the statistics in which we follow individual dimers depends on the relationship between the observation time and the monitoring time for separate dimers. Therefore, when these two intervals of time become equal, a "phase" transition occurs with the appearance of dissipation related to the Boltzmann approach.

This means that if the life span of clusters is so large that we can observe it in equilibrium thermodynamics, then the statistics of undistinguishable particles is no longer applicable.

In 2009, the jamming effect for fluids at sufficiently high pressure was established experimentally and justified theoretically by one of the authors. This effect consists in the smooth passage without a phase jump of the first kind to the state of an amorphic solid. This means that the statistics is that of distinguishable clusters. This implies that we must pass from the statistics of undistinguishable particles, which we regard as fluctuations, to that of a part of the particles, the "clusters", transformed into objects with a long life span.

From the fact that the neighborhood of the critical point remains outside of the basic objects of equilibrium thermodynamics because of the huge fluctuations in it, it follows that it is impossible to indicate exactly where the boundary line between cluster-fluctuations and clusters as distinguishable objects lies. We have already mentioned K_c, which is related to the correlation length (see Fig. 2).

Nevertheless, we can indicate the boundary line in a relatively small neighborhood of which the phase transition from undistinguishable statistics to partially distinguishable occurs. This happens in a small neighborhood of the critical isochore that passes through the critical point. After the isotherm passes through this curve, we must partially take into account the Boltzmann statistics, which leads to the law of energetically preferable states according to which particles tend to occupy lower energy levels, for instance, Fermi-particles tend to fill up the Fermi sphere. And this is the analog of the appearance of viscosity (cf. [8]).

The value of $K(T)$ may be approximately determined from the critical isochore for $\mu = 0$ (see [3]).

From the point of view of parastatistics, this critical isochore is a curve on the (Z, P)-plane, with zero chemical potential and $K = K_c$, and it apparently separates the two statistics. Hence, we can determine the value of $K(T)$ only approximately (see Fig. 2).

Mathematically, this means that, in the passage to a positive chemical potential, we must include in the measure of the intergrals under consideration a factor of the form $e^{-A\xi}$, where A is a parameter depending on the arising viscosity and then normalize the obtained expression so that

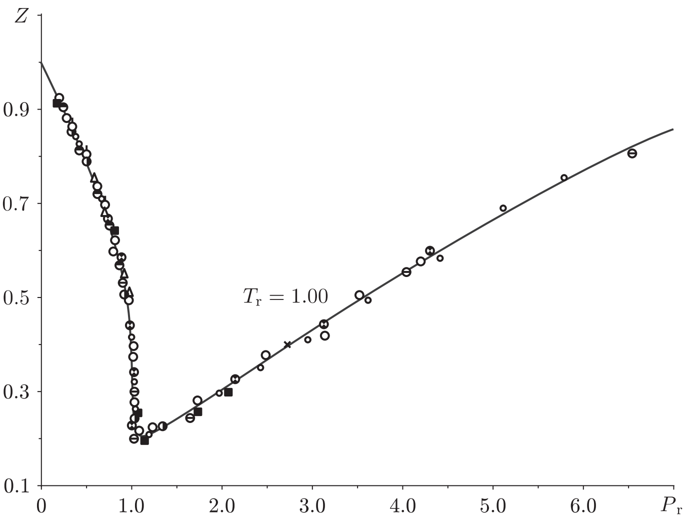

Fig. 4. Experimental isotherm for methane, CO_2, and other gases, indicated by different symbols.

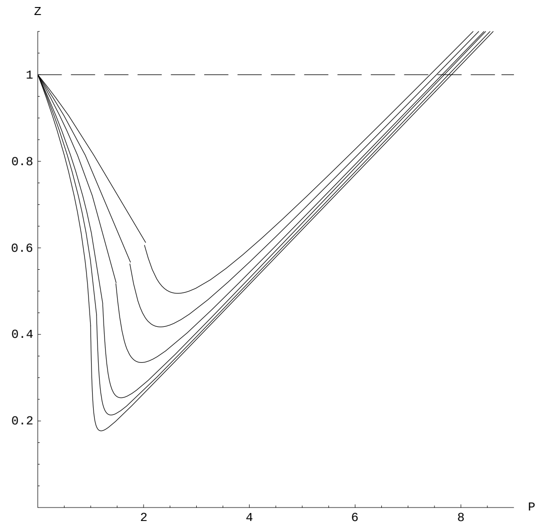

Fig. 5. Theoretical isotherm for methane (for $K_c = 191$) with decreasing dissipation parameter A:$A = 80, A = 54, A = 41, A = 28, A = 24, A = 23$.

will coincide at the point $\mu = 0$ with the statistics for $A = 0$. The smaller are the cluster and the parastatistical parameter K, the lesser will the viscosity be. Therefore, $A(K) \to 0$ as $K \to 0$.

For $\mu > 0$, we pass to the parastatistics of the viscous term. Then for $Z = e^{+\mu/T}$, we have

$$\rho_\sigma(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \left[\frac{1}{z^{-1}e^x - 1} - \frac{K+1}{z^{-(K+1)}e^{(K+1)x} - 1} \right] x^{\sigma-1} e^{-Ax}\, dx, \tag{11}$$

where $A = A(K, \sigma) > 0$.

In Fig. 5, we have chosen $A = 80$ so that the critical isotherm coincides with the experimental isotherm for methane.

Let us consider the economic analogy. As the authors have already indicated [9], to the chemical potential corresponds, in economics, minus the nominal credit rate. The nominal credit rate in Islamic countries equals zero: Islam forbids any gains from lending money. Situations in which the nominal credit rate is negative are very rare. An example is the situation when one has to pay the bank for storing your money, for instance, for renting a box in the bank's safe. The amount of money lying in such boxes is not necessarily known to the bank, but the understanding that there is money in those boxes may influence the financial policy of the bank.

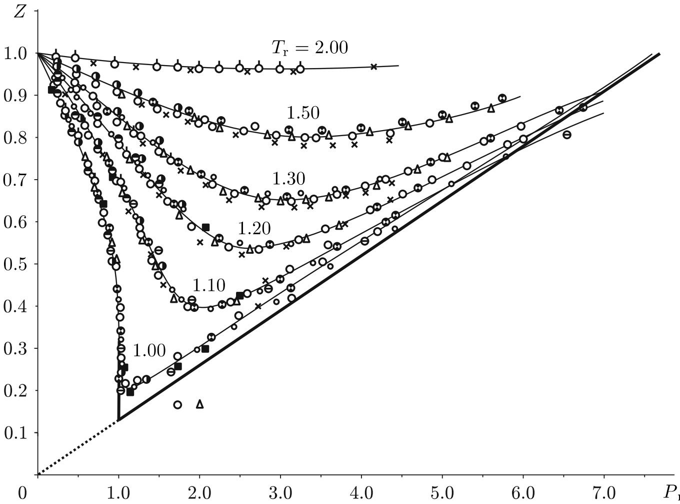

Fig. 6. Experimental isotherms.

Let us note that the money stored in a bank box obeys the Boltzmann statistics: their owner has placed specific bills in the box and expects to get the same bills back, the bank cannot replace the bills by other bills of the same denomination. Thus this example, although it corresponds to negative nominal credit rate, does involve a change in the statistics of undistinguishable objects.

As another example of a negative nominal credit rate, we can consider a fairly reasonable way of calculating the purchasing power of a very poor population (which is, so to speak, in a Bose condensate) if we accept Irving Fisher's point of view and agree that the turnover rate of money corresponds to temperature and energy in thermodynamics.

The giving of credit (lending money) for buying real estate or other goods in order to begin farming and have a regular monthly income could be a reasonable measure for raising the enthusiasm of agricultural workers and increasing their purchasing power. But if the farmer is unable to return his debt in time (within the duration of the loan), this can be regarded as a crime, and the farmer can be punished, e.g., some soft form of compulsory labor.

The Russian poet Maximilian Voloshin once said: "Our property is only what we can give away. What we cannot give away does not belong to us, we belong to it. We are not its owner, it owns us."[2]

This system naturally leads to to the dissipation of riches from the rich to the poor. Here the constant A must be large enough for the loan with negative nominal credit rate not to fall into the hands of people excessively high turn over possibilities.

The following natural question arises: Are these two statistics separated by a barrier? Won't superconducting fluids be a two-liquid model of Tiess–Landau type [10]? Can we capture a second sound in the supercritical statistical system as the mixture of two statistics?

If there are repulsive interactions between the particles of gas, then the potential in the case of the most general Hamiltonian [11] leads to a model of cell-like structure for the "normal" supercritical liquid-fluid through which monomers pass in a superfluid way. But despite the very general character of the Hamiltonian considered in [11], this model cannot be used for ideal noninteracting objects. The general Hamiltonian considered in [11] possibly allows to construct the interaction between subjects of society, where small businesses will move in a superfluid way among the "cells" of big business. The less will this depend on the interactions and the more it will depend on uniting the two statistics, the more feasible will such a scenario be.

In the papers [3] and [12], it was shown that, in the case of interactions, a two-liquid fluid consisting of superfluid monomers and cell-like clusters appears. We conjecture that independently of interaction in the presence of the two entropies – the Boltzmann entropy and the parastatistical entropy – the second sound will always occur.

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[2] *Memories of Maximilian Voloshin*, Moscow, "Sovetskii Pisatel," 1990, p. 450.

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