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Numerical Research of the Optimal Control Problem in the Semi-Markov Inventory Model

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Abstract. This paper is devoted to the numerical simulation of stochastic system for inventory management products using controlled semi-Markov process. The results of a special software for the system's research and finding the optimal control are presented.

Keywords: optimal control, inventory model, numerical approximation
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INTRODUCTION

Stochastic modelling of economic systems designed for temporary storage and delivery directly to consumers of certain products (goods) is a relevant and useful task. Often the optimal control in such systems can be reduced to the determination of optimal values of the parameters of probability distributions or stochastic processes. These values are extremum of some given quality of performance management.

In this paper we investigate inventory management system, in which the consumption of the product occurs at a predetermined constant speed. Control parameter is the time from random replenishment order until the next replenishment. Replenishment depends on the system before and after updating, as well as possible random deviations from the planned volume of delivery. To describe this system two random process are used: the main stochastic process (value is the amount of the stock of product in the system at the time) and the accompanying controlled semi-Markov random process with a finite set of states.

The optimal control strategy is a deterministic. It is defined by specific values of the control parameter corresponding to each state of accompanied by a semi-Markov process. Formally, a set of optimal values of the control parameter is the point of the global extremum of a function of several real variables. One of the most important thing in the studying of such systems is a practical implementation of the theoretical methods. The paper presents a special software created in MATLAB, which allows to simulate inventory management systems and determine the optimal values of the control parameters for different initial characteristics of the model.

MODEL

Consider a system used for storing and delivery of some inventory. The exact value of inventory in the moment $t > 0$ is defined by a stochastic process $x(t) \in (-\infty, \tau]$, where τ is the maximum of a storage capacity. A negative value of inventory is related to inventory deficiency. Inventory consumption rate is a constant and equals $\alpha > 0$.

Discretization procedure is used in the model. The set $(-\infty, \tau]$ as the finite union of its subsets has the following form:

$$H_{N_1}^{(-)} \bigcup \dots \bigcup H_0^{(-)} \bigcup H_0^{(+)} \bigcup \dots \bigcup H_{N_0}^{(+)}, \tag{1}$$

where subsets $H_0^{(-)} = \left(\tau_1^{(-)}, \tau_0^{(-)}\right); H_k^{(-)} = \left(\tau_{k+1}^{(-)}, \tau_k^{(-)}\right], k \in \{1, \dots, N_1\}$ characterize a value of the inventory deficiency and subsets $H_k^{(+)} = \left[\tau_k^{(+)}, \tau_{k+1}^{(+)}\right), k \in \{0, \dots, N_0 - 1\}; H_{N_0}^{(+)} = \left[\tau_{N_0}^{(+)}, \tau_{N_0+1}^{(+)}\right]$ characterize a value of the positive

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real inventory. We assume that $\tau^{(-)}_{N_1+1} = -\infty; \tau^{(-)}_0 = \tau^{(+)}_0 = 0; \tau^{(+)}_{N_0+1} = \tau$.

Let $\{t_n\}_{n=0}^{\infty}$ are random moments of inventory replenishments of the system, $\{t_n^{'}\}_{n=0}^{\infty}$ are random moments of inventory replenishment orders. If $x(t_n) = x \in H_i^{(+)}, i \in \{0,1,\ldots,N_0\}$ then the time, when replenishment order is done, is described by a random variable ξ_i with the cumulative distribution function $G_i(t)$. We assume that $t_n < t_n^{'} < t_{n+1}, n \in \{0,1,\ldots,\}; t_0 = 0$ due to a random time of delivery delay. Namely, if $x(t_n^{'}) = x - \xi_i \in H_k^{(+)}, k \in \{0,1,\ldots,i\}$ and $x(t_{n+1}) \in H_l^{(+)}, l \in \{k,k+1,\ldots,N_0\}$ then the delivery delay time is a random variable $\eta_{kl}^{(+)}$ with given 1-st moment $\mu_{kl}^{(+)} = \mathbf{E}\eta_{kl}^{(+)} < \infty$. Similarly, if $x(t_n^{'}) = x - \xi_i \in H_k^{(-)}, k \in \{0,1,\ldots,N_1\}$ and $x(t_{n+1}) \in H_l^{(+)}, l \in \{0,1,\ldots,N_0\}$, then the delivery delay time is described by a random variable $\eta_{kl}^{(-)}$ with given 1-st moment $\mu_{kl}^{(-)} = \mathbf{E}\eta_{kl}^{(-)} < \infty$.

From the foregoing it is apparent that the inventory replenishment procedure is related to $x(t)$ transition between admissible subsets. The exact subset (as the result of a transition) is defined according to the following probabilistic characteristics of the system:

$\left\{\beta_{kl}^{(+)}\right\}_{l=k}^{N_0}$ are transition probabilities from $H_k^{(+)}$ to $H_l^{(+)}$, where $k \in \{0,1,\ldots,N_0\}$;

$\left\{\beta_{kl}^{(-)}\right._{l=0}^{N_0}$ are transition probabilities from $H_k^{(-)}$ to $H_l^{(+)}$, where $k \in \{0,1,\ldots,N_1\}$.

As is clear from the above, the inventory value after replenishment is always positive. Note that the exact inventory value after replenishment is defined by probabilistic distributions $B_l(x)$, given for each of the subsets $H_l^{(+)}, l \in \{0,1,\ldots,N_0\}$.

For the rules of the system functioning described above, the Markov property holds for the process $x(t)$ in moments $t_n; t_n^{'}, n \in \{0,1,\ldots,\}$.

Denote an auxiliary semi-Markov process with the finite set of states by $\zeta(t), t \geq 0$. Consider a sequence of random variables $\{\zeta_n\}_{n=0}^{\infty}$. Let $\zeta_n = k$ if $x(t_n + 0) \in \left[\tau_k^{(+)}, \tau_{k+1}^{(+)}\right)$, where $k \in \{0,1,\ldots,N_0\}$. The defined sequence $\{\zeta_n\}_{n=0}^{\infty}$ is a Markov chain. The process $\zeta(t)$ related to the sequence $\{\zeta_n\}_{n=0}^{\infty}$ by the following interrelation

$$\zeta(t) = \zeta_n, t_n \leq t < t_{n+1}, n = 0,1,2,\ldots \tag{2}$$

is a controlled semi-Markov process with finite set of states $E = \{0,1,\ldots,N_0\}$.

The process $\zeta(t)$ is controlled in moments $t_n, n = 0,1,2,\ldots$ The control parameter (decision) u_n is a time interval between t_n and $t_n^{'}$. Exactly, $u_n = \xi_k$ if $\zeta_n = k$. Set $U = [0,\infty)$ is a admissible set of actions in moments t_n.

OPTIMAL CONTROL PROBLEM FORMULATION

Consider some additive cost functional related to the process $\zeta(t)$. Denote value of $\zeta(t)$ in moment $t_n + 0, n = 0,1,2,\ldots$ by V_n, and its increment on the interval between decision epochs $(t_n, t_{n+1}]$ by $\Delta V_n = V(t_{n+1}) - V(t_n)$, where $\Delta t_n = (t_{n+1} - t_n)$ is the duration of this interval. Denote the mathematical expectation of additive cost functional for time $t > 0$ by $V(t)$. It is well known [1] that

$$I = \lim_{t\to\infty} \frac{V(t)}{t} = \frac{\sum\limits_{i=0}^{N_0} d_i \pi_i}{\sum\limits_{i=0}^{N_0} T_i \pi_i}, \tag{3}$$

where $\{\pi_0,\ldots,\pi_{N_0}\}$ are steady-state probabilities of Markov chain $\{\zeta_n\}_{n=0}^{\infty}$; $d_i = \mathbf{E}[\Delta V_n | \zeta_n = i]$, $T_i = \mathbf{E}[\Delta t_n | \zeta_n = i], i \in \{0,1,\ldots,N_0\}$ are conditional mathematical expectations.

The average cost functional $I = I(G_0(\cdot), G_1(\cdot), \ldots, G_{N_0}(\cdot))$ defined by (3) is considered as the system control quality indicator. Thus, the optimal control problem for the considered model can be formulated as the extremum problem without restrictions:

$$I = I(G_0(\cdot), G_1(\cdot), \ldots, G_{N_0}(\cdot)) \to extr, \quad G_i \in \Gamma, i \in \{0,1,\ldots,N_0\}, \tag{4}$$

where Γ is a set of probability distribution functions defined on the set of admissible actions $U = [0,\infty)$.

SUMMARIZED THEORETICAL RESULTS

The problem of the optimal control (4) was studied in works [2, 3]. Let us present summarized theoretical results.

The main probabilistic characteristics of the model (in the right part of equation (3)) are obtained in the analytic form. The characteristics are represented as functions by control parameters. The average cost functional (3) is a fractionally linear integral functional of probability distributions $G_0(\cdot), G_1(\cdot), \ldots, G_{N_0}(\cdot)$. In addition, the numerator and denominator integrands of this functional have analytic form based on probabilistic characteristics of the model. We consider ratio between numerator and denominator as the main function of the fractionally linear functional.

From the unconditional extremum theory for fractionally linear functionals it is well known that if main function achieves global extremum at a point $(u_0^*, \ldots, u_{N_0}^*)$ from set of admissible decisions U, then solution of problem (4) exists and corresponds to the probability distributions $G_0(\cdot), G_1(\cdot), \ldots, G_{N_0}(\cdot)$ concentrated at points $u_0^*, u_1^*, \ldots, u_{N_0}^*$ respectively. Thus, to find the global extremum of main function we should proove the existence of solution in problem (4) and find the solution defined by the point $(u_0^*, \ldots, u_{N_0}^*)$.

NUMERICAL EXAMPLES

In this section we present result of numerical calculations of the main function values (3) and try to approximate the point of its global extremum.

The described methods have been implemented in software $Inventory(\cdot)$ created with a help of MATLAB built-in programming language. This software has a high degree of versatility. The number of control parameters (the dimension of the model), the numerical characteristics of probabilistic and deterministic dependency model, kind of cost and gain functions, output parameters can easily be changed.

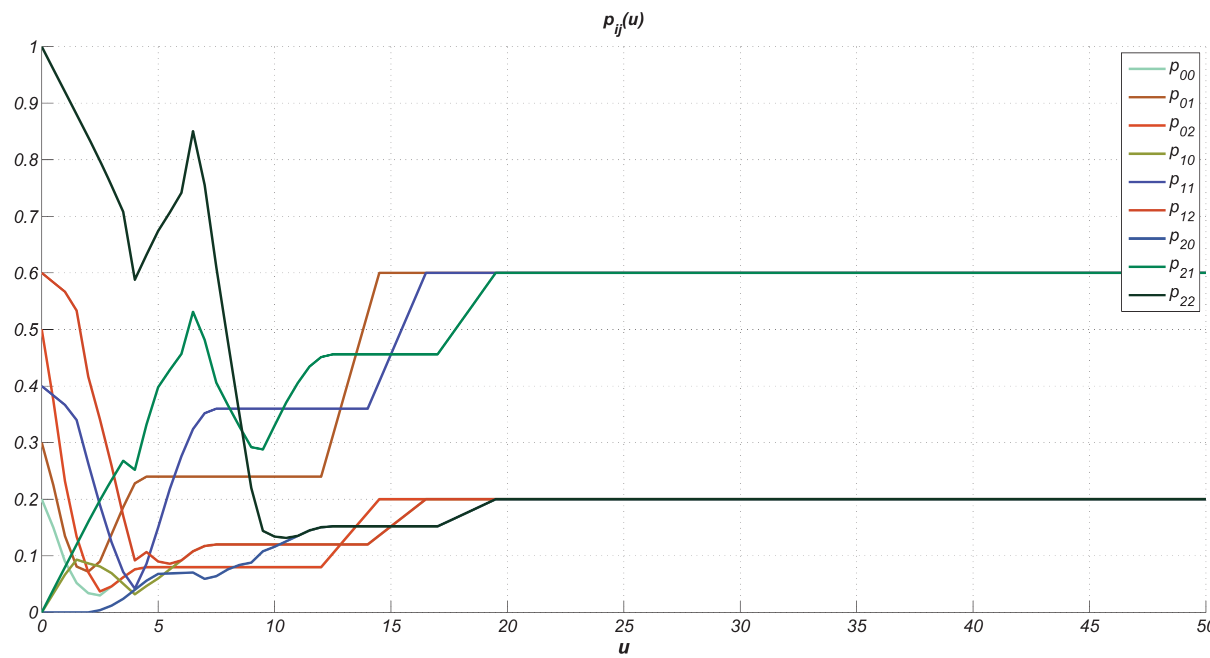

FIGURE 1. Transition probabilities

The program implies not only numerical computations but the symbolic ones too (e.g., functions for the conditional expectations of profit can be found analytically). To realize work with symbolic expression the *int* function of MATLAB symbolic toolbox is used (see `http://www.mathworks.com/help/symbolic/int.html`). One of the most important blocks is a numerical simulation, in particular, the calculation of the integrals by the function *quadgk* [4]. Numerical objective function transfered to the optimization block for finding of extremum (i.e., optimal control parameter values). Output unit can display the graphical results of intermediate functions (Figure 1) and the main function with required profiles (Figure 2).

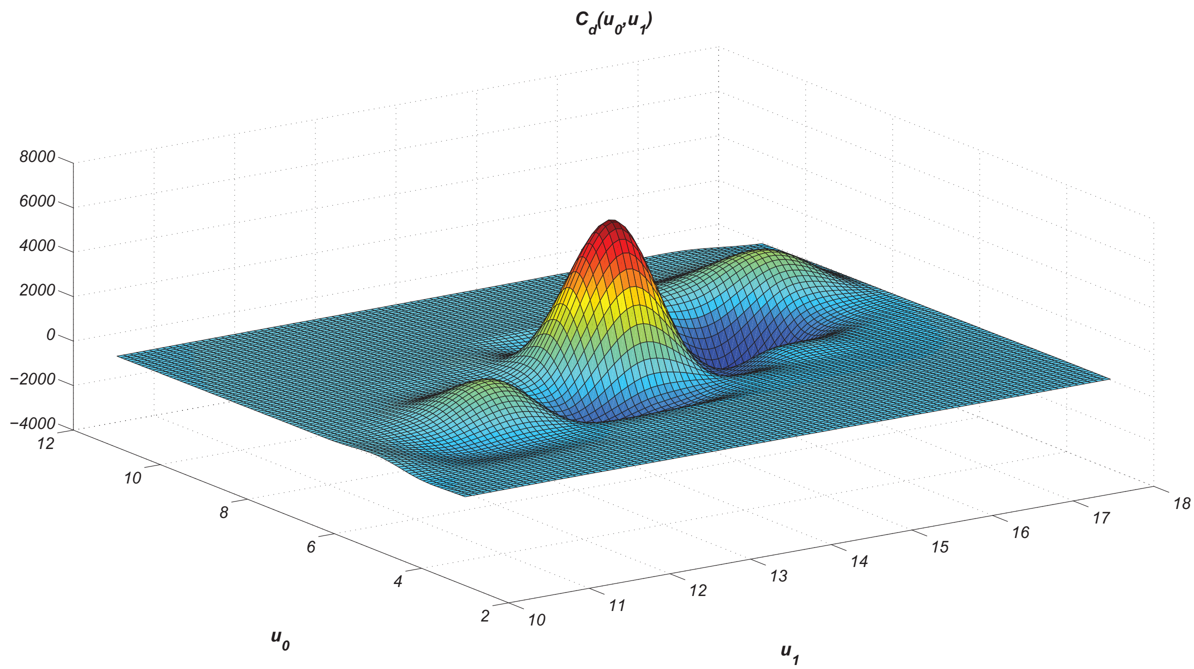

FIGURE 2. Profile of the optimal solution

Investigating the model system it was shown that the objective function had a complicated and weakly ordered shape. So it greatly reduces the efficiency of the well-known and implemented in MATLAB methods. Good results have been obtained for the dimensions of 2 and 3. For next dimensions time of extremum seeking increases significantly. Further work may be related to detailed research of the objective function to produce special optimization techniques for this case. Of course, such an approach in itself represents a separate major challenge.

CONCLUSIONS

For the investigated economic system it was shown that the quality control could be represented in the bilinear functional form. Analytical form of the objective function integrands were found. The finding of global extremum of main objective function determines solution of the problem of optimal control.

The special software developed in MATLAB allows to simulate inventory management system and determine the optimal values of the control parameters for different initial data. It is possible to image the results of computing in numerical, symbolic and graphic forms. The next challenge may relate to development of optimization block for larger dimensions.

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