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A Note on Optimality Conditions for Multi-objective Problems with a Euclidean Cone of Preferences

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Abstract The paper suggests a new—to the best of the author’s knowledge—characterization of decisions, which are optimal in the multi-objective optimization problem with respect to a definite proper preference cone, a Euclidean cone with a prescribed angular radius. The main idea is to use the angle distances between the unit vector and points of utility space. A necessary and sufficient condition for the optimality in the form of an equation is derived. The first-order necessary optimality conditions are also obtained.

Keywords Pareto optimality · Cone of preferences · Scalarization

Mathematics Subject Classification 90C29

1 Introduction

Optimization problems with several objective functions conflicting with one another are encountered in many situations in practice. In analyzing such a problem, the concept of Pareto-optimal decisions, which cannot be improved for each criterion without deteriorating the others, plays an important role. The Pareto optimality notion is used in solving some engineering and finance problems (see, e.g., Steuer [1]), insurance theory problems (e.g., Golubin [2]), etc. There is a large variety of methods for deter-

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mination of Pareto-optimal solutions and their generalizations, in which the optimality is understood with respect to various kinds of preference cones, Miettinen [3] and Jahn [4]. Many of them are based on a scalarization approach [3,5–7] that transforms the initial problem into a single-objective optimization problem. Usually, it involves some parameters that are changed in order to detect different Pareto-optimal points: weights in the linear scalarization function in [6] or, more generally, a composition of the vector objective function and a linear functional from the dual cone of the preference (ordering) cone; and norm parameter for L_p -scalarization in Nikulin et al. [7]. Another group of methods for approximating the Pareto frontier for various decision problems with a small number of objectives (mainly, two) are provided in Ruzika and Wiecek [8]. Makela et al. [9] investigate different types of zero-order geometric conditions for characterization of trade-off curves. In Giannessi et al. [10], equivalence relations between vector variational inequalities and the multi-objective optimization problems, including the existence of their solutions, are discussed.

The present paper suggests a new “angle distance” scalarization technique for the multi-objective problem with a definite kind of preference cones, the so-called Euclidean cones. In distinction to the existing scalarization methods for non-concave multi-objective problems, which detect only a part of all optimal points, we prove that the set of all optimal solutions to the multi-objective problem coincides with the set of all solutions to some equation derived in the paper. Note that the equation does not use any exogenously specified parameters. The first-order necessary conditions for optimality are also derived.

2 Model Description

Formally, the multi-objective optimization problem can be written as

$$\max F(x) := (F_1(x), \dots, F_n(x)) \quad \text{s.t. } x \in X, \tag{1}$$

where X is a decision set or a set of admissible points and $F_i(x)$ are scalar objective functions or utilities defined on X . Remark that $F_i(x)$ are not supposed to be concave.

A generalization of the well known Pareto optimality notion is the following (see, e.g., Boyd and Vandenberghe [11, p. 174]): Let K be a proper cone, i.e., it is convex, closed, the interior $\text{int } K \neq \emptyset$, and K is pointed, (if $x \in K$ and $-x \in K$ then $x = 0$). A point $x^* \in X$ is called optimal with respect to the cone of preferences K (or K -optimal) if and only if (iff) there is no other $x \in X$ such that

$$F(x) \neq F(x^*) \text{ and } F(x) - F(x^*) \in K. \tag{2}$$

A point $x^* \in X$ is called weak K -optimal iff there is no other $x \in X$ such that

$$F(x) - F(x^*) \in \text{int } K. \tag{3}$$

Our goal is to find necessary and/or sufficient conditions for optimality in problem (1) for a concrete kind of the cone K , which is introduced below. The concept of

Pareto optimality has its root in economic equilibrium and welfare theory. In the economic terms, we try to explain specifics of the suggested modification of the Pareto optimality notion. Given a set X of alternative allocations of goods or income for a set of n individuals or members of a community, a change from one allocation x to another y is reckoned as an “ideal” improvement if each member increases his/her own utility by the same quantity. This means that the increment vector $F(y) - F(x)$ lies on the half-line L^I , generated by the unit vector $(1, \dots, 1)^T \in \mathbb{R}^n$ and originated from zero (furthermore, we will use the normalized variant of the unit vector, $r := (1/\sqrt{n}, \dots, 1/\sqrt{n})^T$). A change is considered an improvement if a measure of discrepancy between $F(y) - F(x)$ and the “ideal” improvement is not greater than a prescribed constant a . An allocation is K -optimal when no further improvement can be made.

The measure of discrepancy, which defines the very cone K of preferences, is proposed to be the following. Let p_1 and p_2 be, correspondingly, the orthogonal projections of $F(y) - F(x)$ on the “ideal equality” half-line L^I and on the hyperplane orthogonal to the vector r . The measure of discrepancy is the norm of p_2 per unit of the norm of p_1 , that is, $\|p_2\|/\|p_1\|$, where $\|z\|$ is the Euclidean norm in \mathbb{R}^n , $\|z\| = \sqrt{\sum_1^n z_i^2}$. Passing to the angle distance, in our case, we have that the above-mentioned discrepancy constraint is

$$\tan(F(y) - F(x), r) := \sqrt{1 - \cos^2(F(y) - F(x), r)} / \cos(F(y) - F(x), r) \leq a.$$

Recall, the cosine of the angle between non-zero vectors x and y is given by

$$\cos(x, y) = \frac{\langle x, y \rangle}{\|x\| \|y\|},$$

where $\langle x, y \rangle$ denotes the scalar product, $\langle x, y \rangle = \sum_1^n x_i y_i$. In terms of cosine, the latter inequality is expressed as $\cos(F(y) - F(x), r) \geq s$, where $s = 1/\sqrt{a^2 + 1}$. Now we define the preference cone K as

$$K(s) := \{x \in \mathbb{R}^n : \cos(x, r) \geq s\} \cup \{0\} \tag{4}$$

under a given $s \in]0, 1[$. Thus, $K(s)$ is a set of vectors x such that the angle between the “ideal” direction r and each x is not greater than $\arccos s$ (see Fig. 1). Such a cone is called in Boyd and Vandenberghe [11, p. 449] a Euclidean cone with the axis r and angular radius $\arccos s$.

Next we impose some reasonable lower and upper boundaries on values of s —it is the same as imposing upper and lower boundaries on the discrepancy limit a . The cone of the largest angular radius is supposed to include the n ords $(0, \dots, 0, 1, 0, \dots, 0)^T$ of the Pareto preference cone \mathbb{R}_+^n as boundary points. Hence, the cosine of the angle between r and each ord is $1/\sqrt{n} = s$. The cone of the smallest angular radius is supposed to include the n orthogonal projections of r on the coordinate planes $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ as boundary points. Therefore, the cosine of the angle between r and each projection is

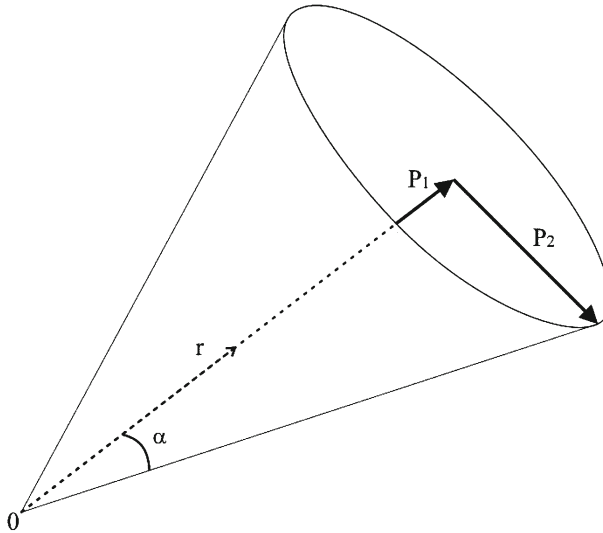


Fig. 1 A preference cone $K(s)$ with angular radius $\alpha = \arccos s$.

$$\frac{\sum_{i=2}^n 1/\sqrt{n}}{\sqrt{\frac{n-1}{n}}} = \frac{\sqrt{n-1}}{\sqrt{n}} = s.$$

Roughly saying, this cone is “inscribed” of \mathbb{R}_+^n , while the previous cone is “circumscribed” of \mathbb{R}_+^n . Then, a collection of the preference cones to be considered is

$$\left\{ K(s), s \in S := \left[1/\sqrt{n}, \sqrt{n-1}/\sqrt{n} \right] \right\},$$

where $K(s)$ is defined by (4). It is worth noting that, if $n = 2$ (bi-objective optimization problem (1)), then the set $\{K(s), s \in S\}$ consists of a single cone \mathbb{R}_+^2 as the interval $S = [1/\sqrt{n}, \sqrt{n-1}/\sqrt{n}]$ converts into a point $1/\sqrt{2}$, so the $K(s)$ optimality coincides with the Pareto optimality notion. This particular case was studied in Golubin [12]. Note also that in the case $n > 2$, none of the cones in $\{K(s), s \in S\}$ coincide with \mathbb{R}_+^n . Indeed, if $s > 1/\sqrt{n}$, then the ort $x = (1, 0, \dots, 0)^T \in \mathbb{R}_+^n$ does not belong to any $K(s)$ as

$$\cos(x, r) = \frac{\langle x, r \rangle}{\|x\|} = \frac{1}{\sqrt{n}} < s.$$

If $s = 1/\sqrt{n}$, then a vector $x = (-(n-2)/2, 1, \dots, 1)^T \notin \mathbb{R}_+^n$ is a boundary point of $K(s)$ as $\cos(x, r) = 1/\sqrt{n}$.

Denote by K_U and K_L the biggest cone $K(1/\sqrt{n})$ and the smallest cone $K(\sqrt{n-1}/\sqrt{n})$ correspondingly. We call a point x^* *upper (lower) optimal* iff x^* is optimal in (1) with respect to K_U (K_L).

From the definition, it is easily seen that all the components of any non-zero $x \in K_L$ are necessarily non-negative, and at most one of them is zero. Indeed, let $x = (0, \dots, 0, x_{i+1}, \dots, x_n)^T$ have $i (\geq 2)$ zero components. Then,

$$\cos(x, r) \leq \frac{\sqrt{n-i}}{\sqrt{n}} < \frac{\sqrt{n-1}}{\sqrt{n}}.$$

As was shown above, for $n > 2$, a vector $x \in K_U$ may have some (not all) negative components.

Returning to the description of $K(s)$ -optimality in economic terms, one can say that a K_L -improvement of an allocation is that necessarily making at least $n - 1$ members of a community better off without making the other member worse off, while a K_U -improvement may involve decreases in the utilities of some members; however, increases in the utilities of the others make the situation better from the view-point of the community as a whole.

Let X_U^* , X_L^* , and $X^*(s)$ denote, respectively, the sets of all upper optimal, lower optimal, and $K(s)$ -optimal points. By construction, $K(s_1) \subset K(s_2)$ for $s_1 > s_2$, $s_i \in S$. Then, according to the $K(s)$ -optimality definition,

$$X_U^* \subseteq X^*(s_2) \subseteq X^*(s_1) \subseteq X_L^*. \tag{5}$$

As compared with the Pareto optimality notion, where the preference cone is \mathbb{R}_+^n , the cone $K_L \subset \mathbb{R}_+^n$ and $\mathbb{R}_+^n \subset K_U$. It leads to that $X_U^* \subseteq X_{PO}^*$ and $X_{PO}^* \subseteq X_L^*$, where X_{PO}^* denotes the set of all Pareto-optimal points.

Furthermore, when deriving optimality conditions, we will need not only the cone $K(s)$, but also the dual cone of it. Recall that the dual of a cone K is the set $K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in K\}$. Below we give a description of the dual of a Euclidean cone with an arbitrary axis,

$$K(q, s) := \{x \in \mathbb{R}^n : \cos(x, q) \geq s\} \cup \{0\},$$

where $s \in]0, 1[$ and $q \in \mathbb{R}^n$ with $\|q\| = 1$. As is known (Boyd and Vandenberghe [11, p. 449]), it can be represented as the image of the second-order cone $\{x = (\tilde{x}, x_n) \in \mathbb{R}^n : x_n \geq \|\tilde{x}\|\}$ under an appropriate nonsingular linear mapping A . Instead of obtaining the form of $K^*(q, s)$ in terms of the matrix A , we give a simple description of it as a cone $K(q, s')$ with a definite cosine value s' .

Lemma 2.1 *The dual of the cone $K(q, s)$ is*

$$K^*(q, s) = K\left(q, \sqrt{1-s^2}\right). \tag{6}$$

Proof By the definition of the cone, it suffices to consider only the vectors of $K(q, s)$ and $K^*(q, s)$ that have a unit norm. Thus, we focus on describing the set

$$E = \{x \in \mathbb{R}^n : \|x\| = 1, \langle x, y \rangle \geq 0 \text{ for all } y \in K(q, s) \text{ such that } \|y\| = 1\}.$$

First, study the two-dimension case $n = 2$. Clearly, $x \in E$ as long as the angle between x and “the worst” vector $y^* \in K(q, s)$ (with a unit length), which has the largest angle distance from x , is not greater than $\pi/2$. The cosine of the angle α between y^* and q is $\cos \alpha = s$, and therefore $\cos(x, q) = \langle x, q \rangle \geq \cos\{\pi/2 - \alpha\} = \sqrt{1 - s^2}$. Thus, (6) is true for $n = 2$.

Now proceed with the case $n > 2$. Note, first of all, that $q \in E$ and $-q \notin E$. Fix any $x \in E$ such that $x \neq q$, and prove that $\langle x, q \rangle \geq \sqrt{1 - s^2}$. Let Π be a two-dimension plane passing through the vectors x, q , and 0 , i.e., the intersection of all hyper-planes in \mathbb{R}^n containing these three points—note that x and q are linearly independent. Denote by y^* any minimum point in the problem

$$\min \langle x, y \rangle \quad \text{s.t.} \quad \langle y, q \rangle \geq s, \quad \|y\| = 1 \tag{7}$$

and show that $y^* \in \Pi$. Consider, at first, an auxiliary problem with a wider and convex set of admissible points:

$$\min \langle x, y \rangle \quad \text{s.t.} \quad \langle y, q \rangle \geq s, \quad \|y\| \leq 1. \tag{8}$$

Due to convexity of the set $\{y : \|y\| \leq 1\}$, linearity of both the goal function $\langle x, y \rangle$ and the inequality $\langle y, q \rangle \geq s$, we have the following (see, e.g., Bazaraa and Shetty [13]): If y' solves (8), then there exists $\lambda \in [0, \infty[$ such that y' solves the problem $\min_{\|y\| \leq 1} \langle x, y \rangle - \lambda \langle y, q \rangle$. Whence, $y' = -(x - \lambda q) / \|x - \lambda q\|$, and therefore y' is a solution to (7) also. Thus, y^* is a linear combination of vectors x and q ; hence, y^* belongs to Π . The latter brings us to the two-dimension case considered above, so $\langle x, q \rangle \geq \sqrt{1 - s^2}$.

By analogous reasonings, it can easily be shown that, if $x \notin E$ and $x \neq -q$, then $\langle x, q \rangle < \sqrt{1 - s^2}$.

To sum up, a vector x (of a unit norm) belongs to E if and only if (iff) $\langle x, q \rangle \geq \sqrt{1 - s^2}$, which completes the proof. \square

By definition, $K_U = K(1/\sqrt{n})$ and $K_L = K(\sqrt{n-1}/\sqrt{n})$. Therefore, Lemma 2.1 gives, in particular, that K_U and K_L are dual cones,

$$K_U^* = K_L \text{ and } K_L^* = K_U.$$

Moreover, the only self-dual cone in the collection $\{K(s), s \in S\}$ is $K(1/\sqrt{2})$. This is a Lorentz cone (Dattorro [14, p. 92]) with the axis r and its aperture or, in other words, its double angular radius equal to a right angle.

3 An “Angle Distance” Scalarization of the Problem

Let us reformulate the above-given definition (3) of weak K -optimality for the case $K = K(s)$: for any $x \in X$ such that $F(x) \neq F(x^*)$, the cosine of the angle between $F(x) - F(x^*)$ and r is not greater than s (see Fig. 2), that is,

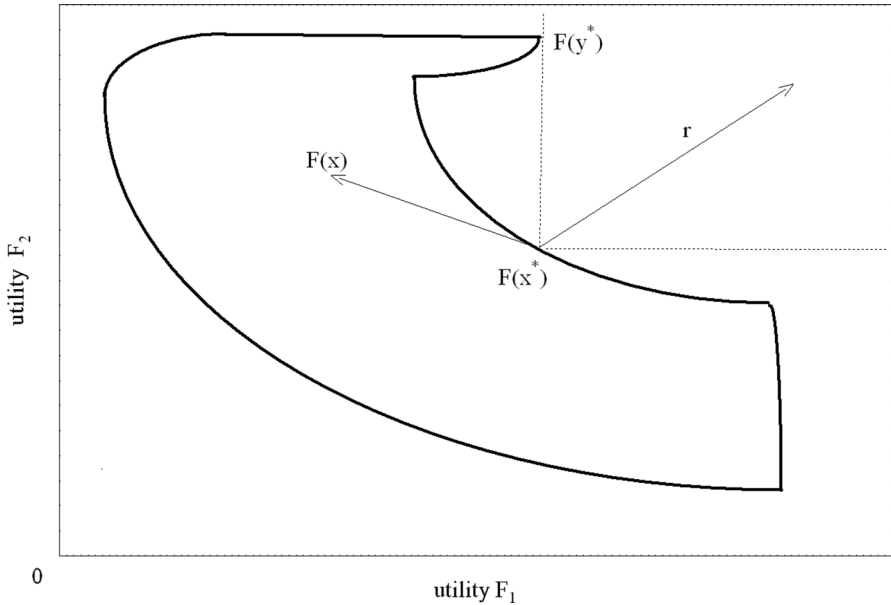


Fig. 2 A non-convex utility space in \mathbb{R}^2 with a weak optimal vector $F(x^*)$.

$$\frac{\sum_1^n (F_i(x) - F_i(x^*)) / \sqrt{n}}{\|F(x) - F(x^*)\|} \leq s. \tag{9}$$

Define a scalar function on X ,

$$G(x) := \sup_{y \in X} \sum_{i=1}^n (F_i(y) - F_i(x)) - s\sqrt{n} \|F(y) - F(x)\|. \tag{10}$$

By construction, $G(x) \geq 0$ for any $x \in X$ and takes values in the extended real half-line $\mathbb{R}_+ \cup \{\infty\}$. Now the necessary and sufficient condition (9) for the weak $K(s)$ -optimality of x^* can be rewritten as $G(x^*) \leq 0$. Taking into account that supremum in the right-hand side of (10) is attained, in particular, at $y = x^*$, the latter inequality is equivalent to $G(x^*) = 0$. Thus, we have proved the following proposition

Proposition 3.1 *A point x^* is weak $K(s)$ -optimal if and only if (iff) x^* is a root of the equation*

$$G(x) = 0, \tag{11}$$

where $G(x)$ is defined in (10).

To find the $K(s)$ -optimal (strong) solutions, return to condition (2). This means that the cone $K(s) + F(x^*)$ has no common point with the utility space $\mathcal{F} := \{F(x) : x \in X\}$, except $F(x^*)$. So, all we need is to find the weak $K(s)$ -optimal point and to exclude the situation like that depicted in Fig. 2.

Proposition 3.2 *A point x^* is $K(s)$ -optimal iff x^* is a root of (11), and maximum in the problem*

$$\max_{y \in X} \sum_{i=1}^n (F_i(y) - F_i(x^*)) - s\sqrt{n}\|F(y) - F(x^*)\| \tag{12}$$

is attained at a “unique” point in the sense that, if y^ gives maximum in (12) then $F(y^*) = F(x^*)$.*

Remark 3.1 One can easily verify that the “angle distance” scalarization introduced in (10) and providing necessary and sufficient conditions for optimality in the multi-objective optimization problem (with respect to the cone $K(s)$) can be replaced in the case of Pareto optimality (where the preference cone is \mathbb{R}_+^n) by a maximin scalarization, $G_1(x) := \sup_{y \in X} \min_{i=1, \dots, n} F_i(y) - F_i(x)$. The equation corresponding to (11) is then $G_1(x) = 0$. Generally saying, the function $G_1(x)$ does not seem convenient for applications because it is not smooth. $G_1(x)$ is different from (10) even in the case $n = 2$, where the set $\{K(s), s \in S\}$ consists of the single Pareto cone \mathbb{R}_+^2 .

4 Zero-order Optimality Conditions

First, we consider conditions for upper optimality, where, recall, the cone of preferences $K_U = K(1/\sqrt{n})$ is the biggest cone of the family $\{K(s), s \in S\}$. Next statement deals with a zero-order condition for weak optimality of some point x^* , i.e., with a condition for solvability of equation (11) with respect to x^* , where $s = 1/\sqrt{n}$. Denote by $\Delta_i^*(x) := F_i(x) - F_i(x^*)$, $i = 1, \dots, n$, where $x^* \in X$ and $x \in X$.

Proposition 4.1 *A point x^* is weak upper optimal iff, for all $x \in X$ such that $\sum_{i=1}^n \Delta_i^*(x) > 0$, if any, it holds that $\sum_{i,j:i \neq j} \Delta_i^*(x)\Delta_j^*(x) \leq 0$.*

Proof A point x^* is a root of (11), where now $s\sqrt{n} = 1$, iff $y = x^*$ is a maximizer in problem (12). In our notation, this is equivalent to

$$\sqrt{\sum_{i=1}^n (\Delta_i^*(x))^2} \geq \sum_{i=1}^n \Delta_i^*(x) \tag{13}$$

for any $x \in X$. If x is such that $\sum_{i=1}^n \Delta_i^*(x) \leq 0$, then (13) holds. If $\sum_{i=1}^n \Delta_i^*(x) > 0$ then, after squaring both parts of (13), we have that (13) holds iff $\sum_{i,j:i \neq j} \Delta_i^*(x)\Delta_j^*(x) \leq 0$. □

Remark 4.1 It is easily seen that the statement of Proposition 4.1 can be reformulated as follows: A point x^* is weak upper optimal iff x^* is a root of an equation $g(x) = 0$, where the function

$$g(x) := \sup_{y \in A(x)} \sum_{i,j:i \neq j} (F_i(y) - F_i(x))(F_j(y) - F_j(x)) \tag{14}$$

and the set $A(x) := \left\{ y \in X : \sum_{i=1}^n F_i(y) - F_i(x) \geq 0 \right\}$. Let us consider the case of linear objective functions. Let $X \subseteq \mathbb{R}^k$ and $F(x) = Cx$, where $C = (c_{i,l})$ is a given $n \times k$ matrix. Then, the function $G(x)$ defined in (10) can be rewritten as

$$G(x) = \sup_{y \in X} \sum_{i=1}^n \sum_{l=1}^k c_{il}(y_l - x_l) - s\sqrt{n} \|C(y - x)\|.$$

The terms $\sum_{i=1}^n \Delta_i^*(x)$ and $\sum_{i,j:i \neq j} \Delta_i^*(x)\Delta_j^*(x)$ used in Propositions 4.1 take the forms

$$\sum_{i=1}^n \Delta_i^*(x) = \sum_{l=1}^k d_l \Delta_l^* \quad \text{and} \quad \sum_{i,j:i \neq j} \Delta_i^*(x)\Delta_j^*(x) = \sum_{1 \leq l,m \leq k} d_{lm} \Delta_l^* \Delta_m^*,$$

where now $\Delta_l^* := x_l - x_l^*$, $d_l := \sum_{i=1}^n c_{il}$, and $d_{lm} := \sum_{i,j:i \neq j} c_{il}c_{jm}$ with $l = 1, \dots, k$ and $m = 1, \dots, k$. Thus, in problem (14), the set $A(x)$ is defined by a linear form $\sum_{i=1}^n F_i(y) - F_i(x) = \sum_{l=1}^k d_l(y_l - x_l)$ (with respect to y), and the goal function is quadratic $\sum_{i,j:i \neq j} (F_i(y) - F_i(x))(F_j(y) - F_j(x)) = \sum_{1 \leq l,m \leq k} d_{lm}(y_l - x_l)(y_m - x_m)$.

Note that the matrix $D = (d_{lm})$ may be neither positive semidefinite nor negative semidefinite, depending on the entries of C .

An analog of Proposition 4.1 with respect to the (strong) upper optimality follows from the fact that in this case, inequality (9) converts into the strict inequality.

Proposition 4.2 *A point x^* is upper optimal iff, for all $x \in X$ such that $F(x) \neq F(x^*)$ and $\sum_{i=1}^n \Delta_i^*(x) \geq 0$, if any, it holds that $\sum_{i,j:i \neq j} \Delta_i^*(x)\Delta_j^*(x) < 0$.*

Return to the general case of the preference cone $K(s)$, $s \in S$. A sufficient condition for $K(s)$ -optimality in the proposition below is a direct consequence of the known result (see, e.g. Boyd and Vandenberghe [11, p. 178]) for our case of the Euclidean cone, where the dual cone $K^*(s) = K(\sqrt{1 - s^2})$.

Proposition 4.3 *Let $\lambda \in \text{int } K(\sqrt{1 - s^2})$ and x^* be a maximizer in the problem*

$$\max \sum_{i=1}^n \lambda_i F_i(x) \quad \text{s.t. } x \in X. \tag{15}$$

Then, x^ is $K(s)$ -optimal.*

Since $K_U^* = K_L$ and $K_L^* = K_U$, next statement directly follows from Proposition 4.3.

Corollary 4.1 *Let $\lambda \in \text{int } K_L$ ($\lambda \in \text{int } K_U$) and x^* be a maximizer in problem (15). Then, x^* is K_U -optimal (K_L -optimal).*

Remark 4.2 According to the definitions of the cones K_L and K_U , any weight vector $\lambda \in \text{int } K_L$ is necessarily positive (component-wise), while $\lambda \in \text{int } K_U$ may have some negative components. For instance, in the case $n = 4$, a vector $\lambda = (1, 1, 2, 2)^T \in \text{int } K_L$ as $\cos(\lambda, r) = 6/\sqrt{40} > s = \sqrt{n-1}/\sqrt{n} = \sqrt{3}/2$; a vector $\lambda = (-\varepsilon, -\varepsilon, 1, 1)^T$, where $\varepsilon \in]0, 2 - \sqrt{3}[$, belongs to $\text{int } K_U$ as $\cos(\lambda, r) > s = 1/\sqrt{n} = 1/2$.

The existence of an upper optimal point (and, therefore, any $K(s)$ -optimal point for $s \in S$ (see (5)) is guaranteed by solvability of problem (15) with positive $\lambda \in \text{int } K_L$, which, in turn, is guaranteed by compactness of X and upper semi-continuity of all $F_i(x)$.

Remark 4.3 Return to the case of linear objective functions described in Remark 4.1. Recall that, if $n = 2$ (bi-objective optimization problem), then the set $\{K(s), s \in S\}$ consists of a single cone \mathbb{R}_+^2 , so the $K(s)$ optimality coincides with the Pareto optimality notion. In order to obtain one Pareto-optimal point, the simplest way is to solve problem

(15) with a weight vector $\lambda = (\lambda_1, \lambda_2)^T > 0$ and $F_i(x) = \sum_{j=1}^k c_{ij}x_j, i = 1, 2$.

From Proposition 4.2 (see also Remark 4.1), it follows that the set of all Pareto-optimal points is determined as follows: Define the function (see (14))

$$g(x) = \sup_{y \in A(x)} \sum_{1 \leq l, m \leq k} d_{lm}(y_l - x_l)(y_m - x_m), \tag{16}$$

with the set $A(x) = \{y \in X : \sum_{l=1}^k d_l(y_l - x_l) \geq 0\}$ for $x \in X$, where now $d_l = c_{1l} + c_{2l}$ and $d_{lm} = c_{1l}c_{2m} + c_{2l}c_{1m}, l = 1, \dots, k$, and $m = 1, \dots, k$. Then, the set of all Pareto-optimal points is the set $\{x^*\}$ of roots of the equation $g(x) = 0$ such that maximum in (16) under $x = x^*$ is attained at a “unique” point in the sense that, if y^* gives maximum in (16), then $Cy^* = Cx^*$. In the case where the decision set X is a polyhedron, $X = \{x \in \mathbb{R}^k : Ax = b, x \geq 0\}$, (15) becomes a linear programming problem, while the maximization problem in (16) is a quadratic programming problem, and the equation $g(x) = 0$ is non-linear.

5 First-Order Necessary Conditions for Optimality

Let x^* be a weak $K(s)$ -optimal point, i.e., a root of (11). Denote by y^* a maximum point in (12). It is easily seen that y^* is also weak $K(s)$ -optimal. We will call such a pair (x^*, y^*) a *weak $K(s)$ -optimal pair*. Of course, if y^* is taken equal to x^* , then (x^*, x^*) is always a weak $K(s)$ -optimal pair. A more interesting situation is that where x^* is not a unique solution to maximization problem (12). In the sequel of this section,

we suppose that the decision set $X \subseteq \mathbb{R}^k$ and utility functions $F_i(x)$, $i = 1, \dots, n$, are differentiable on \mathbb{R}^k .

Proposition 5.1 *Let (x^*, y^*) be a weak $K(s)$ -optimal pair and y^* be an internal point of X . Then,*

$$\sum_{i=1}^n F'_i(y^*) \left[\sum_{j=1}^n \Delta_j^*(y^*) - s^2 n \Delta_i^*(y^*) \right] = 0, \tag{17}$$

$$\sum_{i=1}^n \Delta_i^*(y^*) - s\sqrt{n} \|F(y^*) - F(x^*)\| = 0. \tag{18}$$

Proof Suppose, at first, that $F(y^*) \neq F(x^*)$. Since y^* solves problem (12), the first-order optimality condition is

$$\sum_{i=1}^n F'_i(y^*) - s\sqrt{n} \frac{\sum_{j=1}^n F'_j(y^*) \Delta_j^*(y^*)}{\|F(y^*) - F(x^*)\|} = 0, \tag{19}$$

where, recall, $\Delta_j^*(y^*) = F_j(y^*) - F_j(x^*)$. From Proposition 3.1, it follows that

$$\sum_{j=1}^n \Delta_j^*(y^*) = s\sqrt{n} \|F(y^*) - F(x^*)\|. \tag{20}$$

After substituting the expression for $\|F(y^*) - F(x^*)\|$ into (17), we obtain

$$\sum_{i=1}^n F'_i(y^*) \left[\sum_{j=1}^n \Delta_j^*(y^*) - s^2 n \Delta_i^*(y^*) \right] = 0.$$

The latter relation admits the degenerated case $F(y^*) = F(x^*)$ also. Taking (19) into account, we complete the proof. □

Remark 5.1 The statement of Proposition 5.1 becomes trivial if a maximum point y^* in (12) corresponds to the same point in the utility space as x^* , $F(y^*) = F(x^*)$. Nevertheless, Proposition 5.1 provides an informative necessary condition in the case where x^* is weak $K(s)$ -optimal, but not strong $K(s)$ -optimal, as shown in Fig. 2. Note also that in the case of the upper optimality, with $s = 1/\sqrt{n}$, equation (17) becomes simpler

$$\sum_{i=1}^n F'_i(y^*) \sum_{j \neq i} \Delta_j^*(y^*) = 0. \tag{21}$$

Below we will need the notion of a local $K(s)$ -optimum. A point x^* is called local weak $K(s)$ -optimal iff there exists an ε -neighborhood $O_\varepsilon(x^*)$ of this point such that x^* is weak $K(s)$ -optimal with respect to a smaller decision set $O_\varepsilon(x^*) \cap X$.

Like Proposition 4.3, next proposition is an application of a known general theorem (see [4, p. 166]) on necessary optimality conditions to the considered problem with the Euclidean cone $K(s)$ of preferences.

Proposition 5.2 *Let x^* be a local weak $K(s)$ -optimal point and an internal point of X . Then, there exists a vector $\lambda \in K(\sqrt{1 - s^2}) \setminus \{0\}$ such that*

$$\sum_{i=1}^n \lambda_i F'_i(x^*) = 0. \tag{22}$$

Corollary 5.1 *Let x^* be a local weak upper optimal (lower optimal) point and an internal point of X . Then, there exists a vector $\lambda \in K_L \setminus \{0\}$ ($\lambda \in K_U \setminus \{0\}$) such that (21) holds.*

6 Perspectives

When the decision set X is defined by the constraints

$$X = \{x \in \mathbb{R}^k : f(x) \geq 0, h(x) = 0\},$$

where $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$ and $h : \mathbb{R}^k \rightarrow \mathbb{R}^l$ are given differentiable functions, a Lagrange multiplier analog of Proposition 5.2 easily follows from the above-mentioned general result in [4, p. 166] under an appropriate constraint qualification. At the same time, such a generalization of Proposition 5.1, as well as obtaining the first-order optimality conditions in (14), is not so straightforward, because the directional derivatives of the goal functions have a cumbersome form. Another topic to be investigated is the finding of the properly optimal solutions to (1) (for the definition, see, e.g., [6]). In this connection, the following two approaches seem promising: first, analyzing the objective function equal to a weighted sum of objectives and additional variables (the so-called surplus and slack variables) like in [6, Theorem 3.4]; second, a modification of Proposition 3.1, which is based on the use of the scalar function $G(x)$ introduced in (10). These extensions of Propositions 3.1, 4.1, 5.1, and 5.2, along with finding a method for determination of a solution to equation (11), seem the directions for further research.

7 Conclusions

In this paper, a new form of scalarization technique for solving multi-objective optimization has been proposed. It is shown that the introduced “angle distance” scalarization, which is based on the cosine values of the angles between the unit vector and points of the utility space, gives necessary and sufficient conditions for optimality with respect to the Euclidean cone of preferences of a prescribed angular radius. Making use of a description of the dual to the preference cone obtained in the paper, we derived the first-order necessary conditions for optimality.

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