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NONLINEAR HYDRODYNAMICS

# Nonlinear Gravitational Waves on the Surface of Viscous Fluid: Lagrangian Approach

**A. A. Abrashkin*** **and Yu. P. Bodunova**

*Nizhny Novgorod State Pedagogical University, ul. Ul'yanova 1, Nizhny Novgorod, 603950 Russia*

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**Abstract**—The problems of the asymptotic theory of weakly nonlinear surface waves in viscous fluid are discussed. For standing waves on deep water, the solutions obtained in the first- and second-order approximations in a small parameter—wave steepness—are analyzed. The evolution equation for the amplitude of wave packet envelope is obtained where the inverse Reynolds number is equal to the squared steepness. It is shown that this is a nonlinear Schrödinger equation with linear dissipation.

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## INTRODUCTION

The classical theory of waves on water is based on the assumption of ideal fluid, whereas the consideration of the effect of viscosity on the wave propagation is a complex mathematical problem. Dissipation is large in a thin surface layer, as a result of which the boundary condition formulation for the time-dependent free boundary cannot be applied to the level of unperturbed fluid surface, as this is done for waves in ideal fluid [1]. However, this difficulty of mathematical wave description can be overcome using Lagrangian variables. The vertical coordinate corresponding to the free surface is assumed to be zero in the Lagrangian approach, and the boundary condition is conventionally formulated [2].

The purpose of this study is to develop an asymptotic theory in order to describe the propagation of nonlinear standing waves and the wave packet of traveling waves. The features of their mathematical description are revealed and their solutions for the lower-order approximations are presented.

## 1. STANDING WAVES ON DEEP WATER

Let us consider standing waves on the fluid surface. The two-dimensional hydrodynamics equations for a viscous fluid can be written in the Lagrangian form as

$$[X,Y]=\frac{D(X,Y)}{D(a,b)}=1, \tag{1.1}$$

$$X_{tt}=-\rho^{-1}[p,Y]+\nu\Big\{\big[X,[X,X_t]\big]+\big[Y,[Y,X_t]\big]\Big\},$$

$$Y_{tt}=-g-\rho^{-1}[X,p]+\nu\Big\{\big[X,[X,Y_t]\big]+\big[Y,[Y,Y_t]\big]\Big\}.$$

Here, $X$ and $Y$ are the coordinates of the fluid particle trajectory, $a$ and $b$ are the particle Lagrangian coordinates, $t$ is time, $\rho$ is density, $p$ is pressure, $\nu$ is viscosity, and $g$ is the acceleration of gravity; the brackets denote operation of taking Jacobian over the variables $a$ and $b$. The system of equations (1.1) must be supplemented with the following boundary conditions: there are no both bottom flow ($Y_t=0$ at $b=-\infty$) and viscous stress on the free surface [2]:

$$T_{ik}n_k=-p_0n_i, \quad b=0,$$

$$\mathbf{n}\{n_x,n_y\}=\mathbf{n}\left\{-\frac{Y_a}{\sqrt{X_a^2+Y_a^2}},\frac{X_a}{\sqrt{X_a^2+Y_a^2}}\right\},$$

$$T_{xx}=-p+2\nu\rho[X_t,Y], \tag{1.2}$$

$$T_{yy}=-p-2\nu\rho[Y_t,X],$$

$$T_{xy}=\nu\rho\big([Y_t,Y]-[X_t,X]\big),$$

where $T_{ik}$ is the viscous stress tensor, $p_0$ is the constant external pressure, and $\mathbf{n}$ is the outward normal to the free surface.

Let us introduce a new independent variable $\tau=\mu t$ and represent the constant $\mu$ and all unknown functions as a series in the small parameter—wave steepness $\varepsilon=kA$ ($k$ is the wavenumber and $A$ is the wave amplitude),

*E-mail: abrash@hydro.appl.sci-nnov.ru

$$\begin{aligned} X &= a + \varepsilon\xi_1 + \varepsilon^2\xi_2 + O(\varepsilon^3), \\ Y &= b + \varepsilon\eta_1 + \varepsilon^2\eta_2 + O(\varepsilon^3), \\ p &= p_0 + \rho g b + \varepsilon p_1 + \varepsilon^2 p_2 + O(\varepsilon^3), \\ \mu &= \mu_0 + \varepsilon^2\mu_2 + O(\varepsilon^3). \end{aligned} \tag{1.3}$$

The representation of $\mu$ is based on the fact that deep in the water the desired solution must pass into the solution for potential standing waves [3], where $\mu_1 = 0$. Substitution of relations (1.3) into expressions (1.1) and (1.2) yields an equation for the unknown functions in the corresponding order of perturbation theory.

The equations of the first-order approximation have the form

$$\begin{aligned} &\xi_{1a} + \eta_{1b} = 0 \\ &\mu_0^2\xi_{1\tau\tau} = -\rho^{-1}p_{1a} - g\eta_{1a} + \nu\mu_0\Delta_{\mathrm{L}}\xi_{1\tau}, \\ &\mu_0^2\eta_{1\tau\tau} = -\rho^{-1}p_{1b} + g\xi_{1a} + \nu\mu_0\Delta_{\mathrm{L}}\eta_{1\tau}, \end{aligned} \tag{1.4}$$

where the Laplacian is taken over the Lagrangian variables, with the following boundary conditions on the free surface:

$$\eta_{1a} + \xi_{1b} = 0, \ \ -p_1 + 2\nu\rho\mu_0\eta_{1\tau b} = 0, \ \ b = 0. \tag{1.5}$$

The functions $\xi_1$, $\eta_1$, and $p_1$ are assumed to be spatially periodic. It is convenient to search for the solution to system (1.4) in the form

$$\begin{aligned} \xi_1 &= A(b)\, e^{\tau} \sin(ka), \\ \eta_1 &= B(b)\, e^{\tau} \cos(ka), \\ p_1 &= C(b)\, e^{\tau} \cos(ka), \ \ \mathrm{Re}\,\tau < 0, \end{aligned} \tag{1.6}$$

considering $k$ as real and the functions $A$, $B$, $C$, $\tau$ and constant $\mu_0$ as complex. Only the real parts of expressions (1.6) have a physical meaning. After substitution of (1.6) into system (1.4) and simple transformations, we obtain the equation

$$A^{IV}_{bbbb} - \left(2k^2 + \frac{\mu_0}{\nu}\right) A''_{bb} + k^2\left(k^2 + \frac{\mu_0}{\nu}\right) A = 0. \tag{1.7}$$

Assuming that $A = \exp(lb)$, we have a biquadratic equation for $l$ with the solution

$$l_1^2 = k^2, \quad l_2^2 = k^2 + \frac{\mu_0}{\nu} = m^2. \tag{1.8}$$

The wave perturbations should decrease with penetrating the bulk (at $b \to -\infty$); therefore, the function $A$ should be chosen in the form

$$A = \alpha e^{kb} + \beta e^{mb}, \quad k, \mathrm{Re}\, m > 0. \tag{1.9}$$

The functions $B$ and $C$ are determined by the equalities

$$\begin{aligned} B &= -\left(\alpha e^{kb} + \frac{k}{m}\beta e^{mb}\right), \\ C &= \frac{\rho}{k}\left[(\mu_0^2 + kg)\,\alpha e^{kb} + \frac{k^2}{m}\, g\beta e^{mb}\right]. \end{aligned} \tag{1.10}$$

Substitution of the expressions found for the functions $A$, $B$, and $C$ into the boundary condition (1.5) gives

$$\begin{aligned} &2k\nu m\alpha + \beta(2\nu k^2 + \mu_0) = 0, \\ &(\mu_0^2 + 2\nu k^2\mu_0 + kg)\alpha + \left(\frac{k^2}{m}g + 2\nu\mu_0 k^2\right)\beta = 0. \end{aligned} \tag{1.11}$$

The compatibility condition for Eqs. (1.11) has the form

$$(\mu_0 + 2\nu k^2)^2 + kg = 4\nu^2k^3m. \tag{1.12}$$

Having using the designations $\omega^2 = gk$, $\nu k^2/\omega = \theta$, and $\mu_0 + 2\nu k^2 = s\omega$, where $\omega$ is the frequency of linear gravitational waves, we transform (1.12) as follows:

$$(s^2 + 1)^2 = 16\theta^3(s - \theta). \tag{1.13}$$

This equation exactly coincides with that arising in the consideration of linear waves propagating on the surface of viscous fluid. It has four roots, and only two of them satisfy the condition $\mathrm{Re}\, m > 0$ [4]. The real and imaginary parts of root determine, respectively, the decrement and frequency of wave oscillations. One of the two values, $\alpha$ and $\beta$, can be chosen arbitrarily. Let, for example, $\alpha$ be specified. Then

$$\beta = -\frac{2k\nu m}{\mu_0 + 2\nu k^2}\,\alpha. \tag{1.14}$$

The $\alpha$ value determines the initial wave amplitude.

Within this approximation the wave vorticity is set by the expression

$$\Omega_1 = -\mathrm{Re}\left[\frac{\mu_0^2}{\nu m}\beta e^{mb+\tau}\sin(ka)\right], \tag{1.15}$$

it is concentrated in a surface layer of thickness $2\pi|m|^{-1}$.

As an example, we will consider the decay of fairly long waves with the wavelength $\lambda = 2\pi k^{-1}$ satisfying the condition $\theta = 4\pi^2\nu/\omega\lambda^2 \ll 1$. In this case, one can neglect the right-hand side in Eq. (1.13), and the solution will be the values $s = \pm i$ or, in dimensional units, $\mu_0 = -2\nu k^2 \pm i\omega$. Hence, the decrement is $-2\nu k^2$ and the wave oscillation frequency is $\omega$ (the sign can be arbitrary; we chose plus for definiteness). As follows from (1.8), the $m$ value is approximately $m = (1+i)\Delta$, $\Delta = (\omega/2\nu)^{1/2}$, and relation (1.14) yields $\beta/\alpha = -(1+i)k/\Delta$. The modulus of this ratio is always much smaller than unity. Therefore, within our approximation, $\beta$ can be neglected in comparison with $\alpha$. As a result, the free-surface rise $\eta_1$ is determined by the formula (see (1.5))

$$\eta_1 = \alpha_0 \exp(-2\nu k^2 t + kb)\cos(ka)\sin(\omega t). \quad (1.16)$$

Here, $\alpha = i\alpha_0$ ($\alpha_0$ is real). Then (1.16) is similar to the solution for potential waves [3]. These expressions coincide at zero viscosity. For sufficiently long waves, the effect of viscosity is reduced to an exponential decrease in the oscillation amplitude with time. Note that the flow in the surface layer is vortex, and its vorticity changes according to the law

$$\Omega_1 = 2l\omega\alpha_0 \exp(-2\nu k^2 t + \Delta b)\sin(ka)\sin(\Delta b + \omega t). \quad (1.17)$$

Standing waves are a rare example of analytical representation of a flow with a time-dependent vorticity distribution.

The solution in the second-order approximation is very cumbersome. Therefore, we will restrict ourselves to the qualitative analysis of its most important properties. In the second-order approximation, the expressions for the displacements of fluid particles in potential standing waves have the form [3]

$$\xi_{2\mathrm{pot}} = 0, \quad \eta_{2\mathrm{pot}} = \frac{k^3}{4\omega^2}\, e^{2kb}\big[1-\cos(2\omega t)\big]. \quad (1.18)$$

There is no the horizontal displacement of fluid particles, and the vertical ones are independent of the horizontal Lagrangian coordinate $a$; i.e., within this approximation the surface $b = \mathrm{const}$ oscillates as a whole. The solution for the waves in a viscous fluid can be written as

$$\begin{aligned} \xi_2 &= \operatorname{Re}\Phi_1(b)\, e^{2\tau}\sin(2ka), \\ \eta_2 &= \eta_{2\mathrm{pot}} + \operatorname{Re}\Phi_2(b)\, e^{2\tau} + \operatorname{Re}\Phi_3(b)\, e^{2\tau}\cos(2ka). \end{aligned} \quad (1.19)$$

Here, the functions $\Phi_1$, $\Phi_2$, and $\Phi_3$ depend only on the variable $b$. They exponentially decrease with penetrating the fluid bulk. Thus, the motion of fluid particles in a viscous wave is obviously complicated. First, horizontal displacements arise, which are zero only in the wave nodes, where $ka = \pi n$ ($n$ is an integer). Second, the vertical particle displacements depend on the horizontal coordinate and are inhomogeneous with respect to $x$. Finally, since the functions $\Phi_1$, $\Phi_2$, $\Phi_3$, and $\tau$ are complex, along with the conventional amplitude decay with time, the oscillation phase of separate particles depends on depth.

The asymptotic theory for standing waves in viscous fluid can also be constructed for higher-order approximations. There are no fundamental difficulties because midflows are absent in this case. However, they always exist for progressive waves. Hence, secular terms arise when constructing the perturbation theory in the higher-order approximations. Within the Lagrangian approach for potential waves this problem can be solved by introducing modified Lagrangian coordinates [2]. For waves in viscous fluid the situation is much more difficult [5–9] and requires new concepts.

## 2. PACKET OF PROGRESSIVE WAVES

To study the dynamics of a wave packet traveling along the viscous fluid surface, we will transform the system of hydrodynamic equations (1.1) and (1.2). To this end, we will introduce a complex coordinate of the fluid particle trajectory $W = X + iY$ ($\overline{W} = X - iY$) and a new traveling coordinate $q = a + \sigma t$, where $\sigma$ is a constant velocity. In the new variables the equations of viscous fluid motion in the Lagrangian form can be written as

$$[\overline{W}, W] = \frac{D(\overline{W}, W)}{D(q,b)} = 2i, \quad (2.1)$$

$$W_{tt} + 2\sigma W_{qt} + \sigma^2 W_{qq} = -ig + \frac{i}{\rho}[p, W] + \frac{\nu}{2}\Big\{\big[W, [\overline{W}, W_t + \sigma W_q]\big] + \big[\overline{W}, [W, W_t + \sigma W_q]\big]\Big\}.$$

Here, the brackets indicate taking Jacobian over the variables $q$ and $b$. Let us make system (2.1) dimensionless by introducing the new variables

$$W = LW_n, \quad q = Lq_n, \quad t = \frac{L}{\sigma}\, t_n, \quad p = \rho\sigma^2 p_n, \quad (2.2)$$

where $L$ is some distance scale (the indices are omitted below). The equation of motion and the boundary conditions on the free surface take the form

$$W_{tt}+2\sigma W_{tq}+\sigma^2 W_{qq}=-ig\,\frac{L}{\sigma^2}+i[p,W]+\frac{1}{2\mathrm{Re}}\Big\{\big[W,[\overline{W},W_t+\sigma W_q]\big]+\big[\overline{W},[W,W_t+\sigma W_q]\big]\Big\},$$
$$T_{ik}n_k=-p_0 n_i,\quad \mathbf{n}\{n_x,n_y\}=\mathbf{n}\left\{-\frac{Y_q}{\sqrt{X_q^2+Y_q^2}},\frac{X_q}{\sqrt{X_q^2+Y_q^2}}\right\},\quad b=0,$$
$$T_{xx}=-p+\frac{1}{2i\,\mathrm{Re}}\big([W_t+\sigma W_q,W-\overline{W}]-\mathrm{c.c.}\big),$$
$$T_{xy}=-\frac{1}{2i\,\mathrm{Re}}\big([W_t+\sigma W_q,W]+\mathrm{c.c.}\big),$$
$$T_{yy}=-p-\frac{1}{2i\,\mathrm{Re}}\big([W_t+\sigma W_q,W+\overline{W}]-\mathrm{c.c.}\big). \tag{2.3}$$

The equation of motion contains two dimensionless parameters: the Reynolds number $\mathrm{Re}=\sigma L/\nu$ and $gL/\sigma^2$.

Let us investigate the behavior of a weakly nonlinear train of waves with slowly varying parameters (amplitude, frequency, and wavenumber). To this end, we will use the derivative-expansion (or multiscale [10]) method. We assume that the function $W$ depends on the spatial ($q_0=q$, $q_1=\varepsilon q$, $q_2=\varepsilon^2 q, b$) and temporal ($t_1=\varepsilon t$, $t_2=\varepsilon^2 t$) variables ($\varepsilon$ is a small parameter—steepness of the wave) and can be expanded as

$$W(q_0,q_1,q_2,b,t_1,t_2)=q_0+ib+\varepsilon w_1+\varepsilon^2 w_2+\varepsilon^3 w_3+O(\varepsilon^4) \tag{2.4}$$

and that the fluid pressure is determined by the relation

$$p==p_0-b+\varepsilon p_1+\varepsilon^2 p_2+\varepsilon^3 p_3+O(\varepsilon^4). \tag{2.5}$$

Substitution of the first two terms from (2.5) (which determine the hydrostatic pressure) into the first equation from (2.3) gives the second dimensionless parameter: $gL/\sigma^2=1$. On the assumption that $L=k^{-1}$ and $\sigma=\omega/k$ is the wave phase velocity, this relation coincides with the dispersion relation for waves on deep water.

Generally, the problem of describing the wave packet evolution in viscous fluid is extremely difficult. To simplify it, we assume that the Reynolds number is fairly large: $\mathrm{Re}^{-1}=\varepsilon^2$. This means that viscosity effects manifest themselves only in the third-order approximation.

The equations in the linear approximation have the form

$$w_{1b}+iw_{1q_0}-\mathrm{c.c.}=0,$$
$$w_{1q_0q_0}=i(w_{1q_0}+ip_{1q_0}-p_{1b}). \tag{2.6}$$

The solution to (2.6), which decays into the fluid bulk and satisfies the condition of constant pressure on the free surface, has the form

$$w_1=A(q_1,q_2,t_1,t_2)\,e^{b+iq_0}+\psi_1(q_1,q_2,b,t_1,t_2). \tag{2.7}$$

The first term in (2.7) describes the oscillatory motion of fluid particles, while the second term describes the drift flow. The wave amplitude $A$ is a function of slow variables.

In the second-order approximation the equation for this parameter is exactly the same as for a wave in ideal fluid

$$A_{t_1}+\frac{1}{2}A_{q_1}=0. \tag{2.8}$$

The packet envelope is transported with a group velocity $\sigma/2$, which is a half of the phase velocity.

In the third-order approximation one must take into account the viscosity. However, now the viscous term in the equation of motion is proportional to the Laplacian $\Delta_{\mathrm{L}} w_1$ over the fast variables which is zero. Hence, the solution coincides again with that solution for potential waves. However, this potential solution must satisfy the viscous boundary condition

$$-p_3+(-iw_{1q_0b}-\mathrm{c.c.})=0,\quad b=0. \tag{2.9}$$

If the expression in the parentheses is neglected, this condition will lead to a nonlinear Schrödinger equation [11], while the consideration of this term will supplement the nonlinear Schrödinger equation with a dissipative term

$$iA_\tau+\frac{1}{2}|A|^2A+\frac{1}{8}A_{\zeta\zeta}+iA=0, \tag{2.10}$$
$$\tau=t_2,\quad \zeta=q_1-\frac{1}{2}t_1,$$

or, in the dimensional variables,

$$iA_\tau+\frac{1}{2}\omega k^2|A|^2A+\frac{1}{8}\frac{\omega}{k^2}A_{\zeta\zeta}+i\omega A=0. \tag{2.11}$$

The most important feature of this equation is that it does not contain the drift flow $\psi_1$. This is due to the fact that we considered fairly large Reynolds numbers. If the inverse Reynolds number is of the same order of magnitude or larger than the wave steepness, the evolution equation will contain the midflow. It is noteworthy that the wave frequency plays the role of amplitude decrement.

One must apply numerical methods to study the properties of Eq. (2.11).

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