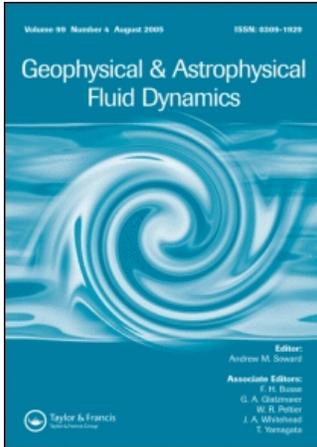


This article was downloaded by:[Grimshaw, Roger]  
On: 25 May 2008  
Access Details: [subscription number 793268203]  
Publisher: Taylor & Francis  
Informa Ltd Registered in England and Wales Registered Number: 1072954  
Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK



## Geophysical & Astrophysical Fluid Dynamics

Publication details, including instructions for authors and subscription information:  
<http://www.informaworld.com/smpp/title-content=t713642804>

### Fission of a weakly nonlinear interfacial solitary wave at a step

Roger Grimshaw<sup>a</sup>; Efim Pelinovsky<sup>b</sup>; Tatiana Talipova<sup>b</sup>  
<sup>a</sup> Department of Mathematical Sciences, Loughborough University, Loughborough, UK  
<sup>b</sup> Department of Nonlinear Geophysical Processes, Institute of Applied Physics, Nizhny, Novgorod, Russia

Online Publication Date: 01 April 2008

To cite this Article: Grimshaw, Roger, Pelinovsky, Efim and Talipova, Tatiana (2008) 'Fission of a weakly nonlinear interfacial solitary wave at a step', *Geophysical & Astrophysical Fluid Dynamics*, 102:2, 179 — 194

To link to this article: DOI: 10.1080/03091920701640115  
URL: <http://dx.doi.org/10.1080/03091920701640115>

PLEASE SCROLL DOWN FOR ARTICLE

Full terms and conditions of use: <http://www.informaworld.com/terms-and-conditions-of-access.pdf>

This article maybe used for research, teaching and private study purposes. Any substantial or systematic reproduction, re-distribution, re-selling, loan or sub-licensing, systematic supply or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.

## Fission of a weakly nonlinear interfacial solitary wave at a step

ROGER GRIMSHAW\*<sup>†</sup>, EFIM PELINOVSKY<sup>‡</sup> and TATIANA TALIPOVA<sup>‡</sup>

<sup>†</sup>Department of Mathematical Sciences, Loughborough University,  
Loughborough, UK

<sup>‡</sup>Department of Nonlinear Geophysical Processes,  
Institute of Applied Physics, Nizhny, Novgorod, Russia

(Received 2 April 2007; in final form 23 July 2007)

The transformation of a weakly nonlinear interfacial solitary wave in an ideal two-layer flow over a step is studied. In the vicinity of the step the wave transformation is described in the framework of the linear theory of long interfacial waves, and the coefficients of wave reflection and transmission are calculated. A strong transformation arises for propagation into shallower water, but a weak transformation for propagation into deeper water. Far from the step, the wave dynamics is described by the Korteweg-de Vries equation which is fully integrable. In the vicinity of the step, the reflected and transmitted waves have soliton-like shapes, but their parameters do not satisfy the steady-state soliton solutions. Using the inverse scattering technique it is shown that the reflected wave evolves into a single soliton and dispersing radiation if the wave propagates from deep to shallow water, and only dispersing radiation if the wave propagates from shallow to deep water. The dynamics of the transmitted wave is more complicated. In particular, if the coefficient of the nonlinear quadratic term in the Korteweg-de Vries equation is not changed in sign in the region after the step, the transmitted wave evolves into a group of solitons and radiation, a process similar to soliton fission for surface gravity waves at a step. But if the coefficient of the nonlinear term changes sign, the soliton is destroyed completely and transforms into radiation. The effects of cubic nonlinearity are studied in the framework of the extended Korteweg-de Vries (Gardner) equation which is also integrable. The higher-order nonlinear effects influence the amplitudes of the generated solitons if the amplitude of the transformed wave is comparable with the thickness of lower layer, but otherwise the process of soliton fission is qualitatively the same as in the framework of the Korteweg-de Vries equation.

*Keywords:* Nonlinear waves in the ocean; Internal waves; Two-layer flow; Solitons; Korteweg-de Vries equation

### 1. Introduction

The fission of a solitary wave (soliton) is well known for surface waves passing through a zone of rapid depth change, when the incident soliton transforms into a group of the secondary solitons (Tappert and Zabusky 1971, Pelinovsky 1971, 1977, Johnson 1972). The same process was studied theoretically for internal waves by Djordevic and Redekopp (1978) and Helfrich and Melville (1986) (see also Zheng *et al.* 2001).

---

\*Corresponding author. Email: R.H.J.Grimshaw@lboro.ac.uk

Fission of internal solitary waves propagating along an inhomogeneous thermocline was observed in the Gulf of Aden (Zheng *et al.* 2001), and while propagating across the continental slope in the South China Sea (Liu *et al.* 2004). Sometimes this process is observed with a changing of polarity of solitons (Liu *et al.* 1998, Orr and Mignerey 2003, Zhao *et al.* 2003). In these cited papers for internal waves, the width of the transition zone is relatively small compared with a characteristic nonlinear length-scale, but is relatively large compared with the wavelength. In this case a WKB-type approximation for linear long waves can be used to describe the wave transformation in the transition zone. This feature can also be built into a variable-coefficient Korteweg-de Vries equation to describe the transformation of internal solitary waves propagating over the continental slope (see, for instance, Grimshaw *et al.* 2004, 2007).

On the other hand, the case when the background state changes very rapidly, for instance at a step in the bottom topography, the incident modal internal wave is transformed into reflected and transmitted internal waves with many modes; this process is more difficult for theoretical analysis even when the waves are linear. But, if the fluid stratification is modeled as a two-layer fluid, there is only one mode present (interfacial waves), and hence there can be no scattering into other modes. This simplified process is studied in this paper, where we analyze the fission of an interfacial soliton incident on a step. In the vicinity of the step, where wave transformation may be quite rapid, we use linear long wave theory to describe the wave transformation, see section 2; this assumes that the wave amplitude is small, and the wavelengths are much greater than the fluid depth, while the width of the transition zone is much shorter than these wavelengths. Then in sections 3 and 4 we allow weakly nonlinear and weakly dispersive effects to come into play to describe the development of the reflected and transmitted wave fields, so that each of these are governed by a Korteweg-de Vries equation. In section 3 we show that only a single soliton forms in the reflected field for the case of a transformation from deep to shallow water over the step; otherwise for a transformation from shallow to deep water, no solitons form in the reflected field which instead contains only dispersing radiation. Note that “deep” is used here and throughout the text only in contrast to “shallow”; in all cases the water depth is much less than the wavelength. Then in section 4 we examine the fission of the transmitted wave, and show that secondary solitons form together with some radiation. But, in contrast with the case of a surface wave, secondary solitary waves may not appear at all if the nonlinearity changes its sign after a step, and only radiation is formed. Some effects on this fission phenomenon of cubic nonlinearity leading to an extended Korteweg-de Vries (Gardner) equation are investigated in section 5. Our results are summarized in the conclusion.

## 2. Linear long interfacial wave transformation at a step

Let us consider the problem sketched in figure 1. The thickness of the upper layer,  $h_1$  is constant, and the thickness of the lower layer,  $h_2$  is varied rapidly from  $h_{2-}$  to  $h_{2+}$  in the vicinity of  $x = 0$ . The vertical displacement of the interface is  $\eta(x, t)$ , and the layer-mean horizontal velocities are  $u_1$  and  $u_2$  respectively. The density jump on the interface is  $\Delta\rho/\rho$ , and the acceleration  $g$  due to gravity is directed down. We will use the Boussinesq

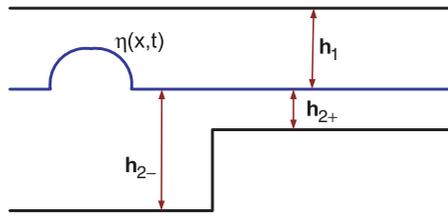


Figure 1. Geometry of the problem.

approximation (that is, the density jump is weak) and the approximation of a rigid upper lid, which are typical for oceanographic applications.

In general, the description of the wave field near the step is a difficult task (see for instance, Baines 1995); it contains propagating waves (reflected as well as transmitted) and non-propagating evanescent modes. But, in the long wave limit, the characteristics of the reflected and transmitted waves can be found from the conservation of pressure (water level) and the mass flux in the lower layer at a step, they are

$$\eta_- = \eta_+, \quad h_{2-}u_{2-} = h_{2+}u_{2+}. \tag{1}$$

In linear long wave theory, each dependent variable satisfies the linear wave equation, with a speed  $c$  given by

$$c = \sqrt{g \frac{\Delta\rho}{\rho} \frac{h_1 h_2}{h_1 + h_2}}. \tag{2}$$

The wave field is then expressed as a superposition of an incident and a reflected wave in  $x < 0$  before the step, and a transmitted wave in  $x > 0$  after the step. After taking into account the relationship between the horizontal velocity and the vertical displacement in each wave

$$u_2 = \pm \frac{c\eta}{h_2}, \tag{3}$$

it is straightforward to find the coefficients of reflection  $R$  and transmission  $T$ :

$$R = \frac{(1 - (c_+/c_-))}{(1 + (c_+/c_-))}, \quad T = \frac{2}{(1 + (c_+/c_-))}. \tag{4}$$

Here  $c_+$  and  $c_-$  are values of the long wave speed (2) in  $x > 0$  after the step and in  $x < 0$  before the step, respectively. As expected, these expressions are the same as for surface waves, and any special features arising for interfacial waves are due to the dependence of the long wave speed (2) on the thickness of the lower layer. More specifically, the reflection and transmission properties are determined by the speed jump

$$\delta = \frac{c_+}{c_-} = \sqrt{\frac{h_+ h_1 + h_-}{h_- h_1 + h_+}}, \tag{5}$$

where for convenience we have denoted  $h_{2-}$  and  $h_{2+}$  by  $h_-$  and  $h_+$  respectively. It is convenient to normalize the depth of the lower layer with the thickness of upper layer and let  $H = h_2/h_1$ . As a result, the speed jump is a function of two parameters, the initial

depth of lower layer  $H_-$  and the depth jump  $\Delta H = h_+/h_-$ :

$$\delta = \sqrt{\Delta H \frac{1 + H_-}{1 + H_- \Delta H}}. \quad (6)$$

The case  $\Delta H < 1$  corresponds to the wave transmission from deep to shallow water, and  $\Delta H > 1$  to wave transmission from shallow to deep water. In contrast to surface waves, the result now depends on the initial depth of the lower layer before the step, as well as on the depth jump. When  $\Delta H$  is small (transmission from deep to shallow water) we have the approximate expression

$$\delta \approx \sqrt{\Delta H(1 + H_-)}. \quad (7)$$

In the opposite case of wave transmission from shallow to deep water, if the depth jump  $\Delta H$  is high, we have another approximate expression

$$\delta = \sqrt{\frac{1 + H_-}{H_-}}, \quad (8)$$

and for a very large initial depth there is no difference in the speed of propagation across the bottom step. This is simply because the speed of propagation of long internal waves is determined by the depth of the narrowest layer if the thicknesses are very different.

The calculated coefficients for the wave transmission and reflection are shown in figure 2. As expected, when a wave passes from deep to shallow water the transmitted wave is increased and the reflected wave is decreased. On the other hand when a wave propagates from shallow to deep water, the transmitted wave is decreased, and the reflected wave has the opposite polarity. This conclusion is similar to the behavior for the surface waves at a step (all formulas for this case are obtained if formally  $H_- = 0$ ). The main new result for an interfacial wave is manifested if the initial depth is large; in this case the wave passes through a step with very little change, as explained in the previous paragraph.

### 3. Reflection of a soliton from a step

First, let us consider the transformation of a Korteweg-de Vries soliton at a step. The Korteweg-de Vries equation (KdV) for an interfacial wave in a two-layer fluid has the following form

$$\frac{\partial \eta}{\partial t} + (c + \alpha \eta) \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0, \quad (9)$$

where

$$\beta = \frac{ch_1^2 H}{6}, \quad \alpha = \frac{3c}{2h_1} \frac{1 - H}{H}, \quad (10)$$

where again we use the normalization of the depth of the lower layer by the thickness of the upper layer, and  $c$  is the long wave speed given by (2). The solitary wave (soliton)

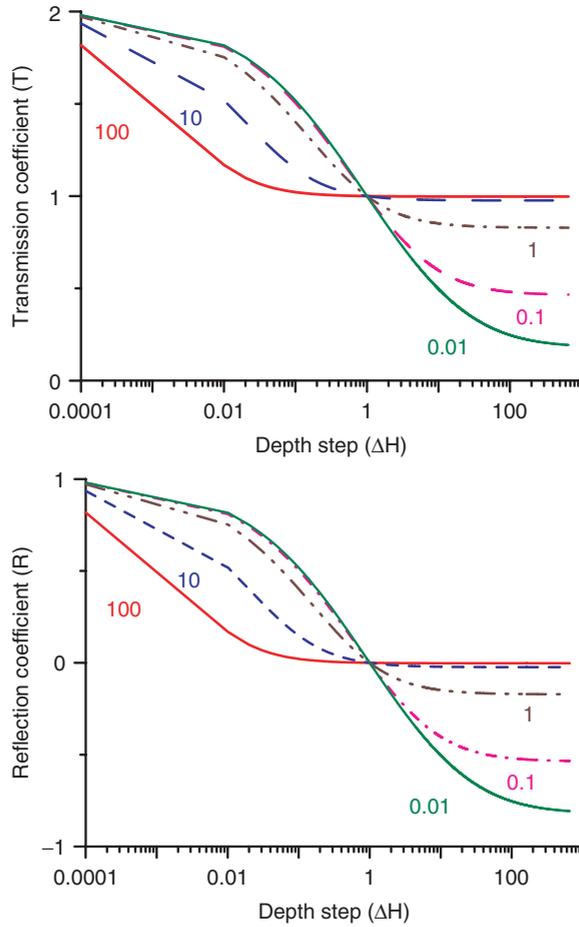


Figure 2. Transmission and reflection coefficients of long internal wave on step (the numbers indicate the initial depth).

solution of (9) is

$$\eta = A \operatorname{sech}^2 \sqrt{\left(\frac{3A}{4h_1}\right) \frac{(1 - H_-)}{H_-^2} [x - (c + \alpha A/3)t]}. \tag{11}$$

The soliton amplitude is positive if  $H < 1$  (elevation soliton), and negative if  $H > 1$  (depression soliton). If  $H = 1$ , as is well-known, the interfacial solitons do not exist. We will assume that (11) describes the incident wave propagating from the left towards the step, and hence  $H = H_-$ .

Taking into account that the soliton amplitude is weak (both the nonlinear and dispersive effects are weak, but in balance), the process of the wave transformation at the step can be described to leading order in the framework of the linear theory of long interfacial waves. The reflection coefficient  $R$  was calculated in section 2, and does not depend on the wave scale. This means that to leading order after reflection, in the

vicinity of the step, the reflected wave has the same shape as the incident wave, but its amplitude is different, so that near the step

$$\eta_{ref} = A_{ref} \operatorname{sech}^2 \left( \sqrt{\frac{3A}{4h_1} \frac{1 - H_-}{H_-^2} \frac{x}{h_1}} \right), \quad A_{ref} = RA. \tag{12}$$

This reflected wave, although it has a soliton-like shape, is now not a steady-state soliton, because its width is determined by the initial soliton amplitude, and not by the amplitude of reflected wave. Instead, the expression (12) should be used now to solve the KdV equation (10) (with  $c$  replaced with  $-c$ ). As is well-known, the KdV equation is exactly integrable using an associated spectral problem and the inverse scattering transform. Indeed, in the case of soliton-like disturbances, the spectral problem has an explicit solution (e.g. Drazin and Johnson 1989). The answer depends on the sign of the reflected coefficient. If the wave passes from shallow into deep water, the reflection coefficient is negative, and polarity of the reflected wave is opposite to polarity of the incident soliton. In this case no solitons are produced and the reflected wave decays into dispersing radiation. If the wave passes from deep into shallow water, the polarity of the reflected wave is unchanged, and the disturbance evolves into a finite number of solitons, and radiation. Using the explicit results from the KdV equation (Drazin and Johnson 1989) it is easy to show that only one soliton is formed in the reflected wave, and its amplitude is

$$\frac{A_{sr}}{A} = \left( \sqrt{2R + \frac{1}{4} - \frac{1}{2}} \right)^2. \tag{13}$$

This expression can be called the ‘‘soliton reflection’’ coefficient, and it does not depend on the incident wave amplitude. If the depth ratio is small, the linear reflection coefficient is small too, and the soliton reflection coefficient is described by the approximate expression

$$\frac{A_{ref}}{A} \approx 4R^2. \tag{14}$$

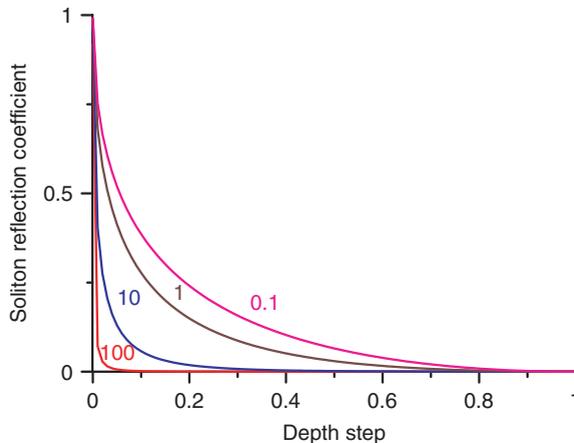


Figure 3. Amplitude of soliton formed in the reflected wave (the numbers indicate the initial depth).

Such a soliton is accompanied by a large dispersing wavetrain. If the depth ratio is high, the linear reflection coefficient tends to 1, and the soliton reflects with the same amplitude. The dependence of the “soliton reflection” coefficient on the depth ratio and the initial depth of the lower layer is shown in figure 3. In deepest water the amplitude of reflected soliton is less than in shallow water. Polarity of the formed soliton is the same as polarity of the incident wave: positive if  $H < 1$  and negative if  $H > 1$ . We again underline that the soliton in the reflected wave forms only if the incident wave approaches to shallow water, in the opposite case the reflected wave transforms into a damped dispersive train.

**4. Transmission of the KdV soliton through a step**

A similar approach can be used to calculate the solitary waves formed in the field of the transmitted wave. The transformed wave in the vicinity of the step has a soliton-like shape, but its parameters are not the same as for the permanent soliton; its amplitude differs by a factor of  $T$ , and its wavelength differs by a factor of  $\delta = c_+/c_-$ , since the temporal structure is not changed over the step. Near the step it is given by

$$\eta_{tr} = A_{tr} \operatorname{sech}^2 \left( \sqrt{\frac{3A}{4h_1} \frac{1 - H_-}{H_-^2} \frac{C_-}{C_+} \frac{x}{h_1}} \right), \quad A_{tr} = TA. \tag{15}$$

In general such a soliton-like disturbance (15) evolves into a group of solitons and a dispersive wave train. But solitons can be formed only if the sign of the nonlinear term is not changed after passage over the step. The possible existence of transmitted solitons and their polarities are displayed in figure 4. In shallow water, when both depths are less then the thickness of the upper layer, the polarities of the transmitted solitons are positive, the same as the polarity of the incident soliton. In deep water, when both depths are bigger than the thickness of the upper layer, all solitons will be solitons of depression.

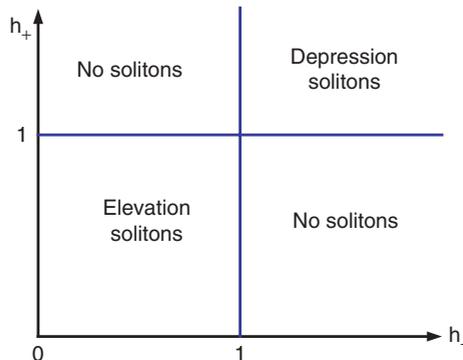


Figure 4. Polarity of solitons formed in the transmitted wave.

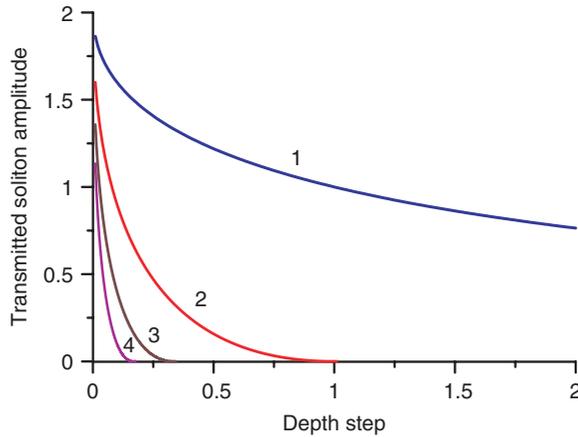


Figure 5. Internal soliton transmission in deep water.

If the sign of the nonlinear coefficient is not changed, secondary solitons can be formed in the transmitted wave. Their amplitudes can be also calculated from the inverse scattering technique using the same scheme as for the reflected wave

$$\frac{A_{str}^m}{A} = \left( \frac{c_-^2 \alpha_- \beta_+}{c_+^2 \alpha_+ \beta_-} \right) \left[ \sqrt{2T \left( \frac{c_+^2 \alpha_+ \beta_-}{c_-^2 \alpha_- \beta_+} \right) + \frac{1}{4}} - \left( m + \frac{1}{2} \right) \right]^2, \quad (16)$$

where  $m = 0, 1, 2, \dots, N - 1$ , and

$$N = \left[ \sqrt{2T \left( \frac{c_+^2 \alpha_+ \beta_-}{c_-^2 \alpha_- \beta_+} \right) + \frac{1}{4}} + \frac{1}{2} \right] \quad (17)$$

is the number of transmitted solitons. All ratios in (16) are functions of the depth ratio  $\Delta H$  and initial depth of the lower layer  $H_-$ :

$$\frac{c_+}{c_-} = \sqrt{\Delta H \frac{1 + H_-}{1 + H_- \Delta H}}, \quad T = 2 / \left( 1 + \frac{c_+}{c_-} \right), \quad \frac{\alpha_+}{\alpha_-} = \frac{1}{\Delta H} \frac{1 - H_- \Delta H}{1 - H_-}, \quad \frac{\beta_+}{\beta_-} = \Delta H. \quad (18)$$

If the fluid is very deep on both sides of the step ( $H_- \gg 1$ ), the wave passes over the step with no change in amplitude ( $T \approx 1$ ), but the dispersion coefficient “feels” the thickness of the lower layer, and the soliton-like pulse nevertheless transforms into solitons with amplitudes described by the simplified asymptotic expressions (see figure 5)

$$N \approx \left[ \sqrt{\frac{2}{\Delta H} + \frac{1}{4}} + \frac{1}{2} \right], \quad \frac{A_{str}^m}{A} \approx \Delta H \left[ \sqrt{\frac{2}{\Delta H} + \frac{1}{4}} - \left( m + \frac{1}{2} \right) \right]^2, \quad (19)$$

If the wave transforms from shallow to deep water ( $\Delta H > 1$ ), only one soliton is formed and its amplitude decreases with the increase of the depth ratio. This decrease of the

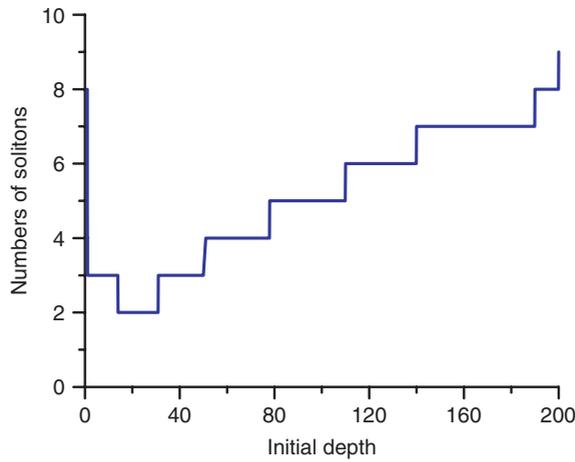


Figure 6. Numbers of secondary solitons formed in the transmitted wave.

soliton amplitude is only due to the increase in the dispersion. If the wave transforms from deep to shallow water ( $\Delta H < 1$ ), but the shallow depth is also large, the number of solitons increases, and their amplitudes are increased (see figure 5).

Next, if the wave approaches from “real” shallow water or is transmitted into shallow water, the result depends also on the initial thickness of the lower layer. In particular, if the thicknesses of both layers after a step are almost the same ( $H_+ \sim 1$ ), the nonlinear coefficient ( $\alpha_+$ ) tends to zero, only one small soliton can appear, and its amplitude follows from (16):

$$\frac{A_{tr}}{A} \approx 4T^2 \frac{c_+^2 \alpha_+ \beta_-}{c_-^2 \alpha_- \beta_+}. \quad (20)$$

On the other hand, if the soliton approaches from an incident zone with a small value of the nonlinear coefficient ( $\alpha_-$ ) to a transmitted zone with “normal” values of the nonlinear coefficient ( $\alpha_+$ ), the number of transmitted solitons is large, and the amplitude of the leading transformed soliton is  $2TA$ . The intermediate general case is determined by two parameters and it is rather complicated. Figure 6 demonstrates the maximum number of secondary solitons formed in the transmitted wave, if the initial depth  $H_- > 1$  (the maximum determined from all depth ratios, but  $H_+ > 1$  also). As indicated above, the number of solitons formed is large if the initial depth is close to 1 (a zone with small values of the nonlinear parameter). It is also large when the initial depth is large, and this is due to passage from deep to shallow water. The amplitudes of the secondary solitons are shown in figure 7 for various values of the initial depth and depth ratio. If the initial depth is close to 1, several solitons are formed (for instance, for  $H_- = 1.05$  there are four solitons but the smallest soliton is not visible). If the initial depth is  $H_- = 2$ , two solitons are formed, but one of them is too small and is not visible. For  $H_- = 5$  a third soliton has appeared, but it is not visible. At  $H_- = 20$  all three solitons are visible.

If the wave passes over a step in the case when both zones are shallow water ( $H_-, H_+ < 1$ ), many solitons are generated if the depth of the lower layer is very small

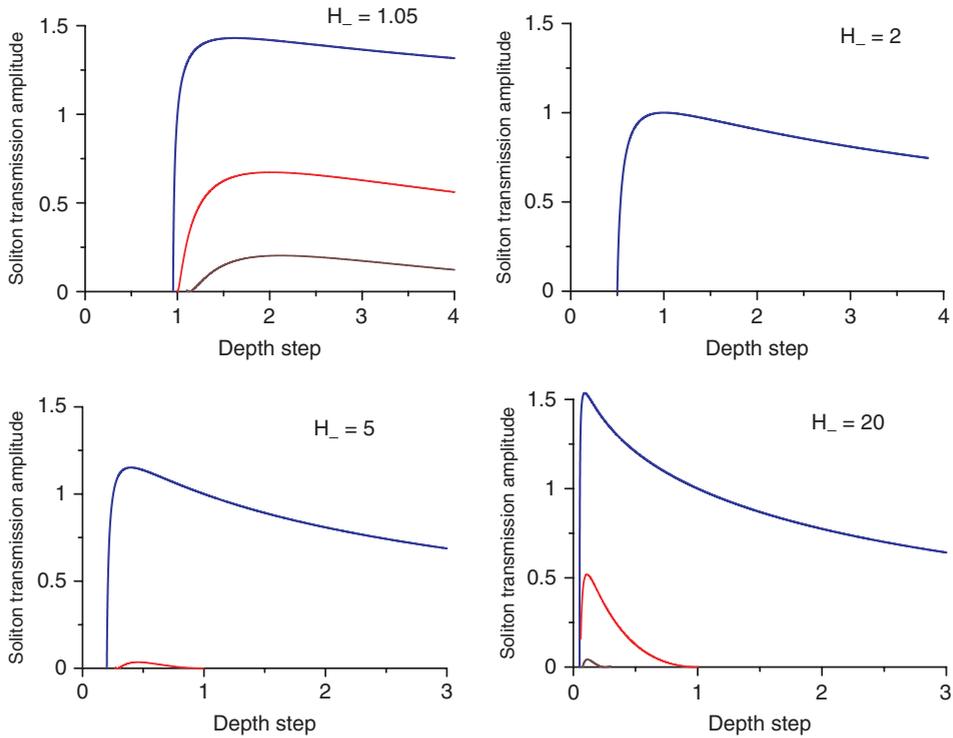


Figure 7. Amplitudes of solitons in the transmitted wave for various initial depths.

( $\Delta H \ll 1$ ), and their number is described by the approximation

$$N \approx \sqrt{\frac{4}{\Delta H} \frac{1 + H_-}{1 - H_-}}. \quad (21)$$

The amplitude of the first soliton tends to 4. The amplitudes of the first four solitons are shown in figure 8. Clearly seen is the increase of the secondary soliton amplitudes if the transmitted depth is small as seen above in (21). If the initial depth is close to 1 (the zone of small values of the nonlinear coefficient) the number of secondary solitons is increased for any depth ratio.

### 5. Higher-order nonlinear effects for soliton transmission over a step

The Korteweg-de Vries equation is valid for weakly nonlinear waves when the amplitude is less than the thickness of both layers. But if the density jump lies near the middle of the fluid depth, the quadratic nonlinear term becomes small and high-order nonlinear effects (cubic nonlinearity) have to be taken into account. The extension of the Korteweg-de Vries equation for two-layer fluid was derived first by Kakutani and Yamasaki (1978), and is given by the extended KdV (or Gardner) equation,

$$\frac{\partial \eta}{\partial t} + (c + \alpha \eta + \alpha_1 \eta^2) \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0, \quad (22)$$

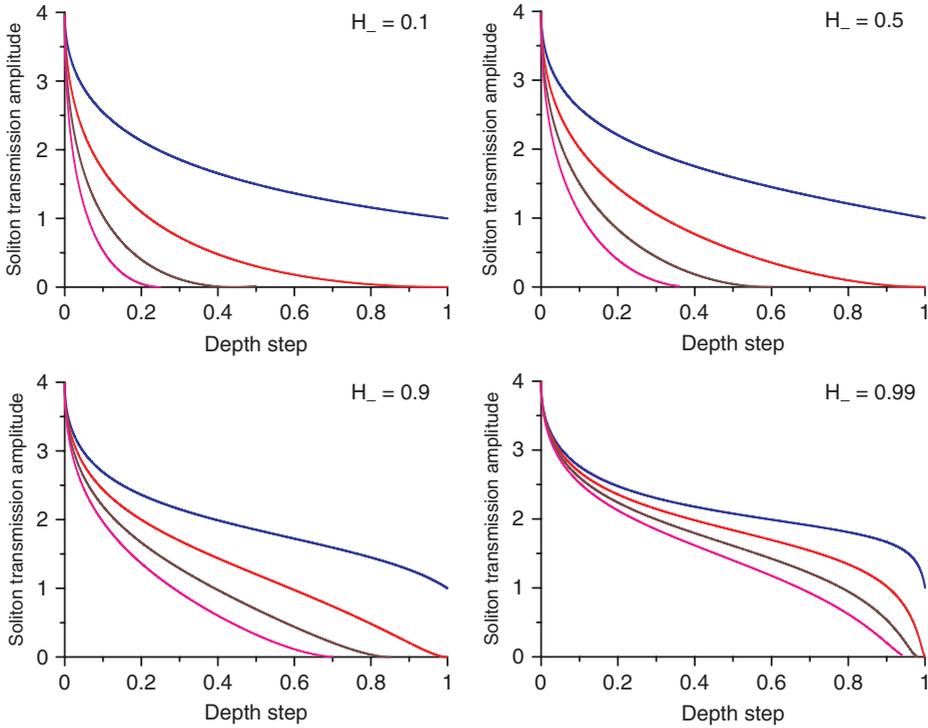


Figure 8. Amplitudes of the secondary solitons formed in the transmitted wave in shallow water for various initial depths.

where the cubic nonlinear coefficient is

$$\alpha_1 = -\frac{3c}{8h_1^2} \frac{1 + H^2 + 6H}{H^2}. \tag{23}$$

It is important to mention that the cubic nonlinear coefficient is negative for all ratios of the thickness of both layers. Steady-state solitary wave solution of the Gardner equation can be easily found

$$\eta(x, t) = \frac{D}{1 + B \cosh[\gamma(x - Vt)]}, \tag{24}$$

$$D = \frac{6\beta\gamma^2}{\alpha}, \quad B^2 = 1 + \frac{6\alpha_1\beta\gamma^2}{\alpha^2}, \quad V = \beta\gamma^2, \tag{25}$$

where  $\gamma$  is a free parameter characterizing the inverse width of the soliton. The soliton amplitude is

$$A = \frac{D}{(1 + B)}, \tag{26}$$

and is positive for positive values of the coefficient of quadratic nonlinearity. For the case here, when  $\alpha_1 < 0$ ,  $0 < B < 1$ . The soliton amplitude varies from small values ( $B \approx 1$ ),

when the Gardner soliton (24) coincides with the Korteweg-de Vries soliton (11), to the limiting value ( $B \approx 0$ )

$$A_{\text{lim}} = \frac{\alpha}{|\alpha_1|}, \quad (27)$$

when the soliton has a ‘‘table-top’’ shape.

The influence of the cubic nonlinear term can be demonstrated by two examples of wave transformation over a step. First is the case when the incident wave propagates in a zone of a small value of the coefficient of the nonlinear quadratic term and so is described by the Gardner soliton (24). After the step the wave has the same shape but with a different amplitude and width

$$\eta_{tr}(x) = DT \left/ \left[ 1 + B \cosh\left(\frac{C_-}{C_+} rx\right) \right] \right., \quad D = \frac{6\beta_- \gamma^2}{\alpha_-}, \quad B^2 = 1 + \frac{6\alpha_1 - \beta_- \gamma^2}{\alpha_-^2}. \quad (28)$$

If the transformed wave propagates in a fluid with a ‘‘normal’’ (that is, not too small) value of the coefficient of the nonlinear quadratic term, expression (28) can be used as an initial condition for the Korteweg-de Vries equation. The solution of the associated spectral problem with the initial condition (28) cannot be obtained in explicit form as for soliton-like disturbances, but the main conclusions are evident. The Gardner soliton (24) is wider than the KdV-soliton at the same amplitude (the same is the case for the transformed pulse after a step), and therefore, the number of secondary solitons is increased. In particular, if the incident wave is a table soliton with amplitude close to the limiting amplitude (27), the number of transmitted solitons can be described by the approximate formula (Drazin and Johnson 1989)

$$N \approx \int_{-\infty}^{+\infty} |x\eta_{tr}(x, 0)| dk, \quad (29)$$

and the amplitudes of the first (leading) soliton is close to

$$A_{\text{sol}} \approx \frac{2T\alpha_-}{|\alpha_{1+}|}. \quad (30)$$

The main difference with the KdV scenario is in the number of transmitted solitons.

The second example is when a KdV-soliton is incident on the step, but after the step there is an anomalously small value of the coefficient of the nonlinear quadratic term. In this case the transmitted wave has the KdV-soliton-like shape (15) again in the vicinity of the step. But its amplitude is significantly bigger than the limiting amplitude (27). As is shown in Grimshaw *et al.* (2002), such a disturbance in general evolves into one table-shape soliton and, perhaps, also smaller KdV-solitons. Their amplitudes cannot exceed the limiting value  $\alpha_+ / |\alpha_{1+}|$  (27), and in this case we may say that the amplitude of table soliton does not depend on the incident soliton amplitude  $A$ , in contrast with the prediction of the KdV theory, see (20). More precisely, the number and amplitude of the transmitted solitons can be found from the discrete eigenvalues of the associated spectral matrix problem

$$\Omega \bar{\Psi} = k \bar{\Psi}, \quad \Omega = \begin{pmatrix} -\partial/\partial z & u(z) \\ u(z) - 1 & \partial/\partial z \end{pmatrix}, \quad (31)$$

where

$$u(z) = B \operatorname{sech}^2\left(\frac{z}{L}\right), \quad B = AT \frac{|\alpha_{1+}|}{\alpha_+}, \quad L = \frac{c_+}{c_-} \sqrt{\frac{2}{A} \left(\frac{\beta_-}{\beta_+}\right) \frac{\alpha_+^2}{\alpha_- |\alpha_{1+}|}}. \quad (32)$$

These parameters are obtained after reducing the Gardner equation (21) to canonical form with coefficients 6 in front of each nonlinear term. The spectrum of (31) can be found numerically, but one limiting case can be analyzed analytically. If the coefficient of the quadratic nonlinear term  $\alpha_+$  tends to zero, the effective wave amplitude  $B$  tends to infinity, but its effective length  $L$  tends to zero, and the disturbance (32) can be considered as the  $\delta$ -function with effective mass

$$M = 2T \left(\frac{c_+}{c_-}\right) \sqrt{2A \left(\frac{\beta_-}{\beta_+}\right) \frac{|\alpha_{1+}|}{\alpha_-}}. \quad (33)$$

As it can be shown from the analytical result for a rectangular box disturbance (Grimshaw *et al.* 2002), only one spectral level exists in (31), and it can be found analytically:

$$k = (\tanh M)/2. \quad (34)$$

The amplitude of the transmitted soliton is expressed through this spectral eigenvalue  $k$  (now in dimensional variables):

$$A_{sol} = \frac{\alpha_+}{|\alpha_{1+}|} \frac{4k^2}{1 + \sqrt{1 - 4k^2}}. \quad (35)$$

If the mass is large, a table soliton is formed; if the mass is small, then a KdV soliton will form; this last example was considered earlier, see (20). If the amplitudes of the KdV and Gardner solitons are normalized by the limiting value (27), the following relation between them can be found (figure 9):

$$\frac{A_{Gar}}{A_{lim}} = \frac{\tanh^2 \sqrt{2A_{KdV}/A_{lim}}}{1 + \operatorname{sech} \sqrt{2A_{KdV}/A_{lim}}}. \quad (36)$$

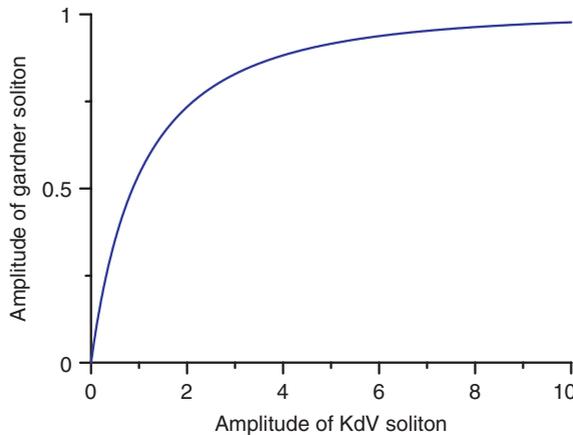


Figure 9. Relation between amplitudes of the Gardner and KdV solitons when the wave transformed to zone with small values of the quadratic nonlinear term; amplitudes are normalized on limiting value (27).

If the amplitude of the KdV soliton is small, the amplitude of the Gardner soliton coincides with it, and the effects of the cubic nonlinearity can be ignored. If the amplitude of the KdV soliton is large (compared with the limiting amplitude), the amplitude of the Gardner soliton tends to the limiting value, and the soliton has a table shape.

We have analyzed as above the soliton part of the wave field for the transmitted wave. A dispersive wavetrain tail is also induced for both the reflected and transmitted waves. But this wave tail attenuates with time, and so the solitons will be the main representatives of the wave field for large distances.

## 6. Numerical simulation of soliton generation in the transmitted wave

To demonstrate the process of the soliton fission, we performed some direct numerical simulations of the transmitted wave in the framework of the Gardner equation (22). We assume that the initial dimensionless thickness of the lower layer is 10, and after the step is 1.5. The transmitted coefficient is  $T=1.1$  according to (4), and therefore the linear amplification of the interfacial wave after a step is not too large. The initial soliton amplitude of the incident wave is varied from 0.1 (KdV) to 1.0 (Gardner). Numerical solution of the Gardner equation with constant coefficients calculated for the case after the step is performed using a finite-difference scheme described by

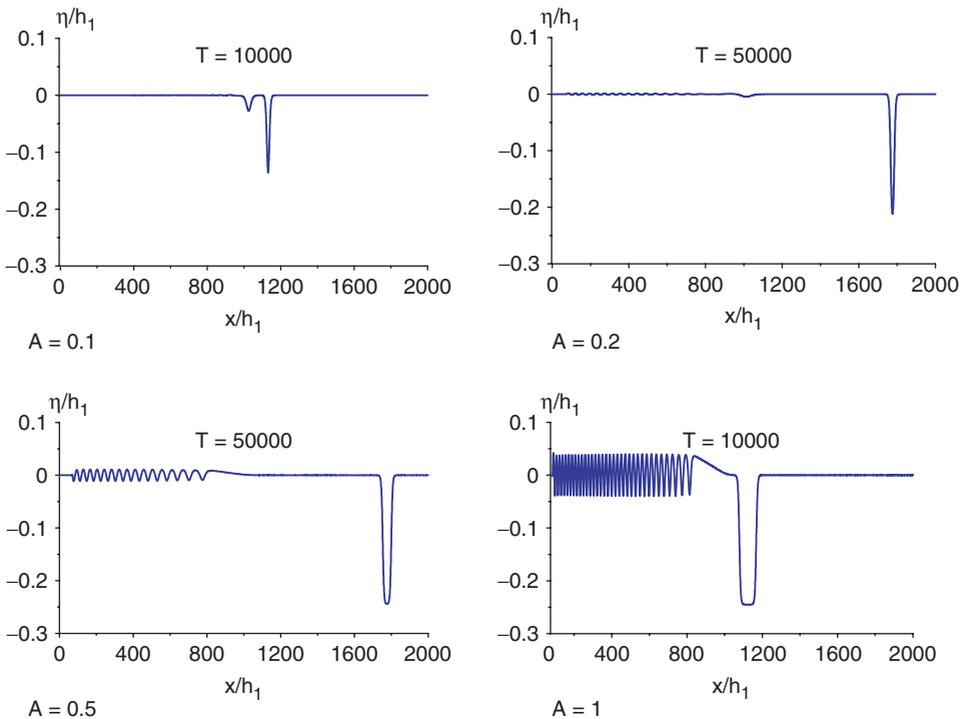


Figure 10. Soliton appearance in the field of transmitted wave for various initial soliton amplitudes.

Berezin (1987). Periodic boundary conditions are applied in the spatial domain,  $0 < x/h_1 < 2000$ , and the number of points is 8000. The time step is chosen from the Courant criterion. Time is normalized by  $h_1/(g'h_1\Delta\rho/\rho)^{1/2}$ .

Figure 10 demonstrates the generation of secondary solitons in the field of the transmitted wave. In the case of weak initial amplitude (0.1) two solitons are generated with amplitudes 0.136 and 0.027. According to the KdV theory (16) the soliton amplitudes are 0.131 and 0.013. The agreement is quite good especially for the first soliton. With an increase in the initial wave amplitude, the second soliton disappears, and the first soliton transforms to a table-top soliton with an amplitude close to the critical value 0.245, according to (27). Such behavior corresponds to the theoretical scenario described in section 5. The amplitude of the dispersive tail is also increased.

## 7. Conclusions

In this paper we have considered the transformation of a weakly nonlinear interfacial solitary wave incident on a step in the framework of a two-layer flow formulation. In the vicinity of the step the wave transformation can be described using the linear theory for long interfacial waves, and the coefficients of wave reflection and transmission are calculated. A strong transformation occurs for transmission into shallower water, and a weak one for transmission into deeper water. Far from the step, the wave dynamics can be described by the Korteweg-de Vries equation, which is fully integrable. The reflected and transmitted waves in the vicinity of a step have soliton-like shapes, but their parameters do not allow them to satisfy the steady-state soliton equations. Using the inverse scattering technique it is shown that the reflected wave evolves into a soliton and dispersive radiation if the incident wave meets a step from deep to shallow water, and conversely, dispersive radiation only if the wave meets a step from shallow to deep water. The dynamics of the transmitted wave is more complicated. In particular, if the coefficient of the nonlinear quadratic term in the Korteweg-de Vries equation is not changed in sign after the step, the transmitted wave evolves into a group of solitons and dispersing radiation, and qualitatively this process is similar to soliton fission for surface gravity waves. If the coefficient of the nonlinear term changes sign, the soliton is destroyed completely and transforms into dispersing radiation. The effects of cubic nonlinearity are studied in the framework of the Gardner equation, which is also integrable. Higher-order nonlinear effects influence the amplitudes of the generated solitons if the amplitude of the transformed wave is comparable with the thickness of the lower layer, but qualitatively the process of soliton fission is the same as in the framework of the Korteweg-de Vries equation.

Although our results are for a very simple two-layer model of the density stratification, and for a very simple representation (a step) of an abrupt change in bottom topography, we expect that our results will form a useful guide to the behaviour of oceanic internal solitary waves, propagating on the oceanic thermocline, incident on the continental shelf from the deep ocean for the situation when the wavelength is significantly greater than the horizontal scale of the topographic change. In this scenario we have identified those wave and physical parameters

which lead to substantial fission of the transmitted wave. The alternative case when the horizontal scale of the topographic change is much greater than the wavelength can be studied in the framework of a variable coefficient KdV, or Garder, equation (see Grimshaw *et al.* 2004, 2007, for instance). Although fission may occur in this case too, the number and amplitudes of the transmitted solitons is quite different from those found here, in part due to the role played by the magnitude of the transmission coefficient. The intermediate case when the horizontal scale of the topographic change is comparable with the wavelength is apparently not readily amenable to analysis, and requires numerical simulation.

### Acknowledgement

Authors thank grants from INTAS (06-1000013-9236 and 03-51-4286), Royal Society (EP), London Mathematical Society (TT) and RFBR (06-05-64232).

### References

- Baines, P.G., *Topographic effects in stratified flows*, 1995 (Cambridge University Press: Cambridge/New York/Melbourne).
- Berezin, Y.A., *Modelling nonlinear wave processes*, 1987 (VNU Science Press: Utrecht).
- Djordjevic, V. and Redekopp, L., The fission and disintegration of internal solitary waves moving over two-dimensional topography. *J. Phys. Ocean*, 1978, **8**, 1016–1024.
- Drazin, P.G. and Johnson, R.S., *Solitons: an introduction*, 1989 (Cambridge University Press: Cambridge/New York/London).
- Grimshaw, R., Pelinovsky, D., Pelinovsky, E. and Slunyaev, A., Generation of large-amplitude solitons in the extended Korteweg-de Vries equation. *Chaos*, 2002, **12**, 1070–1076.
- Grimshaw, R., Pelinovsky, E., Talipova, T. and Kurkin, A., Simulation of the transformation of internal solitary waves on oceanic shelves. *J. Phys. Ocean*, 2004, **34**, 2774–2779.
- Grimshaw R., Pelinovsky, E. and Talipova, T., Modeling internal solitary waves in the coastal ocean. *Surveys in Geophysics*, 2007, **28**(4), 273–298.
- Johnson, R.S., Some numerical solutions of a variable-coefficient Korteweg-de Vries equation (with applications to solitary wave development on a shelf). *J. Fluid Mech.*, 1972, **54**, 81–91.
- Helfrich, K.R. and Melville, W.K., On long nonlinear internal waves over slope-shelf topography. *J. Fluid Mech.*, 1986, **167**, 285–308.
- Liu, A.K., Chang, Y.S., Hsu, M.K. and Liang, N.K., Evolution of nonlinear internal waves in the East and South China Seas. *J. Geophys. Res.*, 1998, **103**(C4), 7995–8008.
- Liu, A.K., Ramp, S.R., Zhao, Y. and Tang, T.Y., A case study of internal solitary wave propagation during ASIAEX 2001. *IEEE J. of Oceanic Engineering*, 2004, **29**, 1144–1156.
- Kakutani, T. and Yamasaki, N., Solitary waves in a two-layer fluid. *J. Phys. Soc. Japan*, 1978, **45**, 674–679.
- Orr, M.H. and Mignerey, P.C., Nonlinear internal waves in the South China Sea: observation of the conversion of depression internal waves to elevation internal waves. *J. Geophys. Res.*, 2003, **108**, No. C3, 3064, doi: 10.1029/2001JC001163.
- Pelinovsky, E.N., On the soliton evolution in inhomogeneous media. *Applied Mechanics and Technical Physics*, 1971, **6**, 80–85.
- Pelinovsky, E.N., Solitary wave transformation on a shelf with horizontal bottom. In: *Theoretical and Experimental Investigations on Tsunami Problem*. Moscow. 1977, 61–63.
- Tappert, F.D. and Zabusky, N.J., Gradient-induced fission of solitons. *Phys. Rev. Letters*, 1971, **27**, 1774–1776.
- Zhao, Z., Klemas, V.V., Zheng, Q. and Yan X.-H., Satellite observation of internal solitary waves converting polarity. *Geophys. Res. Letters*, 2003, **30**(19), 1988, doi: 10.1029/2003GLO18286.
- Zheng, Q., Klemas, Q., Yan, X.-H. and Pan, J., Nonlinear evolution of ocean internal solitons propagating along an inhomogeneous thermocline. *J. Geophys. Res.*, 2000, **106**, 14083–14094.