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Rational Curves on Hyperkähler Manifolds

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Let M be an irreducible holomorphically symplectic manifold. We show that all faces of the Kähler cone of M are hyperplanes H_i orthogonal to certain homology classes, called monodromy birationally minimal (MBM) classes. Moreover, the Kähler cone is a connected component of a complement of the positive cone to the union of all H_i. We provide several characterizations of the MBM classes. We show the invariance of MBM property by deformations, as long as the class in question stays of type (1, 1). For hyperkähler manifolds with Picard group generated by a negative class z, we prove that $\pm z$ is $\mathbb{Q}$-effective if and only if it is an MBM class. We also prove some results toward the Morrison–Kawamata cone conjecture for hyperkähler manifolds.

1 Introduction

1.1 Kähler cone and MBM classes: an introduction

Let M be a hyperkähler manifold, that is, a compact, holomorphically symplectic Kähler manifold. We assume that $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$ (the general case reduces to this by Theorem 1.3). In this paper, we give a description of the Kähler cone of M in terms of a set of cohomology classes $S \subset H^2(M, \mathbb{Z})$ called *monodromy birationally minimal (MBM)*

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classes (Definition 1.13). This set is of topological nature, that is, it depends only on the deformation type of M.

It is known since [28] that the Kähler cone of a Fano manifold is polyhedral. Each of its finitely many faces is formed by classes vanishing on a certain rational curve: one says that the numerical class of such a curve generates an extremal ray of the Mori cone. The notion of an extremal ray also makes sense for projective manifolds which are not Fano: the number of extremal rays then does not have to be finite. However, they are discrete in the half-space where the canonical class restricts negatively.

Huybrechts [17] and Boucksom [7] have studied the Kähler cone of hyperkähler manifolds (not necessarily algebraic). They have proved that the Kähler classes are exactly those positive classes (i.e., classes with positive Beauville–Bogomolov–Fujiki square; see Definition 1.8) which restrict positively to all rational curves (Theorem 1.9).

Our work puts these results in a deformation-invariant setting. Let $\mathrm{Pos}\subset H^{1,1}_{\mathbb{R}}(M)$ be the positive cone, and $S(I)$ the set of all MBM classes which are of type (1,1) on M with its given complex structure I. Then the Kähler cone is a connected component of $\mathrm{Pos}\backslash S(I)^{\perp}$, where $S(I)^{\perp}$ is the union of all orthogonal complements to all $z\in S(I)$.

We describe the MBM classes in terms of the minimal curves on deformations of M, birational maps and the monodromy group action (Sections 4 and 5), and formulate a finiteness conjecture (Conjecture 6.4) claiming that primitive integral MBM classes have bounded square. We deduce the Morrison–Kawamata cone conjecture from this conjecture. For deformations of the Hilbert scheme of points on a K3 surface, our finiteness conjecture is known [1, Proposition 2]. This gives a proof of Morrison–Kawamata cone conjecture for deformations of the Hilbert scheme of K3 (Theorem 1.21). Its proof is independently obtained by Markman and Yoshioka using different methods (forthcoming).

1.2 Hyperkähler manifolds

Definition 1.1. A *hyperkähler manifold* is a compact, Kähler, holomorphically symplectic manifold. □

Definition 1.2. A hyperkähler manifold M is called *simple* if $\pi_1(M)=0$, $H^{2,0}(M)=\mathbb{C}$. □

This definition is motivated by the Theorem 1.3 of Bogomolov.

Theorem 1.3 ([4])**.** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds. □

Remark 1.4. Furthermore, we shall assume (sometimes, implicitly) that all hyperkähler manifolds we consider are simple. □

Remark 1.5. A hyperkähler manifold naturally possesses a whole 2-sphere of complex structures (see Section 2). We shall use the notation (M, I) or M_I to stress that a particular complex structure is chosen, and M to discuss the topological properties (or simply when there is no risk of confusion). □

The Bogomolov–Beauville–Fujiki form was defined in [2, 5], but it is easiest to describe it using the Fujiki theorem, proved in [14].

Theorem 1.6 (Fujiki)**.** Let M be a simple hyperkähler manifold, $\eta \in H^2(M)$, and $n = \frac{1}{2}\dim M$. Then $\int_M \eta^{2n} = cq(\eta, \eta)^n$, where q is a primitive integer quadratic form on $H^2(M, \mathbb{Z})$, and $c > 0$ an integer. □

Remark 1.7. Fujiki formula (Theorem 1.6) determines the form q uniquely up to a sign. For odd n, the sign is also unambiguously determined. For even n, one needs the following explicit formula, which is due to Bogomolov and Beauville.

$$\lambda q(\eta, \eta) = \frac{1}{2}\int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \frac{n-1}{n}\left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n}\right)\left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1}\right) \quad (1.1)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$. □

Definition 1.8. A cohomology class $\eta \in H^{1,1}_{\mathbb{R}}(M, I)$ is called *negative* if $q(\eta, \eta) < 0$, and *positive* if $q(\eta, \eta) > 0$. Since the signature of q on $H^{1,1}_{\mathbb{R}}(M, I)$ is $(1, b_2 - 3)$, the set of positive vectors is disconnected. *The positive cone* $\mathrm{Pos}(M, I)$ is the connected component of the set $\{\eta \in H^{1,1}_{\mathbb{R}}(M, I) \mid q(\eta, \eta) > 0\}$ which contains the classes of the Kähler forms. It is easy to check that the positive cone is convex. □

Theorem 1.9 is crucial for our work.

Theorem 1.9 (Huybrechts, Boucksom; see [7])**.** Let (M, I) be a simple hyperkähler manifold. The Kähler cone $\mathrm{Kah}(M, I)$ can be described as follows

$$\mathrm{Kah}(M, I) = \{\alpha \in \mathrm{Pos}(M, I) | \alpha \cdot C > 0\ \forall C \in RC(M, I)\}$$

where $RC(M, I)$ denotes the set of classes of rational curves on (M, I). □

1.3 Main results

Remark 1.10. Let (M, I) be a hyperkähler manifold, and $\varphi\colon (M, I) \dashrightarrow (M, I')$ a bimeromorphic (also called "birational") map to another hyperkähler manifold (note that by a result of Huybrechts [16], birational hyperkähler manifolds are deformation equivalent; that is why there is the same letter M used for the source and the target of φ). Since the canonical bundle of (M, I) and (M, I') is trivial, φ is an isomorphism in codimension 1 (see for example [16], Lemma 2.6). This allows one to identify $H^2(M, I)$ and $H^2(M, I')$. Clearly, this identification is compatible with the Hodge structure. Furthermore, we call (M, I') "a birational model" for (M, I), and identify $H^2(M)$ for all birational models. □

Definition 1.11. Let M be a hyperkähler manifold. The *monodromy group* of M is a subgroup of $GL(H^2(M, \mathbb{Z}))$ generated by monodromy transforms for all Gauss–Manin local systems. This group can also be characterized in terms of the mapping class group action (Remark 2.21). □

Definition 1.12. Let (M, I) be a hyperkähler manifold. A rational homology class $z \in H_{1,1}(M, I)$ is called $\mathbb{Q}$-*effective* if Nz can be represented as a homology class of a curve, for some $N \in \mathbb{Z}^{>0}$, and *extremal* if for any $\mathbb{Q}$-effective homology classes $z_1, z_2 \in H_{1,1}(M, I)$ satisfying $z_1 + z_2 = z$, the classes z_1, z_2 are proportional. □

In the projective case, a negative extremal class is $\mathbb{Q}$-effective and some multiple of it is represented by a rational curve. Moreover, the negative part of the cone of $\mathbb{Q}$-effective classes is locally rational polyhedral. This is shown by a version of Mori theory adapted to the case of hyperkähler manifolds (see [15]).

The Beauville–Bogomolov–Fujiki form allows one to identify $H^2(M, \mathbb{Q})$ and $H_2(M, \mathbb{Q})$. More precisely, it provides an embedding $H^2(M, \mathbb{Z}) \to H_2(M, \mathbb{Z})$ which is not an isomorphism (indeed, q is not necessarily unimodular), but becomes an isomorphism after tensoring with $\mathbb{Q}$. We thus can talk of extremal classes in $H^{1,1}(M, \mathbb{Q})$, meaning that the corresponding classes in $H_{1,1}(M, \mathbb{Q})$ are extremal. In general, we shall switch from the second homology to the second cohomology as it suits us and transfer all definitions from one situation to the other one by means of the Beauville-Bogomolov-Fujiki (BBF) form, with one important exception, namely the notion of effectiveness. Indeed, an effective class in $H_{1,1}(M, \mathbb{Q})$ is a class of a curve, whereas an effective class in $H^{1,1}(M, \mathbb{Q})$ is a class of a hypersurface.

We shall also extend the BBF form to $H_2(M, \mathbb{Z})$ as a rational-valued quadratic form, and to $H_2(M, \mathbb{Q})$, without further note.

The following property, with which we shall work in this paper, looks stronger than extremality modulo monodromy and birational equivalence, but is equivalent to it whenever the negative part of the Mori cone is locally rational polyhedral.

Recall that *a face* of a convex cone in a vector space V is an intersection of its boundary and a hyperplane which has nonempty interior (Definition 3.1).

Definition 1.13. A nonzero negative rational homology class $z \in H^{1,1}(M, I)$ is called *monodromy birationally minimal (MBM)* if for some isometry $\gamma \in O(H^2(M, \mathbb{Z}))$ belongs to the monodromy group, $\gamma(z)^\perp \subset H^{1,1}(M, I)$ contains a face of the Kähler cone of one of birational models (M, I') of (M, I). □

The definition has an obvious counterpart for homology classes of type $(1,1)$, where one replaces the orthogonality with respect to the BBF form by the usual duality between homology and cohomology, given by integration of forms over cycles.

Remark 1.14. A face of $\operatorname{Kah}(M, I')$ is, by definition, of maximal dimension $h^{1,1}(M, I') - 1$. So the definition of z being MBM means that $\gamma(z)^\perp \cap \partial \operatorname{Kah}(M, I')$ contains an open subset of $\gamma(z)^\perp$. □

The point of Definition 1.13 is, roughly, as follows. As we shall see, rational curves have nice local deformation properties when they are minimal (extremal). However, globally, a deformation of an extremal rational curve on a variety (M, I) does not have to remain extremal on a deformation (M, I'). One can only hope to show the deformation-invariance of the property of z being extremal (as long, of course, as it remains of type $(1,1)$) *modulo monodromy and birational equivalence*. In what follows, we shall solve this problem for MBM classes and apply it to the study of the Kähler cone.

The deformation equivalence is especially useful because when $\operatorname{Pic}(M)$ has rank one, the somewhat obscure notion of MBM classes becomes much more streamlined.

Theorem 1.15. Let (M, I) be a hyperkähler manifold, $\operatorname{rk}\operatorname{Pic}(M, I) = 1$, and $z \in H_{1,1}(M, I)$ a nonzero negative rational class. Then z is MBM if and only if $\pm z$ is $\mathbb{Q}$-effective. □

Proof. See Theorem 5.11. ■

Definition 1.16. Let (M, I) be a hyperkähler manifold. A negative rational class $z \in H^{1,1}_{\mathbb{Q}}(M, I)$ is called *divisorial* if $z = \lambda[D]$ for some effective divisor D and $\lambda \in \mathbb{Q}$. □

Our first main results concern the deformational invariance of these notions.

Theorem 1.17. Let (M, I) be a hyperkähler manifold, $z \in H_{1,1}(M, I)$ an integer homology class, $q(z, z) < 0$, and I' a deformation of I such that z is of type (1,1) with respect to I'. Assume that z is MBM on (M, I). Then z is MBM on (M, I'). The property of $z \in H^{1,1}(M, I)$ being divisorial is likewise deformation-invariant, provided that one restricts oneself to the complex structures with Picard number one (i.e., such that the Picard group is generated by z over $\mathbb{Q}$). □

Proof. See Theorem 4.10, Corollary 4.11, and Theorem 5.13. ■

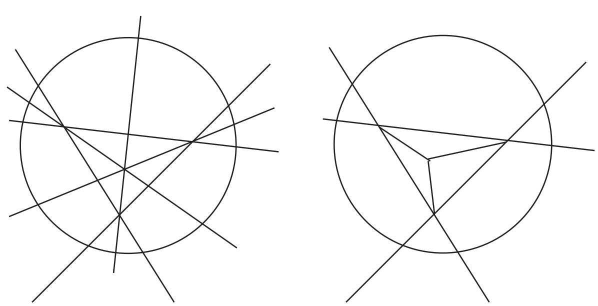
Remark 1.18. We expect that the property of z being divisorial is deformation-invariant as long as z stays of type $(1, 1)$, and plan to return to this question in a forthcoming paper. □

The MBM classes can be used to determine the Kähler cone of (M, I) explicitly.

Theorem 1.19. Let (M, I) be a hyperkähler manifold, and $S \subset H_{1,1}(M, I)$ the set of all MBM classes. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \mid z \in S\}$ in $H^{1,1}_{\mathbb{R}}(M, I)$. Then the Kähler cone of (M, I) is a connected component of $\operatorname{Pos}(M, I) \backslash \cup_{W \in S^\perp} W$, where $\operatorname{Pos}(M, I)$ is the positive cone of (M, I). Moreover, for any connected component K of $\operatorname{Pos}(M, I) \backslash \cup_{W \in S^\perp} W$, there exists $\gamma \in O(H^2(M, \mathbb{Z}))$ in the monodromy group of M, and a hyperkähler manifold (M, I') birationally equivalent to (M, I), such that $\gamma(K)$ is the Kähler cone of (M, I'). □

Proof. See Theorem 6.2. ■

Remark 1.20. In particular, $z^\perp \cap \operatorname{Pos}(M, I)$ either has dense intersection with the interior of the Kähler chambers (if z is not MBM), or is a union of walls of those (if z is MBM); that is, there are no "barycentric partitions" in the decomposition of the positive cone into the Kähler chambers.



Allowed partition Prohibited partition □

Finally, we apply this to the Morrison–Kawamata cone conjecture.

Theorem 1.21. Let M be a simple hyperkähler manifold, and q the Bogomolov–Beauville–Fujiki form. Suppose that there exists $C > 0$ such that $|q(\eta, \eta)| < C$ for all primitive integral MBM classes (or, alternatively, for all extremal rational curves on all deformations of M). Then the Morrison–Kawamata cone conjecture holds for M: the group $\mathrm{Aut}(M)$ acts on the set of faces of the Kähler cone with finitely many orbits. □

Proof. See Theorem 6.6. ■

The condition of the theorem is satisfied for manifolds which are deformation equivalent to the Hilbert scheme of points on a K3 surface, see [1]; for such manifolds, one, therefore, obtains a proof of the Morrison–Kawamata cone conjecture.

2 Global Torelli Theorem, Hyperkähler Structures, and Monodromy Group

In this section, we recall a number of results about hyperkähler manifolds, used further on in this paper. For more details and references, see [3, 34].

2.1 Hyperkähler structures and twistor spaces

Definition 2.1. Let (M, g) be a Riemannian manifold, and I, J, K endomorphisms of the tangent bundle TM satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\mathrm{Id}_{TM}.$$

The triple (I, J, K) together with the metric g is called *a hyperkähler structure* if I, J, and K are integrable and Kähler with respect to g. □

Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ on M:

$$\omega_I(\cdot,\cdot) := g(\cdot, I\cdot), \quad \omega_J(\cdot,\cdot) := g(\cdot, J\cdot), \quad \omega_K(\cdot,\cdot) := g(\cdot, K\cdot).$$

An elementary linear-algebraic calculation implies that the 2-form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is of Hodge type $(2, 0)$ on (M, I). This form is clearly closed and nondegenerate, hence it is a holomorphic symplectic form.

In algebraic geometry, the word “hyperkähler” is essentially synonymous with “holomorphically symplectic”, due to the Theorem 2.2, which is implied by Yau’s solution of Calabi conjecture [2, 3].

Theorem 2.2. Let M be a compact, Kähler, holomorphically symplectic manifold, ω its Kähler form, $\dim_{\mathbb{C}} M = 2n$. Denote by Ω the holomorphic symplectic form on M. Suppose that $\int_M \omega^{2n} = \int_M (\operatorname{Re}\Omega)^{2n}$. Then there exists a unique hyperkähler metric g with the same Kähler class as ω, and a unique hyperkähler structure (I, J, K, g), with $\omega_J = \operatorname{Re}\Omega$, $\omega_K = \operatorname{Im}\Omega$. □

Furthermore, we shall speak of "hyperkähler manifolds" meaning "holomorphic symplectic manifolds of Kähler type", and "hyperkähler structures" meaning the quaternionic triples together with a metric.

Every hyperkähler structure induces a whole two-dimensional sphere of complex structures on M, as follows. Consider a triple $a, b, c \in R$, $a^2 + b^2 + c^2 = 1$, and let $L := aI + bJ + cK$ be the corresponding quaternion. Quaternionic relations imply immediately that $L^2 = -1$, hence L is an almost complex structure. Since I, J, K are Kähler, they are parallel with respect to the Levi–Civita connection. Therefore, L is also parallel. Any parallel complex structure is integrable, and Kähler. We call such a complex structure $L = aI + bJ + cK$ *a complex structure induced by a hyperkähler structure*. There is a two-dimensional holomorphic family of induced complex structures, and the total space of this family is called *the twistor space* of a hyperkähler manifold.

2.2 Global Torelli theorem and monodromy

Definition 2.3. Let M be a compact complex manifold, and $\mathsf{Diff}_0(M)$ a connected component of its diffeomorphism group (*the group of isotopies*). By Kodaira–Spencer stability theorem [21], the space of complex structures of Kähler type on M is an open subset in the space of all complex structures. Denote by Comp the space of complex structures of Kähler type on M, equipped with its structure of a Fréchet manifold, and let $\mathsf{Teich} := \mathsf{Comp}/\mathsf{Diff}_0(M)$. We call it *the Teichmüller space.* □

Remark 2.4. In many important cases, such as for manifolds with trivial canonical class [11], Teich is a finite-dimensional complex space; usually, it is non-Hausdorff. □

Definition 2.5. The universal family of complex manifolds over Teich is defined as $\mathsf{Univ}/\mathsf{Diff}_0(M)$, where Univ is the natural universal family over Comp with its Fréchet manifold structure. Locally, it is isomorphic to the universal family over the Kuranishi space. We are grateful to Claire Voisin for this observation. □

Definition 2.6. The *mapping class group* is $\Gamma = \mathsf{Diff}(M)/\mathsf{Diff}_0(M)$. It naturally acts on Teich (the quotient of Teich by Γ may be viewed as the "moduli space" for M, but in general, it has very bad properties; see below). □

Remark 2.7. Let M be a hyperkähler manifold (as usually, we assume M to be simple). For any $J \in \mathsf{Teich}$, (M, J) is also a simple hyperkähler manifold, because the Hodge numbers are constant in families, and therefore, $H^{2,0}(M, J)$ is one-dimensional. □

Definition 2.8. Let

$$\mathsf{Per}: \ \mathsf{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$$

map J to the line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map Per is called *the period map*. □

Remark 2.9. The period map Per maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}\mathsf{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called *the period domain* of M. Indeed, any holomorphic symplectic form l satisfies the relations $q(l, l) = 0$, $q(l, \bar{l}) > 0$, as follows from (1.1). □

Proposition 2.10. The period domain $\mathbb{P}\mathsf{er}$ is identified with the quotient $\mathrm{SO}(b_2 - 3, 3)/\mathrm{SO}(2) \times \mathrm{SO}(b_2 - 3, 1)$, which is the Grassmannian of positive oriented 2-planes in $H^2(M, \mathbb{R})$. □

Proof. This statement is well known, but we shall sketch its proof for the reader's convenience.

Step 1: given $l \in \mathbb{P}H^2(M, \mathbb{C})$, the space generated by $\operatorname{Im} l$, $\operatorname{Re} l$ is two-dimensional, because $q(l, l) = 0$, $q(l, \bar{l}) > 0$ implies that $l \cap H^2(M, \mathbb{R}) = 0$.

Step 2: This two-dimensional plane is positive, because $q(\operatorname{Re} l, \operatorname{Re} l) = q(l + \bar{l}, l + \bar{l}) = 2q(l, \bar{l}) > 0$.

Step 3: Conversely, for any two-dimensional positive plane $V \in H^2(M, \mathbb{R})$, the quadric $\{l \in V \otimes_{\mathbb{R}} \mathbb{C} \mid q(l, l) = 0\}$ consists of two lines; a choice of a line is determined by the orientation. ■

Definition 2.11. Let M be a topological space. We say that $x, y \in M$ are *nonseparable* (denoted by $x \sim y$) if for any open sets $V \ni x$, $U \ni y$, $U \cap V \neq \emptyset$. □

Theorem 2.12 (Huybrechts; [16]). If two points $I, I' \in \mathsf{Teich}$ are nonseparable, then there exists a bimeromorphism $(M, I) \dashrightarrow (M, I')$. □

Remark 2.13. The converse is not true since many points in the Teichmüller space correspond to isomorphic manifolds, and they are not always pairwise nonseparable (consider orbits of the mapping class group action). But it is true that if two hyperkähler manifolds are bimeromorphic, then one can find nonseparable points in the Teichmüller space representing them. □

Definition 2.14. The space $\mathsf{Teich}_b := \mathsf{Teich}/\sim$ is called *the birational Teichmüller space* of M. □

Remark 2.15. This terminology is slightly misleading since there are nonseparable points of the Teichmüller space which correspond to biregular, not just birational, complex structures. Even for K3 surfaces, the Teichmüller space is non-Hausdorff. □

Theorem 2.16 (Global Torelli theorem; [34]). The period map $\mathsf{Teich}_b \xrightarrow{\mathsf{Per}} \mathbb{P}\mathsf{er}$ is an isomorphism on each connected component of Teich_b. □

Remark 2.17. By a result of Huybrechts [18], Teich has only finitely many connected components. We shall denote by Teich_I the component containing the parameter point for the complex structure I, and by Γ_I the subgroup of the mapping class group Γ fixing this component. Obviously, Γ_I is of finite index in Γ. □

Definition 2.18. Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the *mapping class group* $\mathsf{Diff}(M)/\mathsf{Diff}_0(M)$. The quotient Teich_b/Γ is called *the birational moduli space* of M. Its points are in bijective correspondence with the complex structures of hyperkähler type on M up to a bimeromorphic equivalence. □

Remark 2.19. The word "space", in this context, is misleading. In fact, the quotient topology on Teich_b/Γ is extremely non-Hausdorff, for example, every two open sets would intersect [35]. □

The global Torelli theorem can be stated as a result about the birational moduli space.

Theorem 2.20 ([34, Theorem 7.2, Remark 7.4, and Theorem 3.5]). Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. Then W is isomorphic to $\mathbb{P}\mathsf{er}/\mathsf{Mon}_I$, where $\mathbb{P}\mathsf{er} = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$ and Mon_I is an arithmetic group in $O(H^2(M,\mathbb{R}), q)$, called *the monodromy group* of (M, I). In fact, Mon_I is the image of Γ_I in $O(H^2(M,\mathbb{R}), q)$. □

Remark 2.21. The monodromy group of (M, I) can be also described as a subgroup of the group $O(H^2(M,\mathbb{Z}), q)$ generated by monodromy transform maps for Gauss–Manin local systems obtained from all deformations of (M, I) over a complex base [34, Definition 7.1]. This is how this group was originally defined by Markman [23, 24]. The fact that it is of finite index in $O(H^2(M,\mathbb{Z}), q)$ is crucial for the Morrison–Kawamata conjecture, see next section. □

Remark 2.22. A caution: usually, "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, the Hodge structure on $H^2(M,\mathbb{Z})$ determines the complex structure. For $\dim_{\mathbb{C}} M > 2$, it is false. □

3 Morrison–Kawamata Cone Conjecture for Hyperkähler Manifolds

In this section, we introduce the Morrison–Kawamata cone conjecture, and give a brief survey of what is known, following [24, 32], with some easy generalizations.

Originally, this conjecture was stated in [29] while it has already been known for K3 surfaces from [31]. A relative version for Calabi–Yau 3-fold admitting a holomorphic fibration over a positive-dimensional base has been obtained by Kawamata in [19].

3.1 Morrison–Kawamata cone conjecture

Definition 3.1. Let M be a compact, Kähler manifold, $\mathrm{Kah} \subset H^{1,1}(M,\mathbb{R})$ the Kähler cone, and $\overline{\mathrm{Kah}}$ its closure in $H^{1,1}(M,\mathbb{R})$, called *the nef cone*. A *face* of the Kähler cone is the intersection of the boundary of $\overline{\mathrm{Kah}}$ and a hyperplane $V \subset H^{1,1}(M,\mathbb{R})$, which has nonempty interior. □

Conjecture 3.2 (Morrison–Kawamata cone conjecture, Kähler version)**.** Let M be a Calabi–Yau manifold. Then the group $\mathrm{Aut}(M)$ of biholomorphic automorphisms of M acts on the set of faces of Kah with finite number of orbits. □

Remark 3.3. The actual Morrison–Kawamata conjecture is a stronger statement for projective Calabi–Yau manifolds. Roughly speaking (up to some boundary issues), it affirms that $Aut(M)$ has a finite polyhedral domain on the ample cone. Such a statement obviously cannot be true for the Kähler rather than the ample cone (for instance, a very general irreducible hyperkähler manifold has no automorphisms, whereas its Kähler cone is equal to the positive cone). So our version is as strong as one can expect in the nonprojective case. □

This conjecture has a birational version, which has many important implications. For projective hyperkähler manifolds, the birational version of Morrison–Kawamata cone conjecture was proved by Markman in [24].

Definition 3.4. Let M, M' be compact complex manifolds. Define a *pseudo-isomorphism* $M \dashrightarrow M'$ as a birational map which is an isomorphism outside of codimension $\geqslant 2$ subsets of M, M'. □

Remark 3.5. For any pseudo-isomorphic manifolds M, M', the second cohomologies $H^2(M)$ and $H^2(M')$ are naturally identified. □

As we have already remarked, any birational map of hyperkähler varieties is a pseudo-isomorphism; more generally, this is true for all varieties with nef canonical class.

Definition 3.6. The *movable cone*, also known as *birational nef cone*, is the closure of the union of $\mathrm{Kah}(M')$ for all M' pseudo-isomorphic to M. The union of $\mathrm{Kah}(M')$ for all M' pseudo-isomorphic to M is called *birational Kähler cone*. □

Conjecture 3.7 (Morrison–Kawamata birational cone conjecture, Kähler version)**.** Let M be a Calabi–Yau manifold. Then the group $\mathrm{Bir}(M)$ of birational automorphisms of M acts on the set of faces of its movable cone with finite number of orbits. □

3.2 Morrison–Kawamata cone conjecture for K3 surfaces

In this section, we prove the Morrison–Kawamata cone conjecture for K3 surfaces. Originally, it was proved by Sterk (see [31]). The proof we are giving has two advantages: it works in the nonalgebraic situation, and to some extent, it generalizes to arbitrary dimension. Note that the pseudo-isomorphisms of smooth surfaces are isomorphisms, hence for K3 surfaces, both versions of Morrison–Kawamata cone conjecture are equivalent.

Definition 3.8. A cohomology class $\eta \in H^2(M, \mathbb{Z})$ on a K3 surface is called *a* (-2)*-class* if $\int_M \eta \wedge \eta = -2$. □

Remark 3.9. Let M be a K3 surface, and $\eta \in H^{1,1}(M, \mathbb{Z})$ a (-2)-class. Then either η or $-\eta$ is effective. Indeed, $\chi(\eta) = 2 + \frac{\eta^2}{2} = 1$ by Riemann–Roch. □

Theorem 3.10 is well known.

Theorem 3.10. Let M be a K3 surface, and S the set of all effective (-2)-classes. Then $\operatorname{Kah}(M)$ is the set of all $v \in \operatorname{Pos}(M)$ such that $q(v, s) > 0$ for all $s \in S$. □

Proof. By adjunction formula, a curve C on a K3 surface satisfies $C^2 = 2g - 2$, where g is its genus; hence the cone of effective curves is generated by positive classes and effective (-2)-classes. By [12], the Kähler cone is the intersection of the positive cone and the cone dual to the cone of effective curves, hence it is $v \in \operatorname{Pos}(M)$ such that $q(v, s) > 0$ for all effective s. However, for all $s, v \in \operatorname{Pos}(M)$, the inequality $q(v, s) > 0$ automatically follows from the Hodge index formula. ■

Definition 3.11. *A Weyl chamber*, or *Kähler chamber*, on a K3 surface is a connected component of $\operatorname{Pos}(M)\backslash S^\perp$, where $S^\perp$ is the union of all planes $s^\perp$ for all (-2)-classes $s \in S$. *The reflection group* of a K3 surface is the group W generated by reflections with respect to all $s \in S$. □

Remark 3.12. Clearly, a Weyl chamber is a fundamental domain of W, and W acts transitively on the set of all Weyl chambers. Moreover, the Kähler cone of M is one of its Weyl chambers. □

To a certain extent, this is a pattern which is repeated in all dimensions, as we shall see.

Theorem 3.13 is well known, too, but we want to sketch a proof in some detail since it is also important for what follows.

Theorem 3.13. Let M be a K3 surface, and $\operatorname{Aut}(M)$ the group of all automorphisms of M. Then $\operatorname{Aut}(M)$ is the group of all isometries of $H^2(M, \mathbb{Z})$ preserving the Kähler chamber Kah and the Hodge decomposition. □

Proof. First, let us show that the natural map $\operatorname{Aut}(M) \xrightarrow{\varphi} O(H^2(M, \mathbb{Z}))$ is injective. Our argument is similar to Buchdahl's in [9]. Clearly, $\operatorname{Aut}(M)$ acts on $H^2(M, \mathbb{Z})$; its kernel is formed by automorphisms preserving all Kähler classes, hence acting as isometries on all Calabi–Yau metrics on all deformations of M. Deforming M to a Kummer surface, we find that this isometry must also induce an isometry on the underlying 2-torus, acting trivially on cohomology. Therefore, $\ker \varphi$ acts as identity on all Kummer surfaces, and hence also on all K3 surfaces.

To describe the image of φ, we use the Torelli theorem. Let $\gamma \in O(H^2(M,\mathbb{Z}))$ preserve the Hodge decomposition on (M, I) and the Kähler cone. Global Torelli theorem affirms that the period map $\mathsf{Per}: \mathsf{Teich}_b \longrightarrow \mathbb{P}\mathsf{er}$ for K3 surfaces is an isomorphism. The mapping class group $\Gamma = \mathsf{Diff}(M)/\mathsf{Diff}_0(M)$ acting on Teich embeds as the orientation-preserving subgroup in $O(H^2(M,\mathbb{Z}))$ [6]. We deduce that γ comes from an element $\widetilde{\gamma}$ of the mapping class group. Since γ preserves the Hodge decomposition, the corresponding point in the period space $\mathsf{Per}(I) \in \mathbb{P}\mathsf{er}$ is a fixed point of $\widetilde{\gamma}$. Therefore, $\widetilde{\gamma}$ exchanges the nonseparable points in $\mathsf{Per}^{-1}(\mathsf{Per}(I))$. However, as shown by Huybrechts, two manifolds which correspond to distinct nonseparable points in Teich have different Kähler cones, so that the fibers of the period map parameterize the Kähler (Weyl) chambers [16]. Therefore, $\widetilde{\gamma}$ maps $(M, I) \in \mathsf{Per}^{-1}(\mathsf{Per}(I))$ to itself whenever it fixes the Kähler cone. Hence $\widetilde{\gamma}$, considered as a diffeomorphism of M, is an automorphism of the complex manifold (M, I), and γ is in the image of φ. ■

Now we can prove the Morrison–Kawamata cone conjecture for K3 surfaces. Our argument is based on the following general result about lattices.

Theorem 3.14. Let q be an integer-valued quadratic form on $\Lambda = \mathbb{Z}^n$, $n \geqslant 2$ (not necessarily unimodular) such that its kernel is at most one-dimensional, and $O(\Lambda)$ the corresponding group of isometries. Fix $0 \neq r \in \mathbb{Z}$, and let S_r be the set $\{v \in \Lambda \mid q(v, v) = r\}$. Then $O(\Lambda)$ acts on S_r with finite number of orbits. □

Proof. The nondegenerate case is [20, Satz 30.2]. In the case of one-dimensional kernel $L = \ker q$, we write $\Lambda = L \oplus \Lambda_0$ and the elements of Λ as $a_0 + kl$, where $a_0 \in \Lambda_0$, $k \in \mathbb{Z}$ and l is a fixed generator of L. There are finitely many representatives of $S_r \cap \Lambda_0$ under the action of $O(\Lambda_0)$, say $a_0^1, \dots, a_0^m$. For each j, take a system of representatives $k_1, \dots, k_{t_j}$ of $\mathbb{Z}$ modulo the ideal $\mathrm{Hom}(\Lambda_0, L) \cdot a_0^j$. Then representatives of the orbits in S_r are elements of the form $a_0^j + k_i l$. ■

We shall apply this to study the Picard lattice $H^2(M,\mathbb{Z}) \cap H^{1,1}(M)$. Note that it is nondegenerate when M is projective, and can have at most one-dimensional kernel when M is arbitrary.

Theorem 3.15. Morrison–Kawamata cone conjecture holds for any K3 surface. □

Proof. *Step* 1: *setting a goal.* Let $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ be the group of all oriented isometries of $H^2(M,\mathbb{Z})$ preserving the Hodge decomposition (such isometries are known as *Hodge* *isometries*). We have already remarked that when M is a K3 surface, all these are monodromy operators, hence the notation. Since $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ acts on the Picard lattice Λ as a subgroup of index at most two in $O(\Lambda) = O(H^{1,1}(M, \mathbb{Z}))$, Theorem 3.14 implies that $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ acts with finitely many orbits on the set of (-2)-vectors in Λ.

Our goal in the next steps is to prove that the group $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ acts on the set of faces of the Weyl chambers with finitely many orbits.

Step 2: *describing a face by a flag.* Recall that an orientation of a hyperplane in an oriented real vector space is the choice of a "positive" normal direction. Then a face F of a Weyl chamber is determined by the following data: a hyperplane P_{s-1} it sits on, with the orientation pointing to the interior of the chamber; a hyperplane P_{s-2} in P_{s-1} supporting one of the faces F_1 of F, together with the orientation pointing to the interior of the face F; an oriented hyperplane P_{s-3} in P_{s-2} supporting some face F_2 of F_1; and so forth. Here, s is the dimension of the ambient space $H^{1,1}_{\mathbb{R}}(M)$. In other words, a face is determined by a full flag of linear subspaces, oriented step-by-step as above.

To prove that $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ acts on the set of faces with finitely many orbits, it suffices to prove that it acts with finitely many orbits on flags $P_{s-1} \supset P_{s-2} \supset \cdots \supset P_1$ of the above form, orientation forgotten.

Step 3: *bounding squares.* Note that the hyperplane P_{s-1} is $x^{\perp}$, where x is a (-2)-class in $H^{1,1}_{\mathbb{Z}}(M, I)$, and $P_{s-2} = x^{\perp} \cap y^{\perp}$, where y is another (-2)-class in $H^{1,1}_{\mathbb{Z}}(M, I)$. We claim that P_{s-2} as a hyperplane in P_{s-1} is given by $(y')^{\perp}$, where $y' \in x^{\perp}$ is an integral negative class of square at least -8.

Indeed, write $y = \frac{\langle x,y\rangle}{\langle x,x\rangle} x + \widetilde{y}$ (orthogonal projection to $x^{\perp}$). Obviously, P_{s-2} as a hyperplane in P_{s-1} is just the orthogonal to $\widetilde{y} \in x^{\perp}$. The first summand on the right has nonpositive square since it is proportional to a (-2)-class x. We claim that the second summand has strictly negative square (this square has then to be at least -2). Indeed, since the hyperplanes $x^{\perp}$ and $y^{\perp}$ define a Kähler chamber, the intersection of $x^{\perp}$ and $y^{\perp}$ is within the positive cone. But then the orthogonal to $\widetilde{y}$ in $x^{\perp}$ intersects the positive cone in $x^{\perp}$, so that, since the signature of the intersection form restricted to $x^{\perp}$ is $(+, -, \ldots, -)$, $\widetilde{y}$ is of strictly negative square. To make $\widetilde{y}$ integral, it suffices to replace it by $y' = \langle x, x\rangle \widetilde{y} = -2\widetilde{y}$; one has $0 > \langle y', y'\rangle \geqslant -8$.

Step 4: *monodromy action.* We have just seen that the intersections of $x^{\perp}$ with the other hyperplanes defining the Kähler chambers are given, in $x^{\perp}$, as orthogonals to integer vectors of strictly negative bounded square. By Theorem 3.14, the orthogonal group of $H^{1,1}_{\mathbb{Z}}(M, I) \cap x^{\perp}$ acts with finitely many orbits on such vectors. We deduce that the orthogonal group of $\Lambda = H^{1,1}_{\mathbb{Z}}(M, I)$ acts with finitely many orbits on partial flags $P_{s-1} \supset P_{s-2}$ arising from the faces of Kähler chambers. Iterating the argument of Step 3, we see that $O(\Lambda)$, and thus also $\mathrm{Mon}^{\mathrm{Hdg}}(M, I)$, acts with finitely many orbits on full flags and therefore on the set of faces of Kähler chambers.

Step 5: *conclusion.* For each pair of faces F, F' of a Kähler cone and $w \in \mathrm{Mon}^{\mathrm{Hdg}}(M, I)$ mapping F to F', w maps Kah to itself or to an adjoint Weyl chamber K'. Then $K' = r(K)$, where r is the orthogonal reflection in $H^2(M, \mathbb{Z})$ fixing F'. In the first case, $w \in \mathrm{Aut}(M)$. In the second case, rw maps F to F' and maps Kah to itself, hence $rw \in \mathrm{Aut}(M)$. ■

3.3 Finiteness of polyhedral tessellations

The argument used to prove Theorem 3.15 is valid in an abstract setting, which is worth describing here.

Let $V_{\mathbb{Z}}$ is a torsion-free $\mathbb{Z}$-module of rank $n+1$, equipped with an integer-valued (but not necessarily unimodular) quadratic form of signature $(1, n)$, $\Gamma \subset O(V_{\mathbb{Z}})$ a finite index subgroup, and $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. Let $S_0 \subset V_{\mathbb{Z}}$ be a finite set of negative vectors, $S := \Gamma \cdot S_0$ its orbit, and $Z := \bigcup_{s\in S} s^{\perp}$ the union of all orthogonal complements to s in V.

We associate with V a hyperbolic space $\mathbb{H} = \mathbb{P}V^{+}$ obtained as a projectivization of the set of positive vectors. Let $\{P_i\}$ be the set of connected components of $\mathbb{P}V^{+}\backslash\mathbb{P}Z$. This is a polyhedral tessellation of the hyperbolic space.

Definition 3.16. We call such a tessellation *a tessellation cut out by the set of hyperplanes orthogonal to S.* □

Definition 3.17. Let S_d be an intersection of $n-d$ transversal hyperplanes $s^{\perp}$, with $s \in S$. A *d-dimensional face* of a tessellation is a connected component of $S_d \backslash S_{d-1}$. □

Theorem 3.18. Let $\{P_i\}$ be a tessellation obtained in Definition 3.16, and F_d the set of all d-dimensional faces of the tessellation. Then Γ acts on F_d with finitely many orbits. □

Proof. We encode a d-dimensional face S_d^0 (up to a finite choice of orientations) by a sequence of hyperplanes $s_1^{\perp}, s_2^{\perp}, \dots s_{n-d}^{\perp}$ such that their intersection supports S_d^0, plus an oriented flag as in Theorem 3.15. Using induction, we may assume that up to the action of Γ there are only finitely many $(d+1)$-dimensional faces. The same inductive argument as in Theorem 3.15, Step 3 is used to show that the projection of s_{n-d} to $S_{d+1} := \bigcap_{i=1}^{n-d-1} s_i^{\perp}$ has bounded square after multiplying by the denominator. Therefore, there are finitely many orbits of the group $\Gamma_{S_{d+1}} := \{\gamma \in \Gamma \mid \gamma(S_{d+1}) = S_{d+1}\}$ on the set $s_{n-d}^{\perp} \cap S_{d+1}$ for all $s_{n-d} \in S$, and the reasoning is the same for the oriented flag. ■

3.4 Morrison–Kawamata cone conjecture: birational version

Definition 3.19. The birational Kähler cone of a hyperkähler manifold M is the union of pullbacks of the Kähler cones under all birational maps $M \dashrightarrow M'$, where M' is also a hyperkähler manifold. □

Remark 3.20. These birational maps are actually pseudo-isomorphisms [17, Proposition 4.7; 8, Proposition 4.4], so that the second cohomologies of M and M' are naturally identified. We shall sometimes say that the birational Kähler cone is the union of the Kähler cones of birational models, omitting to mention pullbacks. □

Remark 3.21. All this is a slight abuse of language: the birational Kähler cone is not what one would normally call a cone (but its closure, the moving cone, or birational nef cone, is). □

Definition 3.22. *An exceptional prime divisor* on a hyperkähler manifold is a prime divisor with negative square (with respect to the BBF form). □

The birational Kähler (or, rather, nef) cone is characterized in terms of prime exceptional divisors in the same way as the Kähler cone is characterized in terms of rational curves.

Theorem 3.23 (Huybrechts, [17, Proposition 4.2])**.** Let $\eta \in \mathrm{Pos}(M)$ be an element of a positive cone on a hyperkähler manifold. Then η is birationally nef if and only if $q(\eta, E) \geqslant 0$ for any exceptional divisor E. □

Remark 3.24. In other words, the faces of birational Kähler cone are dual to the classes of exceptional divisors. □

Theorem 3.25 (Markman)**.** For each prime exceptional divisor E on a hyperkähler manifold, there exists a reflection $r_E \in O(H^2(M, \mathbb{Z}))$ in the monodromy group, fixing $E^\perp$. □

Proof. In the projective case, this is [25, Theorem 1.1]. The nonprojective case reduces to this by deformation theory. Indeed, as we shall show in the next section (Theorem 4.10), a prime exceptional divisor deforms locally as long as its cohomology class stays of type $(1, 1)$ (and its small deformations are obviously prime exceptional again): in fact, this also has already been remarked by Markman in [25]), but he restricted himself to the projective case. Any hyperkähler manifold (M, I) has a small deformation (M, I') which is projective. The divisor E deforms to E' on (M, I'), so one obtains a reflexion $r'_E = r_E \in$

$O(H^2(M,\mathbb{Z}))$ in the monodromy group (the monodromy being of topological nature, it is deformation-invariant in a tautological way). ■

Definition 3.26. Such a reflection is called *a divisorial reflection*. □

Definition 3.27. *An exceptional chamber*, or *a Weyl chamber* on a hyperkähler manifold is a connected component of $\mathrm{Pos}(M)\backslash E^\perp$, where $E^\perp$ is a union of all planes $e^\perp$ orthogonal to all prime exceptional divisors e. □

Remark 3.28. An exceptional chamber is a fundamental domain of a group generated by divisorial reflections. The birational Kähler cone is a dense open subset of one of the exceptional chambers, which we shall call *birational Kähler chamber*. □

Theorem 3.29 (Markman). Let (M,I) be a hyperkähler manifold, and $\mathsf{Mon}^{\mathsf{Hdg}}(M,I)$ a subgroup of the monodromy group fixing the Hodge decomposition on (M,I). Then the image of $\mathrm{Bir}(M,I)$ in the orthogonal group of the Picard lattice is the group of all $\gamma \in \mathsf{Mon}^{\mathsf{Hdg}}(M,I)$ preserving the birational Kähler chamber Kah_B. □

Proof. Please see [24, Lemma 5.11 (6)]. The proof of Theorem 3.29 is similar to the proof of Theorem 3.13. ■

Now we can generalize the birational Morrison–Kawamata cone conjecture proved by Markman for projective hyperkähler manifolds [24, Theorem 6.25].

Theorem 3.30. Let (M,I) be a hyperkähler manifold, and $\mathrm{Bir}(M,I)$ the group of birational automorphisms of (M,I). Then $\mathrm{Bir}(M,I)$ acts on the set of faces of the birational nef cone with finite number of orbits. □

Proof. *Step* 1: let δ be the discriminant of a lattice $H^2(M,\mathbb{Z})$, and E an exceptional divisor. Then $|E^2| \leqslant 2\delta$. Indeed, otherwise, the reflection $x \longrightarrow x - 2\frac{q(x,E)}{q(E,E)}E$ would not be integral.

Step 2: the group of isometries of a lattice Λ acts with finitely many orbits on the set $\{l \in \Lambda \mid l^2 = x\}$ for any given x (Theorem 3.14), and the monodromy group is of finite index in the isometry group of the lattice $H^2(M,\mathbb{Z})$. Therefore, $\mathsf{Mon}^{\mathsf{Hdg}}(M,I)$ is of finite index in the isometry group of the Picard lattice and so all classes of exceptional divisors belong to finitely many orbits of $\mathsf{Mon}^{\mathsf{Hdg}}(M,I)$.

Step 3: we repeat the argument of Theorem 3.15 to show that $\mathsf{Mon}^{\mathsf{Hdg}}(M,I)$ acts with finite number of orbits on the set of faces of all exceptional chambers. It suffices to show that it acts with finitely many orbits on the set of full flags $P_{s-1} \supset P_{s-2} \supset \cdots \supset P_1$ (notations as in the proof of Theorem 3.15) formed by intersections of orthogonal hyperplanes to the classes of exceptional divisors. Using the fact that the squares of those classes are bounded in absolute value by $C = 2\delta$, we show that P_{s-2} is described inside P_{s-1} as $y_1^\perp$ where y_1 is integral and $|y_1^2| \leqslant C^3$, P_{s-3} is defined in P_{s-2} as $y_2^\perp$, where y_2 is integral and $|y_1^2| \leqslant C^9$, and so on. We deduce by Theorem 3.14 that $O(\Lambda)$ acts with finitely many orbits on the set of such full flags.

Step 4: thus, $\mathrm{Mon}^{\mathrm{Hdg}}(M, I)$ acts with finite number of orbits on the set of faces of all exceptional chambers. For each pair of faces F, F' of a birational Kähler cone and $w \in O(\Lambda)$ mapping F to F', w maps Kah_B to itself or to an adjoint Weyl chamber K'. Then $K' = r(K)$, where r is the reflection fixing F'. In the first case, $w \in \mathrm{Aut}(M)$. In the second case, rw maps F to F' and maps Kah_B to itself, hence $rw \in \mathrm{Aut}(M)$. Therefore, there are as many orbits of $\mathrm{Mon}^{\mathrm{Hdg}}(M, I)$ on the faces of exceptional chambers as there are orbits of $\mathrm{Bir}(M, I)$. ■

4 Deformation Spaces of Rational Curves

By a rational curve on a manifold X, we mean a curve $C \subset X$ such that its normalization is $\mathbb{P}^1$; in other words, the image of a generically injective map $f : \mathbb{P}^1 \to X$. Let $\mathrm{Mor}(\mathbb{P}^1, X)$ denote the parameter space for such maps. Then by deformation theory (see [22]), one has

$$\dim_{[f]}(\mathrm{Mor}(\mathbb{P}^1, X)) \geqslant \chi(f^*(TX)) = -K_X C + \dim(X),$$

so that if H denotes the space of deformations of C in X, then

$$\dim(H) \geqslant -K_X C + \dim(X) - 3.$$

A rational curve on an m-fold with trivial canonical class must, therefore, move in a family of dimension at least $m - 3$.

The following observation due to Z. Ran states that on holomorphic symplectic manifolds, this estimate can be slightly improved.

Theorem 4.1. Let M be a hyperkähler manifold of dimension $2n$. Then any rational curve $C \subset M$ deforms in a family of dimension at least $2n - 2$. □

Proof. See [30, Corollary 5.2]. Alternatively (we thank Eyal Markman for this argument), one may note that an extra parameter is due to the existence of the twistor space $\mathrm{Tw}(M)$. This is a complex manifold of dimension $n+1$, fibered over $\mathbb{P}^1$ in such a way that M is one of the fibers, and the other fibers correspond to the other complex structures coming from the hyperkähler data on M. The map f, seen as a map from $\mathbb{P}^1$ to $\mathrm{Tw}(M)$, deforms in a family of dimension at least $n+1$. But all deformations have image contained in M since the neighboring fibers contain no curves at all by [33], and the rational curves with dominant projection to $\mathbb{P}^1$ belong to a different cohomology class. ■

Before making the following observation, let us recall a few definitions.

Definition 4.2. A complex analytic subvariety Z of a holomorphically symplectic manifold (M, Ω) is called *holomorphic Lagrangian* if $\Omega|_Z = 0$ and $\dim_{\mathbb{C}} Z = \frac{1}{2}\dim_{\mathbb{C}} M$, and *isotropic* if $\Omega|_Z = 0$ (since Ω is nondegenerate, this implies $\dim_{\mathbb{C}} Z \leqslant \frac{1}{2}\dim_{\mathbb{C}} M$). It is called *coisotropic* if Ω has rank $\frac{1}{2}\dim_{\mathbb{C}} M - \mathrm{codim}_{\mathbb{C}} Z$ on TZ in all smooth points of Z, which is the minimal possible rank for a $(2n-p)$-dimensional subspace in a $2n$-dimensional symplectic space.

Let now Z be a compact Kähler manifold covered by rational curves. By [10], there is an almost holomorphic fibration $R: Z \dashrightarrow Q$, called the *rational quotient* of Z, such that its fiber through a sufficiently general point x consists of all y which can be joined from x by a chain of rational curves. That is, the general fiber of R is *rationally connected*. It is well known that rationally connected manifolds do not carry any holomorphic forms, and it follows from [10] that they are projective. □

Remark 4.3. By a theorem due to Graber, Harris, and Starr in the projective setting, the base Q of the rational quotient is not uniruled; this is also true in the compact Kähler case (the reason being that the total space of a family of rationally connected varieties over a curve is automatically algebraic, so that the arguments of Graber–Harris–Starr apply), but we shall not need this. □

Theorem 4.4. Let M be a hyperkähler manifold, $C \subset M$ a rational curve, and $Z \subset M$ be an irreducible component of the locus covered by the deformations of C in M. Then Z is a coisotropic subvariety of M. The fibers of the rational quotient of the desingularization of Z have dimension equal to the codimension of Z in M. □

Proof. We want to prove that at a general point of Z, the restriction of the symplectic form Ω to TZ has kernel of dimension equal to $k = \mathrm{codim}(Z)$. Recall that this kernel cannot be of dimension greater than k by nondegeneracy of Ω, and the equality means that Z is coisotropic.

Let $h: Z' \to Z$ be the desingularization. The manifold Z' is covered by rational curves which are lifts of deformations of C. By dimension count, through a general point p of Z', there is a family of such curves of dimension at least $k-1$. In fact, any rational curve through p deforms in a family of dimension at least $k-1$, as seen by projecting it to M and applying Theorem 4.1. Take a minimal rational curve C' through p, then its deformations through p cover a subvariety of dimension at least k. This is because by bend-and-break, there is only a finite number of minimal rational curves through two general points (note that bend-and-break applies here since all rational curves through p are in the fiber of the rational quotient, and this fiber is projective). This means that the fibers of the rational quotient fibration $R: Z' \dashrightarrow Q'$ are at least k-dimensional. But any holomorphic form on Z' is a pullback of a holomorphic form on Q' (since the rationally connected varieties do not carry any holomorphic m-forms for $m > 0$). So the tangent space to the fiber of R through p is in the kernel of $h^*(\Omega)$, and thus, the kernel of $\Omega|_{TZ}$ at $h(p)$ is of dimension k, and the same is true for the fiber of R through p. ■

In the above argument, we have called a curve $C \subset Z$ *minimal* if it is a rational curve of minimal degree (say, with respect to the Kähler form from the hyperkähler structure) through a general point of Z.

Corollary 4.5. If deformations of a rational curve C in M cover a divisor Z, then C is a minimal rational curve in this divisor. □

Proof. Indeed by Theorem 4.4, there is no other rational curve through a general point of Z. ■

Corollary 4.6. Minimal rational curves on holomorphic symplectic manifolds deform in a $(2n-2)$-parameter family. □

Proof. This is obvious from the proof of Theorem 4.4. Indeed, if the deformations of a minimal C cover a subvariety Z of codimension k, then these deformations form a family of dimension $k-1+dim(Z)-1 = k-1+2n-k-1 = 2n-2$, since there is a $k-1$-parametric family of them through the general point of Z. ■

Remark 4.7. These results have well-known analogs (which are consequences of the work by Wierzba and Kaledin) in the case when Z is contractible, that is, one has a birational morphism $\pi : M \to Y$ whose exceptional set is Z. The image of Z by π replaces the base of the rational quotient. In the case when, moreover, Z is a divisor, the fibers of π are one-dimensional and therefore are trees of smooth rational curves by Grauert–Riemenschneider theorem. In general, it is not obvious whether the minimal rational curves are smooth. In one important case, though, they are smooth: namely when Z is a negative-square divisor on M. The reason is that one can deform the pair (M, Z) so that M becomes projective (see below), and then use results by Druel from [13] to reduce to the contractible case.

In this case, the inverse image of the tangent bundle T_M on the normalization of a minimal rational curve C splits as follows

$$f^* T_M = \mathcal{O}(-2) \oplus \mathcal{O}(2) \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O}.$$

This is because it should be isomorphic to its dual (by the pullback of the symplectic form), have at most one negative summand, and contain $T_{\mathbb{P}^1}$ as a subbundle. This also remains true when the normalization map of C is an immersion. In general, there are problems related to the singularities (for instance, as soon as there are cusps, $T_{\mathbb{P}^1}$ is only a subsheaf and not a subbundle of $f^* T_M$), and it is not obvious how to avoid them in order to get a similar splitting, which one would like to be $f^* T_M = \mathcal{O}(-2) \oplus \mathcal{O}(2) \oplus \mathcal{O}^{\oplus k} \oplus (\mathcal{O}(-1) \oplus \mathcal{O}(1))^{\oplus l}$. □

Corollary 4.8. If C is minimal, any small deformation M_t of $M = M_0$ such that the dual class z of C stays of type $(1, 1)$ on M_t, contains a deformation of C. □

Proof. Consider the universal family $\mathcal{M} \to \mathrm{Def}(M)$ of small deformations of $M = M_0$. The $t \in \mathrm{Def}(M)$ such that z is of type $(1, 1)$ on M_t form a smooth hypersurface in $\mathrm{Def}(M)$ (if one identifies the tangent space to $\mathrm{Def}(M)$ at 0 with $H^1(M, T_M) \cong H^1(M, \Omega^1_M)$, then the tangent space to this hypersurface is the hyperplane orthogonal to z). Now let $\mathcal{C} \to \mathrm{Def}(M)$ be the family of deformations of C in $\mathcal{M}$: the image of $\mathcal{C}$ in $\mathrm{Def}(M)$ is a subvariety by Grauert's proper mapping theorem, and it suffices to prove that it is a hypersurface. This is a simple dimension count. Indeed, one obtains from Riemann–Roch theorem, as in the proof of Theorem 4.1, that C deforms in a family of dimension at least $2n - 3 + \dim(\mathrm{Def}(M))$. Since the deformations of C inside any M_t form a family of dimension $2n - 2$ (when nonempty), the conclusion follows. ■

Corollary 4.9. If C is minimal, any deformation of $M = M_0$ such that the corresponding homology class remains of type $(1, 1)$, has a birational model containing a rational curve in that homology class. □

Proof. This is the same argument as in the previous corollary, but we have to consider the universal family over $\mathsf{Teich}_z(M)^0$, where $\mathsf{Teich}(M)^0$ is the connected component of $\mathsf{Teich}(M)$ containing the parameter point for our complex manifold M_0, and $\mathsf{Teich}_z(M)^0$ is the part of it where z remains of type $(1,1)$. Birational models appear since $\mathsf{Teich}_z(M)$ is not Hausdorff, so that a subvariety of $\mathsf{Teich}_z(M)^0$ of maximal dimension is not necessarily equal to $\mathsf{Teich}_z(M)^0$; on the other hand, it is known by the work of Huybrechts that unseparable points of $\mathsf{Teich}_z(M)$ correspond to birational complex manifolds. ■

The deformations of a minimal C as above are obviously minimal on the neighboring fibers (indeed, new effective classes only appear on closed subsets of the parameter space). However, globally, a limit of minimal curves does not have to be minimal, at least a priori; it can also become reducible (and in fact does so in many examples). The deformation theory of nonminimal curves is not as nice as described in Corollary 4.8. For instance, in an example from [1], credited by the authors to Claire Voisin, there is a holomorphic symplectic 4-fold X containing a lagrangian quadric Q, and rational curves of type $(1,1)$ on Q deform only together with Q, that is over a codimension-two subspace of the base space, as follows from [36]. However, this problem disappears, at least locally, in the case when the dual class z is *negative divisorial*, that is, when $q(z,z)<0$ and z is $\mathbb{Q}$-effective.

Theorem 4.10 is due to Markman in the projective case.

Theorem 4.10. Let z be a negative $(1,1)$-class on M which is represented by an irreducible divisor D. Then z is effective on any deformation M_t where it stays of type $(1,1)$, and represented by an irreducible divisor on an open part of this parameter space. □

Proof. By [8], D is uniruled. By Theorem 4.4, there is only one dominating family of rational curves on D. Take a general member C in this family. We have seen that it deforms on all neighboring M_t, yielding rational curves C_t. In particular, its dual class is proportional (with some positive coefficient) to that of D, since both stay of type $(1,1)$ on the same small deformations of M; therefore, $C\,D<0$ and all deformations of C in M stay inside D. We claim that the deformations of C_t on M_t also cover divisors on M_t, which have then to be deformations of D. Indeed, let Z_t be the subvariety of M_t covered by the curves C_t. Suppose it is not a divisor for all t. Since the dimension jumps over closed subsets of the parameter space, one has an open subset U of such t. By Theorem 4.4, through a general point of Z_t, $t\in U$, there is a positive-dimensional family of curves C_t. Now consider a parameter space for the following triples over T: $\{(C_t, x, C'_t) : x\in C_t\cap C'_t\}$.

The dimension of its fibers over $t \in U$ is strictly greater than the dimension of the central fiber, and this is a contradiction.

Therefore, D deforms everywhere locally. Now consider the universal family $\mathcal{M}$ over Teich_z, and the "universal divisor" $\mathcal{D}$ in this family. The image of $\mathcal{D}$ is a subvariety in Teich_z, of the same dimension as Teich_z. This means that any deformation of M preserving the type $(1,1)$ of z has a birational model such that z is effective. As birational hyperkähler manifolds differ only in codimension two, the theorem follows. ■

Corollary 4.11. A negative class $z \in H^2(M, \mathbb{Z})$ is effective (or not effective) simultaneously in all complex structures where it is of type $(1,1)$ and generates the Picard group over $\mathbb{Q}$. □

Proof. If the only integral $(1,1)$-classes are rational multiples of z, one cannot have more than one effective divisor on M: indeed, if D and D' are prime effective divisors and $q(D, D') < 0$, then $D = D'$ (if not, remark that by definition $q(D, D')$ is obtained by integration of a positive form over $D \cap D'$). In particular, every effective divisor is irreducible. Now apply the previous theorem. ■

When one wants to deform curves rather than divisors, the notion of a minimal curve that we have used above is not quite natural since it depends, a priori, on the subvariety $Z \subset M$ covered by deformations of the curve, and not only on the complex manifold M itself. In the projective setting, one typically considers rational curves generating an extremal ray of the Mori cone: the class of such a curve is *extremal* in the sense of the Introduction. In the nonprojective setting, the following notion of extremality looks better behaved.

Definition 4.12. Let X be a Kähler manifold and z an integral homology class of type $(1,1)$. We say that z is *minimal* if the intersection of $z^\perp$ with the boundary of the Kähler cone contains an open subset of $z^\perp$. □

An example, due to Markman, [25], shows that a limit of minimal (or extremal) curves does not have to be minimal (extremal).

Example 4.13 ([25, Example 5.3]). Let $\overline{X}_0$ be an intersection of a quadric and a cubic in $\mathbb{P}^4$ with one double point. The resolution $p: X_0 \to \overline{X}_0$ is a $K3$ surface, and $p^*H + 2E$, where H is a hyperplane section and E is the exceptional curve, is not minimal (or extremal). Now deform X_0 to a smooth nonprojective K3 surface X_t in such a way that only the multiples of $p^*H + 2E$ survive in $H^{1,1}$. Since $p^*H + 2E$ is a (-2)-class, it is easy to show that those are effective. They are obviously extremal (and minimal, too). □

In what follows (Definition 5.10), we shall define the *MBM classes* as integral homology classes which are minimal modulo monodromy and birational equivalence. We shall show that this notion is deformation-invariant, and that if a class is MBM, then the intersection of its orthogonal hyperplane and the positive cone is a union of faces of Kähler–Weyl chambers (Theorems 6.2 and 5.13). It is an interesting question whether the notions of minimality and extremality are equivalent for hyperkähler manifolds. Minimal classes are extremal, and it follows from our proofs that minimal classes are effective up to a rational multiple and represented by rational curves, as it is the case of extremal rays in the Mori theory. But extremal classes do not, a priori, have to be minimal, since extremal rays of the cone of curves could accumulate. One can conjecture that in real life it never happens (as this is true in the projective case by [15]).

5 Twistor Lines in the Teichmüller Space

5.1 Twistor lines and three-dimensional planes in $H^2(M, \mathbb{R})$

Recall that any hyperkähler structure (M, I, J, K, g) defines a triple of Kähler forms $\omega_I, \omega_J, \omega_K \in \Lambda^2(M)$ (Section 2.1). A hyperkähler structure on a simple hyperkähler manifold is determined by a complex structure and a Kähler class (Theorem 2.2).

Definition 5.1. Each hyperkähler structure induces a family $S \subset \mathsf{Teich}$ of deformations of complex structures parameterized by $\mathbb{C}P^1$ (Section 2.1). The curve S is called *the twistor line* associated with the hyperkähler structure (M, I, J, K, g). □

We identify the period space $\mathbb{P}\mathsf{er}$ with the Grassmannian of positive oriented 2-planes in $H^2(M, \mathbb{R})$ (Proposition 2.10). For any point $l \in S$ on the twistor line, the corresponding two-dimensional space $\mathsf{Per}(l) \in \mathbb{P}\mathsf{er}$ is a 2-plane in the three-dimensional space $\langle \omega_I, \omega_J, \omega_K \rangle$. Therefore, $\langle \omega_I, \omega_J, \omega_K \rangle$ is determined by the twistor line S uniquely, as the linear span of the planes in $\mathsf{Per}(S)$.

We call two hyperkähler structures *equivalent* if one can be obtained from the other by a homothety and a quaternionic reparameterization:

$$(M, I, J, K, g) \sim (M, hIh^{-1}, hJh^{-1}, hKh^{-1}, \lambda g)$$

for $h \in \mathbb{H}^*$, $\lambda \in \mathbb{R}^{>0}$. Clearly, equivalent hyperkähler structures produce the same twistor lines in Teich. However, a hyperkähler structure is determined by a complex structure, which yields a two-dimensional subspace $\mathsf{Per}(I) = \langle \omega_J, \omega_K \rangle$ in $H^2(M, \mathbb{R})$, and a Kähler structure ω_I, as in Theorem 2.2. The form ω_I can be reconstructed up to a constant from the three-dimensional space $\langle \omega_I, \omega_J, \omega_K \rangle$ and the plane $\mathsf{Per}(I)$. This proves the following result, which is essentially a form of Calabi–Yau theorem.

Claim 5.2. Let (M, I, J, K, g) be a hyperkähler structure on a compact manifold, and $S \subset \mathsf{Teich}$ the corresponding twistor line. Then S is sufficient to recover the equivalence class of (M, I, J, K, g). □

Proposition 5.3. Let $S \subset \mathsf{Teich}$ be a twistor line, $W \subset H^2(M, \mathbb{R})$ be the corresponding three-dimensional plane $W := \langle \omega_I, \omega_J, \omega_K \rangle$, and $z \in H^2(M, \mathbb{R})$ a nonzero real cohomology class. Then S lies in Teich_z if and only if $W \perp z$. □

Proof. By definition, Teich_z is the set of all $I \in \mathsf{Teich}$ such that the 2-plane $\mathsf{Per}(I)$ is orthogonal to z. However, any point of $\mathsf{Per}(S)$ lies in W, hence all points in S belong to Teich_z whenever $W \perp z$. Conversely, the planes corresponding to points in $\mathsf{Per}(S)$ generate W, so that W is orthogonal to z if all those planes are. ■

5.2 Twistor lines in Teich_z

Remark 5.4. Let $z \in H^{1,1}(M, I)$ be a nonzero cohomology class on a hyperkähler manifold (M, I), Teich^I the connected component of the Teichmüller space containing I, and Teich_z the set of all $J \in \mathsf{Teich}^I$ such that z is of type $(1, 1)$ on (M, J). Given a Kähler form ω on (M, I) such that $q(\omega, z) = 0$, consider the corresponding hyperkähler structure (M, I, J, K), and let $S \subset \mathsf{Teich}$ be the corresponding twistor line. By Proposition 5.3, S lies in Teich_z. □

In other words, there is a twistor line on Teich_z through the point I if and only if z is orthogonal to a Kähler form on (M, I).

Definition 5.5. In the assumptions of Remark 5.4, let $W_S := \langle \omega_I, \omega_J, \omega_K \rangle$ be a three-dimensional plane associated with the hyperkähler structure (M, I, J, K), and $S \subset \mathsf{Teich}_z$ the corresponding twistor line. The twistor line S is called *z-GHK* (*z*-general hyperkähler) if $W^\perp \cap H^2(M, \mathbb{Q}) = \langle z \rangle$. □

The utility of z-GHK lines is that they can be lifted from the period space to Teich_z uniquely, if Teich_z contains a twistor line (Proposition 5.12). For other twistor lines, such a lift, even if it exists, is not necessarily unique because of nonseparable points, but a z-GHK line has to pass through a Hausdorff point of Teich_z.

Claim 5.6. Let $z \in H^{1,1}(M, I)$ be a nonzero cohomology class on a hyperkähler manifold (M, I). Assume that (M, I) admits a Kähler form ω such that $q(\omega, z) = 0$. Then (M, I) admits a z-GHK line. □

Proof. Let $z^\perp \subset H_I^{1,1}(M, \mathbb{R})$ be the orthogonal complement of z. The set of Kähler classes is nonempty and open in $z^\perp$. For each integer vector $z_1 \in H^2(M, \mathbb{Z})$, noncollinear with z, the orthogonal complement $z_1^\perp$ intersects with $z^\perp$ transversally. Removing all the hyperplanes $z^\perp \cap z_1^\perp$ from $z^\perp$, we obtain a dense set which contains a Kähler form ω_1, such that $\omega_1^\perp$ contains none of the integer vectors z_1. The hyperkähler structure associated with I and ω_1 satisfies $W^\perp \cap H^2(M, \mathbb{Q}) = \langle z \rangle$, because $W \ni \omega_1$, and $\omega_1^\perp \cap H^2(M, \mathbb{Q}) = \langle z \rangle$. ■

The following result is easy to prove, however, we give a reference to [34], where the proof is spelled out in a situation which is almost the same as ours.

Claim 5.7. Let $S \subset \mathsf{Teich}_z$ be a twistor line, associated with a three-dimensional subspace $W \subset H^2(M, \mathbb{R})$. Then the following assumptions are equivalent.

(i) S is a z-GHK line.

(ii) For all $l \in S$, except a countable number, the space $H^{1,1}(M_l, \mathbb{Q})$ is generated by z, where M_l is the manifold corresponding to l.

(iii) For some $w \in W$, the space of rational classes in the orthogonal complement $w^\perp \subset H^2(M, \mathbb{R})$ is generated by z. □

Proof. [34, Claim 5.4] ■

Given a z-GHK line S, we may chose a point $l \in S$ where $H^{1,1}(M_l, \mathbb{Q}) = \langle z \rangle$. For this point, the Kähler cone coincides with the positive cone by Theorem 1.9. Indeed, M_l contains no curves, as the class z, being orthogonal to a Kähler form, cannot be effective. Choosing a different Kähler form in $\operatorname{Pos}(M_l) \cap z^\perp$, we obtain a different twistor line in Teich_z, intersecting S. It was shown that such a procedure can be used to connect any two points of Teich_z, up to nonseparatedness issues (in [34], it was proved for the whole Teich, but for Teich_z the argument is literally the same).

Proposition 5.8. Let $x, y \in \mathsf{Teich}_z$. Suppose that Teich_z contains a twistor line. Then a point $\tilde{x}$ nonseparable from x can be connected to a point $\tilde{y}$ nonseparable from y by a sequence of at most five sequentially intersecting z-GHK lines. □

Proof. From [34, Proposition 5.8], it follows that any two points $a = \mathsf{Per}(x)$, $b = \mathsf{Per}(y)$ in the period domain $\mathsf{Per}(\mathsf{Teich}_z)$ can be connected by at most five sequentially intersecting z-GHK lines. Moreover, the intersection points can be chosen generic, in particular, separable. Lifting these z-GHK lines to Teich_z, we connect a point $\tilde{x}$ in $\mathsf{Per}^{-1}(x)$ to a point $\tilde{y}$ in $\mathsf{Per}^{-1}(y)$ by a sequence of at most five sequentially intersecting z-GHK lines. However, these preimages are nonseparable from x, y by the global Torelli theorem (Theorem 2.16). ■

Remark 5.9. Clearly, if our twistor line already passes through x, we can connect x itself to a $\tilde{y}$ nonseparable from y. □

5.3 Twistor lines and MBM classes

As we have already mentioned in the Introduction, the Bogomolov–Beauville–Fujiki form identifies the rational cohomology and homology of a hyperkähler manifold M, inducing an injective map $q : H^2(M, \mathbb{Z}) \to H_2(M, \mathbb{Z})$. Since q is not necessarily unimodular, this map is not an isomorphism over $\mathbb{Z}$. However, it is an isomorphism over $\mathbb{Q}$, and we shall identify $H^2(M, \mathbb{Q})$ with $H_2(M, \mathbb{Q})$ without further comment (in particular, we shall often view the classes in $H_2(M, \mathbb{Z})$ as rational classes in H^2). The only exception to this rule is the notion of effectiveness: an effective homology class is a class of a curve, whereas an effective cohomology class is a class of an effective divisor.

Definition 5.10. A negative integral homology class $\eta \in H_2(M, \mathbb{Z})$ is called an MBM class if some image of η under monodromy contains in its orthogonal a face of the Kähler cone of some birational hyperkähler model M'. □

To study the MBM classes, it is convenient to work with hyperkähler manifolds which satisfy $\mathrm{rkPic}(M) = 1$, with $\mathrm{Pic}(M) \subset H^2(M, \mathbb{Z})$ generated over $\mathbb{Q}$ by η. Here, η is considered as an element of $H^2(M, \mathbb{Q})$, identified with $H_2(M, \mathbb{Q})$ as above. In this case, M is clearly nonalgebraic since η is negative.

Theorem 5.11. Let (M, I) be a hyperkähler manifold, such that $\mathrm{Pic}(M, I) = \langle z \rangle$, where $z \in H_{1,1}(M, I)$ is a nonzero negative class. Then the following statements are equivalent.

(i) The class $\pm z$ is $\mathbb{Q}$-effective.
(ii) The class $\pm z$ is extremal.
(iii) The Kähler cone of (M, I) is not equal to its positive cone.

(iv) The class z is minimal, in the sense of Definition 4.12.

(v) The class z is MBM. □

Proof. First, the equivalence of (i) and (ii) is clear since the cone of curves on M consists of multiples of $\pm z$ if it is $\mathbb{Q}$-effective and is empty otherwise.

The equivalence of (i) and (iii) is proved as follows: first, for signature reasons one can find a positive class α with $q(\alpha, z) = 0$, and if $\pm z$ is $\mathbb{Q}$-effective then α cannot be Kähler. The converse follows by Huybrechts–Boucksom's description of the Kähler cone, see Theorem 1.9 (if $\pm z$ is not $\mathbb{Q}$-effective, then there are no curves at all, in particular, no rational curves, and thus any positive class is Kähler).

Finally, let us show the equivalence of (iii)–(v).

If $\mathrm{Kah}(M, I) \neq \mathrm{Pos}(M, I)$, then by Huybrechts–Boucksom the Kähler cone has faces supported on hyperplanes of the form $x^{\perp}$, where x is a class of a rational curve. However, the only integral $(1,1)$-classes on (M, I) are multiples of z, so such a face can only be $z^{\perp}$. In this case, z must be minimal; this proves (iii) $\Rightarrow$ (iv). The implication (iv) $\Rightarrow$ (v) is a tautology. Finally, if $\mathrm{Kah}(M, I) = \mathrm{Pos}(M, I)$, there are no minimal classes and no nontrivial birational models of M, so there cannot be any MBM classes either, which proves (v) $\Rightarrow$ (iii). ■

In general, the notion of an MBM class is much more complicated, but one can hope to study them by deforming to the case we have just described. This is, indeed, what we are going to do, using Proposition 5.8 as a key tool. The following useful proposition, which allows one to identify generic positive three-dimensional subspaces of $H^2(M, \mathbb{R})$ and twistor lines, is an illustration of our method.

Proposition 5.12. Let $z \in H^2(M, \mathbb{Q})$ be a nonzero vector, such that Teich_z contains a twistor line, and $W \subset z^{\perp}$ a three-dimensional positive subspace of $H^2(M, \mathbb{R})$ which satisfies $W^{\perp} \cap H^2(M, \mathbb{Q}) \subset \langle z \rangle$. Then there exists a z-GHK twistor line S such that the corresponding three-dimensional space $W_S \subset H^2(M, \mathbb{R})$ is equal to W. □

Proof. Let $V \subset W$ be a two-dimensional plane which satisfies

$$V^{\perp} \cap H^2(M, \mathbb{Q}) \subset \langle z \rangle$$

(by Claim 5.7(ii), this is true for all planes except at most a countable set). Then $V = \mathsf{Per}(I)$, where $I \in \mathsf{Teich}_z$, because the period map is surjective. The space $H^{1,1}_{\mathbb{Q}}(M, I) = V^{\perp} \cap H^2(M, \mathbb{Q})$ is generated by z. Since Teich_z contains a twistor line, by Proposition 5.8, there must be a twistor line through the point $I' \in \mathsf{Teich}_z$ which is inseparable from I.

Now on I', z cannot be effective since it is orthogonal to a Kähler form. But this means that there are no curves on (M, I'), so the Kähler cone of (M, I') is equal to the positive cone, Teich_z is separated at I', and $I = I'$. Therefore, the Kähler cone of (M, I) coincides with its positive cone, hence there exists a Kähler form ω on (M, I) such that $\langle V, \omega\rangle = W$. The corresponding hyperkähler structure gives a 3-plane $W_S = \langle V, \omega\rangle = W$. ∎

Our strategy in proving the deformation invariance of MBM property is as follows: we first show that the property of being orthogonal to a Kähler form, modulo monodromy and birational transformations, is deformation-invariant, and then show that the MBM classes are exactly those which do not have this property.

Theorem 5.13. Let M be a simple hyperkähler manifold, and $z \in H^2(M, \mathbb{Q})$ a cohomology class. Suppose that z is orthogonal to a Kähler form on (M, I_0), where $I_0 \in \mathsf{Teich}_z$. Then:

(i) for any $I \in \mathsf{Teich}_z$, there is some I' nonseparable from I such that z is orthogonal to a Kähler form on (M, I').

(ii) on the manifold (M, I), the class z is orthogonal to an element of a Kähler–Weyl chamber (i.e., z is orthogonal to $\gamma(\alpha)$ for some γ in the monodromy group and α lying in the birational Kähler cone, i.e., Kähler on a birational model, see Definition 6.1).

(iii) the elements of Kähler–Weyl chambers are dense in the intersection of $z^\perp$ and the positive cone. □

Proof. (i) By Claim 5.7, there is a z-GHK line through I_0. By Proposition 5.8 , I_0 is connected to I' by a sequence of z-GHK lines. Therefore, on (M, I'), z is orthogonal to a Kähler form ω'.

(ii) The group $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ acts transitively on the set of the Weyl chambers (see [24]; in fact, this easily follows from Theorem 3.25). Let $W(I)$ denote the fundamental Weyl chamber, that is, the interior of the birational nef cone. Then there is an element $\gamma \in \mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ such that $W(I) = \gamma W(I')$.

Consider $\widetilde{\gamma}$ which is a lift of γ in the mapping class group Γ_I, and the complex structure $\widetilde{\gamma}(I')$. This is again nonseparable from I and I' and it carries a Kähler form $\widetilde{\gamma}(\omega')$, orthogonal to $\gamma(z)$. Its cohomology class is an element of the birational Kähler cone of I. The inverse image of this class by γ is orthogonal to z, q.e.d.

(iii) We claim that any positive $(1,1)$-class ω which is orthogonal to z but not to any other rational cohomology class is Kähler modulo monodromy and birational transforms. Indeed, consider the twistor line $L \subset \mathbb{P}\mathsf{er}$ corresponding to the 3-subspace generated by the period of I_0 and ω. By Proposition 5.12, it lifts to Teich_z as a z-GHK line. This line does not have to pass through I_0, but it passes through some I_0' nonseparable from I_0. On this I_0', ω is Kähler, and we conclude in the same way as in (ii) that on I_0 itself it is Kähler modulo monodromy and birational equivalence. ■

Note that by definition, the property obtained in (iii) expresses exactly the fact that z is not MBM. We thus obtain the deformation-invariance of MBM classes.

Corollary 5.14. A negative class z is MBM or not simultaneously in all complex structures where it is of type $(1,1)$. □

Corollary 5.15. An MBM class is $\pm\mathbb{Q}$-effective and represented by a rational curve (up to a scalar) on a birational model of (M, I). □

Proof. We have seen that such is the case (even on (M, I) itself) when z generates the Picard group over $\mathbb{Q}$. The rest follows by deformation invariance of MBM property and the results on deformations of minimal rational curves in Section 4. ■

Putting all we have done in this section together, we arrive at the following.

Theorem 5.16. Let M be a simple hyperkähler manifold, and $z \in H_2(M, \mathbb{Q})$ a homology class. Then the following statements are equivalent.

(i) The space Teich_z (the subset of all $I \in \mathsf{Teich}$ such that z lies in $H_{1,1}(M, I)$) contains a twistor line.

(ii) For each $I \in \mathsf{Teich}_z$, there exists $I' \in \mathsf{Teich}_z$ nonseparable from I which is contained in a twistor line.

(iii) For each $I \in \mathsf{Teich}_z$ with Picard number one, $\pm z$ is not effective;

(iv) For some $I \in \mathsf{Teich}_z$ with Picard number one, $\pm z$ is not effective;

(v) z is not MBM.

Moreover, in the items (iii) and (iv), one can replace "$\pm z$ is not effective" by "(M, I) contains no rational curves". □

Proof. The implication (i) $\Rightarrow$ (ii) follows from Proposition 5.8. If (ii) holds, then there is I' nonseparable from I such that z is orthogonal to a Kähler form on I', but as we have already seen, one then has $\mathrm{Kah}(I') = \mathrm{Pos}(I')$ and $I = I'$, so z cannot be effective on I', hence (iii). (iii) $\Rightarrow$ (iv) is obvious. To get (i) from (iv), note that $\mathrm{Kah}(I) = \mathrm{Pos}(I)$, so there are Kähler classes orthogonal to z on (M, I). This implies existence of twistor lines in Teich_z.

The property (v) is equivalent to (iv) since these properties are deformation-invariant, and the equivalence for complex structures with Picard number one has already been verified. Finally, if z is effective on (M, I) with Picard number one, it is automatically represented by a rational curve. Indeed, otherwise, there are no rational curves at all, so the Kähler cone should be equal to the positive cone, but one easily finds a positive $(1,1)$-form orthogonal to z. ■

6 Monodromy Group and the Kähler Cone

In this section, we prove the results on the Kähler cone stated in the Introduction.

6.1 Geometry of Kähler–Weyl chambers

Definition 6.1. Let (M, I) be a hyperkähler manifold, and $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ the group of all monodromy elements preserving the Hodge decomposition on (M, I). A *Kähler–Weyl chamber* of a hyperkähler manifold is the image of the Kähler cone of M' under some $\gamma \in \mathsf{Mon}^{\mathsf{Hdg}}(M, I)$, where M' runs through the set of all birational models of M. □

Theorem 6.2. Let (M, I) be a hyperkähler manifold, and $S \subset H_{1,1}(M, I)$ the set of all MBM classes in $H_{1,1}(M, I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \mid z \in S\}$ in $H^{1,1}(M, I)$. Then the Kähler cone of (M, I) is a connected component of $\operatorname{Pos}(M, I)\backslash S^\perp$, where $\operatorname{Pos}(M, I)$ is a positive cone of (M, I). Moreover, the connected components of $\operatorname{Pos}(M, I)\backslash S^\perp$ are Kähler–Weyl chambers of (M, I). □

Proof. First, none of the classes $v \in W$, $W \in S^\perp$ belong to interior of any of the Kähler–Weyl chambers: indeed, it follows from Theorem 5.13 that if $v \in W$ is in the interior of a Kähler–Weyl chamber, then the classes belonging to the interior of Kähler–Weyl chambers are dense in W so $W \notin S^\perp$.

It remains to show that any connected component of $\operatorname{Pos}\backslash S^\perp$ is a Kähler–Weyl chamber.

Consider now a cohomology class $v \notin S^\perp$, and let $W = \langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, v\rangle$ be the corresponding three-dimensional plane in $H^2(M, \mathbb{R})$. We say that v is KW-generic if $W^\perp \cap H^{1,1}(M, \mathbb{Q})$ is at most one-dimensional. Denote by $\mathfrak{W}$ the set of nongeneric $(1,1)$-classes. Clearly, $\mathfrak{W}$ is a union of codimension two hyperplanes, hence removing $\mathfrak{W}$ from $\operatorname{Pos}\backslash S^\perp$ does not affect the set of connected components.

We are going to show that any $v \in \mathrm{Pos}(M, I)\backslash(\mathfrak{W} \cap S^\perp)$ belongs to the interior of a Kähler–Weyl chamber. This would imply that the connected components of $\mathrm{Pos}(M, I)\backslash S^\perp$ are Kähler–Weyl chambers, finishing the proof of Theorem 6.2.

Since v is KW-generic, and not in $S^\perp$, the space $W^\perp \cap H^{1,1}(M, \mathbb{Q})$ contains no MBM classes. This implies that Teich_z is covered by twistor lines, where z is a generator of $W^\perp \cap H^{1,1}(M, \mathbb{Q})$. By Proposition 5.12, there exists a hyperkähler structure (I', J, K) such that W is the corresponding three-dimensional plane $\langle \omega_{I'}, \omega_J, \omega_K \rangle$ and $\mathsf{Per}(I') = \mathsf{Per}(I)$. Therefore, v is Kähler on (M, I'). As we have already seen in the proof of Theorem 5.13, this implies that v belongs to the interior of a Kähler–Weyl chamber on (M, I). ■

6.2 Morrison–Kawamata cone conjecture and minimal curves

Recall the Theorem 6.3 which follows from the global Torelli theorem.

Theorem 6.3. Let (M, I) be a hyperkähler manifold, and

$$\mathsf{Mon}(M, I) \subset O(H^2(M, \mathbb{Z}))$$

its monodromy group. Let G the image of $\mathrm{Aut}(M)$ in $O(H^2(M, \mathbb{Z}))$. Then G is the set of all $\gamma \in \mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ fixing the Kähler chamber. □

Proof. Similar to the proof of Theorem 3.13 (second part); see [24]. ■

Recall also that the image of the mapping class group is a finite index subgroup in $O(H^2(M, \mathbb{Z}))$, and, accordingly, $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ is of finite index in the group of isometries of the Picard lattice.

Let M be a hyperkähler manifold, and $s \in H_2(M, \mathbb{Z})$ a homology class. The BBF form defines an injection $H_2(M, \mathbb{Z}) \overset{j}{\hookrightarrow} H^2(M, \mathbb{Q})$, hence $q(s, s)$ can be rational. However, the denominators of $\mathrm{im}\, j$ are divisors of the discriminant δ of q, hence $\delta j(s)$ is always integer, and $\delta^2 q(s, s) \in \mathbb{Z}$.

Conjecture 6.4. Let M be a hyperkähler manifold. Then there exists a constant $C > 0$, depending only on the deformation type of M, such that for any primitive MBM class s in $H^2(M, \mathbb{Z})$ one has $|q(s, s)| < C$. □

Remark 6.5. This conjecture is slightly weaker than its following, more algebraic–geometric version, implicitly appearing in [1]: let M be a hyperkähler manifold. Then there exists a constant $C > 0$ such that for any extremal rational curve (of minimal degree) R on any deformation (M, I), $I \in \mathsf{Teich}$, one has $|q(R, R)| < C$.

Indeed, for any primitive integral MBM class s, some integral multiple Ns is represented by an extremal rational curve R on such a deformation (M, I) that s generates its Picard group, by Theorem 5.11. So the boundedness of $|q(R, R)|$ means the boundedness of N plus the boundedness of $|q(s, s)|$. □

Theorem 6.6. Let M be a hyperkähler manifold. Then Conjecture 6.4 implies the Kähler version of the Morrison–Kawamata cone conjecture for all deformations of M. □

Proof. Fix a complex structure I on M, and let $S(I)$ be the set of MBM classes which are of type $(1, 1)$ on (M, I). The faces of the Kähler–Weyl chambers are pieces of $s^{\perp}$, where $s^{\perp}$ runs through $S(I)$. If Conjecture 6.4 is true, the monodromy group $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ acts on $S(I)$ with finitely many orbits (Theorem 3.14). Then the argument as in Theorem 3.15 proves that $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ acts on the set of faces of Kähler–Weyl chambers with finitely many orbits.

Let $\mathfrak{F}$ be the set of all pairs (F, ν), where F is a face of a Kähler–Weyl chamber, and ν is orientation on a normal bundle NF. Then the monodromy acts on $\mathfrak{F}$ with finitely many orbits, as we have already indicated. Each face of a Kähler cone Kah gives an element of $\mathfrak{F}$: we pick orientation determined by the side of a face adjoint to the cone.

Denote by $\mathfrak{F}_0 \subset \mathfrak{F}$ the set of all faces of the Kähler cone with their orientations. There are finitely many orbits of $\mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ acting on $\mathfrak{F}_0$. However, each $\gamma \in \mathsf{Mon}^{\mathsf{Hdg}}(M, I)$ which maps an element $f \in \mathfrak{F}_0$ to an element $\gamma(f) \in \mathfrak{F}_0$ maps the Kähler chamber to itself. Indeed, there are two Kähler–Weyl chambers adjoint to each face, and $\gamma(\mathrm{Kah})$ is one of the two chambers adjoint to $\gamma(f)$. But since the orientation is preserved, $\gamma(\mathrm{Kah}) = \mathrm{Kah}$ and not the other one. Therefore, γ is induced by an automorphism of (M, I). ■

For hyperkähler manifolds which are deformation equivalent to the Hilbert scheme of length n subschemes on a K3 surface, Conjecture 6.4 is easily deduced from [1, Proposition 2]. Indeed, it is shown there that for M as above and projective, any extremal ray of the Mori cone contains an effective curve class R with $q(R, R) \geqslant -\frac{n+3}{2}$. But if we want to bound the "length" (that is, the square of a minimal representative curve) of an MBM class on (M, I), we can look at the extremal rays on projective deformations, because MBM classes are deformation equivalent and the monodromy acts by isometries. We thus obtain the following.

Corollary 6.7. The Kähler version of Morrison–Kawamata cone conjecture holds for deformations of the Hilbert scheme of length n subschemes on a K3 surface. □

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When this paper appeared on arXiv, G. Mongardi informed us that he has obtained in [27] a deformation-invariance result similar to ours from Section 5 (and also by a similar method), for certain divisor classes called *wall divisors*. We thank him for this remark. Mongardi's wall divisors, however, are such that their orthogonal does not intersect the birational Kähler cone, rather then some Kähler chamber, so a priori our results are stronger.

One month later, Eyal Markman has sent us his new preprint, a joint work with Yoshioka [26], where a different proof of the original Morrison–Kawamata cone conjecture for hyperkähler manifolds with bounded length of extremal rays is given in the projective case. As observed in this work, both methods of proof also apply to the deformations of the generalized Kummer varieties.

Finally, the referee has asked whether in the projective hyperkähler case, the strong version of the cone conjecture can be deduced from our argument on the Kähler version. We think that this indeed can be done using some hyperbolic geometry. We refer to the last section of a paper written in the meantime as a sequel to this one (arXiv:1408.3892) and currently undergoing the refereeing process, for details.

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