Mathematical and Computer Modelling 48 (2008) 1308–1325

Contents lists available at ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

A modified "winner-take-all" rule for awarding state electoral votes in US presidential elections and a game model for its analysis

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ARTICLE INFO

Article history:
Received 19 May 2008
Accepted 22 May 2008

Keywords:
Connected strategies
Electoral college
Electoral vote
Games on polyhedral sets
Popular vote

ABSTRACT

A modification of the "winner-take-all" rule for awarding state electoral votes in US presidential elections is proposed. The modified rule has the potential of turning a state that is currently "safe" for either major party candidate into a "battleground" one, forcing presidential candidates to compete in the state and encouraging state voters (interested in the election result) to vote in presidential elections. A model for estimating the chances of a candidate to win state electoral votes in presidential elections with two major party candidates under the modified "winner-take-all" rule is proposed. This model has the form of a two-person game on a polyhedral set of connected strategies with payoff functions each being a sum of two linear functions and a bilinear function. The model is one of the two models applicable for analyzing the proposed rule that are discussed in the article.

1. Introduction

The Electoral-College-based system of electing a President is traditionally considered by political pundits as an election system supporting the existing two-party political system in the US. Despite all well-known drawbacks of the Electoral College, American society seems to go along with this election mechanism, and the two-party political system is widely believed to secure political stability in the country. The existing system of electing a President seems to be well tolerated by a majority of the electorate, at least as long as the election winner is favored by a majority or by a substantial plurality of voting voters.

The existing system of electing a President has two principle components

(a) the Electoral College mechanism for aggregating electoral votes won by slates of presidential electors in the states and in the District of Columbia (DC) in choosing a President, and

(b) the "winner-take-all" rule of awarding state electoral votes based on the popular vote results in a state and in DC.

Currently, all the states, except for Maine and Nebraska, award all their electoral votes to the (electors of the) winner of a popular vote plurality, and DC adheres to the same rule.

Both Maine and Nebraska award one electoral vote to the (elector of the) winner of the popular vote contest in each congressional district and two electoral votes to the (electors of the) winner of the state contest. However, in fact, the Maine-like scheme is no more than a particular manner of applying the "winner-take-all" rule, when only two rather than all state electoral votes are awarded to the (electors of the) winner of the state popular vote plurality, and the (elector of the) winner of the district popular vote plurality takes "all" district electoral votes (one electoral vote in each of the two congressional districts in Maine and in each of the three congressional districts in Nebraska [1]). Though these two states have exercised

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doi:10.1016/j.mcm.2008.05.034

this particular version of the "winner-take-all" rule for awarding state electoral votes for quite a while (Maine has done it since 1969, and Nebraska since 1981), in all presidential elections held during 1972–2004, neither state ever split its electoral votes.

Demographic changes in the country, the Electoral College mechanism, and the "winner-take-all" rule have contributed to a well-known phenomenon in US presidential elections. Namely, a majority of the states are currently "safe" for either major party candidate, since a majority of voting voters in each "safe" state strongly favor one of the candidates. Despite the fact that almost 45% of all eligible voters usually do not vote in US presidential elections, the candidates and their teams strongly believe that "swing voters" in the states – which constitute only 10%–16% of the electorate nationwide – are the only "force" to be concerned with. This does not encourage non-voters to participate in presidential elections, making the voting behavior of majorities of voting voters in many states quite predictable. In fact, presidential election campaigns focus exceptionally on the "battlegrounds", whereas "safe" states, including California and Texas – the two most populated states in the country – are often viewed by the candidates as financial donors, and in the course of the election campaign, the candidates visit them mostly for fundraising purposes.

This "status quo" has moved some "safe" states to explore alternatives to the "winner-take-all" rule, and the most known two such alternatives have been discussed over the years. Moreover, in the 2004 election, the so-called proportional plan of awarding state electoral votes – proportionally to the percentage of the popular vote received by (the electors of) the candidates – was put on the ballot in Colorado and was defeated. In 2007, a proposal to adopt a Maine-like district scheme of awarding California's (currently) 55 electoral votes was actively promoted by several political groups, and, after vigorous debates in the media [2,3], contrary to the expectations of its originators, the proposal was rejected even at the stage of putting it on the ballot in November 2008.

Despite real motives underlying proposals to get rid of the "winner-take-all" rule, it is obvious that "safe" states do not want to be spectators in presidential elections. But this does not mean that the "safe" states are doomed to play only this limited role in presidential elections, since the "winner-take-all" rule can be modified to encourage the candidates to actively campaign in each "safe" state.

The aim of this article is to propose such a modification in the form of a simple rule for awarding state electoral votes, having the potential to make every electoral vote in every "safe" state employing this rule contestable to a much larger degree than it currently is. While the adoption of this rule by the states makes sense mostly for large- and medium-sized states (and, certainly, only for those states that would like to change their "safe" status in presidential elections), in close elections, employing this rule may eventually make sense for small states as well. Adopting the proposed rule would encourage the candidates to campaign in "safe" states, and it would encourage voters (interested in the election results) to vote in presidential elections. Its adoption in a state depends on the state legislature only, and its introduction in a state does not require a constitutional amendment.

A game model for analyzing the modified "winner-take-all" rule, which allows one to estimate the chances of the two major party candidates to win the state contest under both the original "winner-take-all" rule and the modified one, is also proposed. The analysis should detect to what extent the proposed rule can affect the candidates' intention to wage strong campaigns in a particular state employing this rule and, possibly, recommend additional changes in the initial "winner-take-all" rule to encourage this intention.

To simplify the material presentation, throughout the rest of the article, votes cast in favor of the electors of a presidential candidate are construed as those cast in favor of the candidate.

2. The description of the modified "winner-take-all" rule in two-party presidential elections

Let us first consider a situation in which all the state votes are cast in favor of two (major party) presidential candidates only.

Let

$r > 0$ be the number of votes cast in favor of the Republican party candidate in the previous election,

$d > 0$ be the number of voters cast in favor of the Democratic party candidate in the previous election,

$a \geq 3$ be the number of state electoral votes to be awarded in the election,

$R > 0$ be the number of votes cast in favor of the Republican party candidate in the election, and

$D > 0$ be the number of votes cast in favor of the Democratic party candidate in the election.

1. Let $R \neq D$, and for the sake of definiteness, let us assume that the inequality $R > D$ holds.

The proposed rule consists of awarding μ electoral votes to the candidate who won a majority of the cast votes (the Republican party candidate in the case under consideration), where

$$\mu = \begin{cases} \left[\dfrac{R}{R+D}a\right], & \text{if } \dfrac{R}{R+D}a - \left[\dfrac{R}{R+D}a\right] \leq 1/2; \\ \left[\dfrac{R}{R+D}a\right] + 1, & \text{if } \dfrac{R}{R+D}a - \left[\dfrac{R}{R+D}a\right] > 1/2, \end{cases}$$

and $[x]$ is the largest integer not exceeding x, while putting

$$a - \mu$$

electoral votes in play between the candidates. Throughout this section, it is assumed that the inequality $a - \mu \geq 1$ holds.

If

$$R - r > D - d, \tag{1}$$

then all these $a - \mu$ electoral votes are awarded to the Republican party candidate so that the Republican party candidate "takes all" a state electoral votes, and if

$$R - r < D - d, \tag{2}$$

then all these $a - \mu$ electoral votes are awarded to the Democratic party candidate.

If

$$D - d = R - r, \tag{3}$$

and $a - \mu \geq 2$ is an even number, then each candidate receives $\frac{1}{2}(a - \mu)$ out of these $a - \mu$ electoral votes. Otherwise, if $a - \mu \geq 3$, and $a - \mu$ is an odd number, each candidate receives $\frac{1}{2}(a - \mu - 1)$ out of these $a - \mu$ electoral votes, whereas the fate of the remaining electoral vote is decided according to any particular manner to be chosen by the state legislature. For instance, this electoral vote can be awarded to the winner of the state popular vote majority or can even be settled by lot. The same principle is applied if $a - \mu = 1$, and in close elections, both candidates may compete for this one electoral vote.

2. Let us now assume that $R = D$. Then each candidate receives $a/2$ electoral votes if a is an even number, $a \geq 4$ (since each state controls at least three electoral votes, and the District of Columbia controls three electoral votes), and each candidate receives $(a - 1)/2$ electoral votes out of $a - 1$ electoral votes, if $a \geq 3$ is an odd number. In the latter case, if inequality (1) holds, the remaining electoral vote goes to the Republican party candidate, whereas if inequality (2) holds, this vote goes to the Democratic party candidate. Finally, if equality (3) holds, the state legislature decides the fate of the remaining electoral vote, or the matter is settled by lot.

In two-party presidential elections, the idea underlying the proposed rule is especially easy to understand—to make every single vote in a state count, and to force the candidates to compete for a part of the state electoral votes that

(a) under the "winner-take-all" rule, would have been automatically awarded to the state's favorite (and, consequently, automatically lost by the opponent), and

(b) under the proportional rule would have been automatically awarded to the favorite's opponent (and, consequently, automatically lost by the favorite),

which would have discouraged both of them to actively compete (and campaign) in the state under either rule.

The proposed rule encourages both candidates to compete for $a - \mu$ votes rather than to take them for granted by the state's favorite or by the favorite's opponent, and, since the rule makes every cast vote important to both candidates, it encourages every state voter (interested in the election result) to vote.

Thus, if the Republican party candidate receives more state votes in the election, she/he may receive all the electoral votes only if her/his campaign mobilized more voters compared to the previous election than did the campaign of the opponent. Otherwise, the Republican party candidate can lose as many as $a - \mu$ electoral votes though she/he receives more (popular) votes than her/his opponent.

3. The description of the modified "winner-take-all" rule in multi-party elections with two major party presidential candidates

Now let

$s \geq 1$ be the number of non-major parties in the (current and previous) election,

$ng_i > 0$ be the number of votes cast in favor of the candidate from non-major party i in the previous election, $i \in \overline{1, s}$, and

$Ng_i > 0$ be the number of votes cast in favor of the candidate from non-major party i in the election, $i \in \overline{1, s}$.

Let us first assume that the inequalities

$$R > D > \max(Ng_1, Ng_2, \ldots, Ng_s)$$

hold.

Further, let us assume that

$$\sum_{i \in K \subseteq \overline{1,s}} (Ng_i - ng_i) = \max_{I \subseteq \overline{1,s}} \sum_{i \in I} (Ng_i - ng_i)\,,$$

and, for the sake of simplicity, let us assume that $K = \overline{1, k}$, where $k \leq s$.

The proposed rule consists of awarding ν electoral votes to the candidate who won at least a plurality of the cast votes (the Republican party candidate in the case under consideration), where

$$\nu = \begin{cases} \left\lfloor \dfrac{aR}{R+D+\sum_{i=1}^{s} Ng_i} \right\rfloor, & \text{if } \dfrac{aR}{R+D+\sum_{i=1}^{s} Ng_i} - \left\lfloor \dfrac{aR}{R+D+\sum_{i=1}^{s} Ng_i} \right\rfloor \leq 1/2; \\ \left\lfloor \dfrac{aR}{R+D+\sum_{i=1}^{s} Ng_i} \right\rfloor + 1, & \text{if } \dfrac{aR}{R+D+\sum_{i=1}^{s} Ng_i} - \left\lfloor \dfrac{aR}{R+D+\sum_{i=1}^{s} Ng_i} \right\rfloor > 1/2, \end{cases} \tag{4}$$

while putting the remaining $a - \nu$ electoral votes in play among either all the candidates (if the Republican party candidate won a majority of the state popular vote) or first among all $s + 2$ candidates, except for the Republican party candidate (if the Republican party candidate won only the state popular vote plurality), i.e., among only $s + 1$ candidates. As before, throughout this section, it is assumed that the inequality $a - \nu \geq 1$ holds.

3.1

Let the inequality

$$R > D + \sum_{i=1}^{s} Ng_i$$

hold, i.e., let the Republican party candidate win the state popular vote majority.

If

$$\max\left(D - d, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d\right) > R - r,$$

then $a-\nu$ electoral votes will either be won by a Democratic party candidate or will first be put in play among the Democratic party candidate and all s non-major party candidates.

If

$$\max\left(D - d, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d\right) < R - r,$$

then the Republican party candidate receives all a electoral votes, and if

$$\max\left(D - d, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d\right) = R - r,$$

these $a - \nu$ electoral votes will be split between the Republican party candidate and

(a) either the Democratic party candidate, or

(b) a group of $s + 1$ candidates – the Democratic party candidate and all s non-major party candidates.

Remark 1. It is important to emphasize that while all the s non-major party candidates contribute to putting $a - \nu$ electoral votes in play, only candidates from parties with the numbers forming the set K "represent the bloc" of non-major party candidates the best in deciding whether at least one electoral vote will go to non-major party candidates. Indeed, under the proposed rule, only these candidates may make this "bloc" competitive with two major party candidates for any part of these electoral votes, since for any $i \in \overline{1, s}$, the inequality $Ng_i - ng_i < 0$ may hold even if Ng_i is comparable with R and D, and illustrative examples of such situations are presented in the next section of the article. However, if any of $a - \nu$ electoral votes are to be put in play among non-major party candidates, these electoral votes will be put in play among all s rather than among only k (if $k < s$) non-major party candidates. Though this makes the proposed rule more favorable to non-major party candidates than it would have been otherwise, it encourages all these candidates to compete in the state.

In case (b), for the sake of definiteness, let us assume that

$$\max\left(D - d, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d\right) = \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d = R - r,$$

i.e., let us assume that $a - \nu$ electoral votes are split between the Republican party candidate and the group of the other $s + 1$ candidates.

If $a-\nu$ is an odd number, and $a-\nu \geq 3$, then the $a-\nu-1$ electoral votes out of $a-\nu$ electoral votes are equally divided between the Republican party candidate and the group of all the other $s+1$ candidates, whereas the fate of the remaining electoral vote can be decided by several manners. For instance, as before, the state legislature may decide to award this electoral vote at their own discretion, this electoral vote may be awarded to the state popular vote winner, etc., and if $a-\nu=1$, then this electoral vote is awarded in just the same manner.

If $a-\nu \geq 2$ is an even number, then the Republican party candidate receives $\frac{a-\nu}{2}$ electoral votes out of $a-\nu$ electoral votes, whereas the other $\frac{a-\nu}{2}$ electoral votes will be split among the Democratic party candidate and all the non-major party candidates as a result of putting them in play only among these $s+1$ candidates according to the same principle as formalized by relations (4).

The other option of splitting $a-\nu$ votes – between the Republican party candidate and the Democratic party candidate – can be analyzed similarly.

In both considered situations in case (b), all $(a-\nu-1)/2$ or $(a-\nu)/2$ electoral votes are divided among $s+1$ candidates in a manner that implies starting with relations (4) and then continuing according to the procedures to be discussed in Sections 3.1–3.3.

Let us consider the case in which

$$\max\left(D-d, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d\right) > R-r,$$

and, for the sake of definiteness, let us assume that

$$\max\left(D-d, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d\right) = \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d > R-r,$$

and $a-\nu \geq 1$ so that $a-\nu$ electoral votes are first put in play among s non-major party candidates and the Democratic party candidate.

If

$$D > \sum_{i=1}^{s} Ng_i,$$

and $a-\nu=1$, this electoral vote is awarded to the Democratic party candidate, whereas if $a-\nu \geq 2$, then the Democratic party candidate receives

$$\eta_d = \begin{cases} \left[\dfrac{D(a-\nu)}{D+\sum_{i=1}^{s} Ng_i}\right], & \text{if } \dfrac{D(a-\nu)}{D+\sum_{i=1}^{s} Ng_i} - \left[\dfrac{D(a-\nu)}{D+\sum_{i=1}^{s} Ng_i}\right] \leq 1/2; \\ \left[\dfrac{D(a-\nu)}{D+\sum_{i=1}^{s} Ng_i}\right] + 1, & \text{if } \dfrac{D(a-\nu)}{D+\sum_{i=1}^{s} Ng_i} - \left[\dfrac{D(a-\nu)}{D+\sum_{i=1}^{s} Ng_i}\right] > 1/2, \end{cases} \tag{5}$$

out of $a-\nu$ electoral votes, and $a-\nu-\eta_d$ electoral votes are put in play among all the $s+2$ candidates.

Let us assume that the inequality $a-\nu-\eta_d \geq 1$ holds.

If

$$D-d > \max\left(R-r, \sum_{i=1}^{k} (Ng_i - ng_i)\right),$$

then $a-\nu=(a-\nu-\eta_d)+\eta_d$ electoral votes are awarded to the Democratic party candidate.

If

$$R-r > \max\left(D-d, \sum_{i=1}^{k} (Ng_i - ng_i)\right),$$

then all these $a-\nu-\eta_d$ electoral votes are awarded to the Republican party candidate, so the Republican party candidate receives $a-\eta_d=\nu+(a-\nu-\eta_d)$ electoral votes in total.

If

$$\sum_{i=1}^{k} (Ng_i - ng_i) > \max(D-d, R-r),$$

then these $a-\nu-\eta_d$ electoral votes are put in play only among s non-major party candidates in just the same manner as formalized by relations (4).

Remark 2. If the relations among $R-r$, $D-d$, and $Ng_i - ng_i$ are such that $a-\nu-\eta_d$ electoral votes are put in play among s non-major party candidates, all these electoral votes will be awarded to non-major party candidates only.

If

$$D-d = R-r > \sum_{i=1}^{k} (Ng_i - ng_i)\,,$$

and $a-\nu-\eta_d \geq 2$, then these $a-\nu-\eta_d$ electoral votes are divided between the Democratic party candidate and the Republican party candidate in just the same manner as was described earlier in this section. If $a-\nu-\eta_d = 1$, then this electoral vote is awarded in just the same manner as was described for $a-\nu = 1$.

If

$$D-d = \sum_{i=1}^{k} (Ng_i - ng_i) > R-r,$$

or

$$R-r = \sum_{i=1}^{k} (Ng_i - ng_i) > D-d,$$

and $a-\nu-\eta_d \geq 2$, then these $a-\nu-\eta_d$ electoral votes are divided between the corresponding major party candidate and the group of s non-major party candidates in just the same manner as was described for $a-\nu$ electoral votes earlier in this section. If $a-\nu-\eta_d = 1$, then this electoral vote is awarded in just the same manner as was described earlier in this section.

In all the cases in which $a-\nu-\eta_d \geq 1$, and $a-\nu-\eta_d$ electoral votes are put in play among s non-major party candidates, at least one of the non-major party candidates always has a chance to win state electoral votes. If $a-\nu-\eta_d$ electoral votes are put in play among s non-major party candidates, all these electoral votes remain in play among s non-major party candidates only (see Remark 2), and these electoral votes are divided first as is it described by relations (4).

If the inequalities

$$\sum_{i=1}^{s} Ng_i > D > \max_{i\in\overline{1,s}} Ng_i$$

hold, then $a-\nu-\eta_d$ electoral votes are first put in play among $s+1$ candidates (the Republican party candidate and s non-major party candidates), assuming as before that the inequality $a-\nu-\eta_d \geq 1$ holds.

If

$$R-r > \sum_{i=1}^{k} (Ng_i - ng_i)\,,$$

then all these $a-\nu-\eta_d$ electoral votes are awarded to the Republican party candidate, so the Republican party candidate receives $a-\eta_d = \nu + (a-\nu-\eta_d)$ electoral votes in total.

If

$$\sum_{i=1}^{k} (Ng_i - ng_i) > R-r,$$

then these $a-\nu-\eta_d$ electoral votes are first put in play only among s non-major party candidates in just the same manner as formalized by relations (4) and then, if, after receiving at least one electoral vote by a non-major party candidate, there is at least one electoral vote (out of these electoral votes) left, this electoral vote (votes) is (are) put in play among the Democratic party candidate and s non-major party candidates in just the same manner as was described earlier in this section (since $D-d$ may exceed $\sum_{i=1}^{k}(Ng_i - ng_i)$).

Finally, if

$$\sum_{i=1}^{k} (Ng_i - ng_i) = R-r,$$

then these $a-\nu-\eta_d$ electoral votes are divided between the Republican party candidate and s non-major party candidates in just the same manner as was described earlier in this section, and then, if, after receiving at least one electoral vote by a non-major party candidate, there is at least one electoral vote (out of these electoral votes) left, this electoral vote (votes) is (are) put in play among the Democratic party candidate and s non-major party candidates in just the same manner as was described earlier in this section.

If

$$\max\left(D-d, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d\right) = D - d > R - r,$$

then the Democratic party candidate receives $a - \nu$ electoral votes, and if

$$\max\left(D-d, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i, \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i + D - d\right) = \sum_{i=1}^{k} Ng_i - \sum_{i=1}^{k} ng_i > R - r,$$

then $a - \nu - \eta_d$ electoral votes are put in play among s non-major party candidates in just the same manner as formalized by relations (4), and all these electoral votes will be split among s non-major party candidates only.

3.2

Let the inequalities

$$D + \sum_{i=1}^{s} Ng_i > R > \max\left(D, \max_{i\in\overline{1,s}} Ng_i\right)$$

hold, i.e., let the Republican party candidate win only a plurality of the state popular vote. Then, as before, the Republican party candidate receives ν electoral votes. However, $a - \nu$ electoral votes are first put in play among only $s + 1$ candidates—the Democratic party candidate and the non-major party candidates. As before, it is assumed that the inequality $a - \nu \geq 1$ holds.

Let the inequality

$$D > \sum_{i=1}^{s} Ng_i$$

hold. Then if $a-\nu = 1$, this electoral vote is awarded to the Democratic party candidate, whereas if $a-\nu \geq 2$, the Democratic party candidate receives

$$\lambda = \begin{cases} \left[\dfrac{D(a-\nu)}{D + \sum_{i=1}^{s} Ng_i}\right], & \text{if } \dfrac{D(a-\nu)}{D + \sum_{i=1}^{s} Ng_i} - \left[\dfrac{D(a-\nu)}{D + \sum_{i=1}^{s} Ng_i}\right] \leq 1/2; \\ \left[\dfrac{D(a-\nu)}{D + \sum_{i=1}^{s} Ng_i}\right] + 1, & \text{if } \dfrac{D(a-\nu)}{D + \sum_{i=1}^{s} Ng_i} - \left[\dfrac{D(a-\nu)}{D + \sum_{i=1}^{s} Ng_i}\right] > 1/2, \end{cases} \tag{6}$$

electoral votes, whereas $a - \nu - \lambda$ electoral votes are put in play among all the $s + 2$ candidates (the Democratic party candidate, the Republican party candidate, and all the non-major party candidates), assuming that the inequality $a - \nu - \lambda \geq 1$ holds.

In the latter case, similar to what was considered in Section 3.1, if

$$D - d > \max\left(R - r, \sum_{i=1}^{k} (Ng_i - ng_i)\right),$$

then the Democratic party candidate receives all the $a - \nu = \lambda + (a - \nu - \lambda)$ electoral votes, whereas if

$$R - r > \max\left(D - d, \sum_{i=1}^{k} (Ng_i - ng_i)\right),$$

then the Republican party candidate receives $a - \lambda = \nu + (a - \nu - \lambda)$ electoral votes.

If

$$\sum_{i=1}^{k} (Ng_i - ng_i) > \max(D - d, R - r),$$

then $a - \nu - \lambda$ electoral votes are put in play among s non-major party candidates in just the same way as was described earlier in Section 3.1.

If

$$R - r = D - d > \sum_{i=1}^{k} (Ng_i - ng_i),$$

or if

$$D - d = \sum_{i=1}^{k} (Ng_i - ng_i) > R - r,$$

or

$$R - r = \sum_{i=1}^{k} (Ng_i - ng_i) > D - d,$$

then $a - \nu - \lambda$ electoral votes are split either between the two major party candidates, or between the corresponding major party candidate and all s non-major party candidates. In the latter case, the major party candidate and the group of s non-major party candidates receive maximum possible equal shares of the electoral votes (if there are enough electoral votes to share, i.e., if $a - \nu - \lambda \geq 2$) with possibly one electoral vote to be awarded in just the same manner as was described in Section 3.1.

Also, the share of state electoral votes for the group of s non-major party candidates is put in play among these s candidates in just the same manner as formalized by relations (4) (see also Remark 2).

Remark 3. One should emphasize that according to the proposed rule, a major party candidate and the group of s non-major party candidates may "play together" against the other major party candidate (in splitting $a - \nu$ electoral votes) only when this other major party candidate receives a majority of votes in the election. If one major party candidate receives only a plurality of votes among all the candidates, each major party candidate and the group of s non-major party candidates "play" only separately, as two different players, in splitting $a - \nu - \lambda$ electoral votes that are put in play among them.

Let the inequalities

$$D + \sum_{i=1}^{s} Ng_i > R > \max\left(D, \max_{i \in \overline{1,s}} Ng_i\right)$$

and

$$\max_{i \in \overline{1,s}} Ng_i < D < \sum_{i=1}^{s} Ng_i,$$

hold. Then $a - \nu - \lambda$ electoral votes are first put in play among all the $s + 2$ candidates, except for the Democratic party candidate, since this candidate received a plurality rather than a majority of the votes among those cast in favor of $s + 1$ candidates (i.e., all the candidates, except for the Republican party candidate), assuming that the inequality $a - \nu - \lambda \geq 1$ holds. These $a - \nu - \lambda$ electoral votes are awarded in just the same manner as was described in 3.2.

If these $a - \nu - \lambda$ electoral votes turn out to be in play among only s non-major party candidates in virtue of the inequality

$$\sum_{i=1}^{k} (Ng_i - ng_i) > R - r,$$

and if the inequality

$$Ng_{i^*} > \max_{i \in \overline{1,n} \setminus \{i^*\}} Ng_i$$

holds for $i^* \in \overline{1, s}$, then if $a - \nu - \lambda = 1$, this electoral vote is awarded to the candidate from non-major party i^*, whereas if $a - \nu - \lambda \geq 2$, candidate from non-major party i^* receives ω electoral votes, where

$$\omega = \begin{cases} \left[\dfrac{(a - \nu - \lambda) Ng_{i^*}}{\sum_{i=1}^{s} Ng_i}\right], & \text{if } \dfrac{(a - \nu - \lambda) Ng_{i^*}}{\sum_{i=1}^{s} Ng_i} - \left[\dfrac{(a - \nu - \lambda) Ng_{i^*}}{\sum_{i=1}^{s} Ng_i}\right] \leq 1/2; \\ \left[\dfrac{(a - \nu - \lambda) Ng_{i^*}}{\sum_{i=1}^{s} Ng_i}\right] + 1, & \text{if } \dfrac{(a - \nu - \lambda) Ng_{i^*}}{\sum_{i=1}^{s} Ng_i} - \left[\dfrac{(a - \nu - \lambda) Ng_{i^*}}{\sum_{i=1}^{s} Ng_i}\right] > 1/2, \end{cases} \tag{7}$$

and $a - \nu - \lambda - \omega$ electoral votes are put in play among the Democratic party candidate and s non-major party candidates, assuming that the inequality $a - \nu - \lambda - \omega \geq 1$ holds. These $a - \nu - \lambda - \omega$ electoral votes are awarded in just the same manner as was described earlier in Sections 3.1 and 3.2.

If

$$R - r > \sum_{i=1}^{k} (Ng_i - ng_i),$$

then these $a - \nu - \lambda$ electoral votes are awarded to the Republican party candidate.

If

$$\sum_{i=1}^{k} (Ng_i - ng_i) = R - r,$$

then these $a - \nu - \lambda$ electoral votes are split between the Republican party candidate and the group of s non-major party candidates in just the same manner as was described in Section 3.1 for $a - \nu$ votes. Here, if the inequality

$$D - d > \sum_{i=1}^{k} (Ng_i - ng_i),$$

holds, then if, after receiving at least one electoral vote by a non-major party candidate, there is at least one electoral vote (out of those not received by the Republican party candidate) left, this electoral vote (votes) is (are) put in play among $s + 1$ candidates (the Democratic party candidate and s non-major party candidates) in just the same manner as was described earlier in this section.

3.3

If for the Republican party candidate, the Democratic party candidate, and all the non-major party candidates, the equalities

$$R - r = D - d = \sum_{i=1}^{k} (Ng_i - ng_i),$$

hold, and $a - \nu - \lambda \geq 3$, then the Republican party candidate, the Democratic party candidate, and the group of s non-major party candidates each receive equal maximum possible share of electoral votes out of these $a - \nu - \lambda$ electoral votes. If, after that, there are either two or one electoral votes left, these electoral votes can be awarded in a manner to be chosen by the state legislature.

If $a - \nu - \lambda = 2$, then these two electoral votes are awarded in a manner to be chosen by the state legislature.

If $a - \nu - \lambda = 1$, the fate of this electoral vote is decided in just the same manner as was described earlier in Section 3.1 for $a - \nu = 1$.

The same scheme is applied in splitting (a) $a - \nu - \eta_d$ electoral votes (or a portion of them), where η_d is determined by (5), and (b) $a - \nu - \lambda - \omega$, where λ and ω are determined by (6) and (7), respectively.

3.4

Let us now assume that the relations

$$R = D > \max(Ng_1, Ng_2, \ldots, Ng_s)$$

hold.

Then 2ν electoral votes, where ν is determined by relations (4), are awarded to the Republican party candidate and the Democratic party candidate (ν electoral votes to each candidate), whereas the remaining $a - 2\nu$ electoral votes are put in play among either s non-major party candidates only (if either major party candidate receives only a plurality of votes cast in favor of the other major party candidate and all non-major party candidates combined) of among all $s + 2$ candidates (if either major party candidate receives a majority of these votes) in just the same manner as was described in Section 3.2, assuming that the inequality $a - 2\nu \geq 1$ holds.

Let the relations

$$R = D > \max(Ng_1, Ng_2, \ldots, Ng_s)$$

hold, along with the relations

$$Ng_{i^*} = Ng_{i^{**}} > \max_{i \in \overline{1,s} \setminus \{i^*, i^{**}\}} Ng_i,$$

and let $a - 2\nu \geq 1$ electoral votes be put in play among s non-major party candidates. Each of the two candidates from non-major parties i^* and i^{**} may receive $\omega \geq 1$ electoral votes only if the inequality $a - 2\nu \geq 2$ holds, and if $a - 2\nu = 1$, this electoral vote is awarded to one of these non-major party candidates in just the same manner as was described in Sections 3.1–3.3 in the corresponding situations, in particular, it can be awarded at the state legislature discretion or can be settled by lot. The case of receiving the same number of votes by more than two non-major party candidates under the same assumptions is considered analogously.

4. Illustrative examples

1. Let us consider the 2004 presidential election, and, for the sake of simplicity, throughout this section, let us consider that all the votes were cast in favor of three candidates only—two from the major parties and one from a non-major party. This assumption does not affect the generality of the reasoning to follow, since the number of votes in favor of the "non-major" party candidate was too small to affect the distribution of the electoral votes under the proposed rule in any state or in DC.

(a) For the state of California, the parameters R, D, r, d, Ng, ng, and a assumed the following values:

$$R = 5509{,}826, \quad D = 6745{,}485, \quad Ng = 164{,}546, \quad a = 55,$$
$$r = 4576{,}429, \quad d = 5861{,}203, \quad ng = 537{,}224$$

so that $R - r = 933{,}397$, $D - d = 884{,}282$, $Ng - ng = -372{,}678$.

According to (4), the Democratic party candidate would have won 30 electoral votes, since

$$\frac{D \times 55}{D + R + Ng} = \frac{6745{,}485 \times 55}{5509{,}826 + 6745{,}485 + 164{,}546} = 0.5431 \times 55 = 29.872$$

and

$$29.872 - [29.872] > \frac{1}{2},$$

whereas, according to the modified "winner-take-all" rule for awarding California's electoral votes, 25 electoral votes would have been put in play among all the candidates. However, due to the inequality

$$933{,}397 = R - r = \max(R - r, R - r + Ng - ng, Ng - ng) > D - d = 884{,}282$$

all these 25 electoral votes would have been awarded to the Republican party candidate.

(b) For the state of Georgia, the parameters R, D, r, d, Ng, ng, and a assumed the following values:

$$R = 1914{,}254, \quad D = 1366{,}149, \quad Ng = 21{,}472, \quad a = 15,$$
$$r = 1419{,}720, \quad d = 1116{,}230, \quad ng = 60{,}854$$

so that $R - r = 494{,}534$, $D - d = 249{,}919$, $Ng - ng = -43{,}382$.

According to (4), the Republican party candidate would have won 10 electoral votes, since

$$\frac{R}{D + R + Ng} = \frac{1914{,}254 \times 15}{1914{,}254 + 1366{,}149 + 21{,}472} = 9.6962,$$

and

$$9.6962 - [9.6962] > \frac{1}{2},$$

whereas, according to the modified "winner-take-all" rule for awarding Georgia's electoral votes, 5 electoral votes would have been put in play among all the candidates. However, due to the inequality

$$494{,}534 = R - r > \max(D - d, D - d + Ng - ng, Ng - ng) = \max(249{,}919, 206{,}537, -43{,}382)$$

all these remaining 5 electoral votes would also have been awarded to the Republican party candidate.

2. Let us now consider the 1992 presidential election, and for the sake of simplicity, let us consider that all the votes were cast in favor of four candidates only—two from the two major parties, one Independent (R. Perot), and one from a non-major party. As before, the assumption about the number of the non-major party candidates (besides the Independent candidate) does not affect the generality of the reasoning to follow, since the number of votes in favor of the "non-major" party candidate was too small to affect the distribution of the electoral votes under the proposed rule in any state or in DC.

For the state of California, the parameters R, D, r, d, Ng_1, Ng_2, ng_1, ng_2, and a assumed the following values:

$$R = 3630{,}574 \quad D = 5121{,}325, \quad Ng_1 = 2296{,}006 \quad Ng_2 = 83{,}816, \quad a = 54,$$
$$r = 5054{,}917, \quad d = 4702{,}233, \quad ng_1 = 0 \quad ng_2 = 129{,}914$$

so that $R - r = -1424{,}343$, $D - d = 419{,}092$, $Ng_1 - ng_1 = 2296{,}006$, $Ng_2 - ng_2 = -46{,}098$.

According to (4), the Democratic party candidate would have won 25 electoral votes, since the inequalities

$$D > \max(R, Ng_1, ng_1),$$

and

$$R + \sum_{i=1}^{2} Ng_i > D > \max(R, \max_{i\in\{1,2\}} Ng_i),$$

held, along with the equalities

$$\frac{D \times 54}{D + R + Ng_1 + Ng_2} = \frac{5121{,}325 \times 54}{5121{,}325 + 3630{,}574 + 2296{,}006 + 83{,}816} = 0.4601 \times 54 = 24.845$$

and the inequality

$$24.845 - [24.845] > \frac{1}{2}.$$

According to the modified "winner-take-all" rule for awarding California's electoral votes, 29 electoral votes would have been put in play among the Republican party candidate, the Independent candidate, and the non-major party candidate only, since the Democratic party candidate received only a plurality of state votes.

Since

$$3360{,}574 = R > \sum_{i=1}^{2} Ng_i = 2296{,}006 + 83{,}816,$$

according to (6), the Republican party candidate would have won 18 electoral votes due to the equalities

$$\frac{R \times 29}{R + Ng_1 + Ng_2} = \frac{3630{,}574 \times 29}{3630{,}574 + 2296{,}006 + 83{,}816} = 17.517,$$

and the inequality

$$17.517 - [17.517] > \frac{1}{2}.$$

According to the modified "winner-take-all" rule for awarding California's electoral votes, 11 electoral votes would have been put in play among all the candidates (the Democratic party candidate, the Republican party candidate, the Independent candidate and the non-major party candidate).

However, due to the inequality

$$2296{,}006 = Ng_1 - ng_1 > \max(D - d, R - r) = 419{,}092,$$

the equalities

$$\frac{Ng_1}{Ng_1 + Ng_2} = \frac{2296{,}006 \times 11}{2296{,}006 + 83{,}816} = 10.612,$$

and the inequality

$$10.612 - [10.612] > \frac{1}{2},$$

these 11 electoral votes would have been won by the independent candidate.

Thus, according to the modified "winner-take-all" rule for awarding California's electoral votes, the Democratic party candidate would have received 25 electoral votes, the Republican party candidate would have received 18 electoral votes, and the Independent candidate would have received 11 electoral votes.

3. Finally, let us consider the 1996 presidential election, and for the sake of simplicity, let us consider that all the votes were cast in favor of four candidates only—two from the two major parties, one from the Reform party (R. Perot), and one from a non-major party. As before, this assumption does not affect the generality of the reasoning to follow, since the number of votes for the "non-major" party candidate was too small to affect the distribution of the electoral votes under the proposed rule in any state or in DC.

For the state of Colorado, the parameters R, D, r, d, Ng_1, Ng_2, ng_1, ng_2 and a assumed the following values:

$$R = 691{,}848 \quad D = 671{,}152, \quad Ng_1 = 99{,}629 \quad Ng_2 = 48{,}075, \quad a = 8$$
$$r = 562{,}850, \quad d = 629{,}681, \quad ng_1 = 366{,}010 \quad ng_2 = 10{,}639$$

so that $R - r = 128{,}998$, $D - d = 41{,}471$, $Ng_1 - ng_1 = -266{,}381$, $Ng_2 - ng_2 = 37{,}436$.

According to (4), the Republican party candidate would have won 4 electoral votes, since

$$D + \sum_{i=1}^{2} Ng_i > R > \max(D, \max_{i\in\{1,2\}} Ng_i)$$

and

$$\frac{R \times 8}{D + R + Ng_1 + Ng_2} = \frac{691{,}848 \times 8}{1510{,}704} = 3.663,$$

whereas

$$3.663 - [3.663] > \frac{1}{2}.$$

According to the modified "winner-take-all" rule for awarding Colorado's electoral votes, the remaining 4 electoral votes would first have been put in play among the Democratic party candidate, the Reform party candidate, and the non-major party candidate.

Since $D > Ng_2 + Ng_1$, the Democratic party candidate would have won 3 electoral votes due to the equality

$$\frac{D \times 4}{D + Ng_1 + Ng_2} = \frac{671{,}152 \times 4}{671{,}152 + 99{,}629 + 48{,}075} = 3.2785$$

and the inequality

$$3.2785 - [3.2785] < \frac{1}{2}.$$

However, due to the inequality

$$128{,}998 = R - r > \max(D - d, Ng_2 - ng_2, Ng_1 - ng_1) = \max(41{,}471, -266{,}381, 37{,}436),$$

the Republican party candidate would have won the remaining one electoral vote, i.e., would have won 5 electoral votes in total.

5. The modified "winner-take-all" rule for awarding state electoral votes as a part of the system of electing a President

As is known, the Electoral College – an intermediate, independent Congress, as the Founding Fathers devised it – and the mechanism for transforming votes cast in 50 states and in DC into electoral votes to determine the election winner – with the numbers of state electoral votes depending on the state's size population – are two different concepts of a system of electing a US President [1]. Though, traditionally, both concepts are referred to as the Electoral College, in the reasoning to follow, only the Electoral College mechanism for transforming state and DC votes into electoral votes is considered.

Many researchers in the field of political science mistakenly believe that the "winner-take-all" rule is what makes 11 largest states in the country a decisive body, capable of determining the election outcome, despite the will of the rest of the country. However, this "feature" of the Electoral College mechanism does not depend on the manner in which states and DC award their electoral votes. Under both proportional rule and a Maine-like district rule or under any other scheme of awarding state (DC) electoral votes, the Electoral College mechanism gives a monopoly to the 11 largest states (as well as to any relatively small group of states controlling a majority of the electoral votes that are in play in the election) to have the power to decide the election outcome. If the Electoral College mechanism remains, there is no way to eliminate this monopoly without changing the concept of the US Presidency.

The idea of one such change was proposed in [4,1,5], and this idea can be implemented without abolishing the Electoral College mechanism, in the framework of the so-called modified election system [1,4]. This idea consists of

(a) giving priority to a candidate who is the choice of a majority of the states and a majority of voting voters if more than 50% of all eligible voters vote, and

(b) referring the election to the Electoral College mechanism only if there is no such candidate in the election, or if less than 50% of all eligible voters vote in the election.

In both election systems – the existing system and the modified one – the "winner-take-all" rule is present though in different forms. In the existing election system, the rule is used in all the states and DC (though, as mentioned in the Introduction, in two different modifications) to determine the state electoral vote winner. Under the modified election system [4], winning a state by a majority (or a plurality) of votes is, in fact, winning by the "winner-take-all" rule, and the same is true regarding winning the nationwide popular vote. But in the modified election system, the "winner-take-all" rule encourages state contests in all the states, whereas, as mentioned in the Introduction, under the Electoral College mechanism, this rule makes many states "safe" for either major party candidate.

Since the Electoral College mechanism remains a back-up in the modified election system, a presidential candidate may eventually decide to structure her/his campaign to win a majority of all the electoral votes that are in play in the election rather than to win in a majority of the states, along with a majority of the nationwide popular vote. Thus, under both election systems, the problem of finding a mechanism for awarding state electoral votes forcing candidates to campaign

throughout the country remains, and the modified "winner-take-all" rule for awarding state electoral votes seems to address this problem to a considerable degree.

The proposed rule occupies an intermediate position between the "winner-take-all" rule and the proportional rule. To illustrate this statement, in a two-candidate race in a state, this rule works as the "winner-take-all" rule only if the following two conditions are met:

(a) a majority of the state votes is won by a candidate, and

(b) the increment of voters voting in favor of the candidate with this majority of votes compared to the previous year is larger than that of the opponent with whom the candidate competes for a portion of the electoral votes that is put in play between them.

In a multi-candidate race with two major party candidates, if no candidate wins a majority of state votes,

(1) the share of the state electoral votes to be received by a candidate with a plurality of state votes is proportional to the share of her/his votes in the statewide popular vote, and

(2) the share of the state electoral votes to be received by a candidate with the largest number of votes among the remaining candidates is proportional to her/his share of votes among all the remaining candidates (i.e., among all the candidates, except for the candidate with the plurality of the statewide popular vote).

If a candidate wins a majority of state votes but fails to outnumber his opponents by the increment of votes compared to the previous year, this candidate does not "take all," and the scheme of awarding state electoral votes remains proportional in the sense described by (1) and (2). The latter encourages voters to vote and the candidates to campaign, making every vote count.

The proposed rule has tangible advantages over the proportional rule of awarding state electoral votes in US presidential elections. Under the proportional rule, in any state in which there exists a long settled ratio between the voters favoring candidates from the two major parties, either major party candidate is unlikely to change this ratio substantially as a result of campaigning in the state. So winning additional one or two electoral votes – fewer than in any small state – is unlikely to encourage this candidate to campaign in the state other than in close elections, and if all the states employ this scheme. On the contrary, under the modified "winner-take-all" rule, the candidate may eventually compete for a substantial number of electoral votes in the state. As shown in one of the illustrative examples for the state of California, as many as 25 out of 55 electoral votes may become at stake for both major party candidates, no matter who amongst them receives a majority of the state votes.

Interestingly, under the modified "winner-take-all" rule in a three-candidate race, a non-major party candidate may eventually have a chance to win electoral votes even if she/he is substantially behind the major party candidates in the total number of the received votes. This may happen if either nobody wins a majority of state votes, or the majority winner fails to meet a condition similar to condition (b) in conformity to three-candidate races to "take all" the state electoral votes. In either case, the share of votes received by the non-major party candidate that affects her/his ability to win electoral votes increases since it is considered only among votes cast not in favor of the candidate who wins a majority or a plurality of state votes. Eventually, this share may allow the non-major party candidate to receive at least one electoral vote if she/he outnumbers the opponents in the increment of voters compared to the previous year.

By requiring the election winner to win state contests in at least 26 states (or in at least 25 states and DC), the modified election system encourages such contests. The introduction of the modified "winner-take-all" rule for awarding state electoral votes at least in current "safe" states encourages state contests in all these states as well.

It is important to notice that due to the years of gerrymandering, currently, an overwhelming majority of congressional districts in the US are not contestable, so the introduction of the Maine-like scheme in other states would not change the "battleground" character of the election campaigns [6]. "Battleground" districts would likely replace "battleground" states. Despite well-known theories of how to attain a fair districting [7,8], the chances of redistricting according to these theories are slim at least in today's America. So, currently, state contests seem to remain the only practically possible form of contests in presidential elections under any system employing the Electoral College mechanism in any form, and such contests have been held in the country for almost 200 years. Thus, any rules encouraging state contests seem to be a step towards improving the existing system of electing a President in the United States.

6. A game model for estimating the chances of a candidate to win a state contest in a US presidential election

Let us consider a state in which all the voters vote in favor of three candidates in a US presidential election—the Democratic party candidate, the Republican party candidate, and a non-major party candidate. Throughout the rest of this section, eligible voters who usually do not vote in presidential elections are called non-voters [9] though the word "usually" often requires additional clarification in the context.

Further, let

n be the number of counties in the state,

b_i be the number of all eligible non-voters in county i, $i \in \overline{1, n}$,

m be the number of categories of non-voters,

K^0 be the number of categories of voting voters,

x_{ij} be the number of non-voters of category j residing in county i who may come to vote on Election day as a result of the election campaign $j \in \overline{1, m_i}$, $i \in \overline{1, n}$,

p_{ij} be the probability that a non-voter of category j from county i will vote in favor of the Democratic party candidate as a result of the election campaign, $j \in \overline{1, m_i},\ i \in \overline{1, n}$,

q_{ij} be the probability that a non-voter of category j from county i will vote in favor of the Republican party candidate as a result of the election campaign, $j \in \overline{1, m_i},\ i \in \overline{1, n}$,

r_{ij} be the probability that a non-voter of category j from county i will either not vote at all or will vote in favor of the non-major party candidate as a result of the election campaign, $j \in \overline{1, m_i},\ i \in \overline{1, n}$,

y^D_{ik} be the number of voting voters of category k in county i, $i \in \overline{1, n},\ k \in \overline{1, K^0}$, including registered Democrats, who may vote in favor of the Democratic party candidate as a result of the election campaign,

y^R_{ik} be the number of voting voters of category k in county i, $i \in \overline{1, n},\ k \in \overline{1, K^0}$, including registered Republicans, who may vote in favor of the Republican party candidate as a result of the election campaign,

p^D_{ik} be the probability that a voter of category k from county i will vote in favor of the Democratic party candidate, $i \in \overline{1, n},\ k \in \overline{1, K^0}$ as a result of the election campaign,

p^R_{ik} be the probability that a voter of category k from county i will vote in favor of the Republican party candidate, $i \in \overline{1, n},\ k \in \overline{1, K^0}$ as a result of the election campaign,

r^0_{ik} be the probability that a voter of category k from county i will either not vote at all or will vote in favor of the non-major party candidate as a result of the election campaign, $i \in \overline{1, n},\ k \in \overline{1, K^0}$,

$\underline{a}^R_{ik}$ be the estimate of the number of voters about whom one can guarantee that they will vote in favor of the Republican party candidate, $i \in \overline{1, n},\ k \in \overline{1, K^0}$ as a result of the election campaign,

$\overline{a}^R_{ik}$ be the estimate of the maximal number of voters who may vote in favor of the Republican party candidate, $i \in \overline{1, n},\ k \in \overline{1, K^0}$ as a result of the election campaign,

$\underline{a}^D_{ik}$ be the estimate of the number of voters about whom one can guarantee that they will to vote in favor of the Democratic party candidate, $i \in \overline{1, n},\ k \in \overline{1, K^0}$ as a result of the election campaign,

$\overline{a}^D_{ik}$ be the estimate of the maximal number of voters who may vote in favor of the Democratic party candidate, $i \in \overline{1, n},\ k \in \overline{1, K^0}$ as a result of the election campaign.

Let the inequalities

$$\underline{a}^D_{ik} \le y^D_{ik} \le \overline{a}^D_{ik}, \qquad \underline{a}^R_{ik} \le y^R_{ik} \le \overline{a}^R_{ik}, \qquad \underline{\gamma} \le y^D_{ik} + y^R_{ik} \le \overline{\gamma},$$

hold, where $\underline{a}^D_{ik},\ \underline{a}^R_{ik},\ \overline{a}^D_{ik},\ \overline{a}^R_{ik}, \underline{\gamma},\ \overline{\gamma} \in R^1_+$.

In the reasoning to follow, the following assumptions hold on Election day:

(a) non-voters and voting voters within each category residing in a state decide to vote in favor of any candidate or not to vote at all independently of each other,

(b) non-voters and voting voters of each category residing in a state also decide to vote in favor of any candidate or not to vote at all independently of voters from the other categories.

Similarly to [9], one can be certain that the expectation of the number of votes for the Democratic party candidate equals

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} x_{ij}p_{ij} + \sum_{i=1}^{n}\sum_{k=1}^{K^0} y^D_{ik}p^D_{ik} + \sum_{i=1}^{n}\sum_{k=1}^{K^0} y^R_{ik}(1 - p^R_{ik} - r^0_{ik}) + \sum_{i=1}^{n}\sum_{k=1}^{K^0} \underline{a}^D_{ik} \tag{8}$$

whereas the expectation of the number of votes for the Republican party candidate equals

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} x_{ij}q_{ij} + \sum_{i=1}^{n}\sum_{k=1}^{K^0} y^R_{ik}p^R_{ik} + \sum_{i=1}^{n}\sum_{k=1}^{K^0} y^D_{ik}(1 - p^D_{ik} - r^0_{ik}) + \sum_{i=1}^{n}\sum_{k=1}^{K^0} \underline{a}^R_{ik}. \tag{9}$$

Let $x,\ p,\ q,\ y^R,\ y^D,\ p^D,\ p^R$ be vectors of corresponding dimensions formed by the numbers $x_{ij},\ p_{ij},\ q_{ij},\ y^R_{ik},\ y^D_{ik},\ p^D_{ik},\ p^R_{ik}$, respectively. Then (8) and (9) can be rewritten in the vector form

$$f(x, p, q, y^D, y^R, p^D, p^R) = \langle x, p\rangle + \langle y^D, p^D\rangle - \langle y^R, p^R\rangle + \langle y^R, \theta^0\rangle + \mu^D$$

and

$$g(x, p, q, y^D, y^R, p^D, p^R) = \langle x, q\rangle + \langle y^R, p^R\rangle - \langle y^D, p^D\rangle + \langle y^D, \theta^0\rangle + \mu^R,$$

where $\mu^D,\ \mu^R \in R^1_+$, and θ^0 is the vector formed by the numbers $1 - r^0_{ik},\ i \in \overline{1, m}, k \in \overline{1, K^0}$, whereas the vectors $(p, q) \in \Omega^{NV},\ (y^R, y^D) \in \Omega^Y,\ (p^R, p^D) \in \Omega^P$, and $\Omega^{NV}, \Omega^Y, \Omega^P$ are polyhedra.

To win all the state electoral votes, each major party candidate should try to maximize the expectation of the number of votes under the most unfavorable moves that can be undertaken by the candidate's opponent. The interaction of the

candidates can be described by a non-cooperative two-person game on a polyhedral set of connected strategies of the following kind:

$$\begin{aligned} &\min_{(x,y^D,y^R,p^R)} f(x,p,q,y^D,y^R,p^D,p^R) \to \max_{(p^D,p)} \\ &\min_{(x,y^D,y^R,p^D)} g(x,p,q,y^D,y^R,p^D,p^R) \to \max_{(p^R,q)}. \end{aligned} \tag{10}$$

Let us assume that game (10) has an equilibrium point in the sense proposed in [10]. Then if $(x^*, p^*, q^*, y^{D^*}, y^{R^*}, p^{D^*}, p^{R^*})$ is an equilibrium point of the game (10), the expectations of the numbers of votes in favor of each candidate become known, and both the expectation of the margin of voters in favor of each major party candidate and the expectations of the margins of the increment of new voters for each candidate can be easily calculated.

Game (10) is a particular case of the game on a polyhedral set of connected strategies $\tilde{x}$ and $\tilde{y}$

$$\begin{aligned} &\min_{\tilde{x}} \left(\langle p_1, \tilde{x}\rangle + \langle \tilde{x}, A_1\tilde{y}\rangle + \langle q_1, \tilde{y}\rangle\right) \to \max_{\tilde{y}} \\ &\min_{\tilde{y}} \left(\langle p_2, \tilde{x}\rangle + \langle \tilde{x}, A_2\tilde{y}\rangle + \langle q_2, \tilde{y}\rangle\right) \to \max_{\tilde{x}}, \end{aligned} \tag{11}$$

where $\tilde{x} \in M$, $\tilde{y} \in \Omega$, and $(\tilde{x}, \tilde{y}) \in T$, whereas p_1, q_1, p_2, q_2 are vectors, A_1, A_2 are matrices of corresponding dimensions, and M, Ω, T are polyhedra in spaces of corresponding (finite) dimensions. This game can be viewed as a generalization of a bi-matrix game though game (11) is the one on sets of connected strategies.

While necessary and sufficient conditions for games on polyhedral sets of connected strategies with the payoff function being a sum of two linear functions and a bilinear function were established in [10], no such conditions are known for game (11).

At the same time, finding the number

$$\min_{(p^R,q)} \max_{(p^D,p)} \min_{(x,y^D,y^R,p^R)} \left(f(x,p,q,y^D,y^R,p^D,p^R) - g(x,p,q,y^D,y^R,p^D,p^R)\right) \tag{12}$$

can be viewed as a problem formulated with the use of a generalization of the model presented in [9].

Both models (11) and (12) can be used to analyze the effectiveness of the proposed modified "winner-take-all" rule. Namely, since regularities describing how

(a) the probabilities to vote in favor of any candidate, and

(b) the numbers of voters about whom one can guarantee that they will vote on Election day

change over the years can be detected, one can eventually detect regularities describing the expectation of the margin of the votes in favor of a major party candidate, as well as the expectations of the margins of the increment of new voters for participating presidential candidates.

Though models (10) and (12) are mostly applicable in analyzing US presidential elections with only two major party candidates, such elections represent typical presidential elections in the US. At the same time, the analysis of even models (10) and (12) presents considerable difficulties, including those associated with establishing solvability conditions for the corresponding games and minmaxmin problems (necessary or (and) sufficient), as well as developing numerical methods for finding solutions to these problems.

7. Concluding remarks

1. One should clearly understand that the proposed modified "winner-take-all" rule is no more than a particular rule for awarding state electoral votes that should be studied and analyzed. At least currently, there is no mathematical proof or statistical grounds to assert that this rule will work better than any other rule for awarding state electoral votes in presidential elections even though common sense suggests that this is likely to be the case.

Moreover, some approaches to awarding portions of state electoral votes in situations considered in Sections 3.1–3.3 can be different. For instance, when one major party candidate receives a majority of the state popular vote, and $a - \nu$ electoral votes are to be put in play between the other major party candidate and a group of s non-major party candidates (see Section 3.1), one may suggest that $a - \nu - \eta_d$ electoral votes should be split among s non-major party candidates only, rather than being put in play among all the $s + 2$ candidates. This may seem especially reasonable if the inequalities

$$\sum_{i=1}^{k} (Ng_i - ng_i) + D - d > R - r > \max\left(\sum_{i=1}^{k} (Ng_i, -ng_i), D - d\right)$$

hold, i.e., the number $\sum_{i=1}^{k} (Ng_i - ng_i)$ is critical to let these $a - \nu$ electoral votes be put in play rather than being awarded to the major party candidate with a majority of the state popular vote. Similarly, one may suggest that $a - \nu - \lambda - \omega$ electoral votes (see Section 3.3) should be split among s non-major party candidates only rather than be put in play among $s + 1$ candidates.

Finding whether these suggestions can encourage (interested) voters to vote and the candidates to campaign more than suggestions considered in this article should be a subject of further research of the proposed modified "winner-take-all" rule.

2. All the illustrative examples have been chosen only to show how electoral votes can be distributed among the candidates in principle. In all the particular elections considered in these examples, the distribution of cast votes could have been different if voting voters had known that they vote under the modified "winner-take-all" rule rather than under the original "winner-take-all" rule.

3. The proposed modified "winner-take-all" rule implies that state voters vote for slates of electors of the candidates on the ballot in the state. If the modified election system were adopted, and voters voted directly for President, their votes for presidential candidates on the ballot in the state would also be considered as those for slates of electors of the candidates if the Electoral College mechanism were to decide the election outcome (as a back-up system) [1,4].

4. Besides proposing a particular rule for awarding state electoral votes in presidential elections, the aim of this article is to show that US presidential elections are both an excellent source for new mathematical problems and a field in which methods for solving these problems can be successfully applied.

5. The proposed rule for awarding state electoral votes, along with the modified election system proposed in [1,4] seem to "squeeze" the maximum from the use of the Electoral College mechanism for choosing a President. However, one should bear in mind that all the improvements of the Electoral-College-based election system make sense only as long as

(a) the Electoral College mechanism remains the key part of this system or is used at least as a back-up system, and

(b) the idea that, at least in presidential elections, an American voter prefers to choose only one candidate among available ones (currently, among slates of electors submitted by presidential candidates) prevails.

While other voting schemes have been developed and analyzed in the framework of both Arrow's system of axioms [11–13] and a different set of axioms [14], their introduction in US presidential elections seems problematic, since Americans have never had either a right or a chance to vote directly for President [5,15], even under the simplest voting rule—to choose one candidate only. In any case, it seems that discussion of political prospectives of these voting schemes goes beyond the scope of mathematical considerations, and this discussion can be found by the interested readers in numerous publications, in particular, on the Internet.

6. One should emphasize that while under the modified "winner-take-all" rule, a non-major party candidate does have a chance to win at least one electoral vote in a state, the total number of votes cast in favor of such a candidate must be substantial in order for this to happen. For instance, in a three-candidate race, in a state with 10 electoral votes in play in a particular presidential election, and with the leading candidate receiving 60% of all the cast votes, a non-major party candidate must receive more than 5% of the total number of votes to have a chance to win one of the 4 electoral votes, which may eventually be lost by the leading major party candidate. Thus, to have a large increment of votes alone is not sufficient to win an electoral vote under the proposed rule.

7. It seems important to emphasize one more time the difference between the proportional rule, the "winner-take-all" rule, and the proposed modified "winner-take-all" rule. Under a settled ratio of voters favoring presidential candidates from the two major parties in a state, the first rule does not encourage the candidates to compete in the state if the ratio is close to 1, and the second rule does not encourage the candidates to compete there if the ratio is not close to 1. On the contrary, the modified "winner-take-all" rule always encourages the candidate to compete by offering them a chance to either "take all" (if only two candidates have a chance to win state electoral votes) or to take more than their proportional share (if more than two candidates have a chance to win state electoral votes). In the latter case, if, say, three candidates are likely to receive 40%, 35%, and 25% of the cast votes, both the first and the second candidate may compete for a portion of votes that could have been awarded to the third candidate under the proportional rule. At the same time, the third candidate can "save" this portion of the electoral votes if the increment of voters favoring this candidate exceeds the maximum of the increments for the first two candidates. Moreover, in a three-candidate race in which no candidate wins a majority of the state popular vote, only electoral votes that could have been awarded to the candidate with the lowest percentage of the received votes are put in play among all the candidates. Thus, in such a race, candidates who received larger percentages of the votes than the lowest one cannot lose their electoral votes.

8. One can imagine that a state may decide to consider the percentage that the increment of voters represents in the total number of voters instead of considering the value of the increment as such. However, in this case, one cannot guarantee that each individual voter would matter in determining the fate of state electoral votes that are put in play among all the candidates.

Also, the increment of votes can be compared to an average number of voters who voted in favor of each party candidate in the last several elections.

9. The modified election system, proposed in [1,4], names the next President a presidential candidate who is the winner of the nationwide popular vote majority and the winner of state popular contests in at least 26 states or in at least 25 states and in DC. From the viewpoint of combining these two decisive factors – the nationwide popular vote and the number of states won in the state contests – in determining the election outcome, a particular manner of determining the winner of the nationwide popular contest and of the contests in the states does not matter. The idea to consider "one state, one vote" principle as a decisive factor in determining the election outcome, reflects the principles underlying the American Constitution in its current form. Should the American people decide that the winner of the nationwide popular vote plurality must be declared the election winner, no matter how small this plurality is, the Constitution may be changed accordingly.

(Though it is hard to imagine that such an idea can prevail despite the recent effort to introduce the nationwide presidential elections in the US by the so-called National Popular Vote plan [5].)

10. In all the illustrative examples, the data was taken from David Leip's Atlas of US presidential elections [16], which is widely available on the Internet. Also, throughout this article, it was assumed that all the states appoint the maximum number of electors that they are entitled to in the election.

11. The proposed rule of awarding state electoral votes can easily be extended to cover situations in which more than two or even all the candidates are considered as those from major parties (rather than as those from only two major parties and from non-major parties, which is currently the case in the US). In particular, such an extension would imply that in putting $a - \nu$ electoral votes in play among all or among a part of all the candidates, the reasoning presented in the remarks from Section 2 of the article should not apply. While this would not present any principle difficulties in describing corresponding schemes of awarding electoral votes in all the considered cases, it would require more cumbersome considerations compared to those presented in Sections 3.1–3.3.

Also, the proposed rule can be extended to cover situations in which the number of non-major party candidates in the previous and in the current elections are not the same.

However, one should bear in mind that such "generalizations" may present only a limited theoretical interest for the system of electing a US President, at least as long as the Electoral College mechanism remains a pillar for the existing two-party political system in the US.

12. In strong multi-candidate races, the employment of the modified "winner-take-all" rule under the existing election system may contribute to throwing a particular presidential election into the Congress, since more than two candidates are likely to receive electoral votes. Though such a distribution of the electoral votes that are in play in the election may better reflect the will of the electorate, advocates of the current system of electing a President may find it threatening to the political stability in the country [4]. However, the fairness of any election system depends much on how the will of the electorate is reflected by the election results rather than on how effectively the system can suppress or ignore the will of a particular part of the electorate.

While this article considers only a particular rule for awarding state electoral votes that encourages all the candidates to compete in a state employing this rule, the above-mentioned political concern should be taken into consideration in researching the pros and cons of the proposed modified "winner-take-all" rule. Also, a problem of reflecting the will of the people in electing a President in the House of Representatives should eventually be addressed.

Interestingly, an electoral tie – one of the possible scenarios even in two-candidate presidential elections – is usually considered almost a national disaster by both opponents and proponents of the Electoral College mechanism, since choosing a President by members of Congress is widely considered by them as an unfair process. However, the "one state, one vote" constitutional principle in electing a President in the House of Representatives, along with the "winner-take-all" rule offers a simple way to reflect the will of the people in this case. Namely, one of the two tied candidates (with (currently) 269 electoral votes each due to the above-mentioned assumption) inevitably wins in at least 26 states or in at least 25 states and in DC by popular vote. If the state delegations in the House of Representatives followed the will of the voters of their states, the election winner in a tied two-candidate election would emerge immediately. The same would be reasonable under the modified "winner-take-all" rule as well if the candidates tied in the received electoral votes were only two electoral vote recipients. Certainly, to introduce such a manner of determining the election winner in this case requires a constitutional amendment.

13. Some other ideas to better use the "winner-take-all" rule in awarding state electoral votes have recently surfaced. For instance, the idea to form a pool of Maryland, New Jersey, Georgia, and Indiana to collectively award their state electoral votes has recently been proposed [17]. The author of this idea observes that

(a) Maryland and New Jersey control the same number of electoral votes combined (25) that do Indiana and Georgia, and

(b) in the recent elections, in Maryland and New Jersey, approximately 56% of all the cast votes favored the Democrats, whereas in Indiana and Georgia, approximately the same 56% of the cast votes favored the Republicans.

Thus, the author believes that forming a pool of these four states would lead to a "superstate" with 50 electoral votes, which would be highly contested, since both candidates would have almost equal chances to "take all" these electoral votes.

Though this idea may seem reasonable at first glance, it has several logical flaws. First, the underlying belief that forming the pool will necessarily encourage the major party candidates to campaign in these currently "safe" states (Maryland and New Jersey for the Democrats, and Georgia and Indiana for the Republicans) looks doubtful. Intensively campaigning in Georgia and Indiana by the Democratic candidate may mobilize both the candidate's supporters and opponents, even if the Republican candidate does not campaign there. So a substantial reduction of the suggested 12% margin of votes favoring the Republican candidate there seems unlikely. The same is true for the Republican candidate in New Jersey and Maryland. Moreover, the voting-age population in Georgia and Indiana outnumbers that of Maryland and New Jersey according to both the 2000 census (by more than 250,000 people) and the forecasts (by almost 500,000 people). Thus, in a particular election, 56% of the Republican votes in Georgia and Indiana may substantially outnumber 56% of the Democratic votes in New Jersey and Maryland (though, in 2000 and 2004, voting voters in Maryland and New Jersey substantially outnumbered those in Georgia and Indiana). Second, in a multi-candidate race, a candidate may win all fifty electoral votes from the pool by, say, 34% plurality of the popular vote while losing in each state there. This may cause the appearance of "faithless" electors, for whom the will of the state voters may be superior to that of the state legislatures. Third, asking voters to vote for state electors (not for President!) while appointing these electors contrary to the voters' choice contradicts common sense and

may suppress the voter turnout in the states forming the pool. Fourth, while constitutionally, the state legislature can chose any manner of appointing state electors, the Supreme Court may find that appointing electors by state legislatures from several states collectively contradicts the Founding Father's underlying intent.

14. In considering the case of multi-party elections with two major party candidates in which the inequality

$$R = D > \max(Ng_1, Ng_2, \ldots, Ng_s)$$

holds, only the relations

$$Ng_{i^*} = Ng_{i^{**}} > \max_{i \in \overline{1,s} \setminus \{i^*, i^{**}\}} Ng_i,$$

were analyzed.

Though the presented reasoning can be easily extended to the case in which there are more than two non-major party candidates who receive the same number of votes in the election, one should bear in mind that this extension may have only limited value in US presidential elections at least under the currently existing election rules.

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