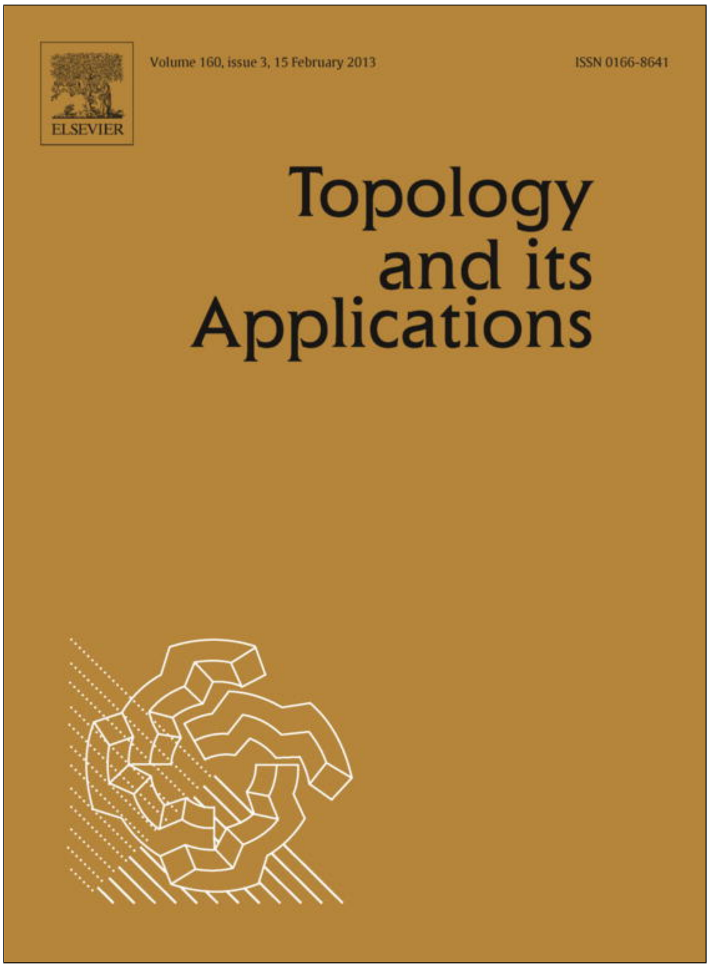
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Morse–Smale systems with few non-wandering points

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ABSTRACT

Let $MS^{\mathrm{flow}}(M^n,k)$ and $MS^{\mathrm{diff}}(M^n,k)$ be Morse–Smale flows and diffeomorphisms respectively the non-wandering set of those consists of k fixed points on a closed n-manifold M^n. For $k=3$, we show that the only values of n possible are $n\in\{2,4,8,16\}$, and M^2 is the projective plane. For $n\geqslant 4$, M^n is simply connected and orientable. We prove that the closure of any separatrix of $f^t\in MS^{\mathrm{flow}}(M^n,3)$ is a locally flat $\frac{n}{2}$-sphere while there is $f^t\in MS^{\mathrm{flow}}(M^n,4)$ such that the closure of separatrix of f^t is a wildly embedded codimension two sphere. This allows us to classify flows from $MS^{\mathrm{flow}}(M^4,3)$. For $n\geqslant 6$, one proves that the closure of any separatrix of $f\in MS^{\mathrm{diff}}(M^n,3)$ is a locally flat $\frac{n}{2}$-sphere while there is $f\in MS^{\mathrm{diff}}(M^4,3)$ such that the closure of any separatrix is a wildly embedded 2-sphere.

0. Introduction

In 1960, Steve Smale [31] introduced a class of dynamical systems (flows and diffeomorphisms) called later Morse–Smale systems. It was proved that Morse–Smale systems are structurally stable and have zero entropy [26,28,30]. In this sense, Morse–Smale systems are simplest structurally stable systems. One can define Morse–Smale systems as being those that are structurally stable and have non-wandering sets that consist of a finite number of orbits. There are deep connections between dynamics and the topological structure of support manifolds [8,18,24,31,32]. On a closed manifold, any Morse–Smale system has at least one attracting orbit and at least one repelling orbit. Thus, the simplest Morse–Smale system has the non-wandering set consisting of two points: a sink and source. In this case, the supporting n-manifold is an n-sphere S^n, and any orientation preserving Morse–Smale systems are conjugate i.e., have the same dynamics (of north–south type) [17,29]. It is natural to study Morse–Smale systems whose non-wandering sets consist of three fixed points. The existence of such systems follows from [15], where one proved the existence of closed manifolds admitting Morse functions with exactly three critical points, and [31] where one proved that any gradient flow can be approximated by Morse–Smale gradient flow.

Our first theorem (that is a generalization of [15]) describes closed manifolds admitting Morse–Smale systems with the non-wandering set consisting of three fixed points.

Theorem 1. *Suppose a closed n-manifold M^n admits a Morse–Smale system $\mathcal{U}$ (diffeomorphism or flow) such that the non-wandering set of $\mathcal{U}$ consists of three fixed points. Then*

- *the only values of n possible are $n\in\{2,4,8,16\}$;*
- *M^2 is the projective plane;*

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- *for $n \geqslant 4$, the homotopy groups $\pi_1(M^n) = \cdots = \pi_{\frac{n}{2}-1}(M^n) = 0$, and hence, M^n is simply connected and orientable*;
- *M^n is a compactification of $\mathbb{R}^n$ by an $\frac{n}{2}$-sphere.*

The first step of the proof is based on the following result.

Theorem 2. *Let $\mathcal{S}$ be a Morse–Smale system on a closed n-manifold M^n, such that the non-wandering set $NW(\mathcal{S})$ of $\mathcal{S}$ consists of three fixed points. Then $NW(\mathcal{S})$ consists of a sink ω, a source α, and a saddle s_0. In addition,* (a) *n is even, and every* (*stable and unstable*) *separatrix of s_0 is $\frac{n}{2}$-dimensional*; (b) *M^2 is the projective plane*; (c) *for $n \geqslant 4$, M^n is a simply connected and orientable manifold. Moreover, the topological closure of the unstable and stable separatrices $Sep^u(s_0)$, $Sep^s(s_0)$ are a topologically embedded $\frac{n}{2}$-spheres $W^u(s_0) \cup \{\omega\} = S_\omega$, $W^s(s_0) \cup \{\alpha\} = S_\alpha$ respectively.*

Later on, the proof of Theorem 1 connects with the possibility for the closure of separatrices to be wildly embedded. Such possibility for simple Morse–Smale systems was discovered by Pixton [25] who constructed the gradient-like Morse–Smale diffeomorphism $f : S^3 \to S^3$ with two sinks, a source, and saddle such that the closure of 2-dimensional separatrix of the saddle is a wildly embedded 2-sphere while the closure of the 1-dimensional separatrix forms a half of the wildly embedded Artin–Fox arc [6]. The similar examples were constructed in [7,10], where the classification of gradient-like Morse–Smale diffeomorphisms was considered. The effect of wildly embedding looks the most interesting when the number of fixed points is minimal, and there are no heteroclinic intersections. It follows from [25,9] that four is the minimal number of fixed points when the effect of wildly embedding holds for 3-dimensional Morse–Smale diffeomorphisms. One can easy prove that the closure of separatrix is locally flat for any 2-dimensional Morse–Smale systems (diffeomorphisms and flows) and 3-dimensional Morse–Smale flows.

The second step of the proof of Theorem 1 is based on the following theorem that is interesting itself.

Theorem 3. *Let $\mathcal{S}$ be a Morse–Smale system on a closed n-manifold M^n, $n \geqslant 6$, such that the non-wandering set $NW(\mathcal{S})$ of $\mathcal{S}$ consists of a sink ω, a source α, and a saddle s_0. Then the spheres $W^u(s_0) \cup \{\omega\} = S_\omega$, $W^s(s_0) \cup \{\alpha\} = S_\alpha$ are locally flat.*

The following theorem shows the difference between flows and diffeomorphisms for the dimension $n = 4$. Denote by $MS^{\mathrm{flow}}(M^n, k)$ the set of Morse–Smale flows the non-wandering set of those consists of k fixed points on a closed n-manifold M^n. Note that a flow $f^t \in MS^{\mathrm{flow}}(M^n, k)$ is gradient one [31]. Similarly, denote by $MS^{\mathrm{diff}}(M^n, k)$ the set of Morse–Smale diffeomorphisms the non-wandering set of those consists of k fixed points.

Theorem 4. (1) *Given any $f^t \in MS^{\mathrm{flow}}(M^4, 3)$, the spheres $W^u(s_0) \cup \{\omega\} = S_\omega$ and $W^s(s_0) \cup \{\alpha\} = S_\alpha$ are locally flat.* (2) *There is $f \in MS^{\mathrm{diff}}(M^4, 3)$ such that the spheres S_ω, S_α are wildly embedded.*

The last theorem allows to us to prove the following theorem concerning the topological equivalence.

Theorem 5. *Any flows $f^t \in MS^{\mathrm{flow}}(M_1^4, 3)$, $g^t \in MS^{\mathrm{flow}}(M_2^4, 3)$ are topologically equivalent. In particular, M_1^4 homeomorphic to M_2^4.*

Remark that Theorem 1 can be obtained from [15,32] for the flows $MS^{\mathrm{flow}}(M^n, 3)$. Indeed, one can prove that a flow $f^t \in MS^{\mathrm{flow}}(M^n, 3)$ is gradient-like. Due to [32], M^n can be endowed with a Riemannian structure such that f^t becomes a gradient flow i.e., f^t is defined by a Morse function. Now the result follows from [15]. However, Theorem 2 shows that there exist Morse–Smale diffeomorphisms $f \in MS^{\mathrm{diff}}(M^n, 3)$ that are not embedded in a flow.

We see that the closure of separatrices for Morse–Smale flows with three fixed points are always locally flat. For Morse–Smale flows with four fixed points, we prove that the closure of separatrices can be wildly embedded. To be precise, the following result holds.

Theorem 6. *Given any $n \geqslant 4$, there are a closed n-manifold M^n and gradient polar flow $f^t \in MS^{\mathrm{flow}}(M^n, 4)$ such that f^t has no heteroclinic intersections, and the closure of some separatrix of f^t is a codimension two wildly embedded sphere.*

The structure of the paper is the following. In Section 1, we formulate the main definitions, give some previous results, and describe the special neighborhood of saddle fixed point. In Section 2, we prove Theorems 1, 2, 3, the first item of Theorem 4, and Theorem 5. At last, in Section 3, we construct the corresponding examples that are essential parts of last item of Theorem 4, and Theorem 6.

1. Main definitions and previous results

Basic definitions of dynamical systems one can find in [4,30,33]. A dynamical system (diffeomorphism or flow) is Morse–Smale if it is structurally stable and the non-wandering set consists of a finitely many periodic orbits (in particular, each periodic orbit is hyperbolic and, stable and unstable manifolds of periodic orbits intersect transversally). Many definitions for Morse–Smale diffeomorphisms and flows are similar. So, we shall give mainly the notation for diffeomorphisms giving the exact notation for flows if necessary.

Let $f: M^n \to M^n$ be a Morse–Smale diffeomorphism of n-manifold M^n. A periodic (in particular, fixed) point σ is called a *saddle* periodic point (in short, *saddle*) if $1 \leqslant \dim W^u(\sigma) \leqslant n-1$, $1 \leqslant \dim W^s(\sigma) \leqslant n-1$ where $W^u(\sigma)$ and $W^s(\sigma)$ are unstable and stable manifolds of σ respectively. A component of $W^u(\sigma) \setminus \sigma$ denoted by $Sep^u(\sigma)$ is called an *unstable separatrix* of σ. If $\dim W^u(\sigma) \geqslant 2$, then $Sep^u(\sigma)$ is unique. The similar notation holds for a stable separatrix. Following [1], one says that the saddle σ *is of type* (μ, ν), if $\mu = \dim W^u(\sigma)$, $\nu = \dim W^s(\sigma)$. The number μ (ν) is called an *unstable* (*stable*) *Morse index*.

1.1. Special neighborhood

Here, we describe some constructions on the boundary of the special neighborhood for later use. Let $\mathbb{R}^n$ be Euclidean space endowed with coordinates $(x_1, \ldots, x_n)$, and a vector field $\vec{V}_s$ defined by the system

$$\dot{x}_1 = -x_1, \quad \ldots, \quad \dot{x}_k = -x_k, \quad \dot{x}_{k+1} = x_{k+1}, \quad \ldots, \quad \dot{x}_n = x_n. \tag{1}$$

We assume that $k \geqslant 2$, $n-k \geqslant 2$. The origin $O = (0, \ldots, 0)$ is a saddle of $\vec{V}_s$ whose k-dimensional stable separatrix $W^s(O)$ and $(n-k)$-dimensional unstable separatrix $W^u(O)$ are the following

$$W^s(O) = \{(x_1, \ldots, x_n) \mid x_{k+1} = 0, \ldots, x_n = 0\} = \mathbb{R}^k \subset \mathbb{R}^n,$$
$$W^u(O) = \{(x_1, \ldots, x_n) \mid x_1 = 0, \ldots, x_k = 0\} \stackrel{\text{def}}{=} \mathbb{R}^n_{k+1} \subset \mathbb{R}^n.$$

Lemma 1. *The function $F(x_1, \ldots, x_n) = \sum_{i=1}^k x_i^2 \sum_{j=k+1}^n x_j^2$ is integral one for the system* (1).

Proof. Taking in mind (1), one gets

$$\frac{dF}{dt} = \sum_{\nu=1}^n \frac{\partial F}{\partial x_\nu} \dot{x}_\nu = \sum_{i=1}^k (2x_i)\dot{x}_i \sum_{j=k+1}^n x_j^2 + \sum_{j=k+1}^n (2x_j)\dot{x}_j \sum_{i=1}^k x_i^2$$
$$= -2\sum_{i=1}^k x_i^2 \sum_{j=k+1}^n x_j^2 + 2\sum_{j=k+1}^n x_j^2 \sum_{i=1}^k x_i^2 = 2F(x_1, \ldots, x_n) - 2F(x_1, \ldots, x_n) \equiv 0. \qquad \square$$

By Lemma 1, $F = 1$ defines an $(n-1)$-manifold, denoted H^{n-1}, that divides $\mathbb{R}^n$ into the two open sets

$$\{\vec{x} = (x_1, \ldots, x_n) \mid F(\vec{x}) < 1\} \stackrel{\text{def}}{=} U_0, \qquad \{\vec{x} = (x_1, \ldots, x_n) \mid F(\vec{x}) > 1\} \stackrel{\text{def}}{=} U_\infty.$$

Clearly, U_0 is an invariant neighborhood of O, called *special*, Fig. 2.

Fix $k \geqslant 2$ and denote by T_r^{n-2} the set of points whose coordinates satisfy the equations

$$x_1^2 + \cdots + x_k^2 = r^2, \qquad r^2(x_{k+1}^2 + \cdots + x_n^2) = 1.$$

Then $T_r^{n-2} \subset H^{n-1}$ and T_r^{n-2} is naturally homeomorphic to the product of the spheres $S_{1,k}^{k-1}(r) \times S_{k+1,n}^{n-k-1}(\frac{1}{r})$ where

$$S_{1,k}^{k-1}(r) = \left\{(x_1, \ldots, x_k, 0, \ldots, 0) \,\middle|\, \sum_{i=1}^k x_i^2 = r^2\right\} \subset \mathbb{R}^k,$$
$$S_{k+1,n}^{n-k-1}\left(\frac{1}{r}\right) = \left\{(0, \ldots, 0, x_{k+1}, \ldots, x_n) \,\middle|\, \sum_{j=k+1}^n x_j^2 = \frac{1}{r^2}\right\} \subset \mathbb{R}^n_{k+1}.$$

The sphere $S_{1,k}^{k-1}(r)$ bounds the disk $D_{1,k}^k = \{(x_1, \ldots, x_k, 0, \ldots, 0) \mid \sum_{i=1}^k x_i^2 \leqslant r^2\} \subset \mathbb{R}^k$. For $r = 1$, denote $S_{1,k}^{k-1}(1)$ by $S_{1,k}^{k-1}$. Similarly, $S_{k+1,n}^{n-k-1}(1) = S_{k+1,n}^{n-k-1}$. One can check that every trajectory of $\vec{V}_s$ belonging to H^{n-1} intersects T_r^{n-2} at a unique point. Therefore, H^{n-1} is homeomorphic to $T_r^{n-2} \times \mathbb{R}$.

Let $H_c^{n-1}(0 \leqslant \tau \leqslant 1)$ be the union of trajectory arcs of $\vec{V}_s$ that start at $S_{1,k}^{k-1} \times S_{k+1,n}^{n-k-1}$ and finish at $S_{1,k}^{k-1}(\frac{1}{\sqrt{e}}) \times S_{k+1,n}^{n-k-1}(\sqrt{e})$. Another words,

$$H^{n-1}(0 \leqslant \tau \leqslant 1) = \bigcup_{0 \leqslant \tau \leqslant 1} f_\tau\left(S_{1,k}^{k-1} \times S_{k+1,n}^{n-k-1}\right).$$

Certainly, $H^{n-1}(0 \leqslant \tau \leqslant 1) \subset H^{n-1}$.

1.2. Flatness and wildness

For $1 \leqslant m \leqslant n$, we presume Euclidean space $\mathbb{R}^m$ to be included naturally in $\mathbb{R}^n$ as the subset whose final $(n-m)$ coordinates each equals 0. Let $e : M^m \to N^n$ be an embedding of closed m-manifold M^m in the interior of n-manifold N^n. One says that $e(M^m)$ is *locally flat at* $e(x)$, $x \in M^m$, if there exists a neighborhood $U(e(x)) = U$ and a homeomorphism $h : U \to \mathbb{R}^n$ such that $h(U \cap e(M^m)) = \mathbb{R}^m \subset \mathbb{R}^n$. Otherwise, $e(M^m)$ is *wild at* $e(x)$ [14]. The similar notation is used for a compact M^m, in particular $M^m = [0; 1]$.

For the reference, we formulate the following lemma proved in [19] (see also [17,20]).

Lemma 2. *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism, and $Sep^\tau(\sigma)$ a separatrix of dimension $1 \leqslant d \leqslant n-1$ of a saddle σ. Suppose that $Sep^\tau(\sigma)$ has no intersections with other separatrices. Then $Sep^\tau(\sigma)$ belongs to unstable (if $\tau = s$) or stable (if $\tau = u$) manifolds of some node periodic point, say N, and the topological closure of $Sep^\tau(\sigma)$ is a topologically embedded d-sphere that equals $W^\tau(\sigma) \cup \{N\}$.*

Note that a separatrix $Sep^\tau(\sigma)$ is a smooth manifold. Hence, $Sep^\tau(\sigma)$ is locally flat at every point [14]. However a priori, $clos\, Sep^\tau(\sigma) = W^\tau(\sigma) \cup \{N\}$ could be wild at the unique point N.

The crucial statement for the proof of Theorem 4 concerning flows, and Theorem 5 is the following statement.

Lemma 3. *Let M^4_* be a compact 4-manifold whose boundary consists of two 3-spheres S^3_1 and S^3_2, $\partial M^4_* = S^3_1 \cup S^3_2$. Suppose that there is a vector field $\vec{V}$ on M^4_* such that:*

1. *$\vec{V}$ has a unique fixed point s_* which is a hyperbolic saddle of type $(2,2)$;*
2. *$\vec{V}$ is transversal to ∂M^4_*, to be precise, with $\vec{V}|_{S^3_1}$ pointing into M^4_* and $\vec{V}|_{S^3_2}$ out of M^4_*;*
3. *Every trajectory of $\vec{V}$, except the trajectories belonging to the separatrices $W^s(s_*)$, $W^u(s_*)$ of the saddle s_*, intersects the both spheres S^3_1, S^3_2;*
4. *The stable separatrix $W^s(s_*)$ and unstable separatrix $W^u(s_*)$ intersects the spheres S^3_1, S^3_2 along the closed curves $W^s(s_*) \cap S^3_1 = C_1$, $W^u(s_*) \cap S^3_2 = C_2$ respectively.*

Then each of the curves C_1, C_2 is unknotted in S^3_1, S^3_2 respectively.

Proof. The curves C_1, C_2 bound in $W^s(s_*)$, $W^u(s_*)$ the closed disks D_1, D_2 respectively. Since S^3_1, S^3_2 are transversal to $\vec{V}$, s_* is inside of each D_1, D_2, and $s_* = D_1 \cap D_2$.

Suppose the contradiction, and assume that C_1 is knotted in S^3_1 (the case when C_2 is knotted in S^3_2 is similar). Due to the extended version of Grobman–Hartman theorem, there is a neighborhood U of $D_1 \cup D_2$ such that $\vec{V}|_U$ is locally equivalent to the vector field defined by the linear part of $\vec{V}$ at s_*. By Proposition 2.15 [27], $\vec{V}|_U$ is equivalent to $\vec{V}_s$ when $n = k = 2$.

By conditions, the exterior of C_1 in S^3_1 is homeomorphic to the exterior of C_2 in S^3_2. Hence, C_2 is knotted in S^3_2, [16]. Moreover, S^3_2 can be considered as a result of knot surgery of S^3_1 along the knot C_1. Because of this surgery is not trivial, S^3_2 is not homeomorphic to a 3-sphere [16,22]. The contradiction follows the statement. □

2. Proof of main results

Proof of Theorem 2. The time-one-shift along the trajectories of a flow $f^t \in MS^{\text{flow}}(M^n, 3)$ gives Morse–Smale diffeomorphism f from $MS^{\text{diff}}(M^n, 3)$. Therefore, it is sufficient to prove Theorem 2 for diffeomorphisms. By a connectedness of M^n, the non-wandering set of f consists of a sink ω, a source α, and a saddle σ.

Obviously, $n \geqslant 2$ because a Morse–Smale diffeomorphism of a circle has an even number of fixed points. First, we consider a 2-manifold M^2 admitting a Morse–Smale diffeomorphism with three fixed points: ω, α, and σ. The stable and unstable manifolds of σ are 1-dimensional. Since a Morse–Smale diffeomorphism has no heteroclinic points, the union of stable manifold $W^s(\sigma)$ and the source α forms the closed simple curve $C \subset M^2$. The set $M^2 \setminus C$ is the basin of ω, $M^2 \setminus C = W^s(\omega)$. By [33], $W^s(\omega)$ is a disk. Hence, M^2 is the projective plane.

Since $f \in MS^{\text{diff}}(M^n, 3)$ has only one saddle, f has no heteroclinic intersections. It follows from [9] that if a Morse–Smale diffeomorphism of a closed 3-manifold has no heteroclinic intersections, then the number of periodic points cannot be three and five. As a consequence, $\dim M^n = n \neq 3$.

Now, $n \geqslant 4$. Recall Morse inequalities for Morse–Smale diffeomorphisms. Let M_j be the number of periodic points $p \in Per(f)$ those stable Morse index equals $j = \dim W^s(p)$, and $\beta_i = rank\, H_i(M^n, \mathbb{Z})$ the Betti numbers. According to [31],

$$M_0 \geqslant \beta_0, \quad M_1 - M_0 \geqslant \beta_1 - \beta_0, \quad M_2 - M_1 + M_0 \geqslant \beta_2 - \beta_1 + \beta_0, \quad \ldots, \tag{2}$$

$$\sum_{i=0}^{n} (-1)^i M_i = \sum_{i=0}^{n} (-1)^i \beta_i. \tag{3}$$

Let $f \in MS^{\text{diff}}(M^n, 3)$, $n \geqslant 4$, and let us show that $2 \leqslant d = \dim Sep^{\tau}(\sigma) \leqslant n-2$. Suppose the contradiction. By Lemma 2, $W^s(\sigma) \cup \{\alpha\} \overset{\text{def}}{=} S^1_\alpha$, $W^u(\sigma) \cup \{\omega\} \overset{\text{def}}{=} S^{n-1}_\omega$ are topologically embedded circle and $(n-1)$-sphere respectively. Since $n \geqslant 4$, there is a neighborhood U_ω of S^{n-1}_ω homeomorphic to $S^{n-1}_\omega \times (-1; +1)$ [11,14]. Without loss of generality, one can assume that $f(U_\omega) \subset U_\omega$. The sphere S^{n-1}_ω does not divide M^n because of S^{n-1}_ω intersects S^1_α at a unique point σ. As a consequence, $M^n_1 = M^n \setminus U_\omega$ is a connected manifold with two boundary components each homeomorphic S^{n-1}_ω. Gluing n-balls to these components, one gets a closed manifold M^n_2. Since $f(U_\omega) \subset U_\omega$, one can extend f to M^n_2 such that f will have a source and two sinks. This is impossible.

By [17], the absence of 1-dimensional separatrices implies that a Morse–Smale diffeomorphism has unique source and unique sink, and M^n is orientable. We show above that $k \neq 1$ and $k \neq n-1$. As a consequence, $M_1 = M_{n-1} = 0$, and M^n is simply connected.

Suppose σ is of type $(n-k, k)$. Then $M_0 = M_n = M_k = 1$. For f^{-1}, one holds $M_0 = M_n = M_{n-k} = 1$. For $j \neq 0, n, k, n-k$, one holds $M_j = 0$. Since the left parts of (3) for f and f^{-1} are equal, $(-1)^k = (-1)^{n-k}$. Hence, $n = 2m$ is even, where $m \geqslant 2$.

Let us show that $k = m$. Suppose the contradiction. Assume for definiteness that $k > m$. It follows from (2) that $\beta_1 = \cdots = \beta_{n-k-1} = 0$ because of $M_1 = \cdots = M_{n-k-1} = 0$. The Poincare duality implies that $\beta_1 = \cdots = \beta_{k-1} = 0$. Hence, $\beta_i = 0$ for all $i = 1, \ldots, n-1$. Then (3) becomes $1 + (-1)^k + (-1)^n = 1 + (-1)^n$. This is impossible. $\square$

Proof of Theorem 3. It remains to prove that $S^k_\omega = W^u(s_0) \cup \{\omega\}$, $S^k_\alpha = W^s(s_0) \cup \{\alpha\}$ are locally flat provided $n \geqslant 6$. It follows from [13] (see [34,12]) that k-manifold has no isolated wild points provided $n \geqslant 5$, $k \neq n-2$. As a consequence, S^k_ω, S^k_α are locally flat k-spheres. This completes the proof of Theorem 3. $\square$

Proof of Theorem 1. Before, we proved that M^2 is the projective plane, and n is even. It follows from Theorem 2 that M^n is a simply connected and orientable manifold provided $n \geqslant 4$. Keeping the notation of Theorem 2, we see that M^n is the union of the basin $W^u(\omega)$ of the sink ω and the topological closure of the separatrix $Sep^s(s_0)$ that is a topologically embedded $\frac{n}{2}$-spheres. Since $W^u(\omega)$ homeomorphic to $\mathbb{R}^n$, M^n is a compactification of $\mathbb{R}^n$ by an $\frac{n}{2}$-sphere.

The existence of closed manifolds M^n admitting a Morse–Smale system (diffeomorphism or flow) the non-wandering set of whose consists of three fixed points follows from [15].

Now, let $n \geqslant 6$. By Theorem 3, the spheres $W^u(s_0) \cup \{\omega\} = S_\omega$, $W^s(s_0) \cup \{\alpha\} = S_\alpha$ are locally flat. Hence, the tubular neighborhood T of S_ω is a locally trivial bundle $\mathcal{B}$ over S_ω and the fiber $\frac{n}{2}$-ball $\mathbb{B}^{\frac{n}{2}}$ [21]. Since the non-wandering set consists of three fixed points, the boundary ∂T belongs to $W^u(\alpha)$ that homeomorphic to $\mathbb{R}^n$. Because of the $\frac{n}{2}$-sphere S_ω is simply connected, the bundle T is orientable. It follows that T admits a vector field that is transversal to ∂T and directed to the center of $\mathbb{B}^{\frac{n}{2}}$ at each fiber of T. This implies that one can define a flow φ^t on $W^u(\alpha)$ such that α is a source and any point of $W^u(\alpha) \setminus \{\alpha\}$ is a wandering point of φ^t. Obviously, there is an $(n-1)$-sphere $\Sigma^{n-1} \subset W^u(\alpha) \setminus \{\alpha\}$ that does not bound a ball in $W^u(\alpha) \setminus \{\alpha\}$. In [23], one considered connected components of wandering set of topological flows, and it was proved that if a connected component contains an embedded $(n-1)$-sphere that does not bound a ball, the connected component is homeomorphic to the prime product $\Sigma^{n-1} \times (0,1)$. As a consequence, ∂T homeomorphic to $(n-1)$-sphere S^{n-1}. Therefore, $\mathcal{B}$ induces the fiber bundle of S^{n-1} over an $\frac{n}{2}$-sphere with a fiber $(\frac{n}{2} - 1)$-sphere. It is well known that for $n \geqslant 6$, such bundles are Hopf bundles existing for $n - 1 = 7$ and $n - 1 = 15$ [2].

It follows from above results that the $\frac{n}{2}$-sphere S_ω is the retract of $M^n \setminus \{\alpha\}$. Hence, the homotopy groups

$$\pi_1(M^n) = \pi_1(M^n \setminus \{\alpha\}) = 0, \quad \ldots, \quad \pi_{\frac{n}{2}-1}(M^n) = \pi_{\frac{n}{2}-1}(M^n \setminus \{\alpha\}) = 0.$$

This completes the proof of Theorem 1. $\square$

The proof of the first item of Theorem 4. Here, we have to prove that given any $f^t \in MS^{\text{flow}}(M^4, 3)$, the spheres $W^u(s_0) \cup \{\omega\} = S_\omega$ and $W^s(s_0) \cup \{\alpha\} = S_\alpha$ are locally flat. There are neighborhoods U_α, U_ω of α and ω respectively homeomorphic to a 4-ball such that $\partial U_\alpha \cap W^s(\sigma)$ (resp., $\partial U_\omega \cap W^u(\sigma)$) is a simple closed curve, say C_α (resp., C_ω). By Lemma 3, C_α (resp., C_ω) is unknotted in ∂U_α (resp., ∂U_ω). This implies that S_ω and S_α are locally flat. $\square$

Proof of Theorem 5. Let ω_f, α_f, σ_f be the sink, source, and saddle of f^t respectively. There are neighborhoods $U(\omega_f)$, $U(\alpha_f)$ of α_f, σ_f respectively such that the boundaries $\partial U(\omega_f)$, $\partial U(\alpha_f)$ are transverse to f^t, and $\sigma_f \notin U(\omega_f) \cup U(\alpha_f)$. Without loss of generality, one can assume that the both $U(\omega_f)$ and $U(\alpha_f)$ homeomorphic to a 4-ball. The similar notation holds for g^t.

By Proposition 2.15 [27], there are neighborhoods $V(\sigma_f)$, $V(\sigma_g)$ of σ_f, σ_g respectively such that each flow $f^t|_{V(\sigma_f)}$, $g^t|_{V(\sigma_g)}$ is equivalent to $f^t_s|_{U_0}$, where U_0 is the special neighborhood. In particular, each intersection $V(\sigma_f) \cap \partial U(\alpha_f) = T_f$, $V(\sigma_g) \cap \partial U(\alpha_g) = T_g$ is a solid torus. We see that there is a homeomorphism $h : V(\sigma_f) \to V(\sigma_g)$ taking the trajectories of $f^t|_{V(\sigma_f)}$ to the trajectories of $g^t|_{V(\sigma_g)}$. Since T_f and T_g are transversal to the flows f^t and f^g respectively, h induces the homeomorphism $T_f \to T_g$ denoted again by h. Obviously, h takes $Sep^u(\sigma_f)$ to $Sep^u(\sigma_g)$. Therefore, h takes $Sep^u(\sigma_f) \cap T_f$ to $Sep^u(\sigma_g) \cap T_g$. According to the flow structure in the special neighborhood U_0, the both $Sep^u(\sigma_f) \cap T_f$ and $Sep^u(\sigma_g) \cap T_g$ are axes of solid tori T_f and T_g respectively.

By Lemma 3, the curves $Sep^u(\sigma_f) \cap T_f$ and $Sep^u(\sigma_g) \cap T_g$ are unknotted in the 3-spheres $\partial U(\alpha_f)$ and $\partial U(\alpha_g)$ respectively. Hence, the complements to T_f and T_g are solid tori, and h can be extended to a homeomorphism $\partial U(\alpha_f) \to \partial U(\alpha_g)$. It

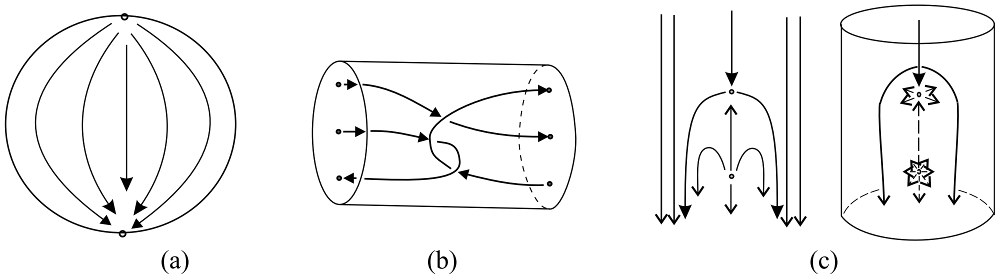


Fig. 1. Pixton–Bonatti–Grines constructions.

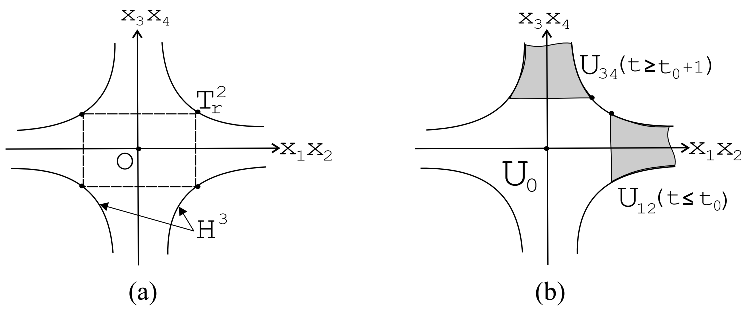


Fig. 2. Special neighborhood.

follows that there is a homeomorphism $h_*: M^4 \setminus (\alpha_f \cup \omega_f) \to M^4 \setminus (\alpha_g \cup \omega_g)$ taking the trajectories to the trajectories. Then h_* is easily extended to M^4 to get a homeomorphism taking the trajectories of f^t to the trajectories of g^t. $\square$

3. Examples

The proof of the second item of Theorem 4. Here we follow [7,25]. For the convenience of the Reader, we briefly review the constructions in [7,25].

Let f^t_{NS} be the north–south type flow on the 3-sphere S^3, Fig. 1(a). If S^3 thought of $\mathbb{R}^3$ completed by the infinity point, f^t_{NS} is defined by the system $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$, $\dot{x}_3 = x_3$. The origin $O = \alpha =$ north is a source, and the infinity point $\omega =$ south is a sink. Let $f_{NS} = f^1_{NS}$ be the shift-time $t = 1$ along the trajectories.

Take the Artin–Fox configuration consisting of three arcs, see Fig. 1(b). One can assume that the Artin–Fox curve l_{AF} is the union of shifts f^m_{NS}, $m \in \mathbb{Z}$, so that l_{AF} connects ω and α, and l_{AF} is invariant under f_{NS}. Well known that the Artin–Fox closed arc $l_{AF} \cup \{\alpha\} \cup \{\omega\}$ is wild at ω and α [6,14]. Let T be a tubular neighborhood of l_{AF} such that T is invariant under f_{NS}. Actually, T is an infinite cylinder that can be thought of the support of Cherry type flow g^t with a saddle, say σ, of type $(2,1)$ and an attracting node, Fig. 1(c). One can assume that the shift-time-one $g^1 = g$ on ∂T coincides with the restriction $f_{NS}|_{\partial T}$. The Pixton–Bonatti–Grines diffeomorphism $f: S^3 \to S^3$ equals f_{NS} on $S^3 \setminus T$ and equals g on T. It is easy to see that f is a gradient-like Morse–Smale diffeomorphism such that the closure of unstable separatrix of σ is a topologically embedded 2-sphere that is wild at ω.

Developing this idea we consider a 4-sphere S^4 being the result after the rotation $\mathcal{R}$ of 3-sphere S^3 such that $\mathcal{R}(\omega) = \omega$, $\mathcal{R}(\alpha) = \alpha$. Instead of T, one takes $\mathcal{R}(T)$, and instead of Cherry type flow g^t we take the flow on the special neighborhood U_0 with a unique saddle of type $(2,2)$.

Here, we keep the notation of Section 1 for $n = 4$, $k = 2$. Given a 2-torus T^2 that is the boundary of solid torus $P^3 = S^1 \times D^2$, $T^2 = \partial P^3 = S^1 \times \partial D^2$, any curve $\{\cdot\} \times \partial D^2$ is called a *meridian*, and $S^1 \times \{\cdot\}$ is a *parallel*. Recall that a 3-sphere S^3 can be obtained after a gluing of two copy of solid torus P^3, $S^3 = P^3 \cup_\nu P^3$, where the glue mapping $\nu: T^2 \to T^2$ takes a meridian to parallel and vise versa. This representation of S^3 is a standard Heegaard splitting of genus 1.

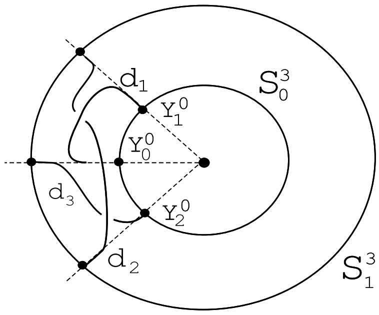
Given any $t \in \mathbb{R}$, we introduce 2-torus

$$\mathbb{T}^2_t = \left\{(x_1, x_2, x_3, x_4) \;\middle|\; x_1^2 + x_2^2 = \exp(-2t),\ x_3^2 + x_4^2 = \exp 2t\right\} \subset H^3$$

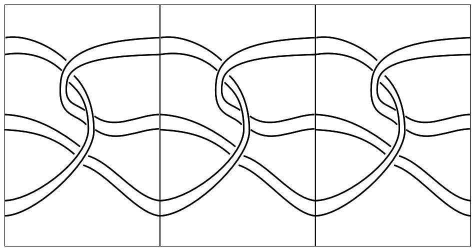
that is the boundary of the following solid tori

$$P^3_{12,t} = \left\{(x_1, x_2, x_3, x_4) \;\middle|\; x_1^2 + x_2^2 = \exp(-2t),\ x_3^2 + x_4^2 \leqslant \exp 2t\right\},$$
$$P^3_{34,t} = \left\{(x_1, x_2, x_3, x_4) \;\middle|\; x_1^2 + x_2^2 \leqslant \exp(-2t),\ x_3^2 + x_4^2 = \exp 2t\right\},$$

that form a standard Heegaard splitting $P^3_{12,t} \cup_{id} P^3_{34,t}$ of genus 1. The 3-sphere $P^3_{12,t} \cup_{id} P^3_{34,t}$ bounds the 4-ball, say B^4_0. Moreover, P^3_{12,t_0} and P^3_{34,t_0} divide U_0 into three domains B^4_0, $U_{12}(t \leqslant t_0)$, $U_{34}(t \geqslant t_0)$, Fig. 2, where



(a)



(b)

Fig. 3. Artin–Fox configuration.

$$U^4_{12}(t \leqslant t_0) = \{(x_1,x_2,x_3,x_4) \mid x_1^2+x_2^2 > \exp(-2t_0),\ x_3^2+x_4^2 < \exp 2t_0,\ (x_1^2+x_2^2)(x_3^2+x_4^2) < 1\},$$
$$U^4_{34}(t \geqslant t_0) = \{(x_1,x_2,x_3,x_4) \mid x_1^2+x_2^2 < \exp(-2t_0),\ x_3^2+x_4^2 > \exp 2t_0,\ (x_1^2+x_2^2)(x_3^2+x_4^2) < 1\}.$$

We see that a 2-torus $\mathbb{T}^2_{t_0}$ divides H^3 into two parts $\mathbb{T}^2_{t\leqslant t_0} = \bigcup_{t\leqslant t_0}\mathbb{T}^2_t$, $\mathbb{T}^2_{t\geqslant t_0} = \bigcup_{t\geqslant t_0}\mathbb{T}^2_t$ such that $\partial U_{12}(t\leqslant t_0) = \mathbb{T}^2_{t\leqslant t_0}$ and $\partial U_{34}(t \geqslant t_0) = \mathbb{T}^2_{t\geqslant t_0}$.

Let us introduce the coordinates (t,u,v) on $H^3 = \partial U_0$ as follows

$$x_1 = e^{-t}\cos 2\pi u, \qquad x_2 = e^{-t}\sin 2\pi u, \qquad x_3 = e^{t}\cos 2\pi v, \qquad x_4 = e^{t}\sin 2\pi v, \tag{4}$$

where $u, v \in [0;1)$ are cyclic coordinates on meridians or parallels on $\mathbb{T}^2_t$. Later on, (t,u,v) becomes (t_2,u_2,v_2).

Take the copy of $\mathbb{R}^4$ endowed with the coordinates (x_1,x_2,x_3,x_4), and the flow f^t_{NS} that is defined by (7) for $n=4$. The diffeomorphism

$$f_{NS} = f^1_{NS} : (x_1,x_2,x_3,x_4) \to (ex_1,ex_2,ex_3,ex_4)$$

is a shift-time-one along the trajectories of f^t_{NS}. Clearly, the family of spheres

$$S^3_m = \{(x_1,\dots,x_4):\ x_2^2+x_2^2+x_3^2+x_4^2 = e^{2m}\}, \qquad S^2_m = S^4_m \cap \{x_4 = 0\}$$

is invariant under f_{NS}.

Now we construct the special Artin–Fox curve as follows. On S^2_0, one takes the points $Y^0_0(\frac12;0;\frac{\sqrt3}{2};0)$, $Y^0_1(0;0;1;0)$, $Y^0_2(\frac{\sqrt3}{2};0;\frac12;0)$. Between the spheres S^3_0 and $f_{NS}(S^3_0) = S^3_1$, one takes arcs d_1, d_2, d_3 forming Artin–Fox configuration such that d_1 connects Y^0_0, Y^0_1, and d_2 connects $f_{NS}(Y^0_1)$, $f_{NS}(Y^0_2)$, and d_3 connects Y^0_2, $f_{NS}(Y^0_0)$, Fig. 3(a). One can assume that the union

$$l^\circ \stackrel{\text{def}}{=} \bigcup_{k\in\mathbb{Z}} f^k_{NS}(d_1\cup d_2\cup d_3)$$

is a simple arc connecting the origin $O = N$ and the infinity point S such that $l = \{S,N\}\cup l^\circ$ is Artin–Fox curve [6]. Here, we consider the 3-sphere S^3 the both as $\mathbb{R}^3$ completed by the infinity point S, and as the natural part of the 4-sphere S^4. Without loss of generality, we can suppose that l intersects transversally all spheres S^3_m, $m\in\mathbb{Z}$.

Let $\mathcal{R}$ be the rotation of the half-space $\mathbb{R}^3_+$ about 2-plane $x_4 = 0 = x_3$ that is defined as

$$\bar{x}_1 = x_1, \qquad \bar{x}_2 = x_2, \qquad \bar{x}_3 = x_3\cos 2\pi v - x_4\sin 2\pi v, \qquad \bar{x}_4 = x_3\sin 2\pi v + x_4\cos 2\pi v, \tag{5}$$

where $v\in[0,1]$. Since S and N are fixed under $\mathcal{R}$ and $l^\circ = l\setminus(\{S,N\})\subset\mathbb{R}^3_+$, $\mathcal{R}(l) = l_{\mathcal{R}}$ is a topologically embedded 2-sphere. It follows from [3,13] that $l_{\mathcal{R}}$ is wild at S and N.

One can introduce the smooth injective parametrization $\theta:\mathbb{R}\to l^\circ$ such that l intersects every S^3_m, $m\in\mathbb{Z}$, at three points $l^\circ(m)$, $l^\circ(m+\frac13)$, $l^\circ(m+\frac23)$ with parameters $t=m, m+\frac13, m+\frac23$ respectively. Since l is invariant under f_{NS}, there is a tubular neighborhood $T(l^\circ)$ of l° such that $T(l^\circ)$ is invariant under f_{NS}, and $T(l^\circ)$ is diffeomorphic to $\mathbb{R}\times D^2$, and $T(l^\circ)$ intersects every S^3_m, $m\in\mathbb{Z}$, at three disks

$$D_{m,0} = \{m\}\times D^2, \qquad D_{m+\frac13,1} = \left\{m+\frac13\right\}\times D^2, \qquad D_{m+\frac23,2} = \left\{m+\frac23\right\}\times D^2,$$

where $Y^0_i \in D_{\frac{i}{3},i}$, $i=1,2,3$.

Clearly, $\mathcal{R}(\mathbb{R}\times D^2)_{AF}$ is a neighborhood of $\mathcal{R}(l^\circ) = l^\circ_{\mathcal{R}}$ which is homeomorphic to $\mathbb{R}\times D^2\times S^1$ the boundary of those is $\mathbb{R}\times S^1\times S^1$. Therefore, this boundary is endowed with the coordinates (t,u,v) defined by (5). Here, we denote (t,u,v) by (t_1,u_1,v_1).

Put by definition, $\mathcal{I} = int(\mathbb{R} \times D^2)_{AF} \cup \{N, S\}$, $M_1 = S^4 \setminus \mathcal{I}$, and $M_2 = clos\, U_0$. We see that ∂M_1 is homeomorphic to $\mathbb{R} \times S^1 \times S^1 \simeq \partial M_2 = H^3$. Recall that H^3 endowed with the coordinates (t_2, u_2, v_2). The mapping $\Xi : \partial M_2 \to \partial M_1$ is defined as follows:

$$t_1 = t_2, \qquad u_1 = u_2 - v_2, \qquad v_1 = v_2. \tag{6}$$

According to [21], the set $M^4_* = M_1 \cup_\Xi M_2$ is a non-compact manifold. Clearly, the set $M'_1 = S^4 \setminus int(\mathbb{R} \times D^2)_{AF}$ is compact, and $M_1 = M'_1 \setminus \{N, S\}$.

The tori $\mathbb{T}^2_0$, $\mathbb{T}^2_1$ divide H^3 into three sets $\mathbb{T}^2_{t\geqslant 1}$, $\mathbb{T}^2_{0\leqslant t\leqslant 1}$, $\mathbb{T}^2_{t\leqslant 0}$ where $\mathbb{T}^2_{0\leqslant t\leqslant 1}$ is compact while the others are non-compact. Denote by $\Xi_{t\geqslant 1}$, $\Xi_{0\leqslant t\leqslant 1}$, $\Xi_{t\leqslant 0}$ the restriction of Ξ on $\mathbb{T}^2_{t\geqslant 1}$, $\mathbb{T}^2_{0\leqslant t\leqslant 1}$, $\mathbb{T}^2_{t\leqslant 0}$ respectively. Similarly, the circles $(\{0\} \times \partial D^2)_{AF}$, $(\{1\} \times \partial D^2)_{AF}$ divide the boundary of $(\mathbb{R} \times D^2)_{AF}$ into three cylinders $C_{0\leqslant t\leqslant 1} = ([0,1] \times S^1)_{AF}$, $C_{t\geqslant 1} = ([1,+\infty) \times S^1)_{AF}$, $C_{t\leqslant 0} = ((-\infty; 0] \times S^1)_{AF}$ where $C_{0\leqslant t\leqslant 1}$ is compact while the others are not. Denote $\mathcal{R}(C_{0\leqslant t\leqslant 1})$, $\mathcal{R}(C_{t\geqslant 1})$, $\mathcal{R}(C_{t\leqslant 0})$ by

$$C_{0\leqslant t\leqslant 1,\mathcal{R}} \simeq [0,1] \times S^1 \times S^1, \qquad C_{t\geqslant 1,\mathcal{R}} \simeq [1,+\infty) \times S^1 \times S^1, \qquad C_{t\leqslant 0,\mathcal{R}} \simeq (-\infty, 0] \times S^1 \times S^1$$

respectively.

Clearly, $\mathbb{R}(S^2_m) = S^3_m \subset \mathbb{R}^4 \setminus \{N, S\}$. Let $K(\geqslant m) = \{(x_1, x_2, x_3, x_4)\colon\ x_2^2 + x_2^2 + x_3^2 + x_4^2 \geqslant e^{2m}\}$ be the exterior of S^3_m, and $K(\leqslant m)$ the interior of S^3_m with the hole N. Denote by $K(m_1, m_2)$ the closed annulus between the spheres $S^3_{m_1}$, $S^3_{m_2}$. Because of (6), $M^4_* = M_1 \cup_\Xi M_2$ is the union of the following sets:

1) $U_{12}(t \leqslant 0) \cup_{\Xi_{t\leqslant 0}} [K(\leqslant 0) \setminus \mathcal{I}] \stackrel{\text{def}}{=} B_N$,
2) $U_{34}(t \geqslant 1) \cup_{\Xi_{t\geqslant 1}} [K(r \geqslant 1) \setminus \mathcal{I}] \stackrel{\text{def}}{=} B_S$,
3) $B^4(0 \leqslant t \leqslant 1) \cup_{\Xi_{0\leqslant t\leqslant 1}} [K(-1, 1) \setminus \mathcal{I}] \stackrel{\text{def}}{=} B_*$.

It follows from (6) that the set

$$S_{*,-m} \cup_{\Xi|_{t\leqslant 0}} \left(P^3_{12,-m} \cup P^3_{12,-m+\frac{1}{3}} \cup P^3_{12,-m+\frac{2}{3}}\right) \stackrel{\text{def}}{=} S^3_{m,*}$$

is a 3-sphere, since the lens $L(1,-1)$, $L(1,1)$ are the 3-sphere $S^3 = L(1,0)$. Note that if l deforms outside of some compact part to being rays, l becomes a locally flat arc. It follows that the set $K_N(-m, -m-1)$ between $S^3_{m,*}$, $S^3_{m+1,*}$ can be embedded in $\mathbb{R}^4$. This implies that $K_N(-m, -m-1)$ homeomorphic to the annulus $S^3 \times [0; 1]$, and hence B_N can be completed by a point to be a smooth manifold. Similarly, B_S. We see that $M^4 = M'_1 \cup_\Xi M_2$ admits a structure of closed smooth 4-manifold.

The shift-time-one diffeomorphisms $f_{NS} : M_1 \to M_1$, $f^1_s : M_2 \to M_2$ induce the diffeomorphism $f : M^4 \to M^4$ with two nodes, say α and ω, and a saddle. By construction, the spheres S_ω, S_α are wildly embedded. This completes the proof of Theorem 4. $\square$

Proof of Theorem 6. First, we introduce the special Morse–Smale flows $MS^{\text{flow}}_{k,n-k}(M^n, 4) \subset MS^{\text{flow}}(M^n, k)$, where $k \geqslant 2$, $n-k \geqslant 2$. In $\mathbb{R}^n$, consider the linear vector field $\vec{V}_n$ defined by the system

$$\dot{x}_1 = x_1, \qquad \dot{x}_2 = x_2, \qquad \dot{x}_{n-1} = x_{n-1}, \qquad \dot{x}_n = x_n. \tag{7}$$

Clearly, $O = (0, \dots, 0)$ is a repelling node, and $(n-1)$-sphere $S^{n-1}_j = \{\vec{x} = (x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 = j^2\}$ is transversal to $\vec{V}_n$ for any $j \in \mathbb{N}$. Let S^{k-1}_1 be a smoothly embedded in S^{n-1}_1 $(k-1)$-sphere. Denote by $T(S^{k-1}_1) \subset S^{n-1}_1$ a closed tubular neighborhood of S^{k-1}_1 diffeomorphic to $S^{k-1}_1 \times D^{n-k}$. Let Q^n be the union of rays starting at $O = (0, \dots, 0)$ through $T(S^{k-1}_1)$. Actually, each ray is the node O and a trajectory through $T(S^{k-1}_1)$. Since $\partial T(S^{k-1}_1)$ is diffeomorphic to $S^{k-1} \times S^{n-k-1}$, the boundary of the set $R \stackrel{\text{def}}{=} \mathbb{R}^n \setminus (O \cup int\, Q^n)$ is diffeomorphic to $S^{k-1} \times S^{n-k-1} \times \mathbb{R}$ where the last factor $\mathbb{R}$ corresponds to the time parameter of (7).

Recall that $\partial U_0 = S^{k-1} \times S^{n-k-1} \times \mathbb{R}$ where the last factor $\mathbb{R}$ corresponds to the time parameter of (1). Let $\eta : \partial U_0 \to \partial R$ be the natural identification. Then $(clos\, U_0) \cup_\eta R$ is a manifold. Because of η is a homotopy identity on the factor $S^{k-1} \times S^{n-k-1}$, one can extend the structure of smooth manifold to $O \cup (clos\, U_0) \cup_\eta R \stackrel{\text{def}}{=} R_n$ such that the set $(S^{n-1}_1 \setminus T(S^{k-1}_1)) \cup_\eta (S^{k-1}_{1,k} \times D^{n-k}_{k+1,n})$ is homeomorphic to S^{n-1}_1 that bounds the neighborhood of O in R_n homeomorphic to $\mathbb{R}^n$.

Let A be a closed annulus bounded by S^{n-1}_1, S^{n-1}_2 in $\mathbb{R}^n$, and $\mathbb{B}^n_2 \subset \mathbb{R}^n$ the closed n-ball bounded by S^{n-1}_2. By construction, η glue $H^{n-1}(0 \leqslant \tau \leqslant 1)$ with $\partial(A \setminus Q^n)$. Therefore, η glue $\partial(S^{n-1}_2 \setminus Q^n)$ with

$$\partial\left(D^k_{1,k}\left(\frac{r}{\sqrt{e}}\right) \times S^{n-k-1}_{k+1,n}\left(\frac{\sqrt{e}}{r}\right)\right) = S^{k-1}_{1,k}\left(\frac{r}{\sqrt{e}}\right) \times S^{n-k-1}_{k+1,n}\left(\frac{\sqrt{e}}{r}\right).$$

Put by definition,

$$D^n(\tau \leqslant 0) = \bigcup_{0\leqslant\tau\leqslant 1} f_\tau\left(D^k_{1,k}\left(\frac{r}{\sqrt{e}}\right) \times S^{n-k-1}_{k+1,n}\left(\frac{\sqrt{e}}{r}\right)\right), \qquad B_n = D^n(\tau \leqslant 0) \cup_\eta \partial\left(\mathbb{B}^n_2 \setminus Q^n\right).$$

The set B_n is a part of R_n with the piecewise smooth boundary

$$\partial B_n = \left(S^{n-1}_2 \setminus Q^n\right) \cup_\eta \left(D^k_{1,k}\left(\frac{r}{\sqrt{e}}\right) \times S^{n-k-1}_{k+1,n}\left(\frac{\sqrt{e}}{r}\right)\right).$$

The vector fields $\vec{V}_s$, $\vec{V}_n$ define the vector field $\vec{v}$ on $int\, B_n$. Smoothing the boundary of B_n and $\vec{v}$ to get a smooth vector field (denoted by $\vec{v}$ again) that is transversal to ∂B_n. By construction, $\vec{v}$ has the repelling node O and the saddle, say s_0, of the type $(n-k,k)$. Note that $S^{k-1}_{1,k} = W^s(s_0) \cap S^{n-1}_1 = S^{k-1}_{1,k}$, $S^{n-k-1}_{k+1,n} = W^u(s_0) \cap \partial B_n = S^{n-k-1}_{k+1,n}$. Take the copy B'_n of B_n with the vector field $-\vec{v}$. Clearly, $-\vec{v}$ has an attracting node, say O', and saddle, say s'_0, of the type $(k,n-k)$. The intersection of $W^s(s'_0)$ with $\partial B'_n$ is a sphere $S^{n-k-1,*}_{k+1,n}$. Without loss of generality, one can assume that $S^{n-k-1,*}_{k+1,n} \cap S^{n-k-1,*}_{k+1,n} = \emptyset$ because of $k \geqslant 2$.

Let $B_n \cup_\psi B'_n \stackrel{\text{def}}{=} M^n$ be the manifold obtained by the identification ψ of the boundaries of B_n, B'_n [21]. The fields $\vec{v}$, $-\vec{v}$ define on M^n the Morse–Smale vector field $\vec{V}$ that induces the Morse–Smale flow denoted by $f^t_{k,n-k}(S^{k-1}_{1,k})$. Obviously, $f^t_{k,n-k}(S^{k-1}_{1,k}) \in MS^t_{k,n-k}(M^n,4)$. For $k=n-2$ and $n \geqslant 4$, take $S^{k-1}_1 = S^{n-3}_1$ to be smoothly embedded and knotted codimension two sphere. Well known that such spheres exist [14]. According to [5,3,13], the spheres $W^s(s_0) \cup O$, $W^u(s'_0) \cup O'$ are wild at O and O' respectively. This completes the proof of Theorem 6. □

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