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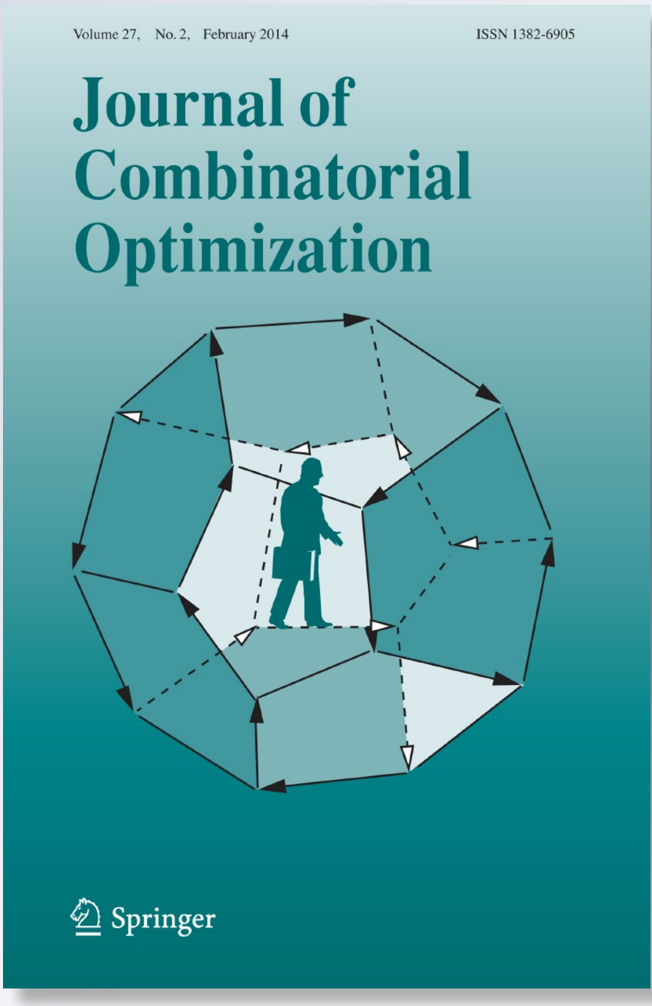

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Boundary graph classes for some maximum induced subgraph problems

Dmitriy S. Malyshev

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Abstract The notion of a boundary graph class was recently introduced for a classification of hereditary graph classes according to the complexity of a considered problem. Two concrete graph classes are known to be boundary for several graph problems. We formulate a criterion to determine whether these classes are boundary for a given graph problem or not. We also demonstrate that the classes are simultaneously boundary for some continuous set of graph problems and they are not simultaneously boundary for another set of the same cardinality. Both families of problems are constituted by variants of the maximum induced subgraph problem.

Keywords Computational complexity · Boundary graph class · Maximum induced subgraph problem

1 Introduction

All graphs considered in this paper are *simple*, i.e. undirected unlabeled graphs without loops and multiple edges. A *class of graphs* is a set of simple graphs. A class of graphs is called *hereditary*, if it is closed under deletions of vertices. In other words, if a hereditary class contains a graph G, then every induced subgraph of G must belong to the class. Many graph classes interesting from theoretical and practical viewpoints are hereditary. For instance, planar graphs, bipartite graphs, perfect graphs, line graphs, graphs of bounded vertex degree hold this property.

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It is known that a hereditary (and only hereditary) class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$, i.e. graphs that do not belong to $\mathcal{X}$. It is denoted by $\mathcal{X} = \mathcal{F}ree(\mathcal{Y})$. For any hereditary class of graphs $\mathcal{X}$ there exists a unique minimal under inclusion set of forbidden induced subgraphs. It is accepted to denote by $\mathcal{F}orb(\mathcal{X})$. The sets $\mathcal{Y}$ and $\mathcal{F}orb(\mathcal{X})$ might be different. For example, if $\mathcal{X}$ is the set of all bipartite graphs, then all graphs with a Hamiltonian odd cycle make up a set of its forbidden induced subgraphs, but the minimal set with this property coincides with the set of all odd chordless cycles (due to the theorem of Kõnig).

The set $\mathcal{F}orb(\mathcal{X})$ can be finite and infinite. If this set is finite, then $\mathcal{X}$ is called *finitely defined*. The classes of line graphs (Harary 1969) and bounded degree graphs are finitely defined. On the other hand, planar and perfect graphs make up classes with infinite sets of minimal forbidden induced subgraphs (Chudnovsky et al. 2006; Kuratowski 1930).

Many graph problems are known to be NP-complete in the set of all graphs and the same remains true under substantial restrictions of this graph class. At the same time, some "areas of tractability" are known, i.e. classes of graphs with polynomial-time solvability of a problem. Sometimes, the computational status of a considered problem can be open for some class. For instance, the maximum independent set problem is NP-complete for planar graphs (Garey et al. 1974), it is solvable in polynomial time for bipartite graphs and its complexity remains open for graphs without induced cordless path and cycle with five vertices (Mosca 2008). When does a difficult problem become easy? The notion of a boundary graph class helps us to answer this question within the family of hereditary graph classes. This notion was originally introduced by Alekseev for the maximum independent set problem (Alekseev 2004). It was applied for the minimum dominating set problem later (Alekseev et al. 2004). Study of boundary graph classes for some graph problems was extended in another papers (Alekseev et al. 2007; Alekseev and Malyshev 2008), where the notion was formulated in its most general form. Let us give the necessary definitions.

Let Π be an NP-complete graph problem. A hereditary graph class $\mathcal{X}$ is called *Π-easy*, if Π is polynomial-time solvable for graphs in $\mathcal{X}$, and *Π-tough*, otherwise. We assume that $P \neq NP$ and this condition is not explicitly included in formulations of statements. An example of such a formulation will be: if a problem Π is NP-complete for graphs in $\mathcal{X}$, then $\mathcal{X}$ is Π-tough. A class of graphs is said to be *Π-limit*, if this class is the limit of an infinite monotonously decreasing chain of Π-tough classes. In other words, $\mathcal{X}$ is Π-limit, if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_1 \supseteq \ldots$ of Π-tough classes, such that $\mathcal{X} = \bigcap_{k=1}^{\infty} \mathcal{X}_k$. A minimal under inclusion Π-limit class is called *Π-boundary*.

The following theorem certifies the significance of the boundary class notion (Alekseev 2004).

Theorem 1 *A finitely defined class is Π-tough if and only if it contains some Π-boundary class.*

The theorem shows that knowledge of all Π-boundary classes leads to a complete classification of finitely defined graph classes with respect to the complexity of Π. Unfortunately, for no one graph problem the structure of boundary classes is completely known. Only separate results on fragments of this sets have been obtained.

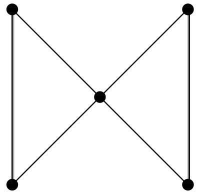

Fig. 1 The butterfly

For instance, Alekseev (2004) proved that some class of graphs is boundary for the independent set problem. Namely, he showed that the class of forests with at most 3 leaves in each connected component is boundary for the problem. We denote this class by $\mathcal{S}$. The class $\mathcal{S}$ is a boundary class for the dominating set problem and the line graphs of graphs in $\mathcal{S}$ constitute a boundary class for this problem (Alekseev et al. 2004). The set of such kind line graphs is denoted by $\mathcal{T}$. Let's notice that one more boundary class has been revealed for the problem in the same paper. The classes $\mathcal{S}$ and $\mathcal{T}$ are known to be simultaneously boundary for several graph problems (see also Alekseev and Malyshev 2008; Malyshev 2009b). On the other hand, there are problems, for which neither $\mathcal{S}$ nor $\mathcal{T}$ are boundary. For example, it is true for the Hamiltonian cycle problem (Korpeilainen et al. 2011) and for the vertex (edge) k-colorability problem under any choice of $k \geq 3$ (Malyshev 2009a, 2009c, 2009d).

Among graph classes, being boundary for at least one graph problem, only $\mathcal{S}$ and $\mathcal{T}$ are boundary for problems of diverse nature. Therefore, it would be informative to know more about common features of such problems. For a systematic study of the problems we formulate a criterion to determine whether $\mathcal{S}$ or $\mathcal{T}$ is boundary for a given graph problem or not. Based on this tool, we answer the interesting question about the "capacity" of graph problems with boundary classes $\mathcal{S}$ and $\mathcal{T}$. Namely, we demonstrate that the classes are simultaneously boundary for a continuous set of variants of the maximum induced subgraph problem. We also prove that these classes are not boundary for a set of variants of the mentioned problem with the same cardinality. We believe that the criterion will be helpful to answer another issues concerning $\mathcal{S}$ and $\mathcal{T}$ as boundary classes.

2 Notation

As usual, C_n, K_n and $K_{p,q}$ stand respectively for the cordless cycle with n vertices, the complete graph with n vertices and the complete bipartite graph with p vertices in the first part and q vertices in second. The graph $K_4 - e$ is a result of an edge deletion from K_4. The *butterfly* is a graph with vertices x_1, x_2, x_3, x_4, x_5 and edges (x_2, x_3), (x_4, x_5), (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5) (see Fig. 1).

The *complemental graph* of G (denoted by $\overline{G}$) is a graph on the same set of vertices and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The disjoint union of k copies of a graph G is denoted by kG.

Each connected component of every graph in $\mathcal{S}$ is a graph of the form $S_{i,j,k}$ ($i \geq 0, j \geq 0, k \geq 0$) and each connected component of every graph in $\mathcal{T}$ is a graph of the form $T_{i,j,k}$ ($i \geq 0, j \geq 0, k \geq 0$) (see Fig. 2).

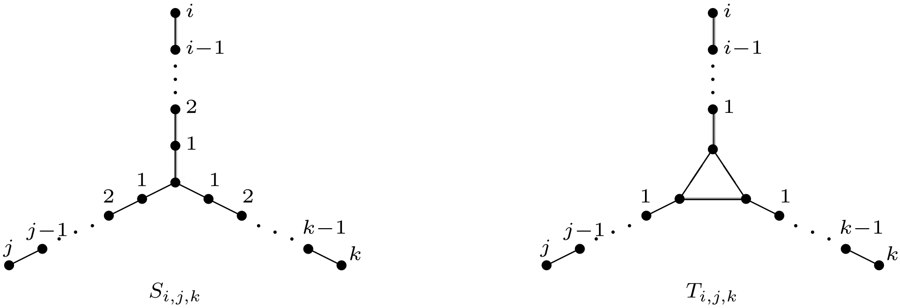

Fig. 2 The graphs $S_{i,j,k}$ and $T_{i,j,k}$

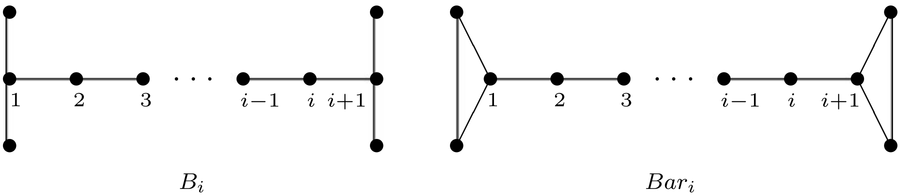

Fig. 3 The bed B_i and the barbell Bar_i

The *bed* B_i $(i \geq 1)$ is a graph with the set of vertices $\{x_1, x_2, y_1, y_2, z_1, z_2, \ldots, z_{i+1}\}$ and the set of edges $\{(x_1, z_1), (z_1, x_2), (z_1, z_2), (z_2, z_3), \ldots, (z_i, z_{i+1}), (y_1, z_{i+1}), (z_{i+1}, y_2)\}$. The *barbell* Bar_i $(i \geq 1)$ is the line graph of B_{i+1}. Both these graphs are drawn in Fig. 3.

The set $\mathcal{F}orb(\mathcal{S})$ consists of all cycles, all beds and the graph $K_{1,4}$. This follows directly from the facts that all graphs in *Free*$(\{K_{1,4}, C_3, C_4, \ldots, B_1, B_2, \ldots\})$ must be *subcubic* (i.e. have the vertex degrees not exceeding 3) forests and every tree in the class can not contain two vertices of the degree 3. From the Theorem 8.4 of the Harary's monograph it follows that all graphs in *Free*$(\{K_{1,3}, K_4 - e\})$ are line. Knowing this and the minimal forbidden subgraphs for $\mathcal{S}$, it is easy to show that the same set for $\mathcal{T}$ is constituted by all cycles of length at least 4, all barbells and the graphs $K_{1,3}, K_4, K_4 - e$, *butterfly*.

We denote the set of graphs without induced beds $B_1, B_2, \ldots, B_i$, induced cycles $C_3, C_4, \ldots, C_i$ and induced graphs $K_{1,4}, kS_{j,j,j}$ by $U(i, j, k)$ $(i \geq 3,\ j \geq 1,\ k \geq 1)$. In other words, $U(i, j, k) = \mathcal{F}ree(\{B_s \mid 1 \leq s \leq i\} \cup \{C_s \mid 3 \leq s \leq i\} \cup \{K_{1,4}, kS_{j,j,j}\})$. The set of graphs $\mathcal{F}ree(\{Bar_s \mid 1 \leq s \leq i-1\} \cup \{C_s \mid 4 \leq s \leq i\} \cup \{K_{1,3}, K_4, K_4 - e, butterfly, kT_{j,j,j}\})$ $(i \geq 4,\ j \geq 1,\ k \geq 1)$ is denoted by $\mathcal{W}(i, j, k)$. It is easy to see that $\mathcal{W}(i, j, k)$ consists of the line graphs of graphs in $\mathcal{U}(i, j+1, k)$. Observe that the inclusions $\mathcal{F}orb(\mathcal{U}(i, j, k)) \subseteq \mathcal{F}orb(\mathcal{S}) \cup \{kS_{j,j,j}\}$ and $\mathcal{F}orb(\mathcal{W}(i, j, k)) \subseteq \mathcal{F}orb(\mathcal{T}) \cup \{kT_{j,j,j}\}$ hold.

The classes of planar and subcubic graphs are denoted by $\mathcal{P}lanar$ and $\mathcal{D}eg(3)$ correspondingly. For a class of graphs $\mathcal{X}$ the formula $co(\mathcal{X})$ stands for the set $\{G \mid \overline{G} \in \mathcal{X}\}$.

All graph definitions and notions, not formulated in this paper, can be found in the books of Bondy and Murty (2008), Diestel (2010) and Harary (1969).

3 The criterion and some auxiliary results

The following statement has been proved by Alekseev and Malyshev (2008).

Theorem 2 *A Π-limit class $\mathcal{B}$ is Π-boundary if and only if for every $G \in \mathcal{B}$ there exists such a finite set of graphs $\mathcal{A} \subseteq \mathcal{F}orb(\mathcal{B})$ that $\mathcal{F}ree(\mathcal{A} \cup \{G\})$ is a Π-easy class.*

The Theorem 2 will be applied further to a proof of necessary and sufficient conditions that describe graph problems, for which $\mathcal{S}$ and $\mathcal{T}$ are boundary. The last theorem provides the following criteria for these classes.

Theorem 3 *The class $\mathcal{S}$ is Π-boundary if and only if it is Π-limit and for any j and k there is a number i, such that the class $\mathcal{U}(i, j, k)$ is Π-easy. The class $\mathcal{T}$ is Π-boundary if and only if it is Π-limit and for any j and k there is a number i, such that the class $\mathcal{W}(i, j, k)$ is Π-easy.*

Proof The proofs for $\mathcal{S}$ and $\mathcal{T}$ are similar. We will concentrate only on the proof for the class $\mathcal{S}$.

Let us prove the necessity. Assume, $\mathcal{S}$ is Π-boundary. Since for any j and k the graph $G = kS_{j,j,j}$ belongs to $\mathcal{S}$, then, by the Theorem 2, there is a finite set of graphs $\mathcal{A} \subseteq \mathcal{F}orb(\mathcal{S})$, such that $\mathcal{F}ree(\mathcal{A} \cup \{G\})$ is Π-easy. Let i be a maximal number among lengths of chordless cycles and beds, contained in $\mathcal{A}$ (if $\mathcal{A}$ does not contain such cycles and beds, then $i = 3$). Then, $\mathcal{U}(i, j, k) \subseteq \mathcal{F}ree(\mathcal{A} \cup \{kS_{j,j,j}\})$. Hence, the class $\mathcal{U}(i, j, k)$ will also be Π-easy.

Let us prove the sufficiency. Any graph $G \in \mathcal{S}$ is an induced subgraph of $kS_{j,j,j}$ for some k and j. Therefore, the inclusions $\mathcal{F}orb(\mathcal{U}(i, j, k)) \setminus \{kS_{j,j,j}\} \subseteq \mathcal{F}orb(\mathcal{S})$ and $\mathcal{F}ree(\mathcal{F}orb(\mathcal{U}(i, j, k)) \cup \{G\}) \subseteq \mathcal{U}(i, j, k)$ are true. From the last inclusion, finiteness of $\mathcal{F}orb(\mathcal{U}(i, j, k))$, Π-easiness of $\mathcal{U}(i, j, k)$ and the Theorem 2 it follows that $\mathcal{S}$ is Π-boundary. □

The Theorem 3 might be a helpful tool for proving or disproving whether the classes $\mathcal{S}$ and $\mathcal{T}$ are boundary or not. It permits to reveal two continuous sets of graph problems, such that both specified classes are boundary for every problem of the first set, but they are not boundary for any problem in the second one. We will define both families of such problems below. They are concrete representatives of the general maximum induced subgraph problem. The essence of its statement in the optimization form consists in finding an induced subgraph with some special properties and the largest possible number of vertices in a given graph. We will consider the problem only in the recognition form further. More formally, we are given a graph G and some graph property (class of graphs) $\mathcal{X}$. An induced subgraph of G is said to be *induced $\mathcal{X}$-subgraph*, if it belongs to $\mathcal{X}$. A maximal number of vertices in $\mathcal{X}$-subgraphs of G is denoted by $n_{\mathcal{X}}(G)$. An $\mathcal{X}$-subgraph of G with the largest value of $n_{\mathcal{X}}(G)$ is called *maximal*. The *maximum induced subgraph problem with respect to $\mathcal{X}$* or the *maximum induced $\mathcal{X}$-subgraph problem* (the ISUBGRAPH[$\mathcal{X}$], for short) is to verify for a given graph G and a natural k the validity of the inequality $n_{\mathcal{X}}(G) \leq k$. Many famous graph problems are maximum induced subgraph problems (under a corresponding choice of $\mathcal{X}$) or polynomially equivalent to them. For instance, it is true for the independent set problem ($\mathcal{X}$ is the set of all empty graphs) and for the maximum induced cycle problem ($\mathcal{X}$ is the set of all chordless cycles).

Recall that an edge in a graph is said to be a *bridge*, if its removal from the graph leads to increasing the number of connected components. An *addition of a bridge* implies insertion of an edge to a graph, connecting vertices from its different connected components. An *s-subdivision* of an edge consists in replacing this edge by a path of length $s+1$. An *s-contraction* of a path with $s+2$ vertices (all intermediate vertices of which have the degrees equal to two) in a graph is to replace the path by an edge. It is obvious that the s-subdivision operation is inverse to the s-contraction operation.

Let $\mathcal{X}$ be an arbitrary hereditary graph class, such that:

- The intersection of $\mathcal{X}$ with the set of subcubic graphs is constituted by all planar subcubic graphs. To put it in another way, $\mathcal{X} \cap \mathcal{D}eg(3) = \mathcal{D}eg(3) \cap \mathcal{P}lanar$.
- The inclusion $\mathcal{X} \subseteq \mathcal{F}ree(\{K_5\})$ holds.

Let us show that there is a continuous set of graph classes with the described property. To do this we consider the set $\mathcal{S}'$ of all homeomorphic to $K_{3,3}$ graphs, the set $\mathcal{S}''$ of all homeomorphic to K_5 graphs and some subset $\tilde{\mathbb{N}}$ of the set of all naturals $\mathbb{N}$. The result of actions of s-subdivisions ($s \in \tilde{\mathbb{N}}$) on edges of K_5 is denoted by $\mathcal{S}(\tilde{\mathbb{N}})$. In other words, a graph G belongs to $\mathcal{S}(\tilde{\mathbb{N}})$ if and only if there exist numbers $n_1, n_2, \ldots, n_k \in \tilde{\mathbb{N}}$ and edges $e_1, e_2, \ldots, e_k \in E(K_5)$, such that applying an n_1-subdivision to e_1, an n_2-subdivision to $e_2, \ldots$, an n_k-subdivision to e_k leads to a graph, isomorphic to G. Obviously, $\mathcal{S}(\tilde{\mathbb{N}}) \subseteq S''$ and $\mathcal{P}lanar = \mathcal{F}ree(S' \cup S'')$. As any graph in $\mathcal{S}(\tilde{\mathbb{N}})$ is not subcubic, then $\mathcal{F}ree(S' \cup \mathcal{S}(\tilde{\mathbb{N}}) \cup \{K_5\}) \cap \mathcal{D}eg(3) = \mathcal{D}eg(3) \cap \mathcal{P}lanar$. Therefore, the set $\{Free(S' \cup \mathcal{S}(\tilde{\mathbb{N}}) \cup \{K_5\})\colon \tilde{\mathbb{N}} \subseteq \mathbb{N}\}$ contains only different classes, satisfying the conditions formulated above. This set is continuous, because the power set of $\mathbb{N}$ is continuous.

The ISUBGRAPH[$\mathcal{X}$] is NP-complete, because the problems ISUBGRAPH[$\mathcal{P}lanar$] and ISUBGRAPH[$\mathcal{X}$] coincide for subcubic graphs and the ISUBGRAPH[$\mathcal{P}lanar$] is NP-complete for these graphs (Faria et al. 2001).

Lemma 1 *If a graph G' is obtained from $G \in \mathcal{D}eg(3)$ by means of an s-subdivision of some edge, then $n_{\mathcal{X}}(G') = n_{\mathcal{X}}(G) + s$.*

Proof The class of planar graphs is closed under additions of isolated vertices, additions of bridges, k-subdivisions and k-contractions for any $k \in \mathbb{N}$. Any induced $\mathcal{X}$-subgraph of G is planar subcubic. These observations will be used in the argumentation below.

Assume that G' is a result of a replacement of $e = (a, b)$ by a path P of length $s+1$ in the graph G. Let $H \in \mathcal{X}$ be an induced subgraph of G with a maximal number of vertices. Create an induced subgraph H' of the graph G' as follows. If $e \in E(H)$, then the whole path P is included in H'. If $e \notin E(H)$ and $a \in V(H)$ ($b \in V(H)$), then P is also included in H', except its end vertex b (a). Finally, if $a \notin V(H), b \notin V(H)$, then P' is included in H', except both end vertices. The graph H' is obtained from H either by an s-subdivision, or by additions of isolated vertices and bridges. Therefore, $H' \in \mathcal{X}$. It is obvious that $n_{\mathcal{X}}(G') \geq |V(H')| \geq |V(H)| + s = n_{\mathcal{X}}(G) + s$, i.e. $n_{\mathcal{X}}(G') \geq n_{\mathcal{X}}(G) + s$.

Let's prove the validity of the inverse inequality. Let H' be a maximal induced $\mathcal{X}$-subgraph of G'. Since the class $\mathcal{P}lanar$ is closed under additions of isolated vertices and bridges and H' has a maximal number of vertices, this subgraph contains either all vertices of P, or all, except one end vertex, or all, except both its end vertices. The path P is s-contracted in H' in the first case. All intermediate vertices of P are deleted from H' in the rest cases and if $a \in V(H'), b \in V(H')$, then any of these vertices is also deleted. Clearly that the obtained graph H belongs to $\mathcal{X}$. Hence, $n_{\mathcal{X}}(G) \geq |V(H)| \geq |V(H')| - s = n_{\mathcal{X}}(G') - s$, i.e. $n_{\mathcal{X}}(G) \geq n_{\mathcal{X}}(G') - s$. This finishes the proof of the Lemma 1. □

The Lemma 1 is a keynote tool at the proof of the following assertion.

Lemma 2 *The classes $\mathcal{S}$ and $\mathcal{T}$ are* ISUBGRAPH[$\mathcal{X}$]*-limit.*

Proof Let G be a graph in $\mathcal{D}eg(3)$. Perform a k-subdivision ($k \geq 4$) of every edge of G. The obtained graph G' does not contain the beds $B_1, B_2, \ldots, B_k$ and the cycles $C_3, C_4, \ldots, C_k$ as induced subgraphs. By the Lemma 1, $n_{\mathcal{X}}(G') = n_{\mathcal{X}}(G) + k$. This implies that for any $k \in \mathbb{N}, k > 3$ the ISUBGRAPH[$\mathcal{X}$] for graphs in $\mathcal{D}eg(3)$ is polynomially reduced to the same problem for graphs in $\mathcal{S}_{k-3} = \mathcal{D}eg(3) \cap \mathcal{F}ree(\{B_1, B_2, \ldots, B_k, C_3, C_4, \ldots, C_k\})$. As $\mathcal{D}eg(3)$ is a tough class for the maximum induced $\mathcal{X}$-subgraph problem, for any $k \geq 1$ the class $\mathcal{S}_k$ is also ISUBGRAPH[$\mathcal{X}$]-tough. Observe that $\mathcal{S}_1 \supseteq \mathcal{S}_2 \supseteq \ldots$ and $\bigcap_{k=1}^{\infty} \mathcal{S}_k = \mathcal{S}$. Therefore, $\{\mathcal{S}_k\}$ is an infinite monotonously decreasing sequence of ISUBGRAPH[$\mathcal{X}$]-tough classes, converging to $\mathcal{S}$. Hence, by the definition, $\mathcal{S}$ is a limit class for the ISUBGRAPH[$\mathcal{X}$].

Let $G \in \mathcal{S}_1$ and H be the line graph of G. Clearly that $H \in Deg(3)$ and $n_{\mathcal{X}}(H)$ is equal to a maximum number of edges in subgraphs of G, belonging to $\mathcal{X}$. Such subgraphs must be only planar. The edge analogue of the ISUBGRAPH[$\mathcal{P}lanar$] (i.e. the maximum planar subgraph problem) is NP-complete for subcubic graphs (Faria et al. 2006). Consequently, the problem ISUBGRAPH[$\mathcal{X}$] is NP-complete for the line graphs of graphs in $\mathcal{S}_1$. Let us denote the set of such line graphs by $\mathcal{T}_1$.

Now, let G be a graph in $\mathcal{T}_1$. It is easy to see that performance of a k-subdivison of every its non-triangular edge (i.e. an edge, not belonging to a triangle) has resulted in the line graph of some graph in S_k. This fact and the reasonings from the first paragraph follow that the considered problem is NP-complete for the line graphs of graphs in $\mathcal{S}_k$ (the set of such graphs is denoted by $\mathcal{T}_k$). So, the class $\mathcal{T}_k$ is ISUBGRAPH[$\mathcal{X}$]-tough, the inclusions $\mathcal{T}_1 \supseteq \mathcal{T}_2 \supseteq \ldots$ hold and the equality $\bigcap_{k=1}^{\infty} \mathcal{T}_k = \mathcal{T}$ is true. Therefore, $\mathcal{T}$ is a limit class for the ISUBGRAPH[$\mathcal{X}$]. □

The classes $co(\mathcal{S})$ and $co(\mathcal{T})$ are ISUBGRAPH[$co(\mathcal{X})$]-limit. This is a trivial corollary from the following general fact and the Lemma 2.

Lemma 3 *For any graph class $\mathcal{Y}$ a class of graphs $\mathcal{B}$ is* ISUBGRAPH[$\mathcal{Y}$]*-boundary if and only if $co(\mathcal{B})$ is* ISUBGRAPH[$co(\mathcal{Y})$]*-boundary.*

Proof For any graph G we have $n_{\mathcal{Y}}(G) = n_{co(\mathcal{Y})}(\overline{G})$. This implies that for any graph class $\mathcal{Z}$ the computational statuses of the ISUBGRAPH[$\mathcal{Y}$] for $\mathcal{Z}$

and the ISUBGRAPH[$co(\mathcal{Y})$] for $co(\mathcal{Z})$ are the same. It follows that an infinite monotonous decreasing chain of ISUBGRAPH[$\mathcal{Y}$]-tough classes, converging to $\mathcal{B}$, exists if and only if there is a chain with the same properties, containing only ISUBGRAPH[$co(\mathcal{Y})$]-tough classes and converging to $co(\mathcal{B})$. Thus, $\mathcal{B}$ is ISUBGRAPH[$\mathcal{Y}$]-boundary if and only if $co(\mathcal{B})$ is ISUBGRAPH[$co(\mathcal{Y})$]-boundary. □

4 Boundary graph classes for the maximum induced $\mathcal{X}$-subgraph and $co(\mathcal{X})$-subgraph problems

The classes $\mathcal{S}$ and $\mathcal{T}$ are known to be simultaneously boundary for many graph problems (Alekseev 2004; Alekseev et al. 2004, 2007; Alekseev and Malyshev 2008; Malyshev 2009b). Two following issues are rather interesting in the view of these circumstances. How many graph problems exist with boundary classes $\mathcal{S}$ and $\mathcal{T}$? Are there exist many problems, for which neither $\mathcal{S}$ nor $\mathcal{T}$ is boundary? In fact, the sets of graph problems with the described properties are continuous. This is proved in the two theorems below.

Theorem 4 *The classes $\mathcal{S}$ and $\mathcal{T}$ are* ISUBGRAPH[$\mathcal{X}$]*-boundary. The classes $co(\mathcal{S})$ and $co(\mathcal{T})$ are* ISUBGRAPH[$co(\mathcal{X})$]*-boundary.*

Proof It is enough to prove only the first part of the statement due to the Lemma 3. By the Lemma 2, both classes $\mathcal{S}$ and $\mathcal{T}$ are ISUBGRAPH[$\mathcal{X}$]-limit. Therefore, by the Theorem 3, it is enough to show that for any j and k there is a number i^*, such that the classes $\mathcal{U}(i^*, j, k)$ and $\mathcal{W}(i^*, j, k)$ are easy for the ISUBGRAPH[$\mathcal{X}$]. We will show that it is true for $i^* = 2j + 4$.

Clearly, any graph G in $\mathcal{U}(i^*, j, k)$ belongs to $\mathcal{D}eg(3)$. A vertex x of the degree 3 will be referred to as *internal* in the graph G, if there is no path, connecting x with a *pendent vertex* (i.e. a vertex of the degree 1), all intermediate vertices of which have the degrees equal to 2. The distance between any two its vertices of the degree 3 is at least $2j + 2$, otherwise G has either an induced cycle of length at most $2j + 4 \leq i^*$ or an induced bed of length at most $2j + 1 \leq i^*$. Any internal vertex x of G belongs to some its induced subgraph, isomorphic to $S_{j,j,j}$. To show this let us consider three shortest paths, connecting x with vertices of the degree 3 (some of such vertices might be coinciding), each of them contains only one neighbor of x. These paths have no common vertices, except x and, maybe, end vertices. Each path has at least $2j + 3 > j + 2$ vertices. Hence, x and some vertices of the paths induce a subgraph of G, isomorphic to $S_{j,j,j}$. Moreover, any two induced subgraphs $S_{j,j,j}$ of the graph have not common vertices and there are no edges, incident to vertices from different these subgraphs, otherwise the distance between some vertices of the degree 3 is at most $2j + 1$. Therefore, G cannot contain k internal vertices, because every such vertex belongs to an induced subgraph, isomorphic to $S_{j,j,j}$, and all these copies of $S_{j,j,j}$ induce $kS_{j,j,j}$.

We can consider that G has not pendent vertices, because any such vertex obligatory belongs to any maximal induced $\mathcal{X}$-subgraph of G (due to the closeness of

$\mathcal{X} \cap \mathcal{D}eg(3)$ with respect to additions of isolated vertices and bridges). If G' is a result of 1-contracting in G, $n_{\mathcal{X}}(G) = n_{\mathcal{X}}(G') + 1$ (by the Lemma 1). Therefore, the maximum induced $\mathcal{X}$-subgraph problem is polynomially reduced to those graphs without pendent vertices, that an 1-contraction can not be applied to them. The quantity of internal vertices in G' and G is the same, all vertices of G' are internal. Therefore, the ISUBGRAPH[$\mathcal{X}$] is solvable in polynomial time for the graphs, described above, because all of them have at most k vertices. This implies that the maximum induced $\mathcal{X}$-subgraph problem for graphs in $\mathcal{U}(i^*, j, k)$ is also solved in polynomial time. So, $\mathcal{S}$ is ISUBGRAPH[$\mathcal{X}$]-boundary.

The proof for the class $\mathcal{T}$ is almost completely analogous to the reasonings from the previous paragraphs. This a cause for omitting it. □

Theorem 5 *The classes $\mathcal{S}$ and $\mathcal{T}$ are not* ISUBGRAPH[$co(\mathcal{X})$]*-boundary. The classes $co(\mathcal{S})$ and $co(\mathcal{T})$ are not* ISUBGRAPH[$\mathcal{X}$]*-boundary.*

Proof By the Lemma 3, it is enough to prove only the first part of the statement. We will show that the classes $\mathcal{S}$ and $\mathcal{T}$ are not ISUBGRAPH[$co(\mathcal{X})$]-limit, hence, they can not be boundary for this problem. Assume the opposite. Then, at least one of such classes is boundary for the ISUBGRAPH[$co(\mathcal{X})$]. Let us consider the class $\mathcal{F}ree(\{K_4\})$. It contains the union of $\mathcal{S}$ and $\mathcal{T}$ and, by the Theorem 1, it is tough for the ISUBGRAPH[$co(\mathcal{X})$].

Since $\mathcal{X} \subseteq \mathcal{F}ree(\{K_5\})$, for any graph $G \in \mathcal{F}ree(\{\overline{K_4}\})$ the inequality $n_{\mathcal{X}}(G) \leq R(4,5) - 1$ holds, where $R(m,n)$ is the Ramsey number, i.e. a minimal quantity of vertices in a graph that contains either $\overline{K_m}$ or K_n as a subgraph. Moreover, $n_{\mathcal{X}}(G) \leq 24$, as $R(4,5) = 25$ (McKay and Radziszowski 1995). Therefore, the ISUBGRAPH[$\mathcal{X}$] can be solved for G by a generation of all its subgraphs with at most 24 vertices and by a verification whether such a subgraph belongs to $\mathcal{X}$. This procedure can be done in polynomial time. Therefore, the class $\mathcal{F}ree(\{\overline{K_4}\})$ is ISUBGRAPH[$\mathcal{X}$]-easy. We have a contradiction. Hence, the assumption was false. □

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