NUMBER OF COMPONENTS OF THE NULLCONE

VLADIMIR L. POPOV*

ABSTRACT. For every pair (G, V) where G is a connected simple linear algebraic group and V is a simple algebraic G-module with a free algebra of invariants, the number of irreducible components of the nullcone of unstable vectors in V is found.

1. We fix as the base field an algebraically closed field k of characteristic zero. Below the standard notation and terminology of the theory of algebraic groups and invariant theory [23] are used freely.

Consider a finite dimensional vector space V over the field k and a connected semisimple algebraic subgroup G of the group $\mathrm{GL}(V)$. Let $\pi_{G,V} \colon V \to V/\!\!/G$ be the categorical quotient for the action of G on V, i.e., $V/\!\!/G$ is the irreducible affine algebraic variety with the coordinate algebra $k[V]^G$ and the morphism $\pi_{G,V}$ is determined by the identity embedding $k[V]^G \hookrightarrow k[V]$. Denote by $\mathcal{N}_{G,V}$ the nullcone of the G-module V, i.e., the fiber $\pi_{G,V}^{-1}(\pi_{G,V}(0))$ of the morphism $\pi_{G,V}$. A point of the space V lies in $\mathcal{N}_{G,V}$ if and only if its G-orbit is nilpotent, i.e., contains in its closure the zero of the space V (see [23, 5.1]).

This article owes its origin to the following A. Joseph's question [8]: may it happen that the nullcone $\mathcal{N}_{G,V}$ is reducible if the group G is simple, its natural action on V is irreducible, and the algebra of invariants $k[V]^G$ is free?

Pairs (G, V) with a free algebra of invariants $k[V]^G$ have been studied intensively in the 70s of the last century (see [23], [17] and the literature cited there). Under the assumptions of simplicity of the group G and irreducibility of its action on V they are completely classified and constitute a remarkable class which admits a number of other important characterizations.

In Theorem 3 proved below we find the number of irreducible components of the nullcone $\mathcal{N}_{G,V}$ for every pair (G, V) from this class. As a corollary we obtain the affirmative answer to A. Joseph's question. The proof is based on the aforementioned classification and characterizations that are reproduced below in Theorems 1 and 2.

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2. Up to conjugacy in $\mathrm{GL}(V)$, the group G is uniquely determined as the image of a representation $\widetilde{G} \to \mathrm{GL}(V)$ of its universal covering group $\widetilde{G}$. The equivalence class on this representation, if it is irreducible, is uniquely determined by its highest weight λ (with respect to a fixed maximal torus and a Borel subgroup of the group $\widetilde{G}$ containing this torus). With this in mind, we shall write $G = (\mathsf{R}, \lambda)$, where R is the type of the root system of the group G. Note that $(\mathsf{R}, \lambda) = (\mathsf{R}, \lambda^*)$, where λ^* is the highest weight of the dual representation. We denote by $\varpi_1, \ldots, \varpi_r$ the fundamental weights of the group $\widetilde{G}$ numbered as in Bourbaki [4]. If $\mathsf{R} = \mathsf{A}_r, \mathsf{B}_r, \mathsf{C}_r, \mathsf{D}_r$, then we assume that, respectively, $r \geqslant 1, 3, 2, 4$.

The following theorem is proved in [10]:

Theorem 1. *All connected nontrivial simple algebraic subgroups G of the group $\mathrm{GL}(V)$ that act on V irreducibly and have a free algebra of invariants $k[V]^G$, are exhausted by the following list:*

(i) (*adjoint groups*):

$$(\mathsf{A}_r, \varpi_1 + \varpi_r); (\mathsf{B}_r, \varpi_2); (\mathsf{D}_r, \varpi_2); (\mathsf{C}_r, 2\varpi_1);$$
$$(\mathsf{E}_6, \varpi_2), (\mathsf{E}_7, \varpi_1); (\mathsf{E}_8, \varpi_8); (\mathsf{F}_4, \varpi_1); (\mathsf{G}_2, \varpi_2)$$

(ii) (*isotropy groups of symmetric spaces*):

$$(\mathsf{B}_r, \varpi_1); (\mathsf{D}_r, \varpi_1); (\mathsf{A}_3, \varpi_2); (\mathsf{A}_1, 2\varpi_1);$$
$$(\mathsf{B}_r, 2\varpi_1); (\mathsf{D}_r, 2\varpi_1); (\mathsf{A}_3, 2\varpi_2); (\mathsf{C}_2, 2\varpi_1); (\mathsf{A}_1, 4\varpi_1);$$
$$(\mathsf{C}_r, \varpi_2); (\mathsf{A}_7, \varpi_4); (\mathsf{B}_4, \varpi_4); (\mathsf{C}_4, \varpi_4); (\mathsf{D}_8, \varpi_8); (\mathsf{F}_4, \varpi_4);$$

(iii) (*groups G with $k[V]^G = k$*):

$$(\mathsf{A}_r, \varpi_1); (\mathsf{A}_r, \varpi_2), r \geqslant 4 \ \textit{even}; (\mathsf{C}_r, \varpi_1); (\mathsf{D}_5, \varpi_5);$$

(iv) (*groups G with* $\operatorname{tr\,deg} k[V]^G = 1$ *not included in* (i) *and* (ii)):

$$(\mathsf{A}_r, 2\varpi_1), r \geqslant 2; (\mathsf{A}_r, \varpi_2), r \geqslant 5 \ \textit{odd};$$
$$(\mathsf{A}_1, 3\varpi_1); (\mathsf{A}_5, \varpi_3); (\mathsf{A}_6, \varpi_3); (\mathsf{A}_7, \varpi_3);$$
$$(\mathsf{B}_3, \varpi_3); (\mathsf{B}_5, \varpi_5); (\mathsf{C}_3, \varpi_3); (\mathsf{D}_6, \varpi_6); (\mathsf{D}_7, \varpi_7);$$
$$(\mathsf{G}_2, \varpi_1); (\mathsf{E}_6, \varpi_1); (\mathsf{E}_7, \varpi_7);$$

(v) (*other groups*):

$$(\mathsf{A}_2, 3\varpi_1); (\mathsf{A}_8, \varpi_3); (\mathsf{B}_6, \varpi_6).$$

Remark 1. There are no repeated groups inside each of these five lists (i)–(v). The unique group included in two different lists (namely, in (i) and (ii)) is $(\mathsf{A}_1, 2\varpi_1)$. The groups G with $\operatorname{tr\,deg} k[V]^G = 1$ included in at least one of the lists (i), (ii) are (B_r, ϖ_1), (D_r, ϖ_1), (A_3, ϖ_2), (C_2, ϖ_2), $(\mathsf{A}_1, 2\varpi_1)$, (B_4, ϖ_4) and only these groups.

3. Recall from [23, 3.8, 8.8], [17, Chap. 5, § 1, 11], [18] that an algebraic subvariety S in V is called *a Chevalley section with the Weyl group* $W(S) := N(S)/Z(S)$, where $N(S) := \{g \in G \mid g \cdot S = S\}$ and $Z(S) := \{g \in G \mid g \cdot s = s \ \forall s \in S\}$, if the homomorphism of k-algebras $k[V]^G \to k[S]^{W(S)}$, $f \mapsto f|_S$, is an isomorphism. A linear subvariety in V that is a Chevalley section with trivial Weyl group (i.e., a linear subvariety intersecting every fiber of the morphism $\pi_{G,V}$ at a single point) is called *a Weierstrass section.* A linear subspace in V that is a Chevalley section with a finite Weyl group is called *a Cartan subspace.*

Recall also (see [23, Thm. 3.3 and Cor. 4 of Thm. 2.3]) that semisimplicity of the group G implies the equality

$$m_{G,V} := \max_{v \in V} \dim G \cdot v = \dim V - \dim V /\!\!/ G. \tag{1}$$

Consider the following properties:

- (FA) $k[V]^G$ is a free k-algebra;
- (FM) $k[V]$ is a free $k[V]^G$-module;
- (ED) all fibers of the morphism $\pi_{G,V}$ have the same dimension;
- (ED_0) $\dim \mathcal{N}_{G,V} = m_{G,V}$ (see (1));
- (FO) every fiber of the morphism $\pi_{G,V}$ contains only finitely many G-orbits;
- (FO_0) $\mathcal{N}_{G,V}$ contains only finitely many G-orbits;
- (NS) G-stabilizers of points in general position in V are nontrivial;
- (CS) there is a Cartan subspace in V;
- (WS) there is a Weierstrass section in V.

The following implications between them hold true:

$$\text{(FM)} \Leftrightarrow \text{(FA)\&(ED)} \quad \text{(see [17, p. 127, Thm. 1]);}$$

$$(\mathrm{ED}_0) \Leftrightarrow \text{(ED)} \Leftarrow (\mathrm{FO}_0) \quad \text{(see [17, p. 128, Thm. 3, Cor.]);}$$

$$(\mathrm{FO}_0) \Leftrightarrow \text{(FO)} \quad \text{(see [23, Cor. 3 of Prop. 5.1]);}$$

$$\text{(CS)} \Rightarrow \text{(FM)} \Leftarrow \text{(WS)} \quad \text{(see [17, p. 133, Thm. 7]).}$$

Theorem 2. *For the connected simple algebraic subgroups G in* $\mathrm{GL}(V)$, *acting on V irreducibly, all nine properties* (FA), (FM), (ED), (ED_0), (FO), (FO_0), (NS), (CS), *and* (WS) *are equivalent*[1].

[1]In [13, p. 207, Thm.], the property (NS) is replaced by the property that the G-stabilizer of *every* point of V is nontrivial. It is a mistake: for instance, the SL_2-module of binary forms in x and y of degree 3 has the property (FA), but the SL_2-stabilizer of the form x^2y is trivial.

Proof. The complete list of the groups G having the property (FA) is obtained in [10]; the one having the property (ED) is obtained in [16], [17, p. 141, Thm. 8] and, in the same papers, that having the property (FM); the one having the property (FO) is obtained in [11]. The results of papers [3], [2], [14], [15] yield the complete list of the groups G having the property (NS). Matching the obtained lists proves the equivalence of the properties (FA), (FM), (ED), (FO), and (NS) (see [23, Thm. 8.8] and [17, p. 127, Thm. 1]). It is proved in [17, p. 142, Thm. 9] that each of the properties (CS) and (WS) is equivalent to the property (ED). $\square$

Remark 2. The conditions of simplicity of the group G and irreducibility of its action on V in Theorem 2 are essential, see [18].

4. Now we turn to finding the number of irreducible components of the nullcone $\mathcal{N}_{G,V}$.

Lemma 1. *If* $\dim V/\!\!/G \leqslant 1$, *then the nullcone* $\mathcal{N}_{G,V}$ *is irreducible. If* $\dim V/\!\!/G = 0$, *then it contains an open dense* G*-orbit.*

Proof. The equality $\dim V/\!\!/G = 0$ means that $\dim V/\!\!/G$ is a single point. By the definition of the nullcone, the latter condition is equivalent to the equality $\mathcal{N}_{G,V} = V$. In particular, in this case the nullcone $\mathcal{N}_{G,V}$ is irreducible. On the other hand, in view of (1), the equality $\dim V/\!\!/G = 0$ is equivalent to that V contains a G-orbit of dimension $\dim V$, i.e., an open and dense orbit.

In view of smoothness of V, the algebraic variety $V/\!\!/G$ is normal (see. [23, Thm. 3.16]). Let dim $V/\!\!/G = 1$. It follows from rationality of the algebraic variety V, dominance of the morphism $\pi_{G,V}$, and Lüroth's theorem that the curve $V/\!\!/G$ is rational. Being normal, it is smooth. Hence $V/\!\!/G$ is isomorphic to an open subset of the affine line. Since every invertible element of the algebra $k[V]$ is a constant, the algebra $k[V]^G$ has the same property. Hence the curve $V/\!\!/G$ is isomorphic to the affine line, and therefore, there is a polynomial $f \in k[V]^G$ such that $f(0) = 0$ and $k[V]^G = k[f]$. Since the group G is connected and has no nontrivial characters, the polynomial f is irreducible (see [23, Thm. 3.17]). Since $\mathcal{N}_{G,V} = \{v \in V \mid f(v) = 0\}$, this implies irreducibility of the nullcone $\mathcal{N}_{G,V}$. $\square$

Theorem 3. *The nullcone* $\mathcal{N}_{G,V}$ *of the connected nontrivial simple algebraic group* $G \subseteq \mathrm{GL}(V)$ *acting irreducibly on* V *and having the equivalent properties listed in Theorem* 2 *is reducible if and only if* G *is contained in the following list:*

$$(\mathsf{D}_r, 2\varpi_1), (\mathsf{A}_3, 2\varpi_2), (\mathsf{A}_7, \varpi_4). \tag{2}$$

For every group G from list (2)*, the number of irreducible components of the nullcone $\mathcal{N}_{G,V}$ is equal to* 2*.*

Proof. From Theorem 2 we obtain the following interpretation of the number of irreducible components of the nullcone $\mathcal{N}_{G,V}$. Using (1) and the fiber dimension theorem (see [7, Chap. II, §3]), we infer that dimension of every irreducible component of the nullcone $\mathcal{N}_{G,V}$ is at least $m_{G,V}$. This and the property (ED_0) imply that dimension of every irreducible component of the nullcone $\mathcal{N}_{G,V}$ is equal to $m_{G,V}$. But in view of the property (FO_0) every irreducible component of the nullcone $\mathcal{N}_{G,V}$ is the closure of some G-orbit. Hence the number of irreducible components of the nullcone $\mathcal{N}_{G,V}$ is equal to the number of $m_{G,V}$-dimensional nilpotent G-orbits in V.

Now we shall use Theorem 1 and find, for every group G listed in it, the number of irreducible components of the nullcone $\mathcal{N}_{G,V}$.

1. If the group G is adjoint, then according to [11, Cor. 5.5], the nullcone $\mathcal{N}_{G,V}$ is irreducible. This conclusion covers all the groups G from list (i) of Theorem 1.

2. In view of Lemma 1, the nullcone $\mathcal{N}_{G,V}$ is irreducible for all the groups G from lists (iii) and (iv) of Theorem 1 and also for the groups with $\mathrm{trdeg}_k k[V]^G = 1$ mentioned in Remark 1.

3. Consider all the groups G from list (v) of Theorem 1.

(3a) The orbits of the group $(\mathsf{A}_2, 3\varpi_1)$ are the orbits of the natural action of the group SL_3 on the space of cubic forms in three variables. According to [23, 5.4, Example 2°], the Hilbert–Mumford criterion implies the existence of a linear subspace L in V such that $\mathcal{N}_{G,V} = G \cdot L$. Hence the nullcone $\mathcal{N}_{G,V}$ is irreducible.

(3b) The orbits of the group (A_8, ϖ_3) are the orbits of the natural action of the group SL_9 on the space of 3-vectors $\bigwedge^3 k^9$. The classification of them is obtained in [24]; it shows (see [24, Table 6, $\dim S = 0$]) that in this case there is a unique nilpotent orbit of dimension $m_{G,V} = 80$. Hence the nullcone $\mathcal{N}_{G,V}$ is irreducible.

(3c) The orbits of the group (B_6, ϖ_6) are the orbits of the natural action of the group Spin_{13} on the space of spinor representation. The classification of them is obtained in [6]; it shows (see [6, Thm. 1(3)]) that in this case there is a unique nilpotent orbit of dimension $m_{G,V} = 62$, and hence the nullcone $\mathcal{N}_{G,V}$ is irreducible.

4. Let us now consider all the groups G from the remaining list (ii) of Theorem 1. By virtue of the Lefschetz principle, we may (and shall) assume that $k = \mathbb{C}$. All these groups are obtained by means of the following general construction.

Consider a semisimple complex Lie algebra $\mathfrak{h}$, it adjoint group $\operatorname{Ad}\mathfrak{h}$, and an involution $\theta \in \operatorname{Aut}\mathfrak{h}$. The decomposition

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{where } \ \mathfrak{k} := \{x \in \mathfrak{h} \mid \theta(x) = x\},\ \mathfrak{p} := \{x \in \mathfrak{h} \mid \theta(x) = -x\}.$$

is a $\mathbb{Z}_2$-grading of the Lie algebra $\mathfrak{h}$, and $\mathfrak{k}$ is its proper reductive subalgebra (see [25]). Let K be the connected algebraic subgroup of $\operatorname{Ad}\mathfrak{h}$ with the Lie algebra $\mathfrak{k}$. The subspace $\mathfrak{p}$ is invariant with respect to the restriction to K of the natural action of the group $\operatorname{Ad}\mathfrak{h}$ on $\mathfrak{h}$. The action of K on $\mathfrak{p}$ arising this way determines a homomorphism $\iota\colon K \to \mathrm{GL}(\mathfrak{p})$.

For every group from list (ii) of Theorem 1, there is a pair $(\mathfrak{h}, \theta)$ such that $V = \mathfrak{p}$ and $G = \iota(K)$.

Next, we use the following facts (see [12], [5], [25], [22]).

In $\mathfrak{h}$, there is a θ-stable real form $\mathfrak{r}$ of the Lie algebra $\mathfrak{h}$, such that $\mathfrak{r} = (\mathfrak{r}\cap\mathfrak{k}) \oplus (\mathfrak{r}\cap\mathfrak{p})$ is its Cartan decomposition (thereby $\mathfrak{r}\cap\mathfrak{k}$ is a compact real form of the Lie algebra $\mathfrak{k}$). The semisimple real Lie algebra $\mathfrak{r}$ is noncompact and the juxtaposition $\mathfrak{r} \rightsquigarrow \theta$ determines a bijections between the noncompact real forms of the Lie algebra $\mathfrak{h}$, considered up to an isomorphism, and the involutions in $\operatorname{Aut}\mathfrak{h}$, considered up to conjugation. By means of this bijection and described construction, every group G from list (ii) of Theorem 1 is determined by some noncompact semisimple real Lie algebra $\mathfrak{s}$; we say that G and $\mathfrak{s}$ *correspond* each other.

The nullcone $\mathcal{N}_{K,\mathfrak{p}}$ for the action of K on $\mathfrak{p}$ contains only finitely many K-orbits, therefore, every its irreducible component contains an open dense K-orbit; the latter is called *principal* nilpotent K-orbit and its dimension is equal to the maximum of dimensions of K-orbits in $\mathfrak{p}$.

Let $\sigma\colon \mathfrak{h} \to \mathfrak{h},\ x + iy \mapsto x - iy,\ x, y \in \mathfrak{r}$. Denote by $\mathcal{N}_{\mathfrak{r}}$ the set of nilpotent elements of the Lie algebra $\mathfrak{r}$. In every nonzero K-orbit $\mathscr{O} \subset \mathcal{N}_{K,\mathfrak{p}}$, there is an element e such that $\{e, f := -\sigma(e), h := [e, f]\}$ is an $\mathfrak{s}l_2$-triple (i.e., $[h, e] = 2e$ and $[h, f] = -2f$). Then the element $(i/2)(e+f-h)$ lies in $\mathcal{N}_{\mathfrak{r}}$, its $\operatorname{Ad}\mathfrak{r}$-orbit $\mathscr{O}'$ does not depend on the choice of e, the equality $2\dim_{\mathbb{C}}\mathscr{O} = \dim_{\mathbb{R}}\mathscr{O}'$ holds, and the map $\mathscr{O} \mapsto \mathscr{O}'$ is a bijection between the set of nonzero K-orbits in $\mathcal{N}_{K,\mathfrak{p}}$ and the set of nonzero $\operatorname{Ad}\mathfrak{r}$-orbits in $\mathcal{N}_{\mathfrak{r}}$.

A nilpotent element of a real semisimple Lie algebra $\mathfrak{s}$ is called *compact* if the reductive Levi factor of its centralizer in $\mathfrak{s}$ is a compact Lie algebra, [22]. For all simple real Lie algebras $\mathfrak{s}$ and their compact elements x, the orbits $(\operatorname{Ad}\mathfrak{s})\cdot x$ are classified (and their dimensions are found) in [22]. If, in the above notation, the elements of an $\operatorname{Ad}\mathfrak{r}$-orbit $\mathscr{O}'$ are compact, then the K-orbit $\mathscr{O}$ is called (-1)-*distinguished*, [19]. All principal nilpotent K-orbits are (-1)-distinguished, [21].

It follows from the aforesaid that the number of irreducible components of the nullcone $\mathcal{N}_{K,\mathfrak{p}}$ is equal to the number of (-1)-distinguished K-orbits of maximal dimension in $\mathfrak{p}$, and also to the number of orbits $(\operatorname{Ad}\mathfrak{r})\cdot x$ of maximal dimension, where x is a compact element in $\mathfrak{r}$.

This reduces the problem to pointing out for every group G from list (ii) of Theorem 1 the simple real Lie algebra $\mathfrak{s}$ corresponding to it, and then to finding the number of orbits $(\operatorname{Ad}\mathfrak{s})\cdot x$, where x is a compact element of $\mathfrak{s}$, such that their dimension is maximal.

Now we shall perform this for every group from list (ii) of Theorem 1, except those from Remark 1 that have already been considered above.

(4a) Let G be one of the groups $(\mathsf{B}_r, 2\varpi_1)$, $(\mathsf{D}_r, 2\varpi_1)$, $(\mathsf{A}_3, 2\varpi_2)$, $(\mathsf{C}_2, 2\varpi_1)$, $(\mathsf{A}_1, 4\varpi_1)$. Therefore, $\mathfrak{k} = \mathfrak{so}_n$, where, respectively, $n = 2r+1$ (with $r \geqslant 3$), $2r$ (with $r \geqslant 4$), 6, 5, 3. Hence the maximal compact subalgebra in $\mathfrak{s}$ is $\mathfrak{so}_{n,0}$ (see [25], [5], [22, Table 1]). In this case, $\mathfrak{s}$ is a real form of the Lie algebra $\mathfrak{sl}_n$ (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and Table 8 in Reference Chapter of [25] that $\mathfrak{s} = \mathfrak{sl}_n(\mathtt{R})$. According to [22, Thm. 8], the number of orbits $(\operatorname{Ad}\mathfrak{s})\cdot x$, where x is a nonzero compact element of $\mathfrak{s}$, is equal to 1 if n is add, and to 2 if n is even, and in the case of even n both of these orbits have the same dimension. Therefore, the nullcone $\mathcal{N}_{G,V}$ is irreducible for odd n and has exactly two irreducible components for even n.

(4b) Let $G = (\mathsf{C}_r, \varpi_2)$. Therefore, $\mathfrak{k} = \mathfrak{sp}_{2r}$, so the maximal compact subalgebra in $\mathfrak{s}$ is $\mathfrak{sp}_{r,0}$ (see [25], [5], [22, Table 1]). In this case, $\mathfrak{s}$ is a real form of the Lie algebra $\mathfrak{sl}_{2r}$ (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and Table 8 in Reference Chapter of [25] that $\mathfrak{s} = \mathfrak{sl}_r(\mathtt{H})$. According to [22, Thm. 8], the number of orbits $(\operatorname{Ad}\mathfrak{s})\cdot x$, where x is a nonzero compact element of $\mathfrak{s}$, is equal to 1. Therefore, the nullcone $\mathcal{N}_{G,V}$ is irreducible.

(4c) Let $G = (\mathsf{A}_7, \varpi_4)$. Then $\mathfrak{k} = \mathfrak{sl}_8$, so the maximal compact subalgebra in $\mathfrak{s}$ is $\mathfrak{su}_8$ (see [25], [5], [22, Table 1]). In this case, $\mathfrak{s}$ is a real form of the Lie algebra E_7 (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and [22, Table 5] that, using E. Cartan's notation, $\mathfrak{s} = \mathrm{E}_{7(7)}$. According to [22, Table 12], for this $\mathfrak{s}$, the number of (-1)-distinguished K-orbits of maximal dimension ($= 63$) in $\mathcal{N}_{K,\mathfrak{p}}$ is equal to 2. Therefore, the number of irreducible componenets of the nullcone $\mathcal{N}_{G,V}$ is equal to 2 as well.

(4d) Let $G = (\mathsf{C}_4, \varpi_4)$. Therefore, $\mathfrak{k} = \mathfrak{sp}_8$, and hence the maximal compact subalgebra in $\mathfrak{s}$ is $\mathfrak{sp}_{4,0}$ (see [25], [5], [22, Table 1]). In this case, $\mathfrak{s}$ is a real form of the Lie algebra E_6 (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and [22, Table 5] that $\mathfrak{s} = \mathrm{E}_{6(6)}$. According to [22, Table 7], for this $\mathfrak{s}$, there is a unique (-1)-distinguished K-orbit of maximal dimension $(= 36)$ in $\mathcal{N}_{K,\mathfrak{p}}$. Therefore, the nullcone $\mathcal{N}_{G,V}$ is irreducible.

(4e) Let $G = (\mathsf{D}_8, \varpi_8)$. Therefore, $\mathfrak{k} = \mathfrak{so}_{16}$, so the maximal compact subalgebra in $\mathfrak{s}$ is $\mathfrak{so}_{16,0}$ (see [25], [5], [22, Table 1]). In this case, $\mathfrak{s}$ is a real form of the Lie algebra E_8 (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and [22, Table 5] that $\mathfrak{s} = \mathrm{E}_{8(8)}$. According to [22, Table 14], for this $\mathfrak{s}$, there is a unique (-1)-distinguished K-orbit of maximal dimension $(= 129)$ in $\mathcal{N}_{K,\mathfrak{p}}$. Hence the nullcone $\mathcal{N}_{G,V}$ is irreducible.

(4f) Let $G = (\mathsf{F}_4, \varpi_4)$. Therefore, $\mathfrak{k} = \mathfrak{f}_4$, so the maximal compact subalgebra in $\mathfrak{s}$ is $\mathrm{F}_{4(-52)}$ (see [22, Sect. 5]). In this case, $\mathfrak{s}$ is a real form of the Lie algebra E_8 (see Summary Table at the end of [23] and Tables 7, 9 in Reference Chapter of [25]). It follows from this and [22, Table 5] that $\mathfrak{s} = \mathrm{E}_{6(-26)}$. According to [22, Table 9], for this $\mathfrak{s}$, there is a unique (-1)-distinguished K-orbit of maximal dimension $(= 24)$ in $\mathcal{N}_{K,\mathfrak{p}}$. Hence in this case the nullcone $\mathcal{N}_{G,V}$ is irreducible. $\square$

Remark 3. In [20] is obtained an algorithm that employs only elementary geometric operations (the orthogonal projection of a finite system of points onto a linear subspace and taking its convex hull) and, starting from the system of weights of the G-module V and the system of roots of the group G, finds a finite set of linear subspaces L in V such that the irreducible components of maximal dimension of the nullcone $\mathcal{N}_{G,V}$ are the varieties $G \cdot L$. In particular, if the property (ED_0) holds (see above the list of properties after formula (1)), this algorithm describes all the irreducible components of the nullcone $\mathcal{N}_{G,V}$. For instance, this is so for every pair (G, V) from Theorem 1. The computer implementation of this algorithm is obtained in [1].

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Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina 8, Moscow, 119991, Russia

E-mail address: `popovvl@mi.ras.ru`