Absolutely trianalytic tori in the generalized Kummer variety

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Abstract

We prove that a generic complex deformation of a generalized Kummer variety contains no complex analytic tori.

Contents

1 Introduction

A Riemannian manifold is called **hyperkähler** if it admits a triple of complex structures I, J, K satisfying quaternionic relations and Kähler with respect to g. A hyperkähler manifold is always holomorphically symplectic. By the Yau's Theorem [Y], a hyperkähler structure exists on a compact complex manifold if and only if it is Kähler and holomorphically symplectic.

Given any triple $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$, the operator $L := aI + bJ + cK$ satisfies $L^2 = -1$ and defines a Kähler structure on (M, g). Such a complex structure is called **induced by the hyperkähler structure**.

Complex subvarieties of such (M, L) for generic (a, b, c) were studied in [V1], [V2].

Definition 1.1: A manifold M is called **holomorphically symplectic** if it is a complex manifold with a closed holomorphic 2-form Ω over M such that $\Omega^n = \Omega \wedge \Omega \wedge ... \wedge \Omega$ is a nowhere degenerate section of a canonical class of M, where $2n = \dim_{\mathbb{C}}(M)$. A manifold M is called **irreducible holomorphically symplectic** if it is simply-connected and $H^0(M, \Omega_M^2) = <\Omega>$.

Definition 1.2: A compact hyperkähler manifold M is called **simple**, or **IHS of maximal holonomy** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Theorem 1.3: (Bogomolov's decomposition, [B]) Any hyperkähler manifold admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.

Definition 1.4: Let M be a hyperkähler manifold, and S the family of all induced complex structures $L := aI + bJ + cK$, where $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$. Then S is called **the twistor family** of complex structures.

Definition 1.5: A closed subset Z of a hyperkähler manifold M is called **trianalytic**, if it is complex analytic with respect to complex structures I, J, K.

Definition 1.6: Let (M, I, J, K) be a compact, holomorphically symplectic, Kähler manifold, and $Z \subset (M, I)$ a complex subvariety, which is trianalytic with respect to any hyperkähler structure compatible with I. Then Z is called **absolutely trianalytic**.

Whenever L is a generic element of a twistor family, all subvarieties of (M, L) are trianalytic (see Theorem 3.6).

Absolutely trianalytic subvarieties were studied in [V3] , where it was shown that a general deformation of a Hilbert scheme of a $K3$ surface has no complex (or, equivalently, no absolutely trianalytic) subvarieties. However, there is an absolutely trianalytic subvariety in a generalized Kummer variety. Recently, Soldatenkov and Verbitsky [SV] have shown that there are no absolutely trianalytic tori in the 6- and 10-dimensional O'Grady examples. Non-existence of absolutely trianalytic subvarieties of known type in 10-dimensional O'Grady manifold M follows from [SV, Corollary 3.17]. Non-existence of absolutely trianalytic tori in a 6-dimensional O'Grady manifold

follows from representation theory of Clifford algebras [SV]. Automorphisms of hyperkähler manifolds acting trivially on the second cohomology group can be characterized as those which have trianalytic graph in $M \times M$ for any given hyperkähler structure on M. The group of such automorphisms is finite [H1]. It has been studied for the generalized Kummer surfaces by Oguiso ([Og]), Boissiere, Nieper-Wisskirchen and Sarti ([BNS]), and for O'Grady examples by Mongardi and Wandel ([MW]).

In the present paper we study absolutely trianalytic tori in the generalized Kummer variety. In Section 2 we give definition of hyperkähler manifolds and recall all known examples. In Section 3 we study general trianalytic subvarieties. In Section 4 we show non-existence of absolutely trianalytic tori in the generalized Kummer variety and prove the following

Theorem 1.7: Let $K_n(T)$ be the generalized Kummer variety and $Z \subset K_n(T)$ be an absolutely trianalytic submanifold of $K_n(T)$. Then Z is not a torus.

2   Preliminaries

Definition 2.1: ([Bes]) A **hyperkähler manifold** is a Riemannian manifold M endowed with three complex structures I, J and K, such that the following holds.

(i) the metric on M is Kähler with respect to these complex structures and

(ii) I, J and K, considered as endomorphisms of a real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

The triple I, J, K is called a **hyperkähler structure** for manifold M.

Definition 2.2: A manifold M is called **holomorphically symplectic** if it is a complex manifold with a closed holomorphic 2-form Ω over M such that $\Omega^n = \Omega \wedge \Omega \wedge ... \wedge \Omega$ is a nowhere degenerate section of a canonical class of M, where $2n = \dim_{\mathbb{C}}(M)$. A manifold M is called **irreducible holomorphically symplectic** if it is simply-connected and $H^0(M, \Omega_M^2) = <\Omega>$.

Remark: It follows from Bochner's vanishing and Berger's classification of holonomy that a hyperkähler manifold has maximal holonomy $\mathrm{Sp}(n)$ whenever $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

A simple algebraic calculation [Bes] shows that the following form

$$\Omega = \omega_J + \sqrt{-1}\,\omega_K \tag{2.1}$$

is of type $(2,0)$. Since it is closed this form is also holomorphic and moreover nowhere degenerate, as another linear algebraic argument shows. It is called **the canonical holomorphic symplectic form of a manifold** M. Thus, the underlying complex manifold (M, L) is holomorphically symplectic for each hyperkähler manifold M, and an induced complex structure L. The converse is also true:

Theorem 2.3: ([Bea], [Bes]) Let M be a compact holomorphically symplectic Kähler manifold with the holomorphic symplectic form Θ, a Kähler class $[\omega] \in H^{1,1}(M)$ and a complex structure I. Let $n = \dim_{\mathbb{C}} M$. Assume that $\int_M \omega^n = \int_M (\mathrm{Re}\,\Omega)^n$. Then there is a unique hyperkähler structure $(I, J, K, (\cdot, \cdot))$ over M such that the cohomology class of the symplectic form $\omega_I = (\cdot, I\cdot)$ is equal to $[\omega]$ and the canonical symplectic form $\omega_J + \sqrt{-1}\,\omega_K$ is equal to Ω.

Two-dimension irreducible holomorphic symplectic manifolds are called $K3$ surfaces. In higher dimensions there are only few examples known. Here is the list of known examples, where compact manifolds of the same deformation type are not distinguished.

(0) *K3 surface.*

(i) *The Hilbert scheme of n points of K3.* If X is a $K3$ surface then the Hilbert scheme $\mathrm{Hilb}^n(X)$ is an irreducible holomorphic symplectic manifold [Bea]. Its dimension is $2n$ and for $n > 1$ its second Betti number is equal to 23. Details of construction of the Hilbert scheme can be found, for example [Bea]. Let X be a $K3$ surface. Take the symmetric product $X^{(r)} = X^r / \mathfrak{S}_r$ which parametrizes subsets of r points in a $K3$ surface X, counted with multiplicities; it is smooth on the open subset X_0 consisting of subsets with r distinct points, but singular otherwise. We blow up singular locus and obtain a smooth compact manifold, which is called the Hilbert scheme $X^{[r]}$. The natural map $X^{[r]} \to X^{(r)}$ is an isomorphism above X_0, and it resolves the singularities of $X^{(r)}$. In other words, the Hilbert scheme parametrizes all 0-dimensional subschemes of the length n.

Let us describe the simplest case $\mathrm{Hilb}^2(X)$ explicitly. For any surface X the Hilbert scheme $\mathrm{Hilb}^2(X)$ is the blow-up $\mathrm{Hilb}^2(X) \to S^2(X)$ of

the diagonal

$$\Delta = \{\{x,x\} \mid x \in X\} \subset S^2(X) = \{\{x,y\} \mid x,y \in X\}.$$

Equivalently, $\mathrm{Hilb}^2(X)$ is the $\mathbb{Z}/2\mathbb{Z}$-quotient of the blow-up of the diagonal in $X \times X$. Since for a $K3$ surface there exists only one $\mathbb{Z}/2\mathbb{Z}$-invariant two-form on $X \times X$, the holomorphic symplectic structure on $\mathrm{Hilb}^2(X)$ is unique.

(ii) *The generalized Kummer variety.* If T is a complex torus of dimension two, then the generalized Kummer variety $\mathrm{K}_n(T)$ is an irreducible holomorphic symplectic manifold [Bea]. Its dimension is $2n$ and for $n > 2$ its second Betti number is 7. Note that the Hilbert scheme $T^{[n]}$ of a two-dimensional torus has the same properties as $K3^{[r]}$, but it is not simply connected. The commutative group structure on the torus T defines a summation map

$$s\,(t_1,\ldots,t_n) = t_1 + \ldots + t_{n+1},$$

$$\Sigma : T^{n+1} \to T,$$

which induces a morphism $\Sigma : T^{[n+1]} \to T$. It is easy to see that Σ coincides with the Albanese map. The generalized Kummer variety $\mathrm{K}_n(T)$ associated to the torus T is the preimage $\Sigma^{-1}(0) \subset T^{[n+1]}$ of the zero $0 \in T$. It is a hyperkähler manifold of dimension $2n$.

(iii) *O'Grady's 10-dimensional example [O1].* Let again X be a $K3$ surface, and M the moduli space of stable rank 2 vector bundles on S, with Chern classes $c_1 = 0, c_2 = 4$. It admits a natural compactification M, obtained by adding classes of semi-stable torsion free sheaves. It is singular along the boundary, but O'Grady [O1] constructs a desingularization of M which is a new hyperkähler manifold, of dimension 10. Its second Betti number is 24 [R]. Originally, it was proved that it is at least 24 [O1].

(iv) *O'Grady's 6-dimensional example [O2].* A similar construction can be done starting from rank 2 bundles with $c_1 = 0, c_2 = 2$ on a 2-dimensional complex torus, this gives new hyperkähler manifold of dimension 6 as in (iii). Its second Betti number is 8.

Thus we have two series, (i) and (ii), and two sporadic examples, (iii) and (iv). All of them have different second Betti numbers. It has been

proved ([KLS], Theorem B) that the moduli spaces for all sets of numerical parameters give $\text{Hilb}^n(K3)$, O'Grady examples, or do not admit a smooth symplectic resolution of singularities. Note that in any given dimension and for any given second Betti number b_2 one knows at most one real manifold carrying the structure of an irreducible holomorphic symplectic manifold.

Definition 2.4: A hyperkähler manifold M is called **simple (or IHS of maximal holonomy)** if $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

One of the main facts about the hyperkähler manifolds is the following

Theorem 2.5: (Bogomolov decomposition theorem, [B]) Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

3 Subvarities in hyperkähler manifolds

3.1 Trianalytic subvarities

Definition 3.1: A closed subset Z of a hyperkähler manifold M is called **trianalytic**, if it is complex analytic with respect to complex structures I, J, K.

Theorem 3.2: (equivalent to the Theorem 1.3)
Let M be a hyperkähler manifold. Then there exists a unique hyperkähler metric in a given Kähler class.

Definition 3.3: Let (M, I, J, K) be a compact, holomorphically symplectic, Kähler manifold, and $Z \subset (M, I)$ a complex subvariety, which is trianalytic with respect to any hyperkähler structure compatible with I. Then Z is called **absolutely trianalytic**.

Theorem 3.4: [SV] For all M, M' be hyperkähler manifolds there is a diffeomorphism which sends absolutely trianalytic subvarieties to absolutely trianalytic.

Definition 3.5: A hyperkähler manifold is called general if all its subvarities are absolutely trianalytic.

Theorem 3.6: ([V2]) Let M be a hyperkähler manifold, S its twistor

family (see Definition 1.4). Then there exists a countable subset $S_1 \subset S$, such that for any complex structure $L \in S \backslash S_1$, all compact complex subvarieties of (M, L) are trianalytic.

3.2 Trianalytic subvarities in the Hilbert scheme and O'Grady examples

It was shown by Verbitsky that a general deformation of a Hilbert scheme of a $K3$ surface has no complex subvarieties [V3]. The same theorem was also claimed (Kaledin, Verbitsky) in the case of generalized Kummer varieties [KV]. However, later ([KV1]) they found that there are counterexamples in the latter case due to involution $\nu : t \to -t$ of a torus. This involution is extended to an involution of the Hilbert scheme $T^{[n+1]}$ and since it commutes with the Albanese map $T^{[n+1]} \longrightarrow T$, the map ν preserves $K_n(T)$. Moreover, ν sends the Kähler class to itself. Hence, the involution ν preserves the hyperkähler structure on $K_n(T)$. For odd $n = 2m - 1$ the map ν fixes the $2m$-tuple

$$(x_1, -x_1, x_2, -x_2, ..., x_m, -x_m) \in T^{(n+1)}$$

When $x_i, -x_i$ are pairwise distinct, they give a point of the Hilbert scheme fixed by ν. Consider the closure X of the set of such points. It is one of components of fixed point set of involution map ν. The submanifold X is birationally equivalent to the Hilbert scheme of a $K3$ surface.

Corollary 3.7: The variety $\mathrm{Sym}^2(T)$ contains a Kummer $K3$ surface.

Non-existence of absolutely trianalytic subvarieties in the Hilbert scheme $\mathrm{Hilb}(K3)$ of $K3$ was used in the book [KV-book] to prove compactness of deformation spaces of certain stable holomorphic bundles on M.

Theorem 3.8: ([V5]) Let M be a hyperkähler manifold, $Z \subset M$ a trianalytic subvariety, and I an induced complex structure. Consider the normalization

$$\widetilde{(Z, I)} \to (Z, I)$$

of (Z, I). Then $\widetilde{(Z, I)}$ is smooth, and the map $\widetilde{(Z, I)} \to M$ is an immersion, inducing a hyperkähler structure on $\widetilde{(Z, I)}$.

This gives that any trianalytic subvariety $Z \to M$ has a smooth hyperkähler normalization $\widetilde{Z}$ immersed to M; this immersion is generically bijective onto its image. Therefore, we can replace any trianalytic cycle by

an immersed hyperkähler manifold. We will consider absolutely trianalytic manifolds whose normalization is the torus.

Theorem 3.9: [SV] Let M be a hyperkähler manifold, $Z \subset M$ an absolutely trianalytic subvariety, and $\widetilde{Z} \to M$ its normalization such that $\widetilde{Z} = T \times \prod_i K_i$, where K_i are IHS of maximal holonomy. Then $b_2(T) \geq b_2(M)$ and $b_2(K_i) \geq b_2(M)$.

Theorem 3.10: [SV] Let M be a hyperkähler manifold of maximal holonomy, T a hyperkähler torus, and $T \to M$ a hyperkähler immersion with absolutely trianalytic image. Then

$$\dim_{\mathbb{C}}(T) \geqslant 2^{\frac{b_2(X)-1}{2}}.$$

Theorem Theorem 3.10 follows from results about dimension of k-symplectic structures on vector spaces

Definition 3.11: A k-symplectic structure on V is a subspace $\Omega \subset \Lambda^2 V^*$ of dimension k, such that for some non-zero quadratic form $q \in S^2\Omega^*$ the following condition is satisfied: for any non-zero $\omega \in \Omega$ we have

$$\dim(\ker \omega) = \begin{cases} 2n, & \text{if } q(\omega) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

A k-symplectic structure is called non-degenerate if the quadratic form q is non-degenerate. A vector space with a k-symplectic structure will be called a k-symplectic vector space.

In the case of Theorem 3.10 the cohomology group $H^1(T, \mathbb{C})$ carries a non-degenerate real k-symplectic structure. The corresponding quadratic form has signature $(k - 3, 3)$ and it is called a Bogomolov–Beauville form [SV].

Non-existence of subvarieties of known type in 10-dimensional O'Grady manifold M follows from Theorem 3.10.

4   Tori in the generalized Kummer varieties

In this section we prove the following

Main (Theorem 1.7): Let $K_n(T)$ be the generalized Kummer variety and $Z \subset K_n(T)$ be an absolutely trianalytic submanifold. Then Z is not a torus.

4.1   Deformations of subvarities

Consider the canonical projection:

$$
\begin{array}{c}
Z \\
\swarrow \quad \downarrow \quad \searrow \\
T^{[n]} \longrightarrow \mathrm{Sym}^n(T) \longleftarrow T^n
\end{array}
\qquad (4.1)
$$

Proposition 4.1: Let $Z \subset T^{[n]}$ be an absolutely trianalytic torus. Then its projection the map from Z to T^n is an embedding.

Proof: The torus Z is projected to $T^{(n)}$, and there exist a map Z into T^n making the diagram commutes. *A priori* the map from Z to T^n could be a ramified covering, by [EV, Theorem 7.7] one can chose T to be general in the sense of Definition 3.5. So Z is a flat torus hence this map is embedding. Indeed, a torus is hyperkähler and if it is general then all its submanifolds are trianalytic. Trianalytic submanifolds are always totally geodesic. Therefore trianalytic submanifolds of flat manifolds are flat, and it completes the proof ([V4, Proposition 3.3]). ∎

Proposition 4.2: Let $Z \subset T^{[n]}$ be an absolutely trianalytic torus. Then its projection π to $T^{(n)}$ is generally finite to its image.

Proof: All fibres of π are rational ([N]) and isotropic ([V3]) and vectors tangent to fibres of projection are orthogonal to vectors of base. The symplectic form on Z is a pull-back of the symplectic form on the base ([Bea]). Therefore Z is symplectic both in $T^{[n]}$ and in $T^{(n)}$. If Z is projected to $\pi(Z)$ with a positive-dimension general fibres, then Z is not symplectic. This gives a contradiction. ∎

Denote the rank of the ramified covering $Z \to \pi(Z)$ by d.

Theorem 4.3: [V3, Theorem 8.1] Let Z be an absolutely trianalytic manifold in a hyperkähler manifold M. Then its deformations $\mathrm{Def}(Z)$ is hyperkähler manifold and $Z \subset \mathrm{Def}(Z)$ is absolutely trianalytic.

Lemma 4.4: Let Z be a absolutely trianalytic submanifold whose normalization is a torus in the generalized Kummer variety $K_n(T)$. Then $\operatorname{Pic}(Z) = 0$.

Proof: By [V4], we can deform Z in the generalized Kummer variety $K_n(T)$, such that T is the general torus. Since Z is absolutely trianalytic as the submanifold, Z is almost geodesic, and hence it is flat. Therefore, Z is a subtorus in $T^{(n)}$ and its Picard group is zero. ∎

4.2   Non-existence of trianalytic tori in the generalized Kummer variety

In this section we prove the Main Theorem 1.7. Consider an absolutely trianalytic torus $Z \subset T^{[n]}$.

First, let us remark that any complex structure on $\operatorname{Sym}^n(T)$ defines a complex structure of Kähler type on $T^{[n]}$. And there is the standard map from $H^2(\operatorname{Sym}^n(T), \mathbb{C}) \oplus \mathbb{C}[E]$ to $H^2(T^{[n]}, \mathbb{C})$, where E is the exceptional divisor of blow up $T^{[n]}$ to $T^{(n)}$. Recall that the class $[E]$ is of type $(1,1)$.

Definition 4.5: The symplectic volume of a holomorphic symplectic manifold (M, Ω) is $\int_M \Omega^{\frac{1}{2} \dim M} \wedge \overline{\Omega}^{\frac{1}{2} \dim M}$.

Definition 4.6: The Kähler volume of a Kähler manifold (M, I, ω) is $\frac{1}{2^{2n}(2n)!} \int_M \omega^{2n}$, where $\dim_{\mathbb{R}}(M) = 2n$.

Remark: They are equal for hyperkähler manifolds ([GV]).

Theorem 4.7: Let $K_n(T)$ be the generalized Kummer variety and $Z \subset K_n(T)$ be an absolutely trianalytic submanifold of $K_n(T)$. Then Z is not a torus.

Proof:

Recall from the Subsection 4.1 that Z is hyperkähler both in $T^{[n]}$ and $T^{(n)}$. Thus, its symplectic volume should be equal to the Kähler one in both spaces.

It is well-known (see e.g. [OVV, Lemma 3.4]) that there exist a Kähler metric with Kähler class $[\omega_{T^{[n]}}] = [\pi^* \omega_{T^{(n)}}] - \epsilon[E]$, where E is the exceptional divisor and $0 < \epsilon \ll 1$.

Hyperkähler triple is given by three forms ω_I, ω_J, and ω_K of the same Kähler volume since there are Kähler forms of three complex structures. Fix a hyperkähler structure (I, J, K) on $T^{(n)}$ such that $\omega_J + i\omega_K = \Omega$ is the symplectic form. Then after blowing up the symmetric power $T^{(n)}$ to the Hilbert scheme of points $T^{[n]}$ forms ω_J and ω_K correspond to their pullbacks. Since $[\omega_{T^{[n]}}] = [\pi^*\omega_{T^{(n)}}] - \epsilon[E]$, volume of $\omega_{T^{[n]}}$ is less then volume of $\pi^*\omega_{T^{(n)}}$. Symplectic volume does not change under the blow-up. Since $T^{[n]}$ is hyperkähler, its Kähler volume has to be equal to the symplectic one. Thus, triple of forms ω_I, ω_J, and ω_K maps to the triple ω_J, ω_K, and $\lambda[\pi^*\omega_{T^{(n)}}] - \epsilon[E]$, where $\lambda > 1$ is determined from the condition that volume does not change.

Note that $\pi^*(\omega) \cup [E] = 0$, and let $\mu = [E]^{2n}$ [EV].

Then,

$$\int_{T^{[n]}} \omega_{T^{[n]}} = (\lambda)^{2n} \cdot \mathrm{Vol}_{\omega_{T^{(n)}}} - \epsilon^{2n} \cdot \mu$$

A constant $\lambda > 1$ could be determined from this equation.

Whenever a manifold is hyperkähler, its symplectic volume is equal to the Kähler one. It follows from Proposition 4.2 that the map from Z to the $T^{(n)}$ is generally finite on its image. Thus the Kähler and symplectic volumes of Z in the symmetric product are multiplied by d, where d is a rank of the ramified covering $Z \to \pi(Z)$.

Let us calculate the Kähler volume of Z in $T^{(n)}$ and $T^{[n]}$. Since Z is absolutely trianalytic in $T^{(n)}$, its volume with respect to ω_J and ω_K is equal to volume with respect to $\omega_{T^{(n)}}$. However, Z is also absolutely trianalytic in $T^{[n]}$, hence this volume is also equal to the volume with respect to $\lambda[\pi^*\omega_{T^{(n)}}] - \epsilon[E]$.

Note that Z does not intersect E. Indeed, consider line bundle $\mathcal{O}(E)$ restricted to Z. Since Z has zero Picard group (Lemma 4.4), then there are no non-trivail line bundles over Z.

Hence, we have

$$\int_Z (\omega_{T^{(n)}})^k = \int_Z (\lambda\pi^*\omega_{T^{(n)}} - \epsilon[E])^k = \lambda^k \int_Z (\omega_{T^{(n)}})^k,$$

On the other hand $\lambda > 1$, that gives a contradiction.
∎

Remark: The proof does not work for other absolutely trianalytic subvarities of generalized Kummer manifold since we highly use that the normalization of our submanifold is a torus to prove that map from Z in $T^{(n)}$ is embedding, and for vanishing of Picard group of Z.

Corollary 4.8: Let Z be an absolutely trianalytic subvariety of the generalized Kummer $K_n(T)$. Then $\mathrm{Pic}(Z) \neq 0$.

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