Title: An expansion of zeta(3) in continued fraction with parameter.

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Abstract

We present here continued fraction for Zeta(3) parametrized by some family of points (F,G) on projective line. This family of points can be obtained if from full projective line would be removed some no more than countable nowhere dense exeptional set of finite points. A countable nowhere dense set, which contains the above exeptional set of finite points, is specified also.

To the thirty fifth anniversary
of Apéry's discovery.

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§0. Foreword .

We say that two infinite continuous fractions are equivalent if the set of their common convergents is infinite. We say that two infinite continuous fractions are essentially distinct if the set of their common convergents is finite or empty.

If $\delta_0 = 1, \delta_\nu \in \mathbb{C}\diagdown\{0\}$,

$$b_\nu^{(2)} = b_\nu^{(1)}\delta_\nu, a_\nu^{(2)} = a_\nu^{(1)}\delta_\nu\delta_{\nu-1}$$

for $\nu \in \mathbb{N}$, then easy induction show that $P_\nu^{(2)} = P_\nu^{(1)} \prod\limits_{\kappa=1}^{\nu} \delta_\kappa, Q_\nu^{(2)} = Q_\nu^{(1)} \prod\limits_{\kappa=1}^{\nu} \delta_\kappa$, and continued fraction

$$b_0^{(i)} + \frac{a_1^{(i)}|}{|b_1^{(i)}} + \frac{a_2^{(i)}|}{|b_2^{(i)}} + ... + \frac{a_\nu^{\vee}|}{|b_\nu^{(i)}} + ..., \tag{0.1}$$

with i=1 is equivalent to continued fraction (0.1) with $i = 2$. According to the famous result of R. Apéry [R.Apéry, 1981],

$$\zeta(3) = b_0^\vee + \frac{a_1^\vee|}{|b_1^\vee} + \frac{a_2^\vee|}{|b_2^\vee} + ... + \frac{a_\nu^\vee|}{|b_\nu^\vee} + ... \tag{0.2}$$

with

$$b_0^\vee = 0,\ b_1^\vee = 5,\ a_1^\vee = 6,\ b_{\nu+1}^\vee = \tag{0.3}$$

$$34\nu^3 + 51\nu^2 + 27\nu + 5,\ a_{\nu+1}^\vee = -\nu^6,$$

for $\nu \in \mathbb{N}$. We denote by $r_{A,\nu}$ the ν-th convergent of continued fraction (0.2). Yu.V. Nesterenko in [Yu.V. Nesterenko, 1996] has offered the following expansion of $2\zeta(3)$ in continuous fraction:

$$2\zeta(3) = 2 + \frac{1|}{|2} + \frac{2|}{|4} + \frac{1|}{|3} + \frac{4|}{|2}..., \tag{0.4}$$

with

$$b_0 = b_1 = a_2 = 2,\ a_1 = 1,\ b_2 = 4, \tag{0.5}$$

$$b_{4k+1} = 2k + 2,\ a_{4k+1} = k(k+1),\ b_{4k+2} = \tag{0.6}$$

$$2k + 4,\ a_{4k+2} = (k+1)(k+2)$$

for $k \in \mathbb{N}$,

$$b_{4k+3} = 2k + 3,\ a_{4k+3} = (k+1)^2,\ b_{4k+4} = \tag{0.7}$$

$$2k + 2,\ a_{4k+4} = (k+2)^2 \text{ for } k \in \mathbb{N}_0.$$

We denote by $r_{N,\nu}$ the ν-th convergent of continued fraction (0.4). The continued fractions (0.4) and (0.2) are equivalent because $2r_{A,\nu} = r_{N,4\nu-2}$ for all $\nu \in \mathbb{N}$.

Elementary proof of Yu.V. Nesterenko result
can be found in [L.A.Gutnik,16, 2010].

Let me to formulate my result now. Let u and v are variables,

$$\tau = \tau(\nu) = \nu + 1,\ \sigma = \sigma(\nu) = \tau(\tau - 1) = \nu(\nu + 1),$$

where $\nu \in \mathbb{N}$. Let further

(0.8)
$$c_{u,v,2}(\nu) = -\tau(\tau+1)^2(2\tau-1)\times$$
$$(-3(2\tau-1)^2u^2 - (10\tau^2 - 10\tau + 3)uv + 2(3\tau^4 - 6\tau^3 + 4\tau^2 - \tau)v^2 =$$
$$-3(4\sigma(\nu)+1)u^2 - (10\sigma(\nu)+3)uv + 2\sigma(3\sigma(nu)+1)v^2 \in \mathbb{N}[u,v].$$

(0.9)
$$c_{u,v,1}(\nu) = -12(68\tau^6 - 45\tau^4 + 12\tau^2 - 1)u^2 -$$
$$8(157\tau^6 - 106\tau^4 + 30\tau^2 - 3)uv +$$
$$4(102\tau^8 - 170\tau^6 + 89\tau^4 - 24\tau^2 + 3)v^2 \in \mathbb{N}[u,v],$$

(0.10)
$$c_{u,v,0}(\nu) = \tau(\tau-1)^2(2\tau+1)\times$$
$$(3(4\tau^2 + 4\tau + 1)u^2 + (10\tau^2 + 10\tau + 3)uv - 2(3\tau^4 + 6\tau^3 + 4\tau^2 + \tau)v^2) =$$
$$\tau(\tau-1)^2(2\tau+1)\times$$
$$(3(4\sigma(\nu+1)+1)u^2 + (10\sigma(\nu+1)+3)uv - 2\sigma(\nu+1)(3\sigma(\nu+1)+1)v^2).$$

(0.11)
$$b_{u,v}(\nu+1) = -c_{u,v,1}(\nu) \in \mathbb{Q}[u,v] \text{ for } \nu \in \mathbb{N},$$

(0.12)
$$a_{u,v}(\nu+1) = -c_{u,v,0}(\nu)c_{u,v),2}(\nu-1) \text{ for } \nu \geq 2,\ \nu \in \mathbb{N},$$

(0.13)
$$a_{u,v}(2) = -c_{u,v,0}(1),$$

(0.14)
$$P_{u,v}(0) = b_{u,v}(0) = 4(3u+2v),\ Q_{u,v}(0) = 1,$$

(0.15)
$$Q_{u,v}(1) = b_{u,v}(1) = (34u + 52v)/(u+v)$$

(0.16)
$$P_{u,v}(1) = (327u + 500v)$$

(0.17)
$$a_{u,v}(1) = P_{u,v}(1) = -b_{u,v}(0)b_{u,v}(1).$$

Calculations described in [L.A.Gutnik,18, 2013] lead to the following continued fraction over the field $\mathbb{Q}(u,v)$:

$$b_{u,v}(0)+\frac{a_{u,v}(1)|}{|b_{u,v}(1)}+\frac{a_{u,v}(2)|}{|b_{u,v}(2)}+\frac{a_{u,v}(3)|}{|b_{u,v}(3)}+\dots. \tag{0.18}$$

We denote by $r_{u,v}(\nu)$ the ν-th convergent of continuous fraction (0.18). We denote by $P_{u,v}(\nu)$ and $Q_{u,v}(\nu)$ the respectively nominator and denominator of $r_{u,v}(\nu)$. Let further $\Delta(x)=18x^2(2x+1)+(7x+3)^2$,

$$\rho_k(x)=\frac{5x+3+(-1)^k\sqrt{\Delta(x)}}{x(3x+2)},$$

where $x=2\nu(\nu+1)$, $\nu\in\mathbb{N}$, $k=1,2$, and let

$$\mathfrak{A}=\{\rho_k(x)\colon x=2\nu(\nu+1),\ \nu\in\mathbb{N},\ k=1,2\}.$$

$$\mathfrak{B}=\Big\{-\beta_2^{*(2)}(1;\nu)/\beta_2^{*(1)}(1;\nu)\colon \nu\in\mathbb{N}_0\Big\},$$

where

$$\beta_2^{*(r)}(z;\nu)=\sum_{k=0}^{\nu+1}\left(\binom{\nu+1}{k}\binom{\nu+k}{k}\right)^2 t^r z^k. \tag{0.19}$$

Then we have

Theorem B. *Let $F+G\neq 0$, $(F+G)G\geq 0$, $G/F\notin\mathfrak{B}\cup\mathfrak{A}$, if $F\neq 0$. Then the specialization of (0.18) for $u=F$, $v=G$ is well defined (i.e all convergents $r_{u,v}(\nu)$ are well defined for $u=F$, $v=G$) and the following equality holds*

$$8(F+G)\zeta(3)=b_{F,G}(0)+\frac{a_{F,G}(1)|}{|b_{F,G}(1)}+\frac{a_{F,G}(2)|}{|b_{F,G}(2)}+\frac{a_{F,G}(3)|}{|b_{F,G}(3)}+\dots. \tag{0.20}$$

moreover $P_{u,v}(\nu)$ and $Q_{u,v}(\nu)$ are homogeneous polynomials in $\mathbb{Z}[u,v]$, and

$$\max(2\nu,1)=\deg_u(P_{u,v}(\nu))=\deg_v(P_{u,v}(\nu))=\deg(P_{u,v}(\nu)), \tag{0.21}$$

$$\max(2\nu-1,0)=\deg_u(Q_{u,v}(\nu))=\deg_v(Q_{u,v}(\nu))=\deg(Q_{u,v}(\nu)), \tag{0.22}$$

where $\nu\in\mathbb{N}_0$.

Remark. The values $\rho_k(x)$ with $k=1,2$ are zeros of the following trinomial $a_0(x)+2a_1(x)\rho+a_2(x)\rho^2$, where

$$a_0(x)=-6(2x+1),\ a_1(x)=-2(5x+3),\ a_2(x)=x(3x+2).$$

Since $-a_0(x)/a_2(x)=3/x+3/(3x+2), -a_1(x)/a_2(x)=3/x+1/(3x+2)$, decrease together with increasing of $x>0$ it follows that

$$\Delta(x)/(a_2(x))^2=(-a_1(x)/a_2(x))^2-(-a_0(x)/a_2(x)),$$

and $\rho_2(x)$ decrease with increasing of x. Moreover, $0 < -\rho_1(x) < \rho_2(x)$ for any $x > 0$ and $\lim_{x\to\infty} r_2(x) = 0$. Consequently, for given $F > 0$ and $G > 0$ the condition of the Theorem B must be checked for finite family of ν; for example, if $G/F > r(4)$, then condition of the Theorem B is fulfilled. We note that $r(4) < 0,36$.

I prove Theorem B in sections 1 – 7. Initial variants of this article can be found in [L.A.Gutnik,17, 2009], [L.A.Gutnik,18, 2013].

§1. Introduction. Begin of the proof of Theorem B.

Let

$$|z| > 1, -3\pi/2 < \arg(z) \le \pi/2, \log(z) = \ln(|z|) + i\arg(z). \tag{1.1}$$

Clearly, $\log(-z) = \log(z) - i\pi$, when $\Re(z) > 0$, and $\log(z) = \log(-z) - i\pi$, when $\Re(z) < 0$. Let

$$f_1^*(z,\nu) := \sum_{k=0}^{\nu+1} (z)^k \binom{\nu+1}{k}^2 \binom{\nu+k}{\nu}^2, \tag{1.2}$$

$$R(t,\nu) = \left(\prod_{j=1}^{\nu}(t-j)\right) \Big/ \left(\prod_{j=0}^{\nu+1}(t+j)\right), \tag{1.3}$$

$$f_2^*(z,\nu) = \sum_{t=1}^{+\infty} z^{-t}(\nu+1)^2(R(\alpha,t,\nu))^2, \tag{1.4}$$

$$f_4^*(z,\nu) = -\sum_{t=1}^{+\infty} z^{-t}(\nu+1)^2 \left(\frac{\partial}{\partial t}(R^2)\right)(t,\nu), \tag{1.5}$$

$$f_3^*(z,\nu) = (\log(z))f_2^*(z,\nu) + f_4^*(z,\nu), \tag{1.6}$$

$$f_k(z,\nu) = f_k^*(z,\nu)/(\nu+1)^2 \tag{1.7}$$

where $k = 1, 2, 3, 4$, $\nu \in \mathbb{N}_0$. Let

$$\tau = \tau(\nu) = \nu + 1,\ \mu = \mu(\nu) = \tau^2 = (\nu+1)^2, \tag{1.8}$$

$$a_{1,1}^*(z,\nu) = \frac{1}{2}(-5\mu + 3\mu - 6\mu^2) - z\mu(1+18\mu) + (3\mu + \mu^2 + z(7\mu + 16\mu^2))\tau, \tag{1.9}$$

(1.10) $$a^*_{1,2}(z;\nu) = -8\mu - 4\mu^2 - z(12\mu + 20\mu^2) + (2 + 10\mu + z(2 + 26\mu))\tau,$$

(1.11) $$a^*_{1,3}(z;\nu) = -1 - 14\mu + z(-1 + 2\mu) + (7 + 8\mu + z(3 - 8\mu))\tau,$$

(1.12) $$a^*_{1,4}(z;\nu) = (z - 1)(2 + 12\mu - 10\tau),$$

(1.13) $$a^*_{2,1}(z;\nu) = -z\mu - z20\mu^2 - z12\mu^3 + \tau(7z\mu + 26z\mu^2),$$

$$z\mu(24 - 22\alpha + 5\alpha^2 + 28\mu - 2\alpha\mu),$$

(1.14) $$a^*_{2,2}(z;\nu) = -\mu - 3\mu^2 + z\mu(-13 - 38\mu)+$$

$$(3\mu + \mu^2 + 2z + 33z\mu + 16z\mu^2)\tau,$$

(1.15) $$a^*_{2,3}(z;\nu) = -8\mu - 4\mu^2 - z - 6z\mu + 4z\mu^2 + (2 + 10\mu + 5z - 2z\mu)\tau$$

(1.16) $$a^*_{2,4}(z;\nu) = (1 + 14\mu - 7\tau - 8\mu\tau)(z - 1),$$

(1.17) $$a^*_{3,1}(z;\nu) = -z\mu - 21z\mu^2 - 26z\mu^3 + (7z\mu + 33z\mu^2 + 8z\mu^3)\tau,$$

(1.18) $$a^*_{3,2}(z;\nu) = -14z\mu - 58z\mu^2 - 12z\mu^3 + (2z + 40z\mu + 42z\mu^2)\tau,$$

(1.19) $$a^*_{3,3}(z;\nu) = -\mu - 3\mu^2 - z - 17z\mu - 6z\mu^2 + (3\mu + \mu^2 + 7z + 17z\mu)\tau$$

(1.20) $$a^*_{3,4}(z;\nu) = (8\mu + 4\mu^2 - 2\tau - 10\mu\tau)(z - 1),$$

(1.21) $$a^*_{4,1}(z;\nu) = -z\mu - 21z\mu^2 - 38z\mu^3 + (7z\mu + 35z\mu^2 + 18z\mu^3)\tau,$$

(1.22) $$a^*_{4,2}(z;\nu) = -15z\mu - 79z\mu^2 - 38z\mu^3+$$

$$(2z + 47z\mu + 75z\mu^2 + 8z\mu^3)\tau,$$

(1.23) $$a^*_{4,3}(z;\nu) = -z - 31z\mu - 48z\mu^2 - 4z\mu^3 + (9z + 53z\mu + 22z\mu^2)\tau,$$

(1.24) $$a^*_{4,4}(z;\nu) = -\mu - 3\mu^2 - z - 9z\mu - 2z\mu^2 + (3\mu + \mu^2 + 5z + 7z\mu)\tau$$

We denote by $A^*(z;\nu)$ the 4×4-matrix with $a^*_{i,k}(z;\nu)$ in its i-th row and k-th column for $i = 1, ..., 4$, $k = 1, ..., 4$. Clearly,

(1.25) $$A^*(z;\nu) = A^*(1;\nu) + (z - 1)V^*(\nu),$$

where the matrix $V^*(\nu)$ does not depend from z. Let

$$(1.26)\qquad X_k(z;\nu)=\begin{pmatrix} f_k(z,\nu)\\ \delta f_k(z,\nu)\\ \delta^2 f_k(z,\nu)\\ \delta^3 f_k(z,\nu)\end{pmatrix},\ X_k^*(z;\nu)=(\nu+1)^2X_k(z;\nu)$$

for $k=1,\,2,\,3,\ |z|>1,\ \nu\in\mathbb{N}_0$. Let further

$$(1.27)\qquad X_k(z;-\nu-2)=X_k(z;\nu),$$

where $\nu\in\mathbb{N}_0$. Let us consider the row

$$(1.28)\qquad R(\nu)=(r_1(\nu),\,r_2(\nu),\,r_3(\nu),\,r_4(\nu)),$$

where

$$(1.29)\qquad r_1(\nu)=\mu(\nu)^2,\ r_2(\nu)=0,\ r_3(\nu)=-2\mu(\nu),\ r_4(\nu)=0.$$

We have the following equalities:

$$A^*(z;\nu)=A^*_{1,0}(z;\nu),\ X_k(z;\nu)=X_{1,0,k}(z;\nu),$$

$$R(\nu)=R_{1,0}(\nu),$$

where $A^*_{\alpha,0}(z;\nu),\ X_{\alpha,0,k}(z;\nu)$
and $R_{\alpha,0}(\nu)$ are studied in [L.A.Gutnik,5, 2006] − [L.A.Gutnik,15, 2006]. We take $\alpha=1$ in (105), [L.A.Gutnik,13, 2007], in (1), [L.A.Gutnik,15, 2006], in §10.1, [L.A.Gutnik,14, 2007], §11.3,[L.A.Gutnik,15, 2006]. Then we have the following Theorem:

Theorem 1. *The column $X_k(z;\nu)$ satisfies to the equation*

$$(1.30)\qquad \nu^5X_k(z;\nu-1)=A^*(z;\nu)X_k(z;\nu),$$

for $\nu\in M_1^*=(-\infty,-2]\cup[1,+\infty))\cap\mathbb{Z},\ k=1,\,2,\,3,\ |z|>1$; moreover, the matrix $A^*(z;\nu)$ has the following property:

$$(1.31)\qquad -\nu^5(\nu+1)^5E_4=A^*(z;-\nu-1)A^*(z;\nu),$$

where E_4 is the 4×4 *unit matrix,* $z\in\mathbb{C},\ \nu\in\mathbb{C}$. The Lemma 11.3.1 in [L.A.Gutnik,15, 2006] have the following formulation for $\alpha=1$:

Theorem 2. *The row $R(\nu)$ has the following property:*

$$(1.32)\qquad R(\nu-1)A^*(1;\nu)=\nu^5R(\nu),\ \textit{where}\ \nu\in\mathbb{C}.$$

§2. Transformation of the system considered in §1.

In view of (1.8), (1.29)

(2.1) $$r_1(\nu) = \mu(\nu)^2 = (\nu+1)^4 = \tau^4,\ r_2(\nu) = 0,\ r_3(\nu) =$$

$$-2\mu(\nu) = -2(\nu+1)^2 = -2\tau^2,\ r_4(\nu) = 0.$$

Let E_4 denotes 4×4-unit matrix, and let $C(\nu)$ is result of replacement of first row of the matrix E_4 by the row in (1.28). Let further $D(\nu)$ denotes the adjoint matrix to the matrix $C(\nu)$. Then

(2.2) $$C(\nu) = \begin{pmatrix} r_1(\nu) & r_2(\nu) & r_3(\nu) & r_4(\nu) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(2.3) $$D(\nu) = \begin{pmatrix} 1 & -r_2(\nu) & -r_3(\nu) & -r_4(\nu) \\ 0 & r_1(\nu) & 0 & 0 \\ 0 & 0 & r_1(\nu) & 0 \\ 0 & 0 & 0 & r_1(\nu) \end{pmatrix}.$$

Clearly,

(2.4) $$C(\nu)D(\nu) = (\mu(\nu))^2 E_4,\ C(-\nu-2) = C(\nu),\ D(-\nu-2) = D(\nu).$$

Let

(2.5) $$A^{**}(z;\nu) = C(\nu-1)A^{*}_{1,0}(z;\nu)D(\nu).$$

Then

(2.6) $$A^{**}(z;-\nu-1) = C(-\nu-2)A^{*}(z;-\nu-1)D(-\nu-1) =$$

$$C(\nu)A^{*}(z;-\nu-1)D(\nu-1),$$

and, in view of (2.4), (1.31), (2.5),

(2.7) $$A^{**}(z;-\nu-1)A^{**}_{1,0}(z;\nu) =$$

$$C(\nu)A^{*}(z;-\nu-1)D(\nu-1)C(\nu-1)A^{*}(z;\nu)D(\nu) =$$

$$-(\mu(\nu)\mu(\nu-1))^2(\nu(\nu+1))^5 E_4.$$

Let

(2.8) $$Y_k(z;\nu) = C(\nu)X_k(z;\nu),$$

where $k = 1, 2, 3$, $|z| > 1$, $\nu \in M_1^{***} = ((-\infty,-2] \cup [0,+\infty)) \cap \mathbb{Z}$. Then, in view of (1.27), (2.4), (1.30),

(2.9) $$Y_k(z;-\nu-2) = Y_k(z;\nu),$$

$$A^{**}(z;\nu)Y_k(z;\nu) = C(\nu-1)A^{*}(z;\nu)D(\nu)C(\nu)X_k(z;\nu) = \tag{2.10}$$

$$\mu(\nu)^2 C(\nu-1)A^{*}(z;\nu)X_k(z;\nu) =$$

$$\mu(\nu)^2\nu^5 C(\nu-1)X_k(z;\nu-1) = \mu(\nu)^2\nu^5 Y_k(z;\nu-1),$$

where $k = 1, 2, 3$, $|z| > 1$, $\nu \in M_1^* = ((-\infty,-2]\cup[1,+\infty))\cap\mathbb{Z}$. Replacing in the equality (2.10) $\nu \in M_1^* = ((-\infty,-2]\cup[1,+\infty))\cap\mathbb{Z}$ by

$$\nu := -\nu - 2 \in M_1^{**} = ((-\infty,-3]\cup[0,+\infty))\cap\mathbb{Z},$$

and taking in account (2.9) we obtain the equality

$$-A^{**}(z;-\nu-2)Y_k(z;\nu) = \mu(\nu)^2(\nu+2)^5 Y_k(z;\nu+1), \tag{2.11}$$

where $k = 1, 2, 3$, $|z| > 1$, $\nu \in M_1^{**} = ((-\infty,-3]\cup[0,+\infty))\cap\mathbb{Z}$.

§3. Calculation of the matrix $A^{**}_{1,0}(z;\nu)$.

We denote by $a^{**}_{i,j}(1;\nu)$, where $i, j = 1, 2, 3, 4$, the expressions, which stand on intersection of i-th row and j-th column in the matrix $A^{ast*}(1;\nu)$. Let

$$V^{**}(\nu) = C(\nu-1)V^{*}(\nu)D(\nu). \tag{3.1}$$

Then, in view of (1.25),

$$A^{**}(z;\nu) = A^{**}(1;\nu) + (z-1)V^{**}(\nu), \tag{3.2}$$

where the matrix $V^{**}(\nu)$ does not depend from z. Clearly, the first row of the matrix $C(\nu-1)A^{*}(1,\nu)$ coincides with the row $R(\nu-1)A^{*}(z,\nu)$ and, according to the Theorem 2 coincides with the row $\nu^5R(\nu)$, i.e. with the first row of the matrix $\nu^5C(\nu)$. Therefore, in view of (2.4), the first row of the matrix $A^{**}(1,\nu)$ is equal to $(\mu_1(\nu)^2)\nu^5\bar{e}_{4,1}$, where $\bar{e}_{4,l}$ denotes the l-th row of the matrix E_4 for $l = 1, 2, 3, 4$. Hence

$$a^{**}_{1,1}(1;\nu) = \tau^4(\tau-1)^5,\ a^{**}_{1,k}(1;\nu) = 0, \text{ where } k = 2, 3, 4. \tag{3.3}$$

Clearly, the second, third and fourth row of the matrix $C(\nu-1)A^{*}(1,\nu)$ coincides with respectively the second, third and fourth row of $A^{*}(1,\nu)$.

In view of (1.13), (1.17) and (1.21),

$$a^{**}_{2,1}(1;\nu) = a^{*}_{2,1}(1;\nu) = -(12\tau^6 - 26\tau^5 + 20\tau^4 - 7\tau^3 + \tau^2) = -\tau^2\times \tag{3.4}$$

$$(\tau-1)(12\tau^3 - 14\tau^2 + 6\tau - 1) = -\tau^2(\tau-1)(2\tau-1)(6\tau^2 - 4\tau + 1).$$

$$a^{**}_{3,1}(1;\nu) = a^{*}_{3,1}(1;\nu) = 8\tau^7 - 26\tau^6 + 33\tau^5 - 21\tau^4 + 7\tau^3 - \tau^2 = \tau^2\times \tag{3.5}$$

$$(\tau-1)(8\tau^4 - 18\tau^3 + 15\tau^2 - 6\tau + 1) = \tau^2(\tau-1)^2\times$$

$$(8\tau^3 - 10\tau^2 + 5\tau - 1) = \tau^2(\tau-1)^2(2\tau-1)(4\tau^2 - 3\tau + 1).$$

(3.6) $$a^{**}_{4,1}(1;\nu) = -\tau^2(\tau-1)^3(2\tau-1)(2\tau^2-2\tau+1) =$$

$$a^{*}_{4,1}(1;\nu) = -\tau^2(\tau-1)(4\tau^5-14\tau^4+20\tau^3-15\tau^2+6\tau-1) =$$
$$-\tau^2(\tau-1)^2(4\tau^4-10\tau^3+10\tau^2-5\tau+1) =$$
$$-\tau^2(\tau-1)^3(4\tau^3-6\tau^2+4\tau-1).$$

In view of (2.2), (2.3), (2.5)

(3.7) $$a^{**}_{k,j}(1;\nu) = -r_j(\nu)a^{*}_{k,1}(1;\nu) + r_1(\nu)a^{*}_{k,j}(1;\nu)$$

where j, $k = 2$, 3, 4.

In view of (1.14), (1.18), (1.22), (1.16), (1.20), (3.7) and (1.29),

(3.8) $$a^{**}_{2,2}(1;\nu) = \tau^4 a^{*}_{2,2}(1;\nu) = \tau^5(\tau-1)(17\tau^3-24\tau^2+12\tau-2) =$$

$$(17\tau^5-41\tau^4+36\tau^3-14\tau^2+2\tau)\tau^4.$$

(3.9) $$a^{**}_{3,2}(1;\nu) = \tau^4 a^{*}_{3,2}(1;\nu) = -2\tau^5(\tau-1)^2(6\tau^3-9\tau^2+5\tau^2-1) =$$

$$-(12\tau^6+42\tau^5-58\tau^4+40\tau^3-14\tau^2+2\tau)\tau^4 =$$
$$-2\tau^5(6\tau^5-21\tau^4+29\tau^3-20\tau^2+7\tau-1) =$$
$$-2\tau^5(\tau-1)(6\tau^4-15\tau^3+14\tau^2-6\tau+1),$$

(3.10) $$a^{**}_{4,2}(1;\nu) = \tau^4 a^{*}_{4,2}(1;\nu) = \tau^5(\tau-1)^3(8\tau^3-14\tau^2+9\tau-2) =$$

$$(8\tau^7-38\tau^6+75\tau^5-79\tau^4+47\tau^3-15\tau^2+2\tau)\tau^4 =$$
$$\tau^5(\tau-1)(8\tau^5-30\tau^4+45\tau^3-34\tau^2+13\tau-2) =$$
$$\tau^5(\tau-1)^2(8\tau^4-22\tau^3+23\tau^2-11\tau+2),$$

(3.11) $$a^{**}_{k,4}(1;\nu) = \tau^4 a^{*}_{1,0,k,4}(1;\nu) = 0$$

for $k = 2$, 3. In view of (1.24), (3.7) and (1.29),

(3.12) $$a^{**}_{4,4}(1;\nu) = \tau^4 a^{*}_{4,4}(1;\nu) = \tau^4\times$$

$$(\tau^5-5\tau^4+10\tau^3-10\tau^2+5\tau-1) = \tau^4(\tau-1)^5.$$

In view of (1.15),

(3.13) $$a^{*}_{2,3}(1;\nu) = 8\tau^3-14\tau^2+7\tau-1 = (\tau-1)(8\tau^2-6\tau+1),$$

In view of (1.19),

(3.14) $$a^{*}_{\alpha,0,3,3}(1;\nu) = \tau^5-9\tau^4+20\tau^3-18\tau^2+7\tau-1 =$$

$$(\tau-1)(\tau^4-8\tau^3+12\tau^2-6\tau+1=$$
$$(\tau-1)^2(\tau^3-7\tau^2+5\tau-1),$$

In view of (1.23),

(3.15) $$a^*_{4,3}(1;\nu)=-4\tau^6+22\tau^5-48\tau^4+53\tau^3-31\tau^2+9\tau-1=$$
$$(\tau-1)(-4\tau^5+18\tau^4-30\tau^3+23\tau^2-8\tau+1)=$$
$$(\tau-1)^2(-4\tau^4+14\tau^3-16\tau^2+7\tau-1)=$$
$$(\tau-1)^3(-4\tau^3+10\tau^2-6\tau+1)$$

In view of (3.4), (3.13), (3.5), (3.14), (3.6), (3.15), (3.7) and (1.29),

(3.16) $$a^{**}_{2,3}(1;\nu)=2\tau^2a^*_{2,1}(1;\nu)+\tau^4a^*_{2,3}(1;\nu)=(\tau-1)\times$$
$$(-2\tau^4(2\tau-1)(6\tau^2-4\tau+1)+\tau^4(2\tau-1)(4\tau-1))=$$
$$\tau^4(\tau-1)(2\tau-1)(-12\tau^2+12\tau-3)=-3\tau^4(2\tau-1)^3,$$

(3.17) $$a^{**}_{3,3}(1;\nu)=2\tau^2a^*_{3,1}(1;\nu)+\tau^4a^*_{3,3}(1;\nu)=(\tau-1)^2\times$$
$$(2\tau^4(8\tau^3-10\tau^2+5\tau-1)+\tau^4(\tau^3-7\tau^2+5\tau-1))=$$
$$(\tau-1)^2(17\tau^3-27\tau^2+15\tau-3)=\tau^4(\tau-1)^2((\tau-1)^3+2(2\tau-1)^3),$$

(3.18) $$a^{**}_{4,3}(1;\nu)=2\tau^2a^*_{4,1}(1;\nu)+\tau^4a^*_{4,3}(1;\nu)=(\tau-1)^3\times$$
$$(-2\tau^4(4\tau^3-6\tau^2+4\tau-1)+\tau^4(-4\tau^3+10\tau^2-6\tau+1))=$$
$$-\tau^4(\tau-1)^3(12\tau^3-22\tau^2+14\tau-3)=$$
$$-\tau^4(\tau-1)^3(2\tau-1)(6\tau^2-8\tau+3).$$

§4. Properties of the functions considered in §1.

The function $t^r(R(t,\nu))^2$ (see (1.3)) is regular at $t=\infty$ for $r=0,\ 1,\ 2$, and has a pole of first order at $t=\infty$ for $r=3$. So, in the case $r=0,\ 1,\ 2$ we have the equalities

(4.1) $$Res(t^r(R(t,\nu))^2, t=\infty)=-[r/3] \text{ for } r=0,\ 1,\ 2,\ 3,$$

(4.2) $$\lim_{t\to\infty} t^r(R(t,\nu))^2=0 \text{ for } r=0,\ 1,\ 2,\ 3.$$

In view of (1.4),

(4.3) $$\delta^r f^*_2(z,\nu)=\sum_{t=1}^{+\infty} z^{-t}(\nu+1)^2(-t)^r(R(t,\nu))^2,$$

where we consider $r = 0, 1, 2, 3$. Expanding $(\nu+1)^2(-t)^r(R(t,\nu))^2$ into partial fractions relatively t, we obtain

$$(4.4)\qquad (\nu+1)^2(-t)^r(R(t;\nu))^2 = \sum_{i=1}^{2}\left(\sum_{k=0}^{\nu+1}\beta^{(r)}_{i,k,\nu}(t+k)^{-i}\right),$$

where $\nu \in \mathbb{N}_0$, $r = 0, 1, 2, 3$,

$$(4.5)\qquad \beta^{(r)}_{2-j,k,\nu} = (\nu+1)^2\frac{1}{j!}\lim_{t\to -k}\left(\frac{\partial}{\partial t}\right)^j((-t)^r(R(t,\nu)(t+k))^2)$$

for $j = 0, 1$. In view of (4.1) and (4.4),

$$(4.6)\qquad \sum_{k=0}^{\nu+1}\beta^{(r)}_{1,k,\nu} = -[r/3](\nu+1)^2 \text{ for } r = 0, 1, 2, 3.$$

In view of (4.4),

$$(4.7)\qquad -(\nu+1)^2\frac{\partial}{\partial t}((-t)^r(R(t;\nu))^2) = \sum_{i=1}^{2}\left(\sum_{k=0}^{\nu+1}\beta^{(r)}_{i,k,\nu}i(t+k)^{-i-1}\right),$$

where $\nu \in \mathbb{N}_0$, $r = 0, 1, 2, 3$. Let

$$(4.8)\qquad S_{i,k}(\nu) = -\left(\sum_{\kappa=k+1}^{\nu+k}1/\kappa^i\right) - \left(\sum_{\kappa=1}^{\nu+1-k}1/\kappa^i\right) + \sum_{\kappa=1}^{k}1/\kappa^i,$$

where $\nu \in \mathbb{N}_0$, $i \in \mathbb{N}$, $k \in [0, \nu+1] \cap \mathbb{Z}$. In particular,

$$(4.9)\qquad S_{1,0}(0) = -1,\ S_{1,1}(0) = 1,$$

$$(4.10)\qquad S_{1,0}(1) = -5/2,\ S_{1,1}(1) = -1/2,\ S_{1,2}(1) = 7/6,$$

$$(4.11)\qquad S_{1,0}(2) = -(1+1/2) - (1+1/2+1/3) = -10/3,$$

$$(4.12)\qquad S_{1,1}(2) = -(1/2+1/3) - (1+1/2) + 1 = -4/3$$

$$(4.13)\qquad S_{1,2}(2) = -(1/3+1/4) - 1 + (1+1/2) = -1/12,$$

$$(4.14)\qquad S_{1,3}(2) = -(1/4+1/5) + (1+1/2+1/3) = 83/60.$$

In view of (4.5), (1.3) and (4.8)

$$(4.15)\qquad \beta^{(0)}_{2,k,\nu} = \left(\frac{(\nu+k)!}{k!}\times\frac{\nu+1}{(\nu+1-k)!k!}\right)^2 = \binom{\nu+1}{k}^2\binom{\nu+k}{k}^2,$$

$$\beta^{(0)}_{1,k,\nu} = 2\beta^{(0)}_{2,k,\nu} S_{1,k}(\nu), \tag{4.16}$$

where $\nu \in \mathbb{N}_0$, $i \in \mathbb{N}$, $k \in [0, \nu+1] \cap \mathbb{Z}$. In particular,

$$\beta^{(0)}_{2,0,0} = \beta^{(0)}_{2,1,0} = \beta^{(0)}_{2,0,1} = 1,\ \beta^{(0)}_{2,1,1} = 16,\ \beta^{(0)}_{2,2,1} = 9, \tag{4.17}$$

$$\beta^{(0)}_{2,0,1} = 1,\ \beta^{(0)}_{2,1,1} = 16,\ \beta^{(0)}_{2,2,1} = 9, \tag{4.18}$$

$$\beta^{(0)}_{2,0,2} = 1,\ \beta^{(0)}_{2,1,2} = 81,\ \beta^{(0)}_{2,2,2} = 324,\ \beta^{(0)}_{2,3,2} = 100. \tag{4.19}$$

$$\beta^{(0)}_{2,2,2} = 324,\ \beta^{(0)}_{2,3,2} = 100. \tag{4.20}$$

In view of (4.16), (4.17) – (4.19) and (4.9) – (4.14),

$$\beta^{(0)}_{1,k,0} = 2\beta^{(0)}_{2,k,0} S_{1,k}(0) = -2 \times 1 \times (-1)^k \text{ for } k = 0,\ 1, \tag{4.21}$$

$$\beta^{(0)}_{1,0,1} = 2\beta^{(0)}_{2,0,1} S_{1,0}(1) = 2 \times 1 \times (-5/2) = -5, \tag{4.22}$$

$$\beta^{(0)}_{1,1,1} = 2\beta^{(0)}_{2,1,1} S_{1,1}(1) = 2 \times 16 \times (-1/2) = -16, \tag{4.23}$$

$$\beta^{(0)}_{1,2,1} = 2\beta^{(0)}_{2,2,1} S_{1,2}(1) = 2 \times 9 \times (7/6) = 21, \tag{4.24}$$

$$\beta^{(0)}_{1,0,2} = 2\beta^{(0)}_{2,0,2} S_{1,0}(2) = 2(-10/3) = -20/3, \tag{4.25}$$

$$\beta^{(0)}_{1,1,2} = 2\beta^{(0)}_{2,1,2} S_{1,1}(2) = 2 \times 81 \times (-4/3) = -216, \tag{4.26}$$

$$\beta^{(0)}_{1,2,2} = 2\beta^{(0)}_{2,2,2} S_{1,2}(2) = 2 \times 324 \times (-1/12) = -54, \tag{4.27}$$

$$\beta^{(0)}_{1,3,2} = 2\beta^{(0)}_{2,3,2} S_{1,3}(2) = 2 \times 100 \times (83/60) = 830/3. \tag{4.28}$$

We put in (4.4) $r = 0$, and multiply both sides of obtained equality by $(-t)^r$ for $r = 0,\ 1,\ 2,\ 3$. Then we see that

$$- t(\nu+1)^2 (R(t;\nu))^2 = \sum_{i=1}^{2} \left(\sum_{k=0}^{\nu+1} \frac{\beta^{(0)}_{i,k,\nu}(-t-k+k)}{(t+k)^i} \right) = \tag{4.29}$$

$$\left(\sum_{k=0}^{\nu+1} \frac{k\beta^{(0)}_{2,k,\nu}}{(t+k)^2} \right) + \left(\sum_{k=0}^{\nu+1} \frac{k\beta^{(0)}_{1,k,\nu} - \beta^{(0)}_{2,k,\nu}}{t+k} \right) - \sum_{k=0}^{\nu+1} \beta^{(0)}_{1,k,\nu},$$

$$(4.30)\qquad (-t)^2(\nu+1)^2(R(\alpha,t;\nu))^2=\sum_{i=1}^{2}\left(\sum_{k=0}^{\nu+1}\frac{\beta^{(0)}_{i,k,\nu}(t+k-k)^2}{(t+k)^i}\right)=$$

$$\left(\sum_{k=0}^{\nu+1}\frac{k^2\beta^{(0)}_{2,k,\nu}}{(t+k)^2}\right)+\left(\sum_{k=0}^{\nu+1}\frac{k^2\beta^{(0)}_{1,k,\nu}-2k\beta^{(0)}_{2,k,\nu}}{t+k}\right)+\sum_{k=0}^{\nu+1}(\beta^{(0)}_{2,k,\nu}+(t-k)\beta^{(0)}_{1,k,\nu}),$$

$$(4.31)\qquad (-t)^3(\nu+1)^2(R(\alpha,t;\nu))^2=\sum_{i=1}^{2}\left(\sum_{k=0}^{\nu+1}\frac{\beta^{(0)}_{i,k,\nu}(-t-k+k)^3}{(t+k)^i}\right)=$$

$$\left(\sum_{k=0}^{\nu+1}\frac{k^3\beta^{(0)}_{2,k,\nu}}{(t+k)^2}\right)+\left(\sum_{k=0}^{\nu+\alpha}\frac{k^3\beta^{(0)}_{1,k,\nu}-3k^2\beta^{(0)}_{2,k,\nu}}{t+k}\right)-$$
$$\left(\sum_{k=0}^{\nu+1}(t-2k)(\beta^{(0)}_{2,k,\nu}\right)-\sum_{k=0}^{\nu+1}(t^2-kt+k^2)\beta^{(0)}_{1,k,\nu}.$$

The equality (4.6) with $r=0$ again follows from (4.2) with $r=1$ and (4.29); moreover, in view of (4.4) with $r=1$, and (4.29),

$$(4.32)\qquad \beta^{(1)}_{2,k,\nu}=k\beta^{(0)}_{2,k,\nu},\ \beta^{(1)}_{1,k,\nu}=k\beta^{(0)}_{1,k,\nu}-\beta^{(0)}_{2,k,\nu}$$

for $k=0,\ ...,\ \nu+1$. The equality (4.6) with $r=1$ again follows from (4.2) with $r=2$, (4.6) with $r=0$, (4.30) and (4.32); moreover, in view of (4.4) with $r=2$, and (4.30),

$$(4.33)\qquad \beta^{(2)}_{2,k,\nu}=k^2\beta^{(0)}_{2,k,\nu},\ \beta^{(2)}_{1,k,\nu}=k^2\beta^{(0)}_{1,k,\nu}-2k\beta^{(0)}_{2,k,\nu}$$

for $k=0,\ ...,\ \nu+1$. The equality (4.6) with $r=2$ again follows from (4.2), (4.6) with both $r\in\{0,\ 1\}$, (4.31), (4.32) and from (4.33); moreover, in view of (4.4) with $r=3$, and (4.31),

$$(4.34)\qquad \beta^{(3)}_{2,k,\nu}=k^3\beta^{(0)}_{2,k,\nu},\ \beta^{(3)}_{1,k,\nu}=k^3\beta^{(0)}_{1,k,\nu}-3k^2\beta^{(0)}_{2,k,\nu}$$

for $k=0,\ ...,\ \nu+1$. In view of (1.4) – (1.6),

$$(4.35)\qquad (\delta^r)f^*_3(z,\nu)=(\log(z))(\delta)^r)f^*_2(z,\nu)+$$
$$r(\delta)^{r-1}f^*_2(z,\nu)+(\delta)^rf^*_4(z,\nu)=(\log(z))(\delta)^r)f^*_2(z,\nu)+$$
$$\sum_{t=1}^{+\infty}z^{-t}(\nu+1)^2\left(r(-t)^{r-1}-(-t)^r\frac{\partial}{\partial t}\right)R^2(t,\nu)=$$
$$(\log(z))(\delta)^r)f^*_2(z,\nu)-\sum_{t=1}^{+\infty}z^{-t}(\nu+1)^2\frac{\partial}{\partial t}((-t)^rR^2)(t,\nu).$$

In view of (4.4), (4.7), (4.3) and (4.35),

$$(4.36)\qquad \delta^rf^*_{2+j}(z;\nu)-j(\log(z))\delta^rf^*_2(z;\nu)=$$

$$\sum_{i=1}^{2}\left(\sum_{t=1}^{+\infty}\left(\sum_{k=0}^{\nu+1}(1-j+ij)\beta_{i,k,\nu}^{(r)}z^k z^{-t-k}(t+k)^{-i-j}\right)\right)=$$
$$\sum_{i=1}^{2}\left(\sum_{k=0}^{\nu+1}(1-j+ij)\beta_{i,k,\nu}^{(r)}z^k\left(\sum_{t=1}^{+\infty}z^{-t-k}(t+k)^{-i-j}\right)\right)=$$
$$\sum_{i=1}^{2}\left(\sum_{k=0}^{\nu}(1-j+ij)\beta_{i,k,\nu}^{(r)}z^k\left(L_{i+1}(1/z)-\sum_{\tau=1}^{k}z^{-\tau}i(\tau)^{-i-j}\right)\right)=$$
$$\left(\sum_{i=1}^{2}(1-j+ij)\beta_i^{*(r)}(z;\nu)L_{i+j}(1/z)\right)-\beta_{3+j}^{*(r)}(z;\nu),$$

where $j=0,\ 1,\ r=0,\ 1,\ 2,\ 3,\ |z|>1$,

(4.37) $$L_s(1/z)=\sum_{n=1}^{\infty}1/(z^n n^s),\ \beta_i^{*(r)}(z;\nu)=\sum_{k=0}^{\nu+1}\beta_{i,k,\nu}^{(r)}z^k,$$

for $s\in\mathbb{Z},\ i\in\{1,2\},\ \nu\in\mathbb{N}_0$,

(4.38) $$\beta_{3+j}^{*(r)}(z;\nu)=\sum_{i=1}^{2}\left(\sum_{k=0}^{\nu+1}(1-j+ij)\beta_{i,k,\nu}^{(r)}\left(\sum_{\tau=1}^{k}z^{k-\tau}(\tau)^{-i-j}\right)\right)=$$
$$\sum_{\sigma=0}^{\nu+\alpha-1}z^{\sigma}\sum_{\tau=1}^{\nu+1-\sigma}\sum_{i=1}^{2}(1-j+ij)\beta_{i,\sigma+\tau,\nu}^{(r)}(\tau)^{-i-j}.$$

In fiew of (1.2),

(4.39) $$f_1^*(z,\nu)=\beta_2^{*(0)}(z;\nu).$$

In view of (4.6) and (4.37), if $r=0,\ 1,\ 2$, then

(4.40) $$\beta_1^{*(r)}(z;\nu)=(z-1)\beta_1^{*\vee(r)}(z;\nu),$$

where $\beta_1^{*\vee(r)}(z;\nu)\in\mathbb{Q}[z]$, when $\nu\in\mathbb{N}_0$. In view of (4.6) and (4.37),

(4.41) $$\beta_1^{*(3)}(z;\nu)=-(\nu+1)^2+(z-1)\beta_1^{*\vee(3)}(z;\nu),$$

where $\beta_1^{*\vee(3)}(z;\nu)\in\mathbb{Q}[z]$, when $\nu\in\mathbb{N}_0$. In view of (4.32) – (4.34), (4.37),

(4.42) $$\beta_2^{*(1)}(z;\nu)=\delta\beta_2^{*(0)}(z;\nu)=\delta f_1^*(z,\nu),\ \beta_1^{*(1)}(z;\nu)=$$
$$\delta\beta_1^{*(0)}(z;\nu)-\beta_2^{*(0)}(z;\nu),$$

(4.43) $$\beta_2^{*(2)}(z;\nu)=\delta^2\beta_2^{*(0)}(z;\nu)=\delta^2 f_1^*(z,\nu),\ \beta_1^{*(2)}(z;\nu)=$$
$$\delta^2\beta_1^{*(0)}(z;\nu)-2\delta\beta_2^{*(0)}(z;\nu),$$

(4.44) $\beta_2^{*(3)}(z;\nu)=\delta^3\beta_2^{*(0)}(z;\nu),\ \beta_1^{*(3)}(z;\nu)=\delta^3\beta_1^{*(0)}(z;\nu)-3\delta^2\beta_2^{*(0)}(z;\nu).$

Clearly,

$$(-\delta)^k L_n(1/z)=L_{n-k}(1/z), \tag{4.45}$$

where $k\in[0,+\infty)\cap\mathbb{Z}$, $n\in\mathbb{Z}$, $|z|>1$,

$$L_1(1/z)=-\log(1-1/z),\ -\delta L_1(1/z)=1/(z-1)= \tag{4.46}$$

$$L_0(1/z),\ \delta^2L_1(1/z)=1/(z-1)+1/(z-1)^2=$$

$$L_{-1}(1/z),\ -\delta^3L_1(1/z)=L_{-2}(1/z)=1/(z-1)+3/(z-1)^2+2/(z-1)^3.$$

We apply the operator δ to the equality (4.36) for $r=0,\,1,\,2$. Then, in view of (4.45), we obtain the equality

$$\delta^{r+1}f^*_{2+j}(z;\nu)-j(\log(z))\delta^{r+1}f^*_2(z,\nu)=j\delta^r f^*_2(z;\nu)+ \tag{4.47}$$

$$\left(\sum_{i=1}^{2}((1-j+ij)\delta\beta_i^{*(r)}(z;\nu))L_{i+j}(1/z)\right)-\delta\beta_{3+j}^{*(r)}(z;\nu)-$$

$$\left(\sum_{i=1}^{2}(1-j+ij)\beta_i^{*(r)}(z;\nu)L_{i+j-1}(1/z)\right)=$$

$$j\left(\left(\sum_{i=1}^{2}\beta_i^{*(r)}(z;\nu)L_i(1/z)\right)-\beta_3^{*(r)}(z;\nu)\right)+$$

$$\left(\sum_{i=1}^{2}((1-j+ij)\delta\beta_i^{*(r)}(z;\nu))L_{i+j}(1/z)\right)-\delta\beta_{3+j}^{*(r)}(z;\nu)-$$

$$\left(\sum_{i=1}^{2}(1-j+ij)\beta_i^{*(r)}(z;\nu)L_{i+j-1}(1/z)\right).$$

It follws from (4.47) with $j=0$ that

$$\delta^{r+1}f^*_2(z;\nu)=-\delta\beta_3^{(r)}(z;\nu)+ \tag{4.48}$$

$$\left(\sum_{i=1}^{2}(\delta\beta_i^{*(r)}(z;\nu))L_i(1/z)-\beta_i^{*(r)}(z;\nu)L_{i-1}(1/z)\right)=$$

$$(\delta\beta_2^{*(r)}(z;\nu))L_2(1/z)+(\delta\beta_1^{*(r)}(z;\nu)-\beta_2^{*(r)}(z;\nu))L_1(1/z)-$$

$$\delta\beta_3^{*(r)}(z;\nu)-\beta_1^{*(r)}(z;\nu)L_0(1/z).$$

In view of (4.36) with $j=0$, (4.48), (4.40),

$$\beta_2^{*(r)}(z;\nu)=\delta\beta_2^{*(r-1)}(z;\nu))=\delta^r\beta_2^{*(0)}(z;\nu), \tag{4.49}$$

$$\beta_1^{*(r)}(z;\nu))=\delta\beta_1^{*(r-1)}(z;\nu)-\beta_2^{*(r-1)}(z;\nu)= \tag{4.50}$$

$$\delta^r\beta_1^{*(0)}(z;\nu) - r\delta^{r-1}\beta_2^{*(0)}(z;\nu),$$

(4.51) $$\beta_3^{*(r)}(z;\nu) = \delta\beta_3^{*(r-1)}(z;\nu) + \beta_1^{*(r-1)}(z;\nu))L_0(1/z) =$$

$$\delta\beta_3^{*(r-1)}(z;\nu) + \beta_1^{*\vee(r-1)}(z;\nu),$$

where $r = 1, 2, 3$. The equalities (4.42) – (4.44) follow from the equalities (4.49) and (4.50) again. In view of (4.15), (4.37), (1.2) and (4.49)

(4.52) $$\beta_2^{*(r)}(z;\nu) = \delta^r f_1^*(z;\nu) \in \mathbb{N}[z],$$

where $\nu \in \mathbb{N}_0$, $r = 0, 1, 2, 3$. It follws from (4.47) with $j = 1$ that

(4.53) $$\delta^{r+1}f_3^*(z,\nu) = (\log(z))\delta^{r+1}f_2^*(z,\nu)+$$

$$\left(\left(\sum_{i=1}^{2}\beta_i^{*(r)}(z;\nu)L_i(1/z)\right) - \beta_3^{*(r)}(z;\nu)\right)+$$

$$\left(\sum_{i=1}^{2}i(\delta\beta_i^{*(r)}(z;\nu))L_{i+1}(1/z)\right) - \delta\beta_4^{*(r)}(z;\nu)-$$

$$\left(\sum_{i=1}^{2}i\beta_i^{*(r)}(z;\nu)L_i(1/z)\right) = (\log(z))\delta^{r+1}f_2^*(z;\nu)+$$

$$\left(\sum_{i=1}^{2}i(\delta\beta_i^{*(r)}(z;\nu))L_{i+1}(1/z)\right) - \delta\beta_4^{*(r)}(z;\nu)-$$

$$\beta_2^{*(r)}(z;\nu)L_2(1/z) - (\beta_3^{*(r)}(z;\nu) + \delta\beta_4^{(r)}(z;\nu)) =$$

$$(\log(z))\delta^{r+1}f_2^*(z;\nu) + 2(\delta\beta_2^{*(r)}(z;\nu))L_3(1/z)+$$

$$(\delta\beta_1^{*(r)}(z;\nu) - \beta_2^{*(r)}(z;\nu))L_2(1/z) - (\delta\beta_4^{*(r)}(z;\nu) + \beta_3^{*(r)}(z;\nu)).$$

In view of (4.36) with $j = 1$, (4.53),

(4.54) $$\beta_2^{*(r+1)}(z;\nu) = \delta\beta_2^{*(r)}(z;\nu) = \delta^{r+1}\beta_2^{*(0)}(z;\nu),$$

(4.55) $$\beta_1^{*(r+1)}(z;\nu) = \delta\beta_2^{*(r)}(z;\nu) - \beta_2^{*(r)}(z;\nu) =$$

$$\delta^{r+1}\beta_1^{(0)}(z;\nu) - \delta^r\beta_2^{(0)}(z;\nu),$$

where $r = 0, 1, 2$, and we obtain (4.49) – (4.49) again. Moreover,

(4.56) $$\beta_4^{*(r+1)}(z;\nu) = \delta\beta_4^{*(r)}(z;\nu) + \beta_3^{*(r)}(z;\nu),$$

where $r = 0, 1, 2$. If we take now $z \in (1, +\infty)$ and will tend z to 1, then, in view of (4.36), (4.40), (4.41) and (4.46)

(4.57) $$\delta^r f_{\alpha,0,2+j}^*(1,\nu) = \lim_{z\to 1+0}\delta^r f_2^*(z,\nu) =$$

$$(1-j+ij)\beta_2^{*(r)}(1;\nu)\zeta(2+j)-\beta_{3+j}^{*(r)}(1;\nu)=$$
$$(1+j)\beta_2^{*(r)}(1;\nu)\zeta(2+j)-\beta_{3+j}^{*(r)}(1;\nu),$$

where $r=0,\,1,\,2,\,i=2,\,j=0,\,1,$

(4.58)
$$\lim_{z\to 1+0}(z-1)\delta^3 f_2^*(z,\nu)=0.$$

In view of (4.37), (4.38), (4.17 – (4.24),

(4.59)
$$\beta_1^{*(0)}(z;0)=\beta_{1,0,0}^{(0)}+\beta_{1,1,0}^{(0)}z=-2+2z,$$

(4.60)
$$\beta_2^{*(0)}(z;0)=\beta_{2,0,0}^{(0)}+\beta_{2,1,0}^{(0)}z=1+z,$$

(4.61)
$$\beta_3^{*(0)}(z;0)=\beta_{1,1,0}^{(0)}+\beta_{2,1,0}^{(0)}=3,$$

(4.62)
$$\beta_4^{*(0)}(z;0)=\beta_{1,1,0}^{(0)}+2\beta_{2,1,0}^{(0)}=4,$$

(4.63)
$$\beta_1^{*(0)}(z;1)=\beta_{1,0,1}^{(0)}+\beta_{1,1,1}^{(0)}z+$$
$$\beta_{1,2,1}^{(0)}z^2=-5-16z+21z^2=(z-1)(21z+5),$$

(4.64)
$$\beta_2^{*(0)}(z;1)=\beta_{2,0,1}^{(0)}+\beta_{2,1,1}^{(0)}z+\beta_{2,2,1}^{(0)}z^2=1+16z+9z^2,$$

(4.65)
$$\beta_3^{*(0)}(z;1)=\beta_{1,1,1}^{(0)}+\beta_{2,1,1}^{(0)}+$$
$$\frac{1}{2}\beta_{1,2,1}^{(0)}+\frac{1}{4}\beta_{2,2,1}^{(0)}+(\beta_{1,2,1}^{(0)}+\beta_{2,2,1}^{(0)})z=$$
$$-16+16+\frac{1}{2}\times 21+\frac{1}{4}\times 9+(21+9)z=\frac{51}{4}+30z,$$

(4.66)
$$\beta_4^{*(0)}(z;1)=\beta_{1,1,1}^{(0)}+2\beta_{2,1,1}^{(0)}+$$
$$\frac{1}{4}\beta_{1,2,1}^{(0)}+2\times\frac{1}{8}\beta_{2,2,1}^{(0)}+(\beta_{1,2,1}^{(0)}+2\beta_{2,2,1}^{(0)})z=$$
$$-16+2\times 16+\frac{1}{4}\times 21+\frac{1}{4}\times 9+(21+18)z=\frac{47}{2}+39z.$$

(4.67)
$$\beta_1^{*(0)}(z;2)=\beta_{1,0,2}^{(0)}+\beta_{1,1,2}^{(0)}z+$$
$$\beta_{1,2,2}^{(0)}z^2+\beta_{1,3,2}^{(0)}z^3=$$
$$-\frac{20}{3}-216z-54z^2+\frac{830}{3}z^3=(z-1)(830z^2+668z+20)/3.$$

(4.68)
$$\beta_2^{*(0)}(z;2)=\beta^{(0)}_{2,0,2}+\beta^{(0)}_{2,1,2}z+\beta^{(0)}_{2,2,2}z^2+$$
$$\beta^{(0)}_{2,3,2}z^3=1+81z+324z^2+100z^3.$$

(4.69)
$$\beta_3^{*(0)}(z;2)=\beta^{(0)}_{1,1,2}+\beta^{(0)}_{2,1,2}+$$
$$\frac{1}{2}\beta^{(0)}_{1,2,2}+\frac{1}{4}\beta^{(0)}_{2,2,2}+\frac{1}{3}\beta^{(0)}_{1,3,2}+\frac{1}{9}\beta^{(0)}_{2,3,2}+$$
$$(\beta^{(0)}_{1,2,2}+\beta^{(0)}_{2,2,2})z+\left(\frac{1}{2}\beta^{(0)}_{1,3,2}+\frac{1}{4}\beta^{(0)}_{2,3,2}\right)z+(\beta^{(0)}_{1,3,2}+\beta^{(0)}_{2,3,2})z^2=$$
$$-216+81-\frac{54}{2}+\frac{324}{4}+\frac{830+100}{9}+(324-54+415/3+100/4)z+$$
$$(830/3+100)z^2=\frac{67}{3}+\frac{1300}{3}z+\frac{1130}{3}z^2.$$

(4.70)
$$\beta_4^{*(0)}(z;2)=\beta^{(0)}_{1,1,2}+2\beta^{(0)}_{2,1,2}+(1/4)\beta^{(0)}_{1,2,2}+2\times(1/8)\beta^{(0)}_{2,2,2}+$$
$$(1/9)\beta^{(0)}_{1,3,2}+2\times(1/27)\beta^{(0)}_{2,3,2}+(\beta^{(0)}_{1,2,2}+2\beta^{(0)}_{2,2,2})z+$$
$$\left(\frac{1}{4}\beta^{(0)}_{1,3,2}+2\times\frac{1}{8}\beta^{(0)}_{2,3,2}\right)z+(\beta^{(0)}_{1,3,2}+2\beta^{(0)}_{2,3,2})z^2=-216+162+$$
$$\frac{-54+324}{4}+\frac{830+200}{27}+((-57+648+(830/3+100)/4)z+$$
$$(830/3+200)z^2=2789/54+4129z/6+1430z^2/3.$$

In view of (4.49) – (4.51), (4.56) and (4.59) – (4.70),

(4.71)
$$\beta_1^{*(1)}(z;0)=\delta\beta_1^{*(0)}(z;0)-\beta_2^{*(0)}(z;0)=2z-1-z=z-1,$$

(4.72)
$$\beta_2^{*(1)}(z;0)=\delta\beta_2^{*(0)}(z;0)=z,$$

(4.73)
$$\beta_3^{*(1)}(z;0)=\delta\beta_3^{*(0)}(z;0)+\beta_1^{*\vee(0)}(z;0)=2$$

(4.74)
$$\beta_4^{*(1)}(z;0)=\delta\beta_4^{*(0)}(z;0)+\beta_3^{*(0)}(z;0)=3,$$

(4.75)
$$\beta_1^{*(1)}(z;1)=\delta\beta_1^{*(0)}(z;1)-\beta_2^{*(0)}(z;1)=-16z+$$
$$42z^2-(1+16z+9z^2)=-1-32z+33z^2=(z-1)(33z+1),$$

(4.76)
$$\beta_2^{*(1)}(z;1)=\delta\beta_2^{*(0)}(z;1)=16z+18z^2,$$

(4.77)
$$\beta_3^{*(1)}(z;1)=\delta\beta_3^{*(0)}(z;1)+\beta_1^{*\vee(0)}(z;1)=30z+21z+5=51z+5,$$

(4.78) $\beta_4^{*(1)}(z;1)=\delta\beta_4^{*(0)}(z;1)+\beta_3^{*(0)}(z;1)=39z+\frac{51}{4}+30z=69z+\frac{51}{4},$

(4.79) $$\beta_1^{*(1)}(z;2)=\delta\beta_1^{*(0)}(z;2)-\beta_2^{*(0)}(z;2)=$$
$$-216z-108z^2+830z^3-(1+81z+324z^2+100z^3)=$$
$$-1-297z-432z^2+730z^3=(z-1)(730z^2+298z+1),$$

(4.80) $$\beta_2^{*(1)}(z;2)=\delta\beta_2^{*(0)}(z;2)=81z+648z^2+300z^3,$$

(4.81) $$\beta_3^{*(1)}(z;2)=\delta\beta_3^{*(0)}(z;2)+\beta_1^{*\vee(0)}(z;2)=1300z/3+$$
$$(2260z^2+830z^2+668z+20)/3=1030z^2+656z+\frac{20}{3},$$

(4.82) $$\beta_4^{*(1)}(z;2)=\delta\beta_4^{*(0)}(z;2)+\beta_3^{*(0)}(z;2)=4129z/6+2860z^2/3+$$
$$(67+1300z+1130z^2)/3=67/3+6729z/6+1330z^2,$$

In view of (4.49) – (4.51), (4.56) and (4.71) – (4.82),

(4.83) $$\beta_1^{*(2)}(z;0)=\delta\beta_1^{*(1)}(z;0)-\beta_2^{*(1)}(z;0)=z-z=0,$$

(4.84) $$\beta_2^{*(2)}(z;0)=\delta\beta_2^{*(1)}(z;0)=z,$$

(4.85) $$\beta_3^{*(2)}(z;0)=\delta\beta_3^{*(1)}(z;0)+\beta_1^{*\vee(1)}(z;0)=1,$$

(4.86) $$\beta_4^{*(2)}(z;0)=\delta\beta_4^{*(1)}(z;0)+\beta_3^{*(1)}(z;0)=2$$

(4.87) $$\beta_1^{*(2)}(z;1)=\delta\beta_1^{*(1)}(z;1)-\beta_2^{*(1)}(z;1)=$$
$$-32z+66z^2-(16z+18z^2)=-48z+48z^2=48z(z-1).$$

(4.88) $$\beta_2^{*(2)}(z;1)=\delta\beta_2^{*(1)}(z;1)=16z+36z^2,$$

(4.89) $$\beta_3^{*(2)}(z;1)=\delta\beta_3^{*(1)}(z;1)+\beta_1^{*\vee(1)}(z;1)=$$
$$51z+33z+1=84z+1,$$

(4.90) $$\beta_4^{*(2)}(z;1)=\delta\beta_4^{*(1)}(z;1)+\beta_3^{*(1)}(z;1)=$$
$$69z+51z+5=120z+5,$$

(4.91) $$\beta_1^{*(2)}(z;2) = \delta\beta_1^{*(1)}(z;2) - \beta_2^{*(1)}(z;2) =$$

$$-297z - 864z^2 + 2190z^3 - (81z + 648z^2 + 300z^3) =$$

$$z(-378 - 1512z + 1890z^2) = 378(z-1)z(5z+1),$$

(4.92) $$\beta_2^{*(2)}(z;2) = \delta\beta_2^{*(1)}(z;2) = 81z + 1296z^2 + 900z^3,$$

(4.93) $$\beta_3^{*(0)}(z;2) = \delta\beta_3^{*(1)}(z;2) + \beta_1^{*\vee(1)}(z;2) =$$

$$2060z^2 + 656z + 730z^2 + 298z + 1 = 2790z^2 + 954z + 1,$$

(4.94) $$\beta_4^{*(2)}(z;2) = \delta\beta_4^{*(1)}(z;2) + \beta_3^{*(1)}(z;2) = 6729z/6 + 2660z^2 +$$

$$1030z^2 + 656z + 20/3 = 3690z^2 + 3555z/2 + 20/3,$$

§5. Auxiliary difference equation.

Let $y_{i,k}(z;\nu)$ denotes i−th element of the column $Y_k(z;\nu)$ in (2.8). Then, in view of (1.26), (2.2), (2.8),

(5.1) $$y_{j+1-\kappa,k}(z,\nu) = \delta^j f_k(z,\nu),\ y_{4,k}(z,\nu) = \delta^3 f_k(z,\nu),$$

where $j = 1, 2,\ k = 1, 2, 3,\ |z| > 1,\ \nu \in \mathbb{N}_0$. We denote $v_{i,j}^{**}(\nu)$ the expression, which stands in the matrix $V^{**}(\nu)$ in intersection of i-th row and j-th column, where $i = 1, 2, 3, 4,\ j = 1, 2, 3, 4$. Let

(5.2) $$D(z,\nu,w) = z(w^2 - \mu)^2 - w^4,\ \mu = (\nu+1)^2$$

In view of (1.29)

(5.3) $$(1/z)D(z,\nu,w) = (1 - 1/z)w^4 + \sum_{k=0}^{3} r_{k+1}(\nu)w^k.$$

It follows from general properties of Mejer's functions that

(5.4) $$D(z,\nu,\delta)f_k(z,\nu) = 0,$$

where $|z| > 1, -3\pi/2 < \arg(z) \le \pi/2, \log(z) = \ln(|z|) + i\arg(z),\ k = 1, 2, 3.$ Therefore, in view of (1.26), (2.2), (2.8),

(5.5) $$y_k(z,\nu) = -(1 - 1/z)\delta^4 f_k(z,\nu)$$

where

$$|z| > 1, -3\pi/2 < \arg(z) \le \pi/2,$$

$$\log(z) = \ln(|z|) + i\arg(z),\ k = 1, 2, 3.$$

In view of (1.2) – (1.6), (4.35),

(5.6)
$$\lim_{z\to 1+0}(z-1)\delta^4 f_2(z,\nu)=$$
$$\lim_{z\to 1+0}(z-1)(O(1)\ln(1-1/z)+1/(z-1))=1,$$

(5.7)
$$\lim_{z\to 1+0}(z-1)\delta^4 f_k(z,\nu)=0,$$

if $k-2=\pm 1$,

(5.8)
$$\lim_{z\to 1+0}(\log(z))\delta^i f_k(z,\nu)=\lim_{z\to 1+0}(z-1)\delta^i f_k(z,\nu)=0,$$

if $i=0,\ 1,\ 2,\ 3, k=1,\ 2,\ 3$. Hence, if we tend $z\in(1,+\infty)$ to 1, then, in view of (1.26), (2.2), we obtain the equalities

(5.9)
$$y_{1,1}(1,\nu)=y_{1,3}(1,\nu)=0, y_{1,2}(1,\nu)=-1.$$

In view of (2.10), (3.3) – (3.18), (5.1), (5.5),

(5.10)
$$-a^{**}_{i,1}(1;\nu)(1-1/z)\delta^4 f_k(z,\nu)+\left(\sum_{j=1}^{2}a^{**}_{i+1,j+1}(1;\nu)\delta^j f_k(z,\nu)\right)-$$
$$(z-1)v^{**}_{i,1}(\nu)(1-1/z)\delta^4 f_k(z,\nu)+$$
$$(z-1)\sum_{j=1}^{2}v^{**}_{i+1,j+1}(\nu)\delta^j f_k(z,\nu)=\mu_1(\nu)^2\nu^5\delta^i f_k(z,\nu-1),$$

where $i=1,\ 2,\ k=1,\ 2,\ 3,\ |z|>1,\ -3\pi/2<\arg(z)\le\pi/2$ and ν run over the set $M_1^*=((-\infty,-2]\cup[1,+\infty))\cap\mathbb{Z}$. We tend $z\in(1,+\infty)$ to 1 now and obtain the equalities

(5.11)
$$a^{**}_{i+1,1}(1;\nu)(k-1)(k-3)+\left(\sum_{j=1}^{2}a^{**}_{i+1,j+1}(1;\nu)(\delta^j f_k)(1,\nu)\right)=$$
$$\mu_1(\nu)^2\nu^5\delta^i f_k(1,\nu-1),$$

where $i=1,\ 2,\ k=1,\ 2,\ 3$ and $\nu\in M_1^*=((-\infty,-2]\cup[1,+\infty))\cap\mathbb{Z}$. Let are given

(5.12)
$$F=\{F(\nu)\}_{\nu=-\infty}^{+\infty}\text{ and }G=\{G(\nu)\}_{\nu=-\infty}^{+\infty}$$

such that

(5.13)
$$F(-\nu-2)=F(\nu),\ G(-\nu-2)=G(\nu),\ F(\nu)\in\mathbb{R},\ G(\nu)\in\mathbb{R}$$

for $\nu\in\mathbb{Z}$. Let further

(5.14)
$$y^{**}_{F,G}(z,\nu)=F(\nu)\delta f_k(z,\nu)+G(\nu)\delta^2 f_k(z,\nu)$$

for $k = 1,\ 2,\ 3$ and $\nu \in M_1^{***} = ((-\infty, -2] \cup [0, +\infty)) \cap \mathbb{Z}$. Since F and G have the property (5.13), it follows from (2.9)) that

$$y^{**}_{F,G}(z, -\nu - 2) = y^{**}_{F,G}(z, \nu) \tag{5.15}$$

for $k = 1,\ 2,\ 3$ and $\nu \in M_1^{***} = ((-\infty, -2] \cup [0, +\infty)) \cap \mathbb{Z}$. Let

$$a^{***}_{F,G,j}(z; \nu) = F(\nu - 1)a^{**}_{2,j}(z; \nu) + G(\nu - 1)a^{**}_{3,j}(z, \nu) \tag{5.16}$$

for $\nu \in M_1^{*} = ((-\infty, -2] \cup [1, +\infty)) \cap \mathbb{Z}$, $j = 1,\ 2,\ 3$. In view of (5.11)

$$a^{***}_{F,G,1}(1; \nu)(k - 1)(k - 3)+ \tag{5.17}$$

$$\left(\sum_{j=1}^{2} a^{***}_{F,G,j+1}(1; \nu)(\delta^j f_{1,0,k})(1, \nu)\right) = \mu_1(\nu)^2 \nu^5 y^{**}_{F,G}(1, \nu - 1),$$

with $\nu \in M_1^{*} = ((-\infty, -2] \cup [1, +\infty)) \cap \mathbb{Z}$, $k = 1,\ 2,\ 3$.

Replacing $\nu \in M_1^{*}$ by $\nu := -\nu - 2 \in M_1^{**} = ((-\infty, -3] \cup [0, +\infty)) \cap \mathbb{Z}$ in (5.17), and taking in account (2.9) we obtain the equalities

$$a^{***}_{F,G,1}(1; -\nu - 2)(k - 1)(k - 3)+ \tag{5.18}$$

$$\left(\sum_{j=1}^{2} a^{***}_{F,G,j}(1; -\nu - 2)(\delta^j f_k)(1, \nu)\right) = -\mu_1(\nu)^2 (\nu + 2)^5 y^{**}_{F,G}(z, \nu + 1),$$

where $k = 1,\ 2,\ 3$ and $\nu \in M_1^{**} = ((-\infty, -3] \cup [0, +\infty)) \cap \mathbb{Z}$. Let

$$\vec{w}_{F,G,j}(\nu) = \begin{pmatrix} a^{***}_{F,G,j+1}(1; -\nu - 2) \\ F(\nu)(2 - j) + G(\nu)(j - 1) \\ a^{***}_{F,G,j+1}(1; \nu) \end{pmatrix}, \tag{5.19}$$

where $j = 1, 2$, $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$,

$$W_{F,G}(\nu) = \begin{pmatrix} a^{***}_{F,G,2}(1; -\nu - 2) & a^{***}_{F,G,3}(1; -\nu - 2) \\ F(\nu) & G(\nu) \\ a^{***}_{F,G,2}(1; \nu) & a^{***}_{F,G,3}(1; \nu) \end{pmatrix} = \tag{5.20}$$

$$\begin{pmatrix} \vec{w}_{F,G,1}(\nu) & \vec{w}_{F,G,2}(\nu) \end{pmatrix},\ Y^{***}_k(\nu) = \begin{pmatrix} (\delta f_k)(1, \nu) \\ (\delta^2 f_k)(1, \nu) \end{pmatrix},$$

$$Y^{****}_{F,G,k}(\nu) = \begin{pmatrix} \mu_1(-\nu - 2)^2 (-\nu - 2)^5 y^{**}_{F,G}(z, -\nu - 3) \\ y^{**}_{F,G}(z, \nu) \\ \mu_1(\nu)^2 \nu^5 y^{**}_{F,G}(z, \nu - 1) \end{pmatrix}, \tag{5.21}$$

where $k = 1,\ 3$, $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$. Let further

$$\vec{w}_{F,G,3}(\nu) = \begin{pmatrix} w_{F,G,3,1}(\nu) \\ w_{F,G,3,2}(\nu) \\ w_{F,G,3,3}(\nu) \end{pmatrix} = [\vec{w}_{F,G,1}(\nu), \vec{w}_{F,G,2}(\nu)]. \tag{5.22}$$

is vector product of $\vec{w}_{F,G,1}(\nu)$ and $\vec{w}_{F,G,2}(\nu)$.

Let $\bar{w}_{F,G,3}(\nu) = (\vec{w}_{F,G,3}(\nu))^t$ is the row conjugate to the column $\vec{w}_{F,G,3}(\nu)$. Then for scalar products $(\vec{w}_{F,G,3}(\nu), \vec{w}_{F,G,j}(\nu))$ we have the equalities

$$\bar{w}_{F,G,3}(\nu)\vec{w}_{F,G,j}(\nu) = (\vec{w}_{F,G,3}(\nu), \vec{w}_{F,G,j}(\nu)) = 0,$$

where $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$, $j = 1, 2$. Therefore

$$\bar{w}_{F,G,3}(\nu)W_{F,G}(\nu) = \begin{pmatrix} 0 & 0 \end{pmatrix}, \tag{5.23}$$

where $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$.

In view of (5.11) (5.18) and (5.23),

$$\bar{w}_{F,G,3}(\nu)Y^{*****}_{F,G,k}(\nu) = \bar{w}_{F,G,3}(\nu)W_{F,G,3}(\nu)Y^{***}_k(\nu) = 0, \tag{5.24}$$

where $k = 1, 3$ and $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$.

In view of (5.16), (5.22) – (5.20), (3.3) – (3.18), and,

since $\tau_{-\nu-2} = -\nu - 1 = -\tau_\nu = -\tau$, it follows that

$$a^{***}_{F,G,1}(1;\nu) = F(\nu-1)a^{**}_{2,1}(1,\nu) + G(\nu-1)a^{**}_{3,1}(1,\nu) = \tag{5.25}$$

$$-F(\nu-1)\tau^2(\tau-1)(2\tau-1)(6\tau^2-4\tau+1)+$$
$$G(\nu-1)\tau^2(\tau-1)^2(2\tau-1)(4\tau^2-3\tau+1),$$

$$a^{***}_{F,G,1}(1;-\nu-2) = \tag{5.26}$$

$$F(\nu+1)a^{**}_{2,1}(1,-\nu-2) + G(\nu+1)a^{**}_{3,1}(1,-\nu-2) =$$
$$-F(\nu+1)\tau^2(\tau+1)(2\tau+1)(6\tau^2+4\tau+1)-$$
$$-G(\nu+1)\tau^2(\tau+1)^2(2\tau+1)(4\tau^2+3\tau+1),$$

$$a^{***}_{F,G,2}(1;\nu) = \tag{5.27}$$

$$F(\nu-1)a^{**}_{1,0,2,2}(1,\nu) + G(\nu-1)a^{**}_{3,2}(1,\nu) =$$
$$F(\nu-1)\tau^5(\tau-1)(\tau^3+2(2\tau-1)^3)-$$
$$2G(\nu-1)\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3),$$

$$a^{***}_{F,G,2}(z;-\nu-2) = \tag{5.28}$$

$$F(\nu+1)a^{**}_{2,2}(1,-\nu-2) + G(\nu+1)a^{**}_{3,2}(1,-\nu-2) =$$
$$-F(\nu+1)\tau^5(\tau+1)(\tau^3+2(2\tau+1)^3)-$$
$$G(\nu+1)2\tau^5(\tau+1)^2(2\tau+1)((\tau+1)^3-\tau^3),$$

$$a^{***}_{F,G,3}(1;\nu) = \tag{5.29}$$

$$F(\nu-1)a^{**}_{2,3}(1,\nu) + G(\nu-1)a^{**}_{3,3}(1,\nu) =$$

$-3F(\nu-1)\tau^4(\tau-1)(2\tau-1)^3+G(\nu-1)\tau^4(\tau-1)^2((\tau-1)^3+2(2\tau-1)^3),$

(5.30)
$$a^{***}_{F,G,3}(z;-\nu-2)=$$
$$F(\nu+1)a^{**}_{2,3}(1,-\nu-2)+G(\nu+1)a^{**}_{3,3}(1,-\nu-2)=$$
$$-3F(\nu+1)\tau^4(\tau+1)(2\tau+1)^3-$$
$$G(\nu+1)\tau^4(\tau+1)^2(2\tau+1)((\tau+1)^3+2(2\tau+1)^3),$$

We consider the case now, when F and G are constant sequences, or, equivalently, real constants. In view of (5.19), (5.27) – (5.30)

(5.31)
$$\vec{w}_{1,0,j}(\nu)=\begin{pmatrix}a^{***}_{1,0,j+1}(1;-\nu-2)\\ 2-j\\ a^{***}_{1,0,j+1}(1;\nu),\end{pmatrix},$$

where $j=1,2$, $\nu\in M_1^{****}=((-\infty,-3]\cup[1,+\infty))\cap\mathbb{Z}$,

(5.32)
$$\vec{w}_{1,0,1}(\nu)=\begin{pmatrix}a^{***}_{1,0,2}(1;-\nu-2)\\ 1\\ a^{***}_{1,0,2}(1;\nu)\end{pmatrix}=$$
$$\begin{pmatrix}-\tau^5(\tau+1)(\tau^3+2(2\tau+1)^3)\\ 1\\ \tau^5(\tau-1)(\tau^3+2(2\tau-1)^3)\end{pmatrix},$$

(5.33)
$$\vec{w}_{1,0,2}(\nu)=\begin{pmatrix}a^{***}_{1,0,3}(1;-\nu-2)\\ 0\\ a^{***}_{1,0,3}(1;\nu)\end{pmatrix}=$$
$$\begin{pmatrix}-3\tau^4(\tau+1)(2\tau+1)^3\\ 0\\ -3\tau^4(\tau-1)(2\tau-1)^3\end{pmatrix},$$

(5.34)
$$\vec{w}_{0,1,j}(\nu)=\begin{pmatrix}a^{***}_{0,1,j+1}(1;-\nu-2)\\ j-1\\ a^{***}_{0,1,j+1}(1;\nu),\end{pmatrix},$$

where $j=1,2$, $\nu\in M_1^{****}=((-\infty,-3]\cup[1,+\infty))\cap\mathbb{Z}$,

(5.35)
$$\vec{w}_{0,1,1}(\nu)=\begin{pmatrix}a^{***}_{0,1,2}(1;-\nu-2)\\ 0\\ a^{***}_{0,1,2}(1;\nu)\end{pmatrix}=$$
$$\begin{pmatrix}-2\tau^5(\tau+1)^2(2\tau+1)((\tau+1)^3-\tau^3)\\ 0\\ -2\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3)\end{pmatrix},$$

(5.36) $$\vec{w}_{0,1,2}(\nu)=\begin{pmatrix}a^{***}_{0,1,3}(1;-\nu-2)\\ 1\\ a^{***}_{0,1,3}(1;\nu)\end{pmatrix}=$$

$$\begin{pmatrix}-\tau^4(\tau+1)^2(2\tau+1)((\tau+1)^3+2(2\tau+1)^3)\\ 1\\ \tau^4(\tau-1)^2((\tau-1)^3+2(2\tau-1)^3)\end{pmatrix},$$

and,since F, G are constants now, it follows that

(5.37) $$\vec{w}_{F,G,j}(\nu)=F\vec{w}_{1,0,j}(\nu)+G\vec{w}_{0,1,j}(\nu)$$

where $j=1,2$, $\nu\in M_1^{****}=((-\infty,-3]\cup[1,+\infty))\cap\mathbb{Z}$. In view of (5.22),

(5.38) $$\vec{w}_{F,G,3}(\nu)=F^2[\vec{w}_{1,0,1}(\nu),\vec{w}_{1,0,2}(\nu)]+$$

$$FG([\vec{w}_{1,0,1}(\nu),\vec{w}_{0,1,2}(\nu)]+[\vec{w}_{0,1,1}(\nu),\vec{w}_{1,0,2}(\nu)])+G^2[\vec{w}_{0,1,1}(\nu),\vec{w}_{0,1,2}(\nu)].$$

For any

$$\vec{a}=\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}$$

we put $(\vec{a})_i=a_i$ **for** $i=1,\,2,\,3$. Let further

(5.39) $$\vec{w}_{i,j,4}(\nu)=[\vec{w}_{i,1-i,1}(\nu),\vec{w}_{j,1-j,2}(\nu)],$$

with $i=0,\,1$, $j=0,\,1$. In view of (5.39),(5.27)– (5.30),

$$\vec{w}_{1,1,4}(\nu)=[\vec{w}_{1,0,1}(\nu),\vec{w}_{1,0,2}(\nu)]=\vec{w}_{1,0,3}(\nu)],$$

(5.40) $$(\vec{w}_{1,1,4}(\nu))_1=(\vec{w}_{1,0,3}(\nu))_1=$$

$$\det\begin{pmatrix}1 & 0\\ a^{***}_{1,0,2}(1;\nu) & a^{***}_{1,0,3}(1;\nu)\end{pmatrix}=$$

$$a^{***}_{1,0,3}(1;\nu)=a^{**}_{2,3}(1,\nu)=-3\tau^4(\tau-1)(2\tau-1)^3,$$

(5.41) $$(\vec{w}_{1,1,4}(\nu))_2=(\vec{w}_{1,0,3}(\nu))_2=$$

$$-\det\begin{pmatrix}a^{***}_{1,0,2}(1;-\nu-2) & a^{***}_{1,0,3}(1;-\nu-2)\\ a^{***}_{1,0,2}(1;\nu) & a^{***}_{1,0,3}(1;\nu)\end{pmatrix}=$$

$$-\det\begin{pmatrix}a^{**}_{2,2}(1;-\nu-2) & a^{**}_{2,3}(1;-\nu-2)\\ a^{**}_{2,2}(1;\nu) & a^{**}_{2,3}(1;\nu)\end{pmatrix}=$$

$$a^{**}_{2,2}(1;\nu)a^{**}_{2,3}(1;-\nu-2)-a^{**}_{2,3}(1;\nu)a^{**}_{2,3}(1;-\nu-2)=$$

$$\tau^5(\tau-1)(\tau^3+2(2\tau-1)^3)(-3\tau^4(\tau+1)(2\tau+1)^3)-$$

$$(-3\tau^4(\tau-1)(2\tau-1)^3)(-\tau^5(\tau+1)(\tau^3+2(2\tau+1)^3))=$$

$$-3\tau^9(\tau^2-1)(\tau^3((2\tau-1)^3+(2\tau+1)^3)+4(4\tau^2-1)^3)=$$

$$-12\tau^9(\tau^2-1)(68\tau^6-45\tau^4+12\tau^2-1),$$

(5.42)
$$(\vec{w}_{1,1,4}(\nu))_3=(\vec{w}_{1,0,3}(\nu))_3=$$
$$\det\begin{pmatrix} a^{***}_{1,0,2}(1;-\nu-2) & a^{***}_{1,0,3}(1;-\nu-2)\\ 1 & 0\end{pmatrix}=$$
$$-a^{***}_{1,0,3}(1;-\nu-2)=-a^{**}_{2,3}(1,-\nu-2)==3\tau^4(\tau+1)(2\tau+1)^3.$$

In view of (5.39),(5.27)– (5.30),

$$\vec{w}_{0,0,4}(\nu)=[\vec{w}_{0,1,1}(\nu),\vec{w}_{0,1,2}(\nu)]=\vec{w}_{0,1,3}(\nu),$$

(5.43)
$$(\vec{w}_{0,0,4}(\nu))_1=(\vec{w}_{0,1,3}(\nu))_1=$$
$$\det\begin{pmatrix} 0 & 1\\ a^{***}_{0,1,2}(1;\nu) & a^{***}_{0,1,3}(1;\nu)\end{pmatrix}=$$
$$-a^{***}_{0,1,2}(1;\nu)=-a^{**}_{3,2}(1,\nu)=2\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3),$$

(5.44)
$$(\vec{w}_{0,0,4}(\nu))_2=(\vec{w}_{0,1,3}(\nu))_2=$$
$$-\det\begin{pmatrix} a^{***}_{0,1,2}(1;-\nu-2) & a^{***}_{0,1,3}(1;-\nu-2)\\ a^{***}_{0,1,2}(1;\nu) & a^{**1*}_{0,1,3}(1;\nu)\end{pmatrix}=$$
$$-\det\begin{pmatrix} a^{**}_{3,2}(1;-\nu-2) & a^{**}_{3,3}(1;-\nu-2)\\ a^{**}_{3,2}(1;\nu) & a^{**}_{3,3}(1;\nu)\end{pmatrix}=$$
$$a^{**}_{3,2}(1;\nu)a^{**}_{3,3}(1;-\nu-2)-a^{**}_{3,3}(1;\nu)a^{**}_{3,2}(1;-\nu-2)=$$
$$-2\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3)\times$$
$$(-\tau^4(\tau+1)^2((\tau+1)^3+2(2\tau+1)^3)-$$
$$(-2\tau^5(\tau+1)^2(2\tau+1)((\tau+1)^3-\tau^3))\times$$
$$(\tau^4(\tau-1)^2((\tau-1)^3+2(2\tau-1)^3)=$$
$$4\tau^9(\tau^2-1)^2(102\tau^6-68\tau^4+21\tau^2-3),$$

(5.45)
$$(\vec{w}_{0,0,4}(\nu))_3=(\vec{w}_{0,1,3}(\nu))_3=$$
$$\det\begin{pmatrix} a^{***}_{0,1,2}(1;-\nu-2) & a^{***}_{0,1,3}(1;-\nu-2)\\ 0 & 1\end{pmatrix}=$$
$$a_(3,2)^{**}(1;-\nu-2)=-2\tau^5(\tau+1)^2(2\tau+1)((\tau+1)^3-\tau)^3),$$

In view of (5.39),(5.27)– (5.30),

$$\vec{w}_{0,1,4}(\nu)=[\vec{w}_{0,1,1}(\nu),\vec{w}_{1,0,2}(\nu)],$$

(5.46)
$$(\vec{w}_{0,1,4}(\nu))_1=([\vec{w}_{0,1,1}(\nu),\vec{w}_{1,0,2}(\nu)])_1=$$

$$\det\begin{pmatrix} 0 & 0 \\ a^{***}_{0,1,2}(1;\nu) & a^{***}_{1,0,3}(1;\nu) \end{pmatrix} = 0,$$

(5.47)
$$(\vec{w}_{0,1,4}(\nu))_2 = ([\vec{w}_{0,1,1}(\nu), \vec{w}_{1,0,2}(\nu)])_2 =$$

$$-\det\begin{pmatrix} a^{***}_{0,1,2}(1;-\nu-2) & a^{***}_{1,0,3}(1;-\nu-2) \\ a^{***}_{0,1,2}(1;\nu) & a^{***}_{1,0,3}(1;\nu) \end{pmatrix} =$$

$$-a^{**}_{3,2}(1;-\nu-2)a^{**}_{2,3}(1;\nu) + a^{**}_{3,2}(1;\nu)a^{**}_{2,3}(1;-\nu-2) =$$

$$-12t^9(\tau^2-1)(4\tau^2-1)(12\tau^4-6\tau^2+1),$$

(5.48)
$$(\vec{w}_{0,1,4}(\nu))_3 = ([\vec{w}_{0,1,1}(\nu), \vec{w}_{1,0,2}(\nu)])_3,$$

$$\det\begin{pmatrix} a^{***}_{0,1,2}(1;-\nu-2) & a^{***}_{1,0,3}(1;-\nu-2) \\ 0 & 0 \end{pmatrix} = 0,$$

In view of (5.39),(5.27)– (5.30),

$$\vec{w}_{1,0,4}(\nu) = [\vec{w}_{1,0,1}(\nu), \vec{w}_{0,1,2}(\nu)],$$

(5.49)
$$(\vec{w}_{1,0,4}(\nu))_1 = ([\vec{w}_{1,0,1}(\nu), \vec{w}_{0,1,2}(\nu)])_1$$

$$\det\begin{pmatrix} 1 & 1 \\ a^{***}_{1,0,2}(1;\nu) & a^{***}_{0,1,3}(1;\nu) \end{pmatrix} =$$

$$a^{**}_{3,3}(1;\nu) - a^{**}_{2,2}(1;\nu) =$$

$$-t^4(t-1)(2t-1)(10t^2-10t+3),$$

(5.50)
$$(\vec{w}_{1,0,4}(\nu))_2 = ([w_{1,0,1}(\nu), w_{0,1,2}(\nu)])_2 =$$

$$-\det\begin{pmatrix} a^{***}_{1,0,2}(1;-\nu-2) & a^{***}_{0,1,3}(1;-\nu-2) \\ a^{***}_{1,0,2}(1;\nu) & a^{***}_{0,1,3}(1;\nu) \end{pmatrix} =$$

$$-a^{**}_{2,2}(1;-\nu-2)a^{**}_{3,3}(1;\nu) + a^{**}_{2,2}(1;\nu)a^{**}_{3,3}(1;-\nu-2) =$$

$$-4t^9(t^2-1)(170t^6-104t^4+30t^2-3),$$

(5.51)
$$(\vec{w}_{1,0,4}(\nu))_3 = ([\vec{w}_{1,0,1}(\nu), \vec{w}_{0,1,2}(\nu)])_3 =$$

$$\det\begin{pmatrix} a^{***}_{1,0,2}(1;-\nu-2) & a^{***}_{0,1,3}(1;-\nu-2) \\ 1 & 1 \end{pmatrix} =$$

$$a^{**}_{2,2}(1;-\nu-2) - a^{**}_{3,3}(1;-\nu-2) =$$

$$t^4(t+1)(2t+1)(10t^2+10t+3),$$

(5.52)
$$(\vec{w}_{0,1,4}(\nu))_2 + (\vec{w}_{1,0,4}(\nu))_2 =$$

$$-8t^9(t^2-1)(157t^6-106t^4+30t^2-3).$$

Therefore,

(5.53)
$$(\vec{w}_{F,G,3}(\nu))_1 = -3\tau^4(\tau-1)(2\tau-1)^3F^2-$$
$$t^4(\tau-1)(2\tau-1)(10\tau^2-10\tau+3)FG+$$
$$2\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3)G^2=$$
$$\tau^4(\tau-1)(2\tau-1)\times$$
$$(-3(2\tau-1)^2F^2-(10\tau^2-10\tau+3)FG+2(\tau-1)(3\tau^3-3\tau^2+\tau)G^2)=$$
$$\tau^4(\tau-1)(2\tau-1)\times$$
$$(-3(2\tau-1)^2F^2-(10\tau^2-10\tau+3)FG+2(3\tau^4-6\tau^3+4\tau^2-\tau)G^2,$$

(5.54)
$$(\vec{w}_{F,G,3}(\nu))_2 = -12\tau^9(\tau^2-1)(68\tau^6-45\tau^4+12\tau^2-1)F^2-$$
$$8t^9(t^2-1)(157t^6-106t^4+30t^2-3)FG+$$
$$4\tau^9(\tau^2-1)^2(102\tau^6-68\tau^4+21\tau^2-3)G^2=$$
$$-12\tau^9(\tau^2-1)(68\tau^6-45\tau^4+12\tau^2-1)F^2-$$
$$8\tau^9(\tau^2-1)(157\tau^6-106\tau^4+30\tau^2-3)FG+$$
$$4\tau^9(\tau^2-1)(102\tau^8-170\tau^6+89\tau^4-24\tau^2+3)G^2,$$

(5.55)
$$(\vec{w}_{F,G,3}(\nu))_3 = 3\tau^4(\tau+1)(2\tau+1)^3F^2+$$
$$t^4(t+1)(2t+1)(10t^2+10t+3)FG-2\tau^5(\tau+1)^2(2\tau+1)((\tau+1)^3-\tau^3)G^2=$$
$$\tau^4(\tau+1)(2\tau+1)\times$$
$$3(2\tau+1)^2F^2+(10\tau^2+10\tau+3)FG-2(\tau+1)(3\tau^3+3\tau^2+\tau)G^2=$$
$$\tau^4(\tau+1)(2\tau+1)\times$$
$$3(4\tau^2+4\tau+1)F^2+(10\tau^2+10\tau+3)FG-2(3\tau^4+6\tau^3+4\tau^2+\tau)G^2.$$

According to (5.14), (5.24), (5.21), (5.53), (5.54), (5.55),

(5.56)
$$-\tau^4(\tau+1)^5w_{F,G,3,1}(\nu)y^{**}_{F,G,k}(\nu+1)+$$
$$w_{F,G,3,2}(\nu)y^{**}_{F,G,k}(\nu)+\tau^4(\tau-1)^5w_{F,G,3,3}(\nu)y^{**}_{F,G,k}(\nu-1)=0.$$

Since
$$f_{1,0,k}(1,\nu)=f^*_{1,0,k}(1,\nu)/(\nu+1)^2,$$
it follows from (5.56), (0.8) – (0.10) that

(5.57)
$$c_{F,G,2}(\nu)x(\nu+1)+c_{F,G,1}(\nu)x(\nu)+c_{F,G,0}(\nu)x(\nu-1)=0$$

for $x(\nu)=x_{F,G,k}(\nu)$, where

(5.58)
$$x_{F,G,k}(\nu)=F\delta f^*_k(1,\nu)+G\delta^2 f^*_k(1,\nu)),\ k=1,\ 3.$$

Let

$$(5.59)\qquad \beta^{**}_{F,G,i}(z;\nu) := F\beta^{*(1)}_{2i}(z;\nu) + G\beta^{*(2)}_{2i}(z;\nu)$$

for $i = 1, 2$. In view of (4.52),

$$\beta^{**}_{F,G,1}(1;\nu) := F\beta^{*(1)}_{2}(1;\nu) + G\beta^{*(2)}_{2}(1;\nu) =$$

$$F\delta f^*_1(1,\nu) + G\delta^2 f^*_1(1,\nu) = x_{F,G,1}(\nu)$$

In view of (4.36) with $j = 1$ and (5.58),

$$(5.60)\qquad x_{F,G,3}(\nu) = 2\zeta(3)\beta^{**}_{F,G,1}(1;\nu) - \beta^{**}_{F,G,2}(1;\nu)$$

Before to complete the proof of Theorem B, we want to check equality (5.57) for $\nu = 1$, $k = 3$. In view of (5.60), to check the equation (5.57) for $\nu = 1$ it is sufficient to check the equalities

$$c_{F,G,2}(1)\beta^{**}_{F,G,i}(2) + c_{F,G,1}(1)\beta^{**}_{F,G,i}(1) + c_{F,G,0}(1)\beta^{**}_{F,G,i}(0) = 0,$$

In view of (5.53) – (5.55),

$$c_{F,G,2}(1) = -54(-27F^2 - 23FG + 28G^2),$$

$$c_{F,G,1}(1) = 12(-3679F^2 - 5646FG + 5459G^2),$$

$$c_{F,G,0}(1) = 10(75F^2 + 63FG - 228G^2) = 30(25F^2 + 21FG - 76F^2),$$

in view of (4.72) – (4.94),

$$\beta^{*(1)}_{1,0,2}(1;2) = 1029,\ \beta^{*(1)}_{1,0,2}(1;1) = 34,\ \beta^{*(1)}_{1,0,2}(1;0) = 1,$$

$$\beta^{*(1)}_{1,0,4}(1;2) = \frac{14843}{6},\ \beta^{*(1)}_{1,0,4}(1;1) = \frac{327}{4},\ \beta^{*(1)}_{1,0,4}(1;0) = 3.$$

$$\beta^{*(2)}_{1,0,2}(1;2) = 2277,\ \beta^{*(2)}_{1,0,2}(1;1) = 52,\ \beta^{*(2)}_{1,0,2}(1;0) = 1,$$

$$\beta^{*(2)}_{1,0,4}(1;2) = \frac{32845}{6},\ \beta^{*(2)}_{1,0,4}(1;1) = 125,\ \beta^{*(2)}_{1,0,4}(1;0) = 2,$$

$$\beta^{**}_{F,G,1}(1,2) = 1029F + 2277G,$$

$$(5.61)\qquad \beta^{**}_{F,G,1}(1,1) = 34F + 52G,\ \beta^{**}_{F,G,1}(1,0) = F + G,$$

$$\beta^{**}_{F,G,2}(1,2) = (14843F + 32845G)/6,$$

$$(5.62)\qquad \beta^{**}_{F,G,2}(1,1) = (327F + 500G)/4, \beta^{**}_{F,G,2}(1,0) = 3F + 2G.$$

Let further

$$\Delta_{i,\nu}(F,G) = c_{F,G,2}(\nu)\beta_{F,G,i}**(\nu+1)+$$

$$c_{F,G,1}(\nu)\beta_{F,G,i}**(\nu) + c_{F,G,0}(\nu)\beta_{F,G,i}**(\nu-1)$$

for $i = 1, 2\,\nu \in \mathbb{N}$. We check now that $\Delta_{i,1}(F,G) = 0$ for $i = 1,2$. We note that $\Delta_{i,1}(F,G)$ is homogenous polynomial relatively variables F,G of degree equal to 3. To establish the equalities $\Delta_{i,1}(F,G) = 0$with $i = 1, 2$ it is sufficient to check them in four points

$$(F,G) = (0,1),\ (1,0),\ (1,1),\ (1,2).$$

We have

$$c_{0,1,2}(1) = -54 \times 28 = -12 \times 126,\ c_{0,1,1}(1) = 12 \times 5521,$$

$$c_{0,1,0}(1) = -12 \times 190,$$

$$\beta^{**}_{0,1,1}(2) = 2277,\ \beta^{**}_{0,1,1}(1) = 52,\ \beta^{**}_{0,1,1}(0) = 1,$$

$$\beta^{**}_{0,1,2}(2) = 32845/6,\ \beta^{**}_{0,1,2}(1) = 125,\ \beta^{**}_{0,1,2}(0) = 2,$$

$$\Delta_{1,1}(0,1) = 12 \times (-126 \times 2277 + 5521 \times 52 - 190) =$$

$$12(-287092 + 286902 - 190) = 0,$$

$$\Delta_{2,1}(0,1) = 12 \times (-21 \times 32845 + 5521 \times 125 - 190) =$$

$$60(-21 \times 6569 + 5521 \times 25 - 76) = 60(-137949 + 138025 - 76) = 0,$$

$$c_{1,0,2}(1) = -54 \times (-27) = 6 \times 243,\ c_{1,0,1}(1) = -12 \times 3679 = -4 \times 11037,$$

$$c_{1,0,0}(1) = 10 \times 75,$$

$$\beta^{**}_{1,0,1}(2) = 1029,\ \beta^{**}_{1,0,1}(1) = 34,\ \beta^{**}_{1,0,1}(0) = 1,$$

$$\beta_{1,0,2}(2) = 14843/6,\ \beta_{1,0,2}(1) = 327/4,\ \beta_{1,0,2}(0) = 3,$$

$$\Delta_{1,1}(1,0) = 6(243 \times 1029 - 7358 \times 34 + 125) =$$

$$6(250047 - 250172 + 125) = 0.$$

$$\Delta_{2,1}(1,0) = 243 \times 14843 - 11037 \times 327 + 2250 =$$

$$3606849 - 3609099 + 2250 = 3609099 - 3609099 = 0.$$

$$c_{1,1,2}(1) = -54 \times (-22) = 36 \times 33,\ c_{1,1,1}(1) = -12 \times 3804 = -36 \times 1268,$$

$$c_{1,1,0}(1) = -900 = -36 \times 25,$$

$$\beta^{**}_{1,1,1}(2) = 2 \times 1653,\ \beta^{**}_{1,1,1}(1) = 2 \times 43,\ \beta^{**}_{1,1,1}(0) = 2,$$

$$\beta^{**}_{1,1,2}(2) = 7948,\ \beta^{**}_{1,1,2}(1) = 827/4,\ \beta^{**}_{1,1,2}(0) = 5,$$

$$\Delta_{1,1}(1,1) = 72(33 \times 1653 - 1268 \times 43 - 25) =$$

$$72(54559 - 54524 - 25) = 0,$$

$$\Delta_{2,1}(1,1) = 36(33 \times 7948 - 317 \times 827 - 125) =$$

$$36(262284 - 262159 - 125) = 0.$$

$$c_{1,2,2}(1) = -54 \times (39) = -18 \times 117\, c_{1,2,1}(1) = 18 \times 4742,$$

$$c_{1,1,0}(1) = -18 \times 395,$$

$$\beta^{**}_{1,2,1}(2) = 3 \times 1861,\ \beta^{**}_{1,2,1}(1) = 3 \times 46,\ \beta^{**}_{1,2,1}(0) = 3,$$

$$\beta^{**}_{1,2,2}(2) = 80533/6,\ \beta^{**}_{1,2,2}(1) = 1327/4,\ \beta^{**}_{1,2,1}(0) = 7,$$
$$\Delta_{1,1}(1,2) = 54(-117 \times 1861 + 4742 \times 46 - 395) =$$
$$54(-217737 + 218132 - 395) = 0.$$
$$\Delta_{2,1}(1,2) = -351 \times 80533 + 9times2371 \times 1327 - 63 \times 790) =$$
$$9(-39 \times 80533 + 271 \times 1327 - 7 \times 790) =$$
$$(-3140787 + 3146317 - 5530 = 0.$$

So $\Delta_{i,1}(F,G) = 0$ for any F and G. Therefore the equality (5.57) holds for $\nu = 1$ and any $\{F,G\} \subset \mathbb{R}$.

§6. Auxilliary continued fraction.

Let

(6.1) $$c^*_{u,v,k}(\nu) = c_{u,v,k}(\nu)/(u+v)^2,$$

for $k = 0,\ 1,\ 2,$

(6.2) $$b^*_{u,v}(\nu+1) = -c^*_{u,v,1}(\nu) \in \mathbb{Q}[u,v]/(u+v)^2$$

for $\nu \in \mathbb{N}$,

(6.3) $$a^*_{u,v}(\nu+1) = -c^*_{u,v,0}(\nu)c^*_{u,v,2}(\nu-1)$$

for $\nu \in [2+\infty) \cap \mathbb{N}$,

(6.4) $$a^*_{u,v}(2) = -c^*_{u,v,0}(1),$$

(6.5) $$P^*_{u,v}(0) = b^*_{u,v}(0) = (3u+2v)/(u+v),\ Q_{u,v}(0) = 1,$$

(6.6) $$Q^*_{u,v}(1) = b^*_{u,v}(1) = (34u+52v)(u+v)$$

(6.7) $$P_{u,v}(1) = (327u+500v)(4u+4v)$$

(6.8) $$a^*_{u,v}(1) = P^*_{u,v}(1) = -b^*_{u,v}(0)b^*_{u,v}(1).$$

Let us consider the continued fraction,

(6.9) $$b_{u,v}(0)^* + \frac{a^*_{u,v}(1)|}{b^*_{u,v}(1)} + \frac{a^*_{u,v}(2)|}{b^*_{u,v}(2)} + \frac{a^*_{u,v}(3)|}{b^*_{u,v}(3)} + \frac{a^*_{u,v}(4)|}{|b^*_{F,G}(4)}....$$

Let $r^*_{u,v}(\nu)$ be the ν-th convergent of this continued fraction. Let $P^*_{u,v}(\nu)$ and $Q^*_{u,v}(\nu)$ be respectively nominator and denominator of convergent $r^*_{u,v}(\nu)$. Let us consider the equations

(6.10) $$c_{F,G,2}(\nu)x_{\nu+1} + c_{F,G,1}(\nu)x_{\nu} + c_{F,G,0}(\nu)x_{\nu-1} = 0,$$

where $\nu \in \mathbb{N}$. If $F + G \neq 0$, then the equation 6.10 is equivalent to the equation

$$c^a_{F,G,2}st(\nu)x_{\nu+1} + c^*_{F,G,1}(\nu)x_\nu + c^*_{F,G,0}(\nu)x_{\nu-1} = 0, \tag{6.11}$$

It follows from (5.57) that

$$x_\nu = x_{F,G,k}(\nu) = F\delta f^*_{1,0,k}(1,\nu) + G\delta^2 f_{1,0,k}(1,\nu)$$

satisfies to the equation (6.10) for $\nu \in \mathbb{N}$ and fixed $k \in \{1, 3\}$.

If $G \neq 0$, then, in view of (0.8) – (0.10),

$$c_{F,G,2}(\nu) = -12\tau^8 G^2(1 + o(1))(\tau \to \infty),$$

$$c_{F,G,1}(\nu) = 408\tau^8 G^2(1 + o(1))(\tau \to \infty),$$
$$c_{F,G,0}(\nu) = -12\tau^8 G^2(1 + o(1))(\tau \to \infty).$$

If $G = 0$, then, in view of (0.8) – (0.10),

$$c_{F,G,2}(\nu) = 24\tau^6 F^2(1 + o(1))(\tau \to \infty),$$

$$c_{F,G,1}(\nu) = -24 \times 64\tau^6 F^2(1 + o(1))(\tau \to \infty),$$
$$c_{F,G,0}(\nu) = 24\tau^6 F^2(1 + o(1))(\tau \to \infty).$$

In any case the equation (6.10) is difference equation of Poincaré type with characteristic polynomial $\lambda^2 - 34\lambda + 1$. Hence, if $\{x_\nu\}_{\nu=1}^{+\infty}$ is a non-zero solution of (6.10), $\varepsilon \in (0, 1)$, then there are $C_1(\varepsilon) > 0$ and $C_2(\varepsilon) > 0$ such that only two possibilities exist:

$$C_1(\varepsilon)\left(1 + \sqrt{2}\right)^{-4\nu(1+\varepsilon)} \leq |x_\nu| \leq C_2(\varepsilon)\left(1 + \sqrt{2}\right)^{-4\nu(1-\varepsilon)} \tag{6.12}$$

for all $\nu \in \mathbb{N}$ or

$$C_1(\varepsilon)\left(1 + \sqrt{2}\right)^{4\nu(1-\varepsilon)} \leq |x_\nu| \leq C_2(\varepsilon)\left(1 + \sqrt{2}\right)^{4\nu(1+\varepsilon)}. \tag{6.13}$$

for all $\nu \in \mathbb{N}$. In view of (4.52), if $x_\nu = \beta^{*(r)}(1;\nu) = \delta^r f^*_1(1,\nu)$ with $r = 0, 1, 2$, then (6.12) is impossible. Therefore

$$C_1(\varepsilon)\left(1 + \sqrt{2}\right)^{4\nu(1-\varepsilon)} \leq \beta^{*(r)}(1;\nu) \leq C_2(\varepsilon)\left(1 + \sqrt{2}\right)^{4\nu(1+\varepsilon)}. \tag{6.14}$$

for $r = 0, 1, 2$, and all $\nu \in \mathbb{N}$.

Lemma 6.1. *The following equalities hold:*

$$\lim_{\nu\to\infty} \beta^{*(1)}_{1,0,2}(1;\nu)) = +\infty, \lim_{\nu\to\infty} \beta^{*(2)}_2(1;\nu)/\beta^{*(1)}_2(1;\nu) = +\infty, \tag{6.15}$$

Proof. The first of equalities (6.15) is obvious. We prove the second of equalities (6.15). According to Stirling's formula:

$$\log(n!) = (n + 1/2)\log(n + 1) - n + O(1).$$

Let $\beta > 0$, $n = \beta\nu + \eta_{1,\nu} \in \mathbb{Z}$,

$$\nu \in [(2(\beta+1)/\beta, +\infty) \cap \mathbb{Z},\ |\eta_\nu| < 2. \tag{6.16}$$

Then (for fixed β)

$$\log(n!) = (\beta\nu + \eta_\nu + 1/2)\log(\beta\nu + \eta_\nu + 1) - \beta\nu + O(1) = \tag{6.17}$$

$$(\beta\nu + \eta_\nu + 1/2)\log(\beta\nu) - \beta\nu + O(1) = (\beta\nu)\log(\beta\nu) - \beta\nu + O(1)\log(\nu).$$

Clearly,

$$\log((\nu+1)!) = \nu\log(\nu) - \nu + O(1)\log(\nu+1).$$

Let further $\gamma \in (0,1)$, $k = [\gamma\nu]$, where $\nu \in [2(\gamma+1)/\gamma, +\infty) \cap \mathbb{Z}$. Then, in view of (6.16) – (6.17) with $\beta = \gamma$ and k in the role of n,

$$\log(k!) = \gamma\nu\log(\gamma\nu) - \gamma\nu + O(1)\log(\nu). \tag{6.18}$$

Further we have $\nu + 1 - k = \nu + 1 - \gamma\nu + \{\gamma\nu\} = (1-\gamma)\nu + 1 + \{\gamma\nu\}$.
If $\nu \in [2(2-\gamma)/(1-\gamma), +\infty) \cap \mathbb{Z}$, then, according to (6.16) – (6.17) with $\beta = 1-\gamma$, and $n+1-k$ in the role of n, we have the equality

$$\log((\nu+1-k)!) = (1-\gamma)\nu\log((1-\gamma)\nu) - (1-\gamma)\nu + O(1)\log(\nu). \tag{6.19}$$

Since $\nu + k = \nu + \gamma\nu - \{\gamma\nu\} = (1+\gamma)\nu - \{\gamma\nu\}$ it follows that, if $\nu \in [2(2+\gamma)/(1+\gamma), +\infty) \cap \mathbb{Z}$, then, in view of (6.16) – (6.17) with $\beta = 1+\gamma$ and $n+k$ in the role of n, we have the equality

$$\log((\nu+k)!) = (1+\gamma)\nu\log((1+\gamma)\nu) - (1+\gamma)\nu + O(1)\log(\nu). \tag{6.20}$$

Let $\nu \in [2(1/\min(\gamma, 1-\gamma) + 1)$. Then (6.18) – (6.20) hold, and, moreover,

$$\log((\nu+1)!) = \nu\log(\nu) - \nu + O(1)\log(\nu),$$

$$\log(\nu!) = \nu\log(\nu) - \nu + O(1)\log(\nu).$$

So, if $\nu \in [2/\min(\gamma, 1-\gamma) + 2, +\infty), k = [\gamma\nu]$, then

$$\log\left(\binom{\nu+1}{k}\right) = \nu\log(\nu) - \nu - (\gamma\nu\log(\gamma\nu) - \gamma\nu) -$$

$$((1-\gamma)\nu\log((1-\gamma)\nu) - (1-\gamma)\nu) + O(1)\log(\nu) =$$

$$\nu\log(\nu) - (\gamma\nu\log(\gamma\nu) - (\gamma\nu\log(\gamma) -$$

$$((1-\gamma)\nu\log(\nu) - ((1-\gamma)\nu\log(1-\gamma) + O(1)\log(\nu) =$$

$$(\gamma\log(1/\gamma) + (1-\gamma)\log(1/(1-\gamma)))\nu + O(1)\log(\nu),$$

and, analogously,

$$\log\left(\binom{\nu+k}{k}\right) = ((1+\gamma)\log(1+\gamma) + \gamma\log(1/\gamma))\nu + O(1)\log(\nu)$$

Therefore, if $\nu \in [2/\min(\gamma, 1-\gamma)+2, +\infty), k = [\gamma\nu]$, then

$$\log\left(\binom{\nu+1}{k}\binom{\nu+k}{k}\right) = \psi_1(\gamma)\nu + O(1)\log(\nu), \tag{6.21}$$

where $\psi_1(\gamma) = (1+\gamma)\log(1+\gamma) - (1-\gamma)\log(1-\gamma) - 2\gamma)\log(\gamma)$. Clearly,

$$\left(\frac{d}{d\gamma}\psi_1\right)(\gamma) = \log((1-\gamma^2)/\gamma^2)),$$

and

$$\psi_1(\gamma) < \psi_1\left(1/\sqrt{2}\right) =$$
$$\log\left(\left(1+1/\sqrt{2}\right)/\left(1-1/\sqrt{2}\right)\right) +$$
$$\left(1/\sqrt{2}\right)\log\left(\left(1+1/\sqrt{2}\right)\left(1-1/\sqrt{2}\right)\right) -$$
$$\left(1/\sqrt{2}\log(1/2)\right) = 2\log\left(\left(1+\sqrt{2}\right)\right),$$

if $\gamma \neq 1/\sqrt{2}$. We can rewrite (6.14) in the form

$$C_1(\varepsilon)\exp\left(2\psi_1\left(1/\sqrt{2}\right)\nu(1-\varepsilon)\right) \leq \beta^{*(r)}(1;\nu) \leq \tag{6.22}$$
$$C_2(\varepsilon)\exp\left(2\psi_1\left(1/\sqrt{2}\right)\nu(1+\varepsilon)\right)$$

for $r = 0, 1, 2$, and all $\nu \in \mathbb{N}$.

In view of (4.15), let

$$\psi_{2,\nu}(k) = \beta^{(0)}_{2,k+1,\nu}/\beta^{(0)}_{2,k,\nu} = (\nu+k+1)(\nu-k+1)/(k+1)^2.$$

Then $\psi_{2,\nu}(k) = 1$ if and only if $2k^2 + 2k - \nu(\nu+2) = 0$. Let

$$r(\nu) = -1/2 + \sqrt{1/4 + \nu(\nu+2)/2} = (1 + O(1)/\nu)\nu/\sqrt{2}$$

Clearly, $\beta^{(0)}_{2,k,\nu}$ increases together with increasing of $k \in [0, r(\nu)] \cap \mathbb{Z}$ and $\beta^{(0)}_{2,k,\nu}$ decreases together with increasing of $k \in (r(\nu), \nu+1] \cap \mathbb{Z}$.

We fix $\gamma_1 \in \left(0, 1/\sqrt{2}\right)$ and $\gamma_2 \in \left(1/\sqrt{2}, 1\right)$. Clearly, there exists $\nu_0 \in \mathbb{N}$ such that

$$\gamma_1 < r(\nu)/\nu < \gamma_2$$

for all $\nu \in [\nu_0, +\infty) \cap \mathbb{N}$. Let $\nu > \max(\nu_0, 1/\gamma_1, 1/(1-\gamma_2))$. Then, in view of (6.21) we have

$$\sum_{0 \leq k \leq \gamma_1\nu} \beta^{(r)}_{2,k,\nu} = \sum_{0 \leq k \leq \gamma_1\nu} k^r \beta^{(0)}_{2,k,\nu} < [\gamma_1\nu]^{r+1}\beta^{(0)}_{2,[\gamma_1\nu],\nu} =$$
$$\exp(2\psi_1(\gamma_1)\nu + O(1)\log(\nu)),$$

and analogously

$$\sum_{\gamma_2\nu \leq k \leq \nu+1} \beta^{(r)}_{2,k,\nu} = \exp(2\psi_1(\gamma_2)\nu + O(1)\log(\nu)).$$

Let
$$\gamma_3=\min(\psi_1(1/\sqrt{2})-\psi_1(\gamma_1),\psi_1(1/\sqrt{2})-\psi_1(\gamma_2)).$$
Then, in view of (6.22)
$$\beta_2^{(r)}(1,\nu)=\left(\sum_{\gamma_1\nu<k<\gamma_2\nu}k^r\beta_{2,k,\nu}^{(0)}\right)\times$$
$$(1+O(1)\exp(-2\gamma_3\nu+O(\log(\nu)).$$
Hence, there exists $\nu_1\in\mathbb{N}$ such that for all $\nu\in[\nu_1,+\infty)cap\mathbb{Z}$ we have
$$\beta_2^{(2)}(1,\nu)=\left(\sum_{\gamma_1\nu<k<\gamma_2\nu}k^2\beta_{2,k,\nu}^{(0)}\right)\times$$
$$(1+O(1)\exp(-2\gamma_3\nu+O(\log(\nu))\geq$$
$$[\gamma_1\nu]\left(\sum_{\gamma_1\nu<k<\gamma_2\nu}k\beta_{2,k,\nu}^{(0)}\right)\times$$
$$(1+O(1)\exp(-2\gamma_3\nu+O(\log(\nu)),$$
and
$$\beta_2^{(1)}(1,\nu)=\left(\sum_{\gamma_1\nu<k<\gamma_2\nu}k\beta_{2,k,\nu}^{(0)}\right)(1+O(1)\exp(-2\gamma_3\nu+O(\log(\nu)).$$

So, $\beta_2^{(2)}(1,\nu)\geq[\gamma_1\nu]\beta_2^{(2)}(1,\nu)(1+o(1))$, when $\nu\to+\infty$. $\square$.

Let conditions Theorem B are fulfilled. Then, in view of (0.19),

(6.23) $$\beta_{F,G,1}^{**}(1;\nu)=(F+G)\beta_{1,0,2}^{*(1)}(1;\nu)+G(\beta_{1,0,2}^{*(2)}(1;\nu)-\beta_{1,0,2}^{*(1)}(1;\nu))\neq 0$$

for $\nu\in\mathbb{N}_0$, and therefore, in view of (6.15),

(6.24) $$\beta_{F,G,1}^{*}(1;\nu)=\beta_2^{*(1)}(1;\nu)(F+G\beta_2^{*(2)}(1;\nu)/\beta_2^{*(1)}(1;\nu))\to\infty,$$

when $\nu\to\infty$. Moreover, if $F\neq 0$ and $G/F\notin\mathfrak{B}$ then (6.23) and (6.24) hold, and, if in this case $x_\nu=x_{F,G,1}(\nu)=\beta_{F,G,1}^{**}(1;\nu)$, then (6.12) is impossible.

In view of (1.3) with $\alpha=1$, (4.3) and (4.36) with $j=1$,
$$\delta^r f_{1,0,3}^*(1,\nu)=(\nu+1)^2O(1);$$
hence, if $x_\nu=x_{F,G,3}(\nu)=F\delta f_{1,0,3}^*(1,\nu)+G\delta^2 f_{1,0,3}^*(1,\nu)$, then (6.13) is impossible. Therefore

(6.25) $$\frac{C_1(\varepsilon)/C_2(\varepsilon)}{(1+\sqrt{2})^{8\nu(1+\varepsilon)}}\leq\left|2\zeta(3)-\frac{\beta_{F,G,2}^{**}(1,\nu)}{\beta_{F,G,1}^{**}(1,\nu)}\right|\leq\frac{C_2(\varepsilon)/C_1(\varepsilon)}{(1+\sqrt{2})^{-8\nu(1-\varepsilon)}}.$$

Let

$$\delta^*_{u,v}(\nu) = \prod_{j=1}^{\nu-1} c^*_{u,v,2}(j) \text{ for } \nu \in \mathbb{N}_0. \tag{6.26}$$

So, $\delta^*_{F,G}(1) = \delta^*_{F,G}(0) = 1$. Let $c^{**}_{F,G,1}(\nu) = c^*_{F,G,1}(\nu)$ for all $\nu \in \mathbb{N}$, let

$$c^{**}_{F,G,0}(\nu) = c^*_{F,G,0}(\nu)c^*_{F,G,0}(\nu-1)$$

for all $\nu \in [2, +\infty) \cap \mathbb{N}$, and let $c^{**}_{F,G,0}(1) = c^*_{F,G,0}(1)$. If conditions Theorem B are fulfilled, then $\delta_{F,G}(\nu) \neq 0$ far all the $\nu \in \mathbb{N}$, and the equation (6.10) and the system

$$\begin{cases} y_{\nu+1} + c^*_{F,G,1}(\nu)y(\nu) + c^*_{F,G,0}(\nu)c^a_{F,G,2}st(\nu-1)y(\nu-1) = 0 \\ y_\nu = \delta_{F,G}(\nu)^* x_\nu,\ \nu \in \mathbb{N} \end{cases}$$

are equivalent. Moreover, $P^*_{F,G}(\nu)$ and $\delta^*_{F,G}(\nu)\beta^{**}_{F,G,2}(1,\nu)/\beta^{**}_{F,G,1}(1,0)$ satisfy to the first of equations (6.10) and the same initial conditions. Therefore

$$P^*_{F,G}(\nu) = \delta_{F,G}(\nu)\beta^{**}_{F,G,2}(1,\nu)/\beta^{**}_{F,G,1}(1,0), \tag{6.27}$$

Analogously, we have

$$Q^*_{F,G}(\nu) = \delta_{F,G}(\nu)\beta^{**}_{F,G,1}(1,\nu)/\beta^{**}_{F,G,1}(1,0), \tag{6.28}$$

$r^*_{F,G}(\nu) = \beta^{**}_{F,G,2}(1,\nu)/\beta^{**}_{F,G,1}(1,\nu)$, for all $\nu \in \mathbb{N}_0$. In view of (6.25)

$$\lim_{\nu\to\infty} r^*_{F,G}(\nu) = 2\zeta(3).$$

§7. End of the proof of Theorem B.

Lemma 7.1. *The following equalities hold:*

$$P^*_{u,v}(\nu) = P^*_{u/v,1}(\nu),\ Q^*_{u,v}(\nu) = Q^*_{u/v,1}(\nu), \delta_{u,v}(\nu) = \delta d_{u/v,1}(\nu). \tag{7.1}$$

Proof. In view of (0.16) If $\nu = 0,\ 1$, then the equalities (7.1) directly follows from (6.5) – (6.7) and (6.26) In view of (6.1),

$$c^*_{u,v,k}(\nu) = c^*_{u/v,1,k}(\nu) \tag{7.2}$$

for $k = 0,\ 1,\ 2, \nu \in \mathbb{N}$. Therefore the last equality in (7.1) holds for all $\nu \in \mathbb{N}$. In view (6.8)), (6.5), (6.6), (6.3), (6.2) and (7.2),

$$a_{u,v}(\nu) = a_{u/v,1}(\nu),\ b_{u,v}(\nu) = b_{u/v,1}(\nu)$$

for $\nu \in [2, +\infty) \cap \mathbb{Z}$. Let $\nu \in [2, +\infty) \cap \mathbb{Z}$, and let (7.1) hold for all $\nu - \kappa$ with $\kappa \in [0, \nu - 1] \cap \mathbb{Z}$. Then we have

$$P^*_{u,v}(\nu) = b^*_{u,v}(\nu)P^*_{u,v}(\nu-1) + a^*_{u,v}(\nu)P^(_{u,v}\nu-2) =$$

$$b^*_{u/v,1}(\nu)P^*_{u/v,1}(\nu-1)+a^*_{u/v,1}(\nu)P^*_{u/v,1}(\nu-2)=P^*_{u/v,1}(\nu)$$
$$Q^*_{u,v}(\nu)=b^*_{u,v}(\nu)Q^*_{u,v}(\nu-1)+a^*_{u,v}(\nu)Q^*_{u,v}(\nu-2)=$$
$$b^*_{u/v,1}(\nu)Q^*_{u/v,1}(\nu-1)+a^*_{u/u,1}(\nu)Q^*_{u/v,1}(\nu-2)=Q_{u/v,1}(\nu).$$

□

Lemma 7.2. *If conditions of the Theorem B are fulfilled, then*

(7.3) $$\delta^*_{u,v}(\nu)=\delta_{u,v}(\nu)/(u+v)^{\max 2\nu-2,0}$$

where $\delta^*_{u,v}(\nu)$ *is homogeneous polynomial in* $\mathbb{Z}[u,v]$, *and*

(7.4) $$\max(2\nu-2,0)=\deg_u(\delta_{u,v}(\nu))=\deg_v(\delta_{u,v}(\nu))=\deg(\delta_{u,v}(\nu)).$$

Proof. We have to prove the last equality in (7.4), because other assertions of the Lemma are obvious. In view of (0.10) and (0.8),

$$c_{F,G,0}(\nu)=-frac(\tau-1)^2(2\tau+1)(\tau+1)^2(2\tau-1)\times$$
$$\tau(\tau+1)c_{F,G,2}(\nu+1)\neq 0$$

for all $\nu\in\mathbb{N}_0$. Therefore the last equality in (7.4) holds also.

In view of (5.61) – (5.62), (6.27) – (6.28) and (7.3), the following equalities hold:

$$P_{u,v}(\nu)=4P^*_{u,v}(\nu)(u+v)^{2\nu}=$$
$$16(u+v)\delta_{u,v}(\nu)\beta^{**}_{u,v,2}(1,\nu),$$
$$Q_{u,v}(\nu)=Q^*_{u,v}(\nu)(u+v)^{2\nu-1}=$$
$$4\delta_{u,v}(\nu)\beta{*}{*}_{u,v,1}(1,\nu)$$
$$\max(2\nu,1)=\deg_u(P_{u,v}(\nu))=\deg_v(P_{u,v}(\nu))=\deg(P_{u,v}(\nu)),$$
$$\max(2\nu-1,0)=\deg_u(Q_{u,v}(\nu))=\deg_v(Q_{u,v}(\nu))=\deg(Q_{u,v}(\nu)),$$

where $\nu\in\mathbb{N}_0$. □

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