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Linearization of the Riemann problem for a triangular system of conservation laws and delta shock wave formation process

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Using the weak asymptotic method, we approximate a triangular system of conservation laws arising from the so-called generalized pressureless gas dynamics by a diagonal linear system. Then, we apply the usual method of characteristics to find approximate solution to the original system. As a consequence, we shall see how the delta shock wave naturally arises along the characteristics.

Also, we propose a procedure that could be applied to more general systems of conservation laws.

Keywords: linearization; delta shock wave formation; non-strictly hyperbolic system; global approximate solution; weak asymptotic method

1. Introduction

In recent years, the weak asymptotic method has been applied to many problems involving formation and interaction of nonlinear waves. For instance, using this method, we are able to find explicit formulas describing the interaction of solitons in the case of generalized KdV equations [1, 2], interaction of Sine-Gordon solitons [3, 4], evolution of nonlinear waves in the case of scalar conservation laws [5], interaction [6] and formation [7, 8] of δ-shock waves in the case of a triangular system of conservation laws, confluence of free boundaries in the Stefan problem with underheating [9], different interactions of the shock waves appearing on the gas dynamics [10–12], etc. Here, we want to cast different, we believe, interesting and important light on the possibilities of the method.

The subject of the current paper is the following triangular system of conservation laws:

$$u_t+(f(u))_x=0 \tag{1}$$

$$v_t+(vg(u))_x=0 \tag{2}$$

supplemented with the Riemann initial data:

$$u|_{t=0}=\begin{cases} U_L, & x<0\\ U_R, & x\geqslant 0\end{cases} \tag{3}$$

$$v|_{t=0}=\begin{cases} V_L, & x<0\\ V_R, & x\geqslant 0\end{cases} \tag{4}$$

where $U_R<U_L$.

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This Riemann problem has been intensively investigated in recent years [6–8, 13–26] (the list is far from being complete). The reason for this lies in the applicability of the system: it arises from the (generalized) pressureless gas dynamics [27, 28]. Another purely mathematical reason is the fact that under the following assumptions on f and g (see, e.g. [7] for derivation of the conditions):

$$\begin{array}{cc} f\in C^2([U_R,U_L]), & g\in C^1([U_R,U_L]) \\ f''>0 & \text{on } [U_R,U_L] \\ g'-f''\geqslant 0 & \text{on } [U_R,U_L] \\ \exists \hat{U}\in(U_R,U_L) \text{ such that} & g(\hat{U})=f'(\hat{U}) \end{array} \tag{5}$$

the system, in general, does not admit the classical BV solution. Clearly, this raises challenging mathematical questions.

In the current contribution, we continue our work from [7] by proposing another method for constructing explicit formulas that are smooth in $t\in\mathbb{R}^+$ and represent the global approximate solution to (1), (2) and whose weak limit contains the Dirac δ-distribution. We stress that, before paper [7], no method for constructing explicit formulas representing an approximate solution to the problem was proposed.

We believe that the procedure to be presented here is more comprehensive than that in [7], and that it could be applied to more general systems of conservation laws (see Section 5 for a possible approach).

Concerning problem (1)–(4), informally speaking, we have the Dirac δ distribution as a component of the solution to the problem under study. In order to formalize this situation, several concepts allowing the Dirac δ distribution as a solution to the problem were introduced. We gave a detailed description of different concepts in [7]. Also, one can find lots of information on this issue in [6]. Here, we just mention possible approaches used in formalizing the existence of very singular objects (such as the Dirac distribution) as solutions or even as coefficients of an equation.

Chronologically, the first approach is the famous vanishing viscosity method. In [15], for the Riemann initial data, the author proves that system (1), (2) with vanishing viscosity (the vanishing term was of the form $\varepsilon t(u,v)_{xx}$) admits a solution converging to the Dirac δ distribution. The author obtains the result by using various relations that are satisfied by a family of approximate solutions. Still, no explicit form of the approximate solution is given, and no formal definition of a solution containing the δ distribution is proposed.

One of the first solution concepts allowing the δ distribution as a solution candidate for (1)–(4) was the one obtained by extending the definition of Radon measures (they are defined on the set of continuous functions) to the set of *BV* functions as the set of test functions (see, e.g. [14, 16–19, 24, 26]).

In the second proposed framework (the Colombeau generalized algebra framework [29]), the δ distribution is considered as a family of smooth functions weakly converging to the δ distribution, and these functions are treated as ordinary smooth functions [21, 22].

The third framework is due to V. Danilov and V. Shelkovich. In [6, 30], the authors gave a definition of a solution to a Cauchy problem for system (1), (2) that allows the Dirac distribution as well as its derivatives ($\delta,\delta',\ldots$) to be solutions to the problem. Such approach rather naturally generalizes the classical definition of the weak solution.

In the previous three frameworks, the existence of the solution was proved by using *ad hoc* methods (so no procedure explaining how to construct the solution is given). The uniqueness of the solution is an open question in any of the frameworks.

In the remainder of the introduction, we will focus our attention on the method of linearization that we propose here.

The notion of linearization can be understood in two senses.

The first one consists in finding the linear properties of nonlinear operations. Such a principle is applied in the case of famous compensated compactness [31, 32]. The corner stone of this method is the *div-curl* lemma providing necessary conditions under which two sequences (u_n), $(v_n)\subset L^2(\Omega)$, $\Omega\subset\mathbb{R}^d$, weakly converging in $L^2(\Omega)$ to u, $v\in L^2(\Omega)$, satisfy

$$u_n v_n \rightharpoonup uv \quad \text{in } \mathscr{D}'(\Omega)$$

Similar logic, that is finding the linear properties of nonlinear objects, is used to derive the nonlinear superposition law (see Theorem 3).

The other meaning of linearization, which is interesting to us at the moment, is reducing a problem of solving a nonlinear equation to the problem of solving a linear equation.

For instance, in the case of the Dirichlet problem for quasilinear elliptic equations (see, e.g. [33]), one first finds *a priori* inequalities for solutions of appropriate linear equations and then uses the fixed point theorems to conclude about the existence of the solution to the original nonlinear problem. So, the nonlinearity, which appears in the equation, is replaced by the linear equation and a problem of finding the fixed point.

Another example, corresponding more to our situation, is the reduction of a scalar conservation law to a transport equation. In the famous kinetic approach [34], the nonlinearity is replaced by a 'bad' right-hand side by using the entropy admissibility conditions [35]. More precisely, it can be shown that an entropy admissible solution to the scalar conservation law

$$\partial_t u+\partial_x f(u)=0$$

simultaneously satisfies the linear equation

$$\partial_t h(t,x,\xi)+\partial_x f'(\xi)h(t,x,\xi)=-\partial_\xi m(t,x,\xi)$$

where m is a positive measure and

$$h(t,x,\xi)=\begin{cases}1, & 0\leqslant\xi\leqslant u(x,t)\\ -1, & u(x,t)\leqslant\xi\leqslant 0\\ 0 & \text{else}\end{cases}$$

So, we see that we have lost the nonlinearity but we still have a problematic term on the right-hand side of the new equation. Still, it appears that it is a much easier end efficient to operate with such a linear equation than with the nonlinear one (see [36–38]).

A similar approach exists in the case of hyperbolic systems, and it permits achieving substantially new results [39–41]. But, as in the case of a scalar conservation law, in order to linearize the system, we need the existence of infinitely many entropies corresponding to the system.

Here, we propose a method for linearization, which is independent of the existence of entropies, but it can be applied only in the case of special initial data. Similarly as in the kinetic approach, we 'replace' (or, more precisely, approximate) the nonlinearity by a linear term and a 'bad' right-hand side.

We shall briefly describe the procedure which we shall apply.

First, we consider Equation (1). In general, classical solutions to hyperbolic conservation laws can be obtained by the method of characteristics. Such solutions exist until the characteristics intersect each other. The moment of the first intersection of characteristics is usually called the gradient catastrophe and at that moment the singularity of the solution is formed. Also, at that moment we have to pass to the concept of weak solution.

But, if we somehow succeeded in avoiding intersection of the characteristics and then wrote the equation to which those new characteristics correspond, then the new equation should be linear. In order to explain this idea more precisely, consider Equation (1) with the following perturbed Riemann initial data:

$$u|_{t=0}=\hat{u}(x)=\begin{cases}U_L, & x<a_2\\ u_0(x), & a_2\leqslant x\leqslant a_1\\ U_R, & a_1<x\end{cases} \tag{6}$$

where $a_1=\varepsilon^{1/2}$ and $a_2=-\varepsilon^{1/2}$ for a small parameter ε, while u_0 is such that

$$f'(u_0(x))=-K_1x+b_1 \tag{7}$$

for constants $K_1>0$ and b_1.

The characteristics of Cauchy problem (1), (6) are plotted in Figure 1. Notice that all the characteristics issuing from $(-\varepsilon^{1/2},\varepsilon^{1/2})$ intersect at the same point. In order to avoid their intersection, a natural idea is to smear the discontinuity line, that is to take an ε neighborhood of the discontinuity line and to dispose the characteristics in that neighborhood in a way that they do not intersect each other. Of course, as $\varepsilon\to 0$ all of them should lump together into the discontinuity line. Of course, this will not be the standard characteristics for problem (1), (3). Nevertheless, the approximate solution to our problem will remain constant along them. We call such lines the 'new characteristics' (see Figure 2 and [7, 42]). Also notice that we replaced nonlinear Equation (1) by a family of (almost) linear equations indexed by the small parameter $\varepsilon\to 0$ (they are nonlinear only for a short period of time).

This idea is formalized in [7]. Therefore, in Section 3, we shall just briefly describe the method that we used in [7] in order to linearize and solve (1). At the end of the section, we formulate the main theorem.

In Section 2, we give auxiliary notions and notations.

In Section 4, we consider Equation (2). It is linear with respect to v, but, for the sake of consistency (compare also with Section 5), we will use the same terminology as for Equation (1)—we call the term $g(u)v$ nonlinear and the procedure that we apply on (2) we call linearization. Actually, we use completely the same procedure as in Section 3; we replace nonlinear Equation (2) by a family of (almost) linear equations (27) admitting the family of solutions $(v_\varepsilon)_\varepsilon$. But, unlike Equation (1), we linearize (2) sacrificing zero on the right-hand side (more precisely, the right-hand side will be a regularized δ distribution). As a consequence, we will obtain

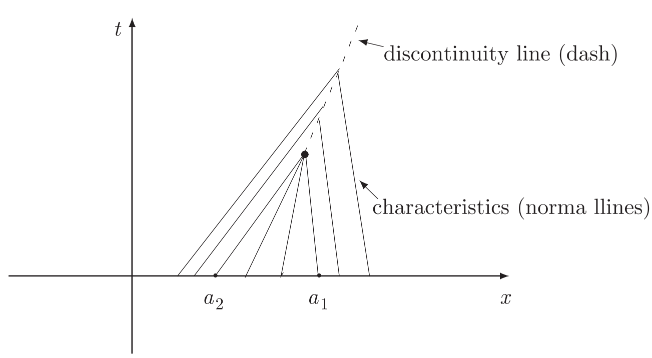

Figure 1. Standard characteristics for (1), (21). Dotted point in (t,x) plane is (t^*,x^*).

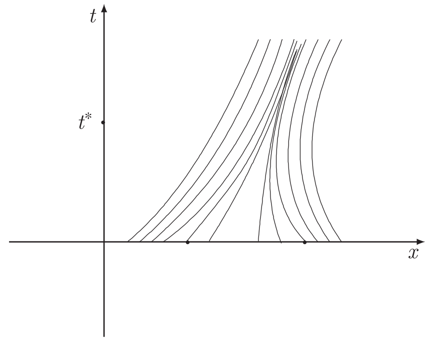

Figure 2. System of characteristics for u_ε defined in Theorem 4. The points $a_1+\varepsilon A((a_1-a_2)/2)$ and $a_2-\varepsilon A((a_1-a_2)/2)$ are dotted on the x axis.

explicit formulas representing the global approximate solution to problem (2), (4), and we will see how the δ-shock wave naturally arises along the 'new characteristics'. More precisely, we will prove that $(v_\varepsilon)_\varepsilon$ converges to a distribution containing the Dirac δ distribution. This is in accordance with the previous result on this subject (see [6, 15–17, 22] among many others).

In Section 5, we give a proposal concerning a possible use of the method presented here for an arbitrary system of conservation laws. We stress that the section does not bring any technical result but only shows general directions for a possible application.

2. Auxiliary notions and statements

This section provides basic notions and statements of the weak asymptotic method.

Definition 1 (*Danilov and Shelkovich* [6])
We denote by $O_{\mathcal{D}'}(\varepsilon^\alpha)\in\mathcal{D}'(\mathbb{R})$, $\alpha\in\mathbb{R}$, a family of distributions depending on $\varepsilon\in(0,1)$ and $t\in\mathbb{R}^+$ such that for any test function $\eta(x)\in C_0^1(\mathbb{R})$, it holds

$$\langle O_{\mathcal{D}'}(\varepsilon^\alpha),\eta(x)\rangle = O(\varepsilon^\alpha),\quad \varepsilon\to 0$$

where the estimate on the right-hand side is understood in the usual Landau sense and locally uniformly in t, that is, $|O(\varepsilon^\alpha)|\leqslant C_T\varepsilon^\alpha$ for $t\in[0,T]$.

Now, we can give a definition of our approximating solution:

Definition 2 (*Danilov and Mitrovic* [7])
The family of pairs of functions $(u_\varepsilon, v_\varepsilon)=(u_\varepsilon(x,t), v_\varepsilon(x,t))$, $\varepsilon>0$, is called the weak asymptotic solution of problem (1)–(4) if for an $\alpha>0$

$$\begin{aligned} u_{\varepsilon t}+(f(u_\varepsilon))_x &= \mathcal{O}_{\mathcal{D}'}(\varepsilon^\alpha)\\ v_{\varepsilon t}+(v_\varepsilon g(u_\varepsilon))_x &= \mathcal{O}_{\mathcal{D}'}(\varepsilon^\alpha)\\ u_\varepsilon|_{t=0}-u|_{t=0} &= \mathcal{O}_{\mathcal{D}'}(\varepsilon^\alpha),\quad v_\varepsilon|_{t=0}-v|_{t=0}=\mathcal{O}_{\mathcal{D}'}(\varepsilon^\alpha),\quad \varepsilon\to 0 \end{aligned} \tag{8}$$

The following theorem is the basic one in our construction. It is called the nonlinear superposition law. Actually, it is a statement about linear properties of the operation of superposition of nonlinear functions.

Theorem 3 (*Danilov and Mitrovic* [42])
Let $\omega_i\in C_0^\infty(\mathbb{R})$, $i=1,2$, where $\lim_{z\to+\infty}\omega_i(z)=1$,

$$\lim_{z\to-\infty}\omega_i(z)=0$$

and

$$\frac{\mathrm{d}\omega(z)}{\mathrm{d}z}\in\mathcal{S}(\mathbb{R})$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions. For bounded functions a,b,c defined on $\mathbb{R}^+\times\mathbb{R}$ and bounded functions φ_i, $i=1,2$, defined on $\mathbb{R}^+$, it holds

$$\begin{aligned} f\left(a+b\omega_1\left(\frac{\varphi_1-x}{\varepsilon}\right)+c\omega_2\left(\frac{\varphi_2-x}{\varepsilon}\right)\right) &= f(a)+H(\varphi_1-x)(f(a+b+c)B_1+f(a+b)B_2-f(a+c)B_1-f(a)B_2)\\ &\quad +H(\varphi_2-x)(f(a+b+c)B_2-f(a+b)B_2+f(a+c)B_1-f(a)B_1)+\mathcal{O}_{\mathcal{D}'}(\varepsilon) \end{aligned} \tag{9}$$

where H is the Heaviside function and the functions $B_i=B_i((\varphi_2-\varphi_1)/\varepsilon)$, $i=1,2$, satisfy for every $\rho\in\mathbb{R}$

$$B_1(\rho)=\int\dot{\omega}_1(z)\omega_2(z+\rho)\,\mathrm{d}z \quad \text{and} \quad B_2(\rho)=\int\dot{\omega}_2(z)\omega_1(z-\rho)\,\mathrm{d}z \tag{10}$$

and

$$B_1(\rho)+B_2(\rho)=1$$

Furthermore:

$$\begin{aligned} B_1(\rho) &= 1-B_2(\rho)\to 1 \quad \text{as } \rho\to+\infty \\ B_1(\rho) &= 1-B_2(\rho)\to 0 \quad \text{as } \rho\to-\infty \end{aligned} \tag{11}$$

3. Linearization of Riemann problem (1), (3)

We explain in this section how to find the approximate solution to problem (1), (3) by linearizing Equation (1). The theorem is formulated at the end of the section and we leave it without proof. The proof can be found in [7].

We first perturb initial data in order to accomplish the linearization properly. The perturbation is given in (6). So, problem (1), (3) is initially replaced by the family of problems (1), (6). The following notation that we shall use here and in the sequel is actually motivated by (1), (6):

$$\begin{aligned} H_i &= H(\varphi_i-x), \quad \delta_i=\delta(\varphi_i-x), \quad x\in\mathbb{R},\ t\in\mathbb{R}^+ \\ B_i &= B_i(\rho), \quad \rho=\frac{\varphi_2-\varphi_1}{\varepsilon}, \quad \varphi_i=\varphi_i(t,\varepsilon), \quad i=1,2 \\ \tau &= \frac{f'(U_L)t+a_2-f'(U_R)t-a_1}{\varepsilon}=\frac{\psi_0(t)}{\varepsilon} \\ t^* &= \frac{a_1-a_2}{f'(U_L)-f'(U_R)}=\frac{2\varepsilon^{1/2}}{f'(U_L)-f'(U_R)} \\ x^* &= f'(U_L)t^*+a_2=f'(U_R)t^*+a_1 \end{aligned} \tag{12}$$

where H is the Heaviside function and δ Dirac distribution. We remind that $a_2=-\varepsilon^{1/2}$ and $a_1=\varepsilon^{1/2}$.

The function τ is so-called 'fast variable'. It is equal to the ε^{-1}-scaled difference of the standard characteristics corresponding to Equation (1) issuing from a_2 and a_1, respectively. Since we assumed $a_2<a_1$ it follows that while we are in the domain of existence of classical solution to (1), (6) we have $\tau\to-\infty$ as $\varepsilon\to 0$, while we are in the domain where (the weak) solution to (1), (6) is discontinuous (i.e. in the form of the shock wave) we have $\tau\to\infty$ as $\varepsilon\to 0$.

The point (t^*,x^*) is the point at which the classical solution to (1), (6) blows up.

The functions φ_i, $i=1,2$, are the 'new characteristics' that issue from $a_1+\varepsilon A((a_1+a_2)/2)$ and $a_2-\varepsilon A((a_1+a_2)/2)$, respectively. They are given by the following globally solvable Cauchy problems (see [7] and (17)):

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(t,\varepsilon)=(B_2(\rho)-B_1(\rho))f'(U_R)+cB_1(\rho), \quad \varphi_1(0,\varepsilon)=a_1+A\varepsilon\frac{a_1-a_2}{2} \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_2(t,\varepsilon)=(B_2(\rho)-B_1(\rho))f'(U_L)+cB_1(\rho), \quad \varphi_2(0,\varepsilon)=a_2-A\varepsilon\frac{a_1-a_2}{2} \tag{14}$$

for a large enough constant A. The function $\rho=\rho(\psi_0(t)/\varepsilon)$ appearing here is the classical solution to the following globally solvable Cauchy problem:

$$\rho_\tau=1-2B_1(\rho), \quad \left.\frac{\rho}{\tau}\right|_{\tau\to-\infty}=1 \tag{15}$$

It is well known that problem (1), (6) will have classical solution up to the moment t^* given by:

$$t^*=\max_{x\in(a_2,a_1)}-\frac{1}{f''(u_0(x))u_0'(x)}=\frac{1}{K_1}=\frac{2\varepsilon^{1/2}}{f'(U_L)-f'(U_R)} \tag{16}$$

for K_1 from (7). The choice of our initial data is such that the shock wave will be formed in the moment of blow up of the classical solution and it will not change its shape for any $t>t^*$. This is because all the characteristics issuing from $[a_2,a_1]$ intersect in the same point (t^*,x^*) (see Figure 1).

As we have already explained in the Introduction, in order to linearize Equation (1), we need to perturb the characteristics so that we avoid their intersection. More precisely, we will dispose the characteristics for every $\varepsilon>0$ in an ε neighborhood of the discontinuity line so that they do not intersect with each other (see Figure 2). We will call this perturbed characteristics the 'new characteristics'.

Then, we will write down an equation corresponding to such non-intersecting characteristics. Exactly this equation will be 'almost' linear (i.e. nonlinearity will disappear after a negligible time).

Another question that arises here is how to distribute the 'new characteristics' in the ε neighborhood of the discontinuity line. The obvious way to accomplish this is to distribute the 'new characteristics' uniformly in the mentioned area, that is in a way that each of them is parallel to the discontinuity line.

In [7], we used Theorem 3 and the 'switch' functions B_i, $i=1,2$, appearing there to obtain the following equation of the 'new characteristics' corresponding to (1):

$$
\begin{aligned}
&\dot{x} = f'(u_\varepsilon)(B_2(\rho)-B_1(\rho))+cB_1(\rho), \quad \dot{u_\varepsilon}=0\\
&x(0) = x_0+\varepsilon A\left(x_0-\frac{a_1+a_2}{2}\right), \quad u_\varepsilon(0)=\hat{u}(x_0), \quad x_0\in\mathbb{R}
\end{aligned}
\tag{17}
$$

where A is a large constant, and, as usual, the function $\rho=\rho(\psi_0(t)/\varepsilon)$ is determined by (15). The constant c from (17) was determined so that the function $u_\varepsilon=\hat{u}(x_0(x,t,\varepsilon))$ represents the weak asymptotic solution to problem (1), (3). It was shown that

$$
c=2\frac{[f]}{[u]}=2\frac{f(U_R)-f(U_L)}{U_R-U_L}
\tag{18}
$$

that is $c/2$ was the velocity of the shock wave given by the Rankine–Hugoniot conditions.

Furthermore, the constant A is such that for $x_0\in\mathbb{R}$ and every $t>0$:

$$
\frac{\partial x}{\partial x_0}=\begin{cases}\dfrac{\varphi_1-\varphi_2}{a_1-a_2}, & x_0\in[a_2,a_1]\\ 1 & \text{else}\end{cases} \quad >0
\tag{19}
$$

The latter equality is easily deducible; concerning the inequality, we address a reader to [7].

The Cauchy problem corresponding to the system of characteristics (17) is

$$
\partial_t u_\varepsilon+\partial_x((B_2-B_1)f(u_\varepsilon)+cB_1u_\varepsilon)=0
\tag{20}
$$

$$
u|_{t=0}=\hat{u}\left(x+\varepsilon\left(x-\frac{a_1+a_2}{2}\right)\right)=\hat{u}_0(x)+\mathcal{O}_{\mathcal{D}'}(\varepsilon)
\tag{21}
$$

for the function $\hat{u}$ given by (6).

According to the Inverse Function Theorem, relation (19) means that the 'new characteristics' do not mutually intersect, which in turn means that there exists the solution $x_0=x_0(x,t,\varepsilon)$ of the implicit equation:

$$
x(x_0,t,\varepsilon)=x
\tag{22}
$$

for the function x which solves (17). Thus, the solution to Cauchy problem (20), (21) can be written in the form:

$$
u_\varepsilon(x,t)=\hat{u}(x_0(x,t,\varepsilon))
$$

Roughly speaking, the blow up of the gradient $\partial_x u_\varepsilon$ will be neutralized by the term B_2-B_1. As we have seen in [7] (see also (24) below and the proof of Corollary 5), the term B_2-B_1 will be close to zero after the gradient catastrophe thus eliminating the influence of the nonlinearity $f(u_\varepsilon)$ appearing in (20).

We formalize the previous considerations in the following theorem, which is proved in [7].

Theorem 4

The family of classical solutions (u_ε) to Cauchy problems (20), (21) is the weak asymptotic solution to Cauchy problem (1), (3) and it is given by

$$
u_\varepsilon(x,t)=\hat{u}(x_0(x,t,\varepsilon))
\tag{23}
$$

where x_0 is the inverse function to the function $x=x(x_0,t,\varepsilon)$, $t>0$, $\varepsilon>0$ of the 'new characteristics' defined through Cauchy problem (17).

The function $\rho=\rho(\psi_0(t)/\varepsilon)$ from (17) is the solution of Cauchy problem (15). Furthermore, it holds

$$
\begin{aligned}
&B_1(\rho(\tau))\to 1/2, \quad \tau\to\infty \quad \text{and} \quad B_1(\rho(\tau))\to 0, \quad \tau\to-\infty, \quad |\rho B_1(\rho)|\leqslant\text{const}<\infty, \quad \rho\in\mathbf{R}\\
&\int_0^\infty (1-2B_1(\rho(\tau)))\,\mathrm{d}\tau<\infty, \quad \int_{-\infty}^0 (B_1(\rho(\tau)))\,\mathrm{d}\tau<\infty
\end{aligned}
\tag{24}
$$

Corollary 5

The weak asymptotic solution (u_ε) of problem (1), (3) satisfies for every fixed $t>0$:

$$
u_\varepsilon(x,t)\rightharpoonup\begin{cases}U_L, & x<\dfrac{ct}{2}\\ U_R, & x>\dfrac{ct}{2}\end{cases} \quad \text{in } \mathcal{D}'(\mathbf{R}) \text{ as } \varepsilon\to 0
$$

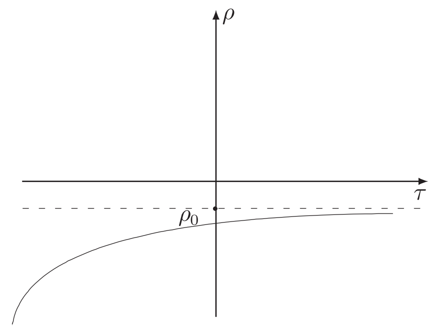

Figure 3. The curve represents the solution of (15). The dot on the ρ axis, denoted by ρ_0, is the smallest (and in this case unique) root of the equation $1-2B_1(\rho)=0$.

Furthermore, we have for $i=1,2$:

$$\varphi_i \to \frac{ct}{2}, \quad \varepsilon \to 0 \tag{25}$$

Proof
One can see from the classical ODE theory that the solution ρ of problem (15) satisfies

$$\rho \to \rho_0 \quad \text{as } \tau \to +\infty \tag{26}$$

where ρ_0 is a stationary solution of (15) (see Figure 3), that is the constant such that $1-2B_1(\rho_0)=0$ (and therefore $B_1(\rho_0)=B_2(\rho_0)=\frac{1}{2}$; we remind that $B_1+B_2=1$).

Furthermore, notice that from (5) it follows $f'(U_L)-f'(U_R)>0$ and therefore

$$\tau = \frac{(f'(U_L)-f'(U_R))t - 2\varepsilon^{1/2}}{\varepsilon} \to \infty \quad \text{as } \varepsilon \to 0$$

for every fixed $t>0$. From here and (26):

$$\rho = \rho(\tau(t)) \to \rho_0 \text{ i.e. } 1-2B_1(\rho) \to 0 \text{ as } \varepsilon \to 0$$

Therefore, after letting $\varepsilon \to 0$ in (13) and (14):

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(t,0) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_2(t,0) = \frac{c}{2}$$

or, since $a_i \to 0$, $i=1,2$:

$$\varphi_1(t,0) = \varphi_2(t,0) = \frac{c}{2}t$$

Since $u_\varepsilon(x,t)=U_L$ for $x<\varphi_2$ and $u_\varepsilon(x,t)=U_R$ for $x>\varphi_1$, we obtain in the limit:

$$u_\varepsilon(x,t) \rightharpoonup \begin{cases} U_L, & x<\dfrac{ct}{2} \\ U_R, & x>\dfrac{ct}{2} \end{cases} \quad \text{in } \mathcal{D}'(\mathbf{R}) \text{ as } \varepsilon \to 0$$

4. Linearization of Riemann problem (2), (4)

We will linearize problem (2), (4), similarly as (1), (3), by replacing it with the following family of problems:

$$\partial_t v_\varepsilon + \partial_x (g(u_\varepsilon)(B_2-B_1)v_\varepsilon + cB_1 v_\varepsilon) = F(x,t,\varepsilon) \tag{27}$$

$$v_\varepsilon|_{t=0} = \hat{v}_0(x) = \begin{cases} V_L, & x<-\varepsilon^{1/2}-\varepsilon^{1/3} \\ v_0(x), & -\varepsilon^{1/2}-\varepsilon^{1/3} \leqslant x < \varepsilon^{1/2}+\varepsilon^{1/3} \\ V_R, & \varepsilon^{1/2}+\varepsilon^{1/3} \leqslant x \end{cases} \tag{28}$$

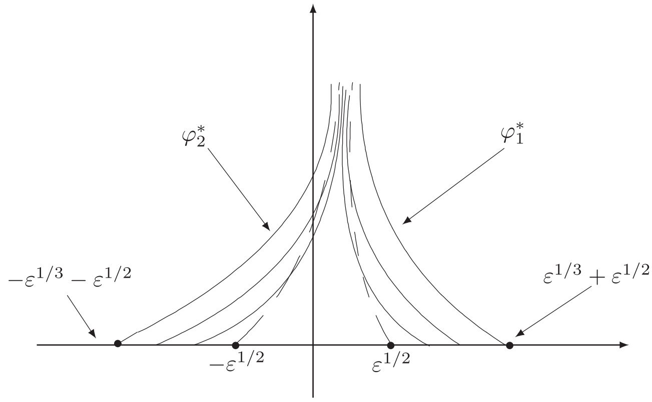

Figure 4. Dashed curves are the functions φ_i, $i=1,2$. Normal curves are characteristics defined by (30). If they issue out of the interval $(-\varepsilon^{1/2},\varepsilon^{1/2})$ they have the same slopes till the intersection with φ_i, $i=1,2$.

where v_0 is an arbitrary smooth function such that $\hat{v}_0$ is Lipschitz function, the constant c is given by (18), and F is chosen so that the classical solution v_ε to (28), (27) satisfies

$$Lv_\varepsilon = v_{\varepsilon t}+(g(u_\varepsilon)v_\varepsilon)_x=\mathcal{O}_{\mathcal{D}'}(\varepsilon^{1/6}) \tag{29}$$

On the first step (Theorem 8) we shall determine the function F. Then, we shall solve problem (28), (27), (Theorem 9) and, finally, we shall determine the weak limit of the solution (Theorem 11).

We begin with the system of characteristics for (27), (28):

$$\dot{X}=g(u_\varepsilon)(B_2-B_1)+B_1 c, \quad X(0)=x_0\in\mathbb{R} \tag{30}$$

$$\dot{v}_\varepsilon=-g'(u_\varepsilon)\partial_x u_\varepsilon(B_2-B_1)v_\varepsilon+F(x,t,\varepsilon), \quad v_\varepsilon(0)=\hat{v}_0(x_0) \tag{31}$$

where $u_\varepsilon=u_\varepsilon(X,t)$.

We have the following lemma:

Lemma 6

The solution $X=X(x_0,t,\varepsilon)$ to (30) satisfies

$$\frac{\partial X}{\partial x_0}>0 \tag{32}$$

The characteristics $\varphi_1^*:=X(\varepsilon^{1/2}+\varepsilon^{1/3},t,\varepsilon)$ and $\varphi_2^*:=X(-\varepsilon^{1/2}-\varepsilon^{1/3},t,\varepsilon)$ issuing from $\varepsilon^{1/2}+\varepsilon^{1/3}$ and $-\varepsilon^{1/2}-\varepsilon^{1/3}$, $\varepsilon>0$ (see Figure 4), satisfy for every $\varepsilon>0$ small enough:

$$\begin{aligned}\varphi_1^*(t,\varepsilon) &= \int_0^t (g(U_R)(B_2-B_1)+cB_1)\,\mathrm{d}t'+\varepsilon^{1/2}+\varepsilon^{1/3}\\ \varphi_2^*(t,\varepsilon) &= \int_0^t (g(U_L)(B_2-B_1)+cB_1)\,\mathrm{d}t'-\varepsilon^{1/2}-\varepsilon^{1/3}\end{aligned} \tag{33}$$

Proof

First, recall that from Theorem 4 follows:

$$u_\varepsilon(x,t)=\hat{u}(\tilde{x}_0(x,t,\varepsilon))$$

where the function $\hat{u}$ is given by (6) and $\tilde{x}_0=\tilde{x}_0(x,t,\varepsilon)$ is the inverse function to the function $x=x(x_0,t,\varepsilon)$ given by (17).

Having this in mind, we get after differentiating (30) in x_0 and using $B_2-B_1=1-2B_1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X}{\partial x_0}=(1-2B_1)g'(\hat{u}(\tilde{x}_0))\hat{u}'(\tilde{x}_0)\frac{\partial\tilde{x}_0}{\partial x}\frac{\partial X}{\partial x_0} \tag{34}$$

where $B_1=B_1(\psi_0(t)/\varepsilon)$.

Then, integrating (34) with respect to the unknown function $\partial X/\partial x_0$ and having in mind that $\partial\tilde{x}_0/\partial x<\infty$ (see (19)):

$$\frac{\partial X}{\partial x_0}=\exp\left(\int_0^t g'(\hat{u})\hat{u}'(\tilde{x}_0)(B_2-B_1)\frac{\partial\tilde{x}_0}{\partial x}\,\mathrm{d}t'\right)>0 \tag{35}$$

which proves (32).

Next, we prove (33).

We have according to the definition of characteristics which issue from $x_0 = -\varepsilon^{1/2} - \varepsilon^{1/3}$ and $x_0 = \varepsilon^{1/2} + \varepsilon^{1/3}$:

$$\begin{aligned} \dot{\varphi}_1^* &= g(\hat{u}(\tilde{x}_0(\varphi_1^*, t, \varepsilon)))(B_2 - B_1) + B_1 c \\ \varphi_1^*(0, \varepsilon) &= \varepsilon^{1/2} + \varepsilon^{1/3} \end{aligned} \tag{36}$$

for $\varphi_1^* = \varphi_1^*(t, \varepsilon)$ and

$$\begin{aligned} \dot{\varphi}_2^* &= g(\hat{u}(\tilde{x}_0(\varphi_2^*, t, \varepsilon)))(B_2 - B_1) + B_1 c \\ \varphi_2^*(0, \varepsilon) &= -\varepsilon^{1/2} - \varepsilon^{1/3} \end{aligned} \tag{37}$$

for $\varphi_2^* = \varphi_2^*(t, \varepsilon)$.

Next, recall that for $x \notin (\varphi_2(t, \varepsilon), \varphi_1(t, \varepsilon))$:

$$u_\varepsilon(x, t) = \hat{u}(\tilde{x}_0(x, t, \varepsilon)) = \begin{cases} U_L, & x \leqslant \varphi_2(t, \varepsilon) \\ U_R, & x \geqslant \varphi_1(t, \varepsilon) \end{cases}$$

Therefore, if we prove:

$$\varphi_2^*(t, \varepsilon) \leqslant \varphi_2(t, \varepsilon), \quad \varphi_1(t, \varepsilon) \leqslant \varphi_1^*(t, \varepsilon) \tag{38}$$

along entire temporal axis, then $U_R = \hat{u}(\tilde{x}_0(\varphi_1^*, t, \varepsilon))$ and $U_L = \hat{u}(\tilde{x}_0(\varphi_2^*, t, \varepsilon))$. Actually, if (38) is true then it holds for $x \notin (\varphi_2^*(t, \varepsilon), \varphi_1^*(t, \varepsilon))$ (since the characteristics X are non-intersecting; see Figure 4):

$$u_\varepsilon(x, t) = \hat{u}(\tilde{x}_0(x, t, \varepsilon)) = \begin{cases} U_L, & x \leqslant \varphi_2^*(t, \varepsilon) \\ U_R, & x \leqslant \varphi_1^*(t, \varepsilon) \end{cases} \tag{39}$$

which together with (36) and (37) immediately gives (33).

We shall prove only $\varphi_1(t, \varepsilon) \leqslant \varphi_1^*(t, \varepsilon)$. The other inequality from (38) is proven analogously.

The solution of (36) is given by

$$\varphi_1^* = \varphi_1^*(t, \varepsilon) = \int_0^t (g(u_\varepsilon)(B_2 - B_1) + cB_1)\,\mathrm{d}t' + \varepsilon^{1/2} + \varepsilon^{1/3} \tag{40}$$

while from (14) it follows:

$$\varphi_1 = \varphi_1(t, \varepsilon) = \int_0^t (f'(U_R)(B_2 - B_1) + cB_1)\,\mathrm{d}t' + \varepsilon^{1/2} + A\varepsilon^{3/2} \tag{41}$$

So, we have to prove that $\varphi_1 - \varphi_1^* \leqslant 0$, or, according to (40) and (41):

$$\varphi_1 - \varphi_1^* = \int_0^t (f'(U_R) - g(u_\varepsilon))(B_2 - B_1)\,\mathrm{d}t' - \varepsilon^{1/3} + A\varepsilon^{3/2} \leqslant 0$$

Since $B_2 - B_1 = 1 - 2B_1$ and $B_1 = B_1(\psi_0(t)/\varepsilon)$, the previous expression can be transformed into

$$\varphi_1 - \varphi_1^* = \int_0^t (f'(U_R) - g(u_\varepsilon)) \left(1 - 2B_1\left(\rho\left(\frac{\psi_0(t')}{\varepsilon}\right)\right)\right) \mathrm{d}t' - \varepsilon^{1/3} + A\varepsilon^{3/2} \tag{42}$$

Next, taking the following change of variables

$$\begin{aligned} \frac{\psi_0(t')}{\varepsilon} = z &\Longrightarrow (f'(U_L) - f'(U_R))\,\mathrm{d}t' = \varepsilon\,\mathrm{d}z \\ 0 < t' < t &\Longrightarrow 0 < z < \frac{\psi_0(t')}{\varepsilon} \end{aligned} \tag{43}$$

we have

$$\begin{aligned} &\left|\int_0^t \left(1 - 2B_1\left(\rho\left(\frac{\psi_0(t')}{\varepsilon}\right)\right)\right) \mathrm{d}t'\right| = \left|\left(\int_0^{t^*} + \int_{t^*}^t\right)\left(1 - 2B_1\left(\rho\left(\frac{\psi_0(t')}{\varepsilon}\right)\right)\right) \mathrm{d}t'\right| \\ &= \left|\int_0^{t^*} \mathrm{d}t' - \varepsilon \frac{1}{f'(U_L) - f'(U_R)} \int_{\psi_0(0)/\varepsilon}^0 B_1(\rho(z))\,\mathrm{d}z \right. \\ &\left. \quad + \varepsilon \frac{1}{f'(U_L) - f'(U_R)} \int_0^{\psi_0(t)/\varepsilon} (1 - 2B_1(\rho(z)))\,\mathrm{d}z\right| < C_3 \varepsilon^{1/2} + |C_4| \varepsilon \end{aligned} \tag{44}$$

where (in what follows, see (16) for C_3 and (24) for C_4):

$$C_3\varepsilon^{1/2}=\int_0^{t^*}\mathrm{d}t'=\frac{2\varepsilon^{1/2}}{f'(U_L)-f'(U_R)}$$

$$C_4=-\frac{1}{f'(U_L)-f'(U_R)}\left(\int_{\psi_0(0)/\varepsilon}^{0}B_1(\rho(z))\,\mathrm{d}z-\int_0^{\psi_0(t)/\varepsilon}(1-2B_1(\rho(z)))\,\mathrm{d}z\right)<\infty$$

From (42) and (44) follow that for an ε small enough

$$\varphi_1-\varphi_1^*<-\varepsilon^{1/3}+A\varepsilon^{3/2}+C_3\varepsilon^{1/2}+|C_4|\varepsilon<0$$

Similarly, we have $\varphi_2(t,\varepsilon)\geqslant\varphi_2^*(t,\varepsilon)$ for an ε small enough.

This proves (39) and concludes the proof of the lemma. □

A very important corollary of (32) and of the Inverse Function Theorem is:

Corollary 7
Cauchy problem (28), (27) has globally defined classical solution.

Now, we can determine the function F so that (29) is satisfied. We have the following theorem:

Theorem 8
Let $\Phi\in C_0^1(\mathbb{R})$ such that supp$\Phi\subset(-1,1)$ and:

$$\int_{-1}^{0}\Phi(z)\,\mathrm{d}z=A_1,\quad \int_0^1\Phi(z)\,\mathrm{d}z=A_2,\quad A_1+A_2=1,\quad A_1(g(U_L)-2c)+A_2(g(U_R)-2c)=0 \tag{45}$$

Denote

$$F(x,t,\varepsilon)=B_1[cV_R-2V_Rg(U_R)+2V_Lg(U_L)-cV_L]\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{x-\bar{\phi}}{\varepsilon^{1/3}}\right) \tag{46}$$

where $\bar{\phi}=(\varphi_1^*+\varphi_2^*)/2$.

Then, the classical solution v_ε to problem (28), (27) is a weak asymptotic solution to (2), (4), that is it satisfies (29) and (27).

Proof
First, we shall compute the distance between φ_i^*, $i=1,2$. We have from (33):

$$\varphi_1^*-\varphi_2^*=\int_0^t(g(U_R)-g(U_L))(B_2-B_1)\,\mathrm{d}t'+2(\varepsilon^{1/2}+\varepsilon^{1/3})=\mathcal{O}(\varepsilon^{1/3}) \tag{47}$$

again relying on (44).

Similarly,

$$\begin{aligned}\varphi_1-\varphi_1^*&=\int_0^t(f'(U_R)-g(U_R))(B_2-B_1)\,\mathrm{d}t'-\varepsilon^{1/3}+A\varepsilon^{3/2}=\mathcal{O}(\varepsilon^{1/3})\\ \varphi_2-\varphi_2^*&=\int_0^t(f'(U_L)-g(U_L))(B_2-B_1)\,\mathrm{d}t'+\varepsilon^{1/3}-A\varepsilon^{3/2}=\mathcal{O}(\varepsilon^{1/3})\end{aligned} \tag{48}$$

From here and (25) we also conclude that:

$$\varphi_i^*-\frac{ct}{2}=\mathcal{O}(\varepsilon^{1/3}) \tag{49}$$

Furthermore, since supp$F(x,t,\varepsilon)\subset(\varphi_2^*,\varphi_1^*)$ for every $t\in\mathbf{R}^+$, we conclude from (33) and (28) for $x\notin(\varphi_2^*,\varphi_1^*)$ that:

$$v_\varepsilon(x,t)=\begin{cases}V_L, & x<\varphi_2^*\\ V_R, & x>\varphi_1^*\end{cases} \tag{50}$$

We finally pass to the proof of (29).

After adding and subtracting appropriate terms in (29) and using $B_1+B_2=1$, we find:

$$\begin{aligned}Lv_\varepsilon&=v_{\varepsilon t}+(g(u_\varepsilon)(B_2-B_1)v_\varepsilon+cB_1v_\varepsilon)_x-F(x,t,\varepsilon)+(2B_1g(u_\varepsilon)v_\varepsilon-cB_1v_\varepsilon)_x+F(x,t,\varepsilon)\\ &\stackrel{(27)}{=}B_1(cv_\varepsilon-2g(u_\varepsilon)v_\varepsilon)_x-F(x,t,\varepsilon)\end{aligned} \tag{51}$$

Now, we multiply (51) with $\eta(x)\in C_0^1(\mathbb{R})$, integrate over $\mathbb{R}$, and use partial integration to obtain

$$\int Lv_\varepsilon\eta\,dx=B_1\int(2g(u_\varepsilon)v_\varepsilon-cv_\varepsilon)\eta'\,dx-\int F\eta\,dx$$

where $F=F(x,t,\varepsilon)$. Taking into account (39) and (50), we get from the last equality:

$$\int Lv_\varepsilon\eta\,dx=B_1\left(\int_{-\infty}^{\varphi_2^*}(2V_Lg(U_L)-cV_L)\eta'(x)\,dx+\int_{\varphi_2^*}^{\varphi_1^*}(2g(u_\varepsilon)v_\varepsilon-cv_\varepsilon)\eta'(x)\,dx\right.$$
$$\left.+\int_{\varphi_1^*}^{+\infty}(2V_Rg(U_R)-cV_R)\eta'(x)\,dx\right)-\int_{-\infty}^{+\infty}F\eta(x)\,dx \tag{52}$$

We shall prove later that (see Lemma 10):

$$\int_{\varphi_2^*}^{\varphi_1^*}(2g(u_\varepsilon)v_\varepsilon-cv_\varepsilon)\eta'(x)\,dx=\mathcal{O}(\varepsilon^{1/6}) \tag{53}$$

Next, from

$$B_1\int_{-\infty}^{\varphi_2^*}(2V_Lg(U_L)-cV_L)\eta'(x)\,dx=B_1(2V_Lg(U_L)-cV_L)\eta(\varphi_2^*)$$
$$B_1\int_{\varphi_1^*}^{+\infty}(2V_Rg(U_R)-cV_R)\eta'(x)\,dx=-B_1(2V_Rg(U_R)-cV_R)\eta(\varphi_1^*)$$

and taking (53) for granted, we conclude from (52):

$$\int Lv_\varepsilon\eta\,dx=B_1[(2V_Lg(U_L)-cV_L)\eta(\varphi_2^*)-(2V_Rg(U_R)-cV_R)\eta(\varphi_1^*)]-\int F\eta(x)\,dx+\mathcal{O}(\varepsilon^{1/6})$$

We rewrite the last expression in the form:

$$\int Lv_\varepsilon\eta\,dx=B_1[(cV_R-2V_Rg(U_R)+2V_Lg(U_L)-cV_L)]\eta(\varphi_1^*)+B_1(2V_Lg(U_L)-cV_L)(\eta(\varphi_2^*)-\eta(\varphi_1^*))-\int F\eta(x)\,dx+\mathcal{O}(\varepsilon^{1/6}) \tag{54}$$

Now, recall that B_1 is bounded and notice that $\eta(\varphi_2^*)-\eta(\varphi_1^*)=\eta'(\tilde\varphi)(\varphi_2^*-\varphi_1^*)=\mathcal{O}(\varepsilon^{1/6})$ for appropriate $\tilde\varphi\in(\varphi_2^*,\varphi_1^*)$. So, after denoting

$$K=[(cV_R-2V_Rg(U_R)+2V_Lg(U_L)-cV_L)] \tag{55}$$

we derive from (54) that F should satisfy

$$B_1K\eta(\varphi_1^*)=\int F\eta(x)\,dx+\mathcal{O}(\varepsilon^{1/6}) \tag{56}$$

which is true for the function F from (46). Indeed, by using the following change of variables

$$\frac{x-\tilde\phi}{\varepsilon^{1/3}}=z\Rightarrow dx=\varepsilon^{1/3}\,dz$$

we have

$$\int F\eta(x)\,dx=B_1K\int\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{x-\tilde\phi}{\varepsilon^{1/3}}\right)\eta(x)\,dx=B_1K\int\Phi(z)\eta(\varepsilon^{1/3}z+\tilde\phi)\,dz=B_1K\int\Phi(z)\left(\eta(\varphi_1^*)+\left(\varepsilon^{1/3}z+\frac{\varphi_2^*-\varphi_1^*}{2}\right)\eta'(\tilde z)\right)dz$$
$$=B_1K\eta(\varphi_1^*)\int\Phi(z)dz+B_1K\int\Phi(z)\left(\varepsilon^{1/3}z+\frac{\varphi_2^*-\varphi_1^*}{2}\right)\eta'(\tilde z)\,dz$$
$$=B_1K\eta(\varphi_1^*)+\mathcal{O}(\varepsilon^{1/6})$$

where the last equality holds due to (47). This concludes the proof. □

Theorem 9
The classical solution v_ε to problem (28), (27) has the form:

$$v_\varepsilon(x,t)=\begin{cases} V_L, & x\leqslant\varphi_2^* \\ \hat{v}(x_0)+\displaystyle\int_0^t B_1K\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{x-\tilde{\phi}}{\varepsilon^{1/3}}\right)\mathrm{d}t', & \varphi_2^*\leqslant x<\varphi_2 \\ \dfrac{1}{\frac{\partial X}{\partial x_0}}\left(\hat{v}(x_0)+\displaystyle\int_0^t B_1K\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{x-\tilde{\phi}}{\varepsilon^{1/3}}\right)\frac{\partial X}{\partial x_0}(x_0,t',\varepsilon)\,\mathrm{d}t'\right), & \varphi_2\leqslant x<\varphi_1 \\ \hat{v}(x_0)+\displaystyle\int_0^t B_1K\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{x-\tilde{\phi}}{\varepsilon^{1/3}}\right)\mathrm{d}t', & \varphi_1\leqslant x<\varphi_1^* \\ V_R, & x\geqslant\varphi_1^* \end{cases} \tag{57}$$

where $x_0=-\int_0^t(g(u_\varepsilon)(B_2-B_1)+cB_1)\,\mathrm{d}t'+x$, and the constant K is defined in (55).

Proof
We shall use the standard method of characteristics.

Accordingly, we need to solve the system of characteristics (30), (31). We have from there:

$$\begin{aligned} X &= \int_0^t(g(u_\varepsilon)(B_2-B_1)+B_1c)\,\mathrm{d}t'+x_0 \\ v_\varepsilon &= \exp\left(-\int_0^t g'(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x}(B_2-B_1)\,\mathrm{d}t'\right)\left(\hat{v}(x_0)+\int_0^t B_1K\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{X-\tilde{\phi}}{\varepsilon^{1/3}}\right)\exp\left(\int_0^t g'(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x}(B_2-B_1)\,\mathrm{d}t''\right)\mathrm{d}t'\right) \end{aligned} \tag{58}$$

Then, notice that it follows from (30):

$$\exp\left(-\int_0^t g'(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x}(B_2-B_1)\,\mathrm{d}t'\right)=\exp\left(-\int_0^t\left(\frac{\mathrm{d}X}{\mathrm{d}t'}\right)_x\mathrm{d}t'\right)=\exp\left(-\int_0^t\frac{\partial}{\partial x_0}\frac{\mathrm{d}X}{\mathrm{d}t'}\frac{\partial\tilde{x}_0}{\partial X}\,\mathrm{d}t'\right)=\exp\left(-\int_0^t\frac{\frac{\mathrm{d}}{\mathrm{d}t'}\frac{\partial X}{\partial x_0}}{\frac{\partial X}{\partial x_0}}\,\mathrm{d}t'\right)=\frac{1}{\frac{\partial X}{\partial x_0}}$$

From here and (58):

$$\begin{aligned} X &= \int_0^t(g(u_\varepsilon)(B_2-B_1)+cB_1)\,\mathrm{d}t'+x_0 \\ v_\varepsilon &= \frac{1}{\frac{\partial X}{\partial x_0}}\left(\hat{v}(x_0)+\int_0^t B_1K\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{X-\tilde{\phi}}{\varepsilon^{1/3}}\right)\frac{\partial X}{\partial x_0}(x_0,t',\varepsilon)\,\mathrm{d}t'\right) \end{aligned} \tag{59}$$

Next, notice that $x_0\notin(X^{-1}(\varphi_2(t,\varepsilon)),X^{-1}(\varphi_1(t,\varepsilon)))$:

$$X(x_0,t',\varepsilon)=\begin{cases} \displaystyle\int_0^{t'}(g(U_L)(B_2-B_1)+cB_1)\,\mathrm{d}t''+x_0, & x_0<X^{-1}(\varphi_2(t,\varepsilon)) \\ \displaystyle\int_0^{t'}(g(U_R)(B_2-B_1)+cB_1)\,\mathrm{d}t''+x_0, & x_0>X^{-1}(\varphi_1(t,\varepsilon)) \end{cases}$$

and thus

$$\frac{\partial X}{\partial x_0}(x_0,t',\varepsilon)=1,\quad x_0\notin(X^{-1}(\varphi_2(t,\varepsilon)),X^{-1}(\varphi_1(t,\varepsilon))),\qquad t\in[0,t) \tag{60}$$

From here, (50), and (58), formula (57) immediately follows. □

Now, we can prove (53). We have the following lemma.

Lemma 10
It holds that

$$B_1\int_{\varphi_2^*}^{\varphi_1^*}(2g(u_\varepsilon)v_\varepsilon-cv_\varepsilon)\eta'(x)\,\mathrm{d}x=\mathcal{O}(\varepsilon^{1/6})$$

Proof
By using the following change of variables $x=X(x_0,t,\varepsilon)\Rightarrow \mathrm{d}x=\frac{\partial X}{\partial x_0}\mathrm{d}x_0$, we have

$$\begin{aligned}
B_1\int_{\varphi_2^*}^{\varphi_1^*}(2g(u_\varepsilon)v_\varepsilon-cv_\varepsilon)\eta'(x)\,\mathrm{d}x &= B_1\int_{\varphi_2^*}^{\varphi_2}(2g(U_L)-c)v_\varepsilon\eta'(x)\,\mathrm{d}x+B_1\int_{\varphi_1}^{\varphi_1^*}(2g(U_R)-c)v_\varepsilon\eta'(x)\,\mathrm{d}x+B_1\int_{\varphi_2}^{\varphi_1}(2g(u_\varepsilon)-c)v_\varepsilon\eta'(x)\,\mathrm{d}x\\
&= B_1\int_{-\varepsilon^{1/2}-\varepsilon^{1/3}}^{X^{-1}(\varphi_2)}(2g(U_L)-c)v_\varepsilon(x_0,t)\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0\\
&\quad +B_1\int_{X^{-1}(\varphi_1)}^{\varepsilon^{1/2}+\varepsilon^{1/3}}(2g(U_R)-c)v_\varepsilon(x_0,t)\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0\\
&\quad +B_1\int_{X^{-1}(\varphi_2)}^{X^{-1}(\varphi_1)}(2g(u_\varepsilon)-c)v_\varepsilon(x_0,t)\eta'(X(x_0,t,\varepsilon))\frac{\partial X}{\partial x_0}(x_0,t,\varepsilon)\,\mathrm{d}x_0 \qquad (61)
\end{aligned}$$

where we use (60) in the last equality.

Next, we compute $X^{-1}(\varphi_i)$, $i=1,2$, where X is defined by (30). We focus our attention on $X^{-1}(\varphi_1)$. The function $X^{-1}(\varphi_2)$ is computed analogously.

It follows from (17) and (30), respectively, that φ_1 satisfies at the same time:

$$\varphi_1=\int_0^t((B_2-B_1)f'(U_R)+cB_1)\,\mathrm{d}t'+\varepsilon^{1/2}+A\varepsilon^{3/2}$$

$$\varphi_1=\int_0^t((B_2-B_1)g(U_R)+cB_1)\,\mathrm{d}t'+X^{-1}(\varphi_1)$$

Comparing the latter two expressions we get as before (see (44)):

$$X^{-1}(\varphi_1)=\int_0^t(B_2-B_1)(f'(U_R)-g(U_R))\,\mathrm{d}t'+\varepsilon^{1/2}+A\varepsilon^{3/2}=\left(\frac{2(f'(U_R)-g(U_R))}{f'(U_L)-f'(U_R)}+1\right)\varepsilon^{1/2}+\mathcal{O}(\varepsilon) \qquad (62)$$

Similarly,

$$X^{-1}(\varphi_2)=\left(\frac{2(f'(U_L)-g(U_L))}{f'(U_L)-f'(U_R)}-1\right)\varepsilon^{1/2}+\mathcal{O}(\varepsilon) \qquad (63)$$

Now, consider the integral

$$\begin{aligned}
&B_1\int_{X^{-1}(\varphi_2)}^{X^{-1}(\varphi_1)}(2g(u_\varepsilon)-c)v_\varepsilon(t,x_0)\eta'(X(x_0,t,\varepsilon))\frac{\partial X}{\partial x_0}(x_0,t,\varepsilon)\,\mathrm{d}x_0\\
&=B_1\int_{X^{-1}(\varphi_2)}^{X^{-1}(\varphi_1)}\left(\hat{v}(x_0)+\int_0^t B_1K\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{X(x_0,t,\varepsilon)-\tilde{\phi}}{\varepsilon^{1/3}}\right)\frac{\partial X}{\partial x_0}(x_0,t',\varepsilon)\,\mathrm{d}t'\right)\eta'(X)\,\mathrm{d}x_0\leqslant C\frac{X^{-1}(\varphi_1)-X^{-1}(\varphi_2)}{\varepsilon^{1/3}}=\mathcal{O}(\varepsilon^{1/6}) \qquad (64)
\end{aligned}$$

where the last equality follows from (62) and (63). The constant C is independent of ε.

Then, consider the other two integrals in (61). First,

$$\begin{aligned}
&B_1\int_{X^{-1}(\varphi_1)}^{\varepsilon^{1/2}+\varepsilon^{1/3}}(2g(U_R)-c)v_\varepsilon(x_0,t)\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0=B_1\int_{X^{-1}(\varphi_1)}^{\varepsilon^{1/2}+\varepsilon^{1/3}}(2g(U_R)-c)\\
&\times\left(\hat{v}(x_0)+\int_0^t\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{x_0+\frac{1}{2}\int_0^{t'}(g(U_L)-g(U_R))(B_2-B_1)\,\mathrm{d}t''}{\varepsilon^{1/3}}\right)\frac{\partial X}{\partial x_0}(x_0,t,\varepsilon)\,\mathrm{d}t'\right)\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0\\
&=B_1\int_{X^{-1}(\varphi_1)}^{\varepsilon^{1/2}+\varepsilon^{1/3}}(2g(U_R)-c)\hat{v}(x_0)\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0+B_1\int_{X^{-1}(\varphi_1)}^{\varepsilon^{1/2}+\varepsilon^{1/3}}(2g(U_R)-c)\\
&\times\int_0^t\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{x_0+\frac{1}{2}\int_0^{t'}(g(U_L)-g(U_R))(B_2-B_1)\,\mathrm{d}t''}{\varepsilon^{1/3}}\right)\mathrm{d}t'\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0\\
&=\mathcal{O}(\varepsilon^{1/3})+B_1\int_{X^{-1}(\varphi_1)}^{\varepsilon^{1/2}+\varepsilon^{1/3}}(2g(U_R)-c)\int_0^t\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{x_0+\frac{1}{2}\int_0^{t'}(g(U_L)-g(U_R))(B_2-B_1)\,\mathrm{d}t''}{\varepsilon^{1/3}}\right)\mathrm{d}t'\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0 \qquad (65)
\end{aligned}$$

where the last two equalities hold due to (60). Then, denote:

$$a(t,\varepsilon)=\frac{1}{2}\int_0^t (g(U_L)-g(U_R))(B_2-B_1)\,\mathrm{d}t'$$

We introduce the following change of variables:

$$\frac{x_0+a(t',\varepsilon)}{\varepsilon^{1/3}}=z\Rightarrow \mathrm{d}x_0=\varepsilon^{1/3}\,\mathrm{d}z$$

$$x_0\in(-\varepsilon^{1/2}-\varepsilon^{1/3},X^{-1}(\varphi_2))\Rightarrow z\in\left(-\varepsilon^{1/6}-1,\frac{X^{-1}(\varphi_2)+a(t,\varepsilon)}{\varepsilon^{1/3}}\right)$$

We get from here and (62):

$$\begin{aligned}B_1\int_{X^{-1}(\varphi_1)}^{\varepsilon^{1/2}+\varepsilon^{1/3}}(2g(U_R)-c)v_\varepsilon(x_0,t)\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0 &= B_1(2g(U_R)-c)\int_0^t\int_{(X^{-1}(\varphi_1)+a(t,\varepsilon))/\varepsilon^{1/3}}^{\varepsilon^{1/6}+1}\Phi(z)\eta'(X(x_0(z,t',\varepsilon),t,\varepsilon))\,\mathrm{d}z\,\mathrm{d}t'+\mathcal{O}(\varepsilon^{1/3})\\ &= B_1(2g(U_R)-c)\int_0^t\int_{\mathcal{O}(\varepsilon^{1/6})}^{\varepsilon^{1/6}+1}\Phi(z)\eta'(X(x_0(z,t',\varepsilon,),t,\varepsilon))\,\mathrm{d}z\,\mathrm{d}t'+\mathcal{O}(\varepsilon^{1/3})\end{aligned}\tag{66}$$

where the last equality holds due to (63). From (61), (65) and (66), we get

$$\begin{aligned}B_1\int_{\varphi_1}^{\varphi_1^*}(2g(U_R)-c)v_\varepsilon\eta'(x)\,\mathrm{d}x &= B_1\int_{\varepsilon^{1/2}+\varepsilon^{1/3}}^{X^{-1}(\varphi_1)}(2g(U_R)-c)v_\varepsilon(x_0,t)\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0+\mathcal{O}(\varepsilon^{1/3})\\ &= tB_1(2g(U_R)-c)\eta'\left(\frac{ct}{2}\right)\int_0^1\Phi(z)\,\mathrm{d}z+\mathcal{O}(\varepsilon^{1/6})\end{aligned}\tag{67}$$

since

$$\eta'(X(x_0,t,\varepsilon))=\eta'\left(\frac{ct}{2}\right)+\left(X(x_0,t,\varepsilon)-\frac{ct}{2}\right)\eta''(\tilde X)=\eta'\left(\frac{ct}{2}\right)+\mathcal{O}(\varepsilon^{1/3})\tag{68}$$

where $\tilde X$ is a point in a neighborhood of $ct/2$. Indeed, taking into account (49) and the fact that $x=X\in(\varphi_2^*,\varphi_2)$, we see that:

$$\varphi_2^*<X(x_0,t,\varepsilon)<\varphi_2\Rightarrow X(x_0,t,\varepsilon)-\frac{ct}{2}=\mathcal{O}(\varepsilon^{1/3})\tag{69}$$

and this immediately gives (68). Similarly,

$$B_1\int_{-\varepsilon^{1/2}-\varepsilon^{1/3}}^{X^{-1}(\varphi_2)}(2g(U_L)-c)v_\varepsilon(x_0,t)\eta'(X(x_0,t,\varepsilon))\,\mathrm{d}x_0=B_1(2g(U_L)-c)\int_{-1}^0\Phi(z)\,\mathrm{d}z\,\eta'\left(\frac{ct}{2}\right)+\mathcal{O}(\varepsilon^{1/6})\tag{70}$$

Finally, the conclusion of the lemma follows from (45), (61), (67) and (70).

Now, we shall investigate the distributional limit of the family (v_ε). We have the following theorem.

Theorem 11
The function v_ε given by (57), satisfies for every $t\geqslant 0$ as $\varepsilon\to 0$

$$v_\varepsilon\rightharpoonup v=\frac{t}{2}K\delta\left(x-\frac{c}{2}t\right)+\begin{cases}V_L, & x<\dfrac{c}{2}t\\ V_R, & x\geqslant\dfrac{c}{2}t\end{cases}\quad\text{in }\mathcal{D}'(\mathbb{R})\tag{71}$$

for the constant K defined in (55).

Proof
Take an arbitrary $\eta\in C_0^1(\mathbb{R})$. Using (57) we have:

$$\begin{aligned}\int v_\varepsilon(x,t)\eta(x)\,\mathrm{d}x &= \int_{-\infty}^{\varphi_2^*}V_L\eta(x)\,\mathrm{d}x+\int_{\varphi_1^*}^{\infty}V_R\eta(x)\,\mathrm{d}x+\int_{\varphi_2^*}^{\varphi_1^*}\frac{1}{\dfrac{\partial X}{\partial x_0}}\hat v(x_0)\eta(x)\,\mathrm{d}x\\ &+\int_{\varphi_2^*}^{\varphi_1^*}\left(\frac{1}{\dfrac{\partial X}{\partial x_0}}\int_0^t B_1K\frac{1}{\varepsilon^{1/3}}\Phi\left(\frac{X(x_0(x,t,\varepsilon),t)-\tilde\phi}{\varepsilon^{1/3}}\right)\frac{\partial X}{\partial x_0}\,\mathrm{d}t'\right)\eta(x)\,\mathrm{d}x\end{aligned}\tag{72}$$

The following change of variables

$$x = X(x_0, t, \varepsilon) \Rightarrow dx = \frac{\partial X}{\partial x_0} dx_0$$

and (72) imply that

$$\int v_\varepsilon(x,t)\eta(x)\,dx = \int_{-\infty}^{\varphi_2^*} V_L \eta(x)\,dx + \int_{\varphi_1^*}^{\infty} V_R \eta(x)\,dx + \int_{-\varepsilon^{1/2}-\varepsilon^{1/3}}^{\varepsilon^{1/2}+\varepsilon^{1/3}} \hat{v}(x_0)\eta(X)\,dx_0$$
$$+K \int_{-\varepsilon^{1/2}-\varepsilon^{1/3}}^{\varepsilon^{1/2}+\varepsilon^{1/3}} \int_0^t B_1 \frac{1}{\varepsilon^{1/3}} \Phi\left(\frac{X(x_0,t,\varepsilon)-\tilde{\phi}}{\varepsilon^{1/3}}\right) \frac{\partial X}{\partial x_0} dt'\eta(X)\,dx_0 \tag{73}$$

We consider the two last terms from (73).
Since $\hat{v}(x_0)$ is bounded:

$$\int_{-\varepsilon^{1/2}-\varepsilon^{1/3}}^{\varepsilon^{1/2}+\varepsilon^{1/3}} \hat{v}(x_0)\eta(X)\,dx_0 \to 0 \quad \text{as } \varepsilon \to 0 \tag{74}$$

Consider now the last term from (73):

$$K \int_{-\varepsilon^{1/2}-\varepsilon^{1/3}}^{\varepsilon^{1/2}+\varepsilon^{1/3}} \int_0^t B_1 \frac{1}{\varepsilon^{1/3}} \Phi\left(\frac{x_0-\tilde{\phi}}{\varepsilon^{1/3}}\right) \frac{\partial X}{\partial x_0} dt'\eta(X)\,dx_0$$
$$= K \int_{-\varepsilon^{1/2}-\varepsilon^{1/3}}^{X^{-1}(\varphi_2)} \int_0^t B_1 \frac{1}{\varepsilon^{1/3}} \Phi\left(\frac{X(x_0,t,\varepsilon)-\tilde{\phi}}{\varepsilon^{1/3}}\right) \frac{\partial X}{\partial x_0} dt'\eta(X)\,dx_0$$
$$+K \int_{X^{-1}(\varphi_2)}^{X^{-1}(\varphi_1)} \int_0^t B_1 \frac{1}{\varepsilon^{1/3}} \Phi\left(\frac{X(x_0,t,\varepsilon)-\tilde{\phi}}{\varepsilon^{1/3}}\right) \frac{\partial X}{\partial x_0} dt'\eta(X)\,dx_0$$
$$+K \int_{X^{-1}(\varphi_2)}^{\varepsilon^{1/2}+\varepsilon^{1/3}} \int_0^t B_1(\rho(\tau)) \frac{1}{\varepsilon^{1/3}} \Phi\left(\frac{X(x_0,t,\varepsilon)-\tilde{\phi}}{\varepsilon^{1/3}}\right) \frac{\partial X}{\partial x_0} dt'\eta(X)\,dx_0 \tag{75}$$

Then, repeating the procedure between formulas (64) and (70):

$$K \int_{-\varepsilon^{1/2}-\varepsilon^{1/3}}^{X^{-1}(\varphi_2)} \int_0^t B_1 \frac{1}{\varepsilon^{1/3}} \Phi\left(\frac{X(x_0,t,\varepsilon)-\tilde{\phi}}{\varepsilon^{1/3}}\right) \frac{\partial X}{\partial x_0} dt'\eta(X)\,dx_0 \to \frac{t}{2} K\eta\left(\frac{c}{2}t\right) \int_0^1 \Phi(z)\,dz$$
$$K \int_{X^{-1}(\varphi_2)}^{X^{-1}(\varphi_1)} \int_0^t B_1 \frac{1}{\varepsilon^{1/3}} \Phi\left(\frac{X(x_0,t,\varepsilon)-\tilde{\phi}}{\varepsilon^{1/3}}\right) \frac{\partial X}{\partial x_0} dt'\eta(X)\,dx_0 \to 0$$
$$K \int_{X^{-1}(\varphi_2)}^{\varepsilon^{1/2}+\varepsilon^{1/3}} \int_0^t B_1 \frac{1}{\varepsilon^{1/3}} \Phi\left(\frac{X(x_0,t,\varepsilon)-\tilde{\phi}}{\varepsilon^{1/3}}\right) \frac{\partial X}{\partial x_0} dt'\eta(X)\,dx_0 \to \frac{t}{2} K\eta\left(\frac{c}{2}t\right) \int_{-1}^0 \Phi(z)\,dz$$

as $\varepsilon \to 0$, where we used

$$X(x_0,t,\varepsilon) \to \frac{c}{2}t \quad \text{(cf. (69))}$$

and

$$B_1 = B_1(\rho(\tau(t))) \to 1/2 \quad \text{for every } t>0 \quad \text{(cf. (24))}$$

Since $\int_{-1}^0 \Phi(z)\,dz + \int_0^1 \Phi(z)\,dz = 1$ (see (45)), it is now clear that we obtain (71) after letting $\varepsilon \to 0$ in (73).

5. On a linearization of an arbitrary 2×2 hyperbolic system of conservation laws

In this section, we will give directions for a possible application of the method presented above on a Riemann problem for an arbitrary 2×2 system of conservation laws. At the moment, we are not able to carry out the proposed procedure due to serious technical obstacles. Still, we believe that it is possible to accomplish the program at least in cases of special systems.

So, consider the following system in one dimension space:

$$\begin{aligned} &\partial_t u + \partial_x f(u,v) = 0 \\ &\partial_t v + \partial_x g(u,v) = 0, \quad t \in \mathbb{R}^+, \ \ x \in \mathbb{R} \end{aligned} \tag{76}$$

where $f, g \in C^1(\mathbf{R}^+ \times \mathbf{R})$, with the following Riemann initial data:

$$\begin{aligned} u|_{t=0} = u_0(x) = \begin{cases} U_L, \ x<0 \\ U_R, \ x\geqslant 0 \end{cases} \\ v|_{t=0} = v_0(x) = \begin{cases} V_L, \ x<0 \\ V_R, \ x\geqslant 0 \end{cases} \end{aligned} \tag{77}$$

Passing to the Riemann invariants [43, Definition 7.3.1] (see also [44] for a simple characterization of Riemann invariants), as long as (76) admits classical solution, we can rewrite system (76) in the form:

$$\partial_t \omega + \lambda_1(u,v)\partial_x \omega = 0 \tag{78}$$

$$\partial_t \eta + \lambda_2(u,v)\partial_x \eta = 0 \tag{79}$$

where λ_i, $i=1,2$, are eigenvalues of the matrix $\nabla(f(u,v), g(u,v))$, and $\omega = \omega(u,v)$ and $\eta = \eta(u,v)$, $(u,v) \in \mathbf{R}^2$ are the Riemann invariants.

It is clear that it is much easier to work with the former diagonal system than with original system (76). But, with initial data (77), system (76) in general does not admit the classical solution. Therefore, in order to connect properly (76) and (78), instead of the initial data $\omega(u_0, v_0)$ and $\eta(u_0, v_0)$, we augment (76) with the following smooth perturbations of $\omega(u_0, v_0)$ and $\eta(u_0, v_0)$, respectively,

$$\omega|_{t=0} = \omega_{0\varepsilon}(x) = \begin{cases} \omega(U_L, V_L), & x > -A(\varepsilon) \\ \omega(u^1_{0\varepsilon}(x), v^1_{0\varepsilon}(x)), & -A(\varepsilon) \leqslant x < A(\varepsilon) \\ \omega(U_R, V_R), & x \geqslant A(\varepsilon) \end{cases} \tag{80}$$

$$\eta|_{t=0} = \eta_{0\varepsilon}(x) = \begin{cases} \eta(U_L, V_L) & x < -B(\varepsilon) \\ \eta(u^2_{0\varepsilon}(x), v^2_{0\varepsilon}(x)), & -B(\varepsilon) \leqslant x < B(\varepsilon) \\ \eta(U_R, V_R), & x \geqslant B(\varepsilon) \end{cases} \tag{81}$$

where A and B are appropriate positive functions tending to zero as $\varepsilon \to 0$ while $u^i_{0\varepsilon}$ and $v^i_{0\varepsilon}$, $i=1,2$, are such that $\omega|_{t=0}$ and $\eta|_{t=0}$ are Lipschitz continuous functions.

According to the standard theory of scalar conservation laws, Cauchy problem (78), (81) will have classical solution till the moment $\min\{t^*, t^{**}\}$, where

$$\begin{aligned} t^* &= \frac{1}{\max_{x\in(-A(\varepsilon),A(\varepsilon))} \partial_x \lambda_1(u^1_{0\varepsilon}(x), v^1_{0\varepsilon}(x))} \\ t^{**} &= \frac{1}{\max_{x\in(-B(\varepsilon),B(\varepsilon))} \partial_x \lambda_2(u^2_{0\varepsilon}(x), v^2_{0\varepsilon}(x))} \end{aligned} \tag{82}$$

Actually, t^* and t^** are the moments when the characteristics corresponding to (78), (80) and (79), (81), respectively, start to intersect. As before, the idea is to modify the characteristics for a small parameter ε so that its intersection is avoided and then to continue classical solution along such 'new characteristics'. Letting $\varepsilon \to 0$ we should recover a singularity that solves our original Riemann problem.

So, instead of system (78), (81), we should solve

$$\partial_t \omega_\varepsilon + \lambda_1(u,v)\partial_x((B^\omega_2 - B^\omega_1)\omega_\varepsilon + c_1(t,x)B^\omega_1 \omega_\varepsilon) = F(t,x,\varepsilon) \tag{83}$$

$$\partial_t \eta_\varepsilon + \lambda_2(u,v)\partial_x((B^\eta_2 - B^\eta_1)\eta_\varepsilon + c_2(t,x)B^\eta_1 \eta_\varepsilon) = G(t,x,\varepsilon) \tag{84}$$

with initial data (81) for appropriately chosen $u^i_{0\varepsilon}$ and $v^i_{0\varepsilon}$, $i=1,2$ (see (85)).

Furthermore,

$$B^\omega_i = B_i(\rho_\omega), \quad B^\eta_i = B_i(\rho_\eta), \quad i=1,2$$

where the functions B_i, $i=1,2$, are defined in Theorem 3, and

$$\rho_\omega = \frac{\varphi^\omega_2(t,\varepsilon) - \varphi^\omega_1(t,\varepsilon)}{\varepsilon}, \quad \rho_\eta = \frac{\varphi^\eta_2(t,\varepsilon) - \varphi^\eta_1(t,\varepsilon)}{\varepsilon}$$

where φ_1^ω and φ_1^η are characteristics corresponding to Cauchy problems (84), (80) and (83), (81), respectively, which issue from $A(\varepsilon)$ and $B(\varepsilon)$, respectively. Similarly, φ_2^ω and φ_2^η are characteristics corresponding to Cauchy problems (84), (80) and (83), (81), respectively, which issue from $-A(\varepsilon)$ and $-B(\varepsilon)$, respectively.

Remark 12

Notice that we can take ρ_η instead of ρ_ω or vice versa since we expect to have blow up of both Riemann invariants at the same time.

Other functions appearing in (84) need to be determined so that the family of solutions to Cauchy problem (84), (81) represents the weak asymptotic solution to system (76), (77).

What we hope is that the situation will be the same as in the case of system (1), (2), at least in the case of special values for U_L, U_R, V_L and V_R. Namely, after the intersection of characteristics, the function B_2-B_1 should eliminate a nonlinearity with expense to a 'bad' right-hand side. Thus, instead of nonlinear conservation law we would obtain two transport equations.

As we can see, everything depends on the function B_i, $i=1,2$, and these functions depend only on extremal characteristics (i.e. those issuing from $\pm A(\varepsilon)$ and $\pm B(\varepsilon)$). Therefore, if we want to have smooth solution all the way until the moment of the intersection of extremal characteristics, we must determine the functions u_0 and v_0 from (81) so that all the characteristics corresponding to ω_ε and η_ε and issuing from the intervals $(-A(\varepsilon),A(\varepsilon))$ and $(-B(\varepsilon),B(\varepsilon))$, respectively, intersect at the same point.

From (82) we have an effective way for determining the functions $u_{0\varepsilon}^i$ and $v_{0\varepsilon}^i$, $i=1,2$. It is clear that it has to be:

$$\begin{aligned} &\partial_x\lambda_1(u_{0\varepsilon}^1(x),v_{0\varepsilon}^1(x))=\bar{K},\quad x\in(-A(\varepsilon),A(\varepsilon))\\ &\partial_x\lambda_2(u_{0\varepsilon}^2(x),v_{0\varepsilon}^2(x))=\bar{K},\quad x\in(-B(\varepsilon),B(\varepsilon)) \end{aligned}\tag{85}$$

for a positive constant $\bar{K}$.

Realization of the procedure described in the current section will be the subject of further investigation. We believe that it could give new results in the case of hyperbolic systems whose Riemann problems do not admit solutions consisted from admissible elementary wave combinations, that is combinations of Lax admissible shock and rarefaction waves.

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