

Macdonald polynomials, Laumon spaces and perverse coherent sheaves

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ABSTRACT. Let G be an almost simple simply connected complex Lie group, and let G/U_- be its base affine space. In this paper we formulate a conjecture, which provides a new geometric interpretation of the Macdonald polynomials associated to G via perverse coherent sheaves on the scheme of formal arcs in the affinization of G/U_- . We prove our conjecture for $G = SL(N)$ using the so called Laumon resolution of the space of quasi-maps (using this resolution one can reformulate the statement so that only “usual” (not perverse) coherent sheaves are used). In the course of the proof we also give a K -theoretic version of the main result of Negut (2009).

1. Introduction

1.1. Notations. Let \mathfrak{g} be a semi-simple Lie algebra over \mathbb{C} and let G be the corresponding simply connected group. Let $B, B_- \subset G$ be a pair of opposite Borel subgroups with unipotent radicals U, U_- and let $T = B \cap B_-$ be the corresponding maximal torus. We denote by Λ the lattice of cocharacters of T (this is also the coroot lattice of G , since G is simply connected) and by $\check{\Lambda}$ the lattice of characters of T . We denote by Λ_+ the cone consisting of sums of positive coroots of G with non-negative coefficients. Similarly, we denote by $\check{\Lambda}^+$ the cone of dominant weights.

We denote by \mathcal{B} the flag variety of G . It can be identified with the quotient G/B . The choice of B_- gives a point in the open B -orbit in \mathcal{B} .

For a pair of variables p, q and for any $n \in \mathbb{N} \cup \infty$ we set

$$(p; q)_n := (1 - p)(1 - qp) \dots (1 - q^{n-1}p).$$

1.2. Quasi-maps and Laumon spaces. For $\alpha \in \Lambda_+$ we denote by ${}_{\mathfrak{g}}\mathcal{M}^\alpha$ the moduli space of maps $\mathbb{P}^1 \rightarrow \mathcal{B}$ of degree α and by ${}_{\mathfrak{g}}\mathcal{QM}^\alpha$ its quasi-maps compactification (cf. [2] for a survey on quasi-maps); we shall sometimes omit the subscript \mathfrak{g} when it does not lead to a confusion. The scheme \mathcal{QM}^α possesses a natural stratification

$$\mathcal{QM}^\alpha = \bigsqcup_{0 \leq \beta \leq \alpha} \mathcal{M}^\beta \times \text{Sym}^{\alpha-\beta}(\mathbb{P}^1),$$

where $\text{Sym}^{\alpha-\beta}(\mathbb{P}^1)$ stands for the space of all formal linear combinations $\sum \gamma_i x_i$ where $\gamma_i \in \Lambda_+$, $x_i \in \mathbb{P}^1$ and $\sum \gamma_i = \alpha$. The points $\{x_i\}$ are called the points of defect of the corresponding quasi-map.

Similarly, we denote by Z^α the space of based quasi-maps of degree α (i.e. those quasi-maps, which have no defect at $\infty \in \mathbb{P}^1$ and which send ∞ to B_- regarded as a point in \mathcal{B}). The space \mathcal{QM}^α has a natural action of $PGL(2) \times G$; here the first factor acts on \mathbb{P}^1 and the second on \mathcal{B} . This action does not preserve Z^α ; however, $\mathbb{G}_m \times T$ still acts on Z^α .

It is well-known that the space \mathcal{QM}^α is usually singular, but when $G = SL(N)$ it has a natural small resolution of singularities by means of Laumon's quasiflags' space \mathcal{Q}^α . By the definition, it consists of flags

$$0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \dots \subset \mathcal{W}_N = \mathcal{O}_{\mathbb{P}^1}^N,$$

where \mathcal{W}_i is a locally free sheaf on \mathbb{P}^1 of rank i and such that

$$\text{deg } \mathcal{W}_i = -\langle \alpha, \check{\omega}_i \rangle.$$

We shall denote by \mathcal{Q}^α the corresponding "based" version of \mathcal{Q}^α .

As before, \mathcal{Q}^α has a natural action of $PGL(2) \times G$ and \mathcal{Q}^α has a natural action of $\mathbb{G}_m \times T$.

1.3. Geometric interpretation of the "Macdonald function" for $G = SL(N)$. In the case $G = SL(N)$ we identify Λ_+ with \mathbb{N}^{N-1} by using the simple coroots α_i as a basis of Λ . Similarly, we identify $\check{\Lambda}^+$ with \mathbb{N}^{N-1} by using the fundamental weights $\check{\omega}_i$ as a basis. Also we have the natural isomorphism $T \simeq \mathbb{G}_m^{N-1}$.

For any $\alpha \in \Lambda_+$ let us set

$$(1.1) \quad \mathfrak{J}_\alpha(q, t, z) = [H^\bullet(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^\bullet)] := \sum_{i,j} (-1)^{i+j} t^j [H^i(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^j)].$$

Here $[H^i(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^j)]$ means the character of $H^i(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^j)$ as a representation of $\mathbb{G}_m \times T$; in other words, it is a function of $q \in \mathbb{G}_m$ and $z \in T$. More precisely, the coordinate functions z_i , $i = 1, \dots, N-1$, satisfy $\check{\omega}_i = z_1 \cdots z_i$.

We would like to organize all the \mathfrak{J}_α into a generating function. Namely, let us set:

$$J(q, t, z, x) = \sum_{\alpha \in \mathbb{N}^{N-1}} x^\alpha \mathfrak{J}_\alpha(q, t, z); \quad \mathfrak{J}(q, t, z, x) = \prod_{i=1}^{N-1} x_i^{\log(\check{\omega}_i)/\log q} J(q, t, z, x).$$

Also, for $1 \leq i \leq N$, we consider the difference operator $T_{i,q^{\pm 1}}$ defined as follows: $T_{i,q^{\pm 1}} F(q, t, z, x_1, \dots, x_{N-1}) := F(q, t, z, x_1, \dots, x_{i-2}, q^{\mp 1} x_{i-1}, q^{\pm 1} x_i, x_{i+1}, \dots, x_{N-1})$. Our first main result is the following

THEOREM 1.4. (1) Define the function z_N on the Cartan torus T of $SL(N)$ by $z_N := z_1^{-1} \cdots z_{N-1}^{-1}$. Then we have

$$D\mathfrak{J}(q, t, z, x) = (z_1 + \dots + z_N)\mathfrak{J}(q, t, z, x),$$

where

$$(2) \quad D := \sum_{i=1}^N \prod_{j < i} \frac{1 - q^{-1} t^{i-j-1} x_j \cdots x_{i-1}}{1 - t^{i-j} x_j \cdots x_{i-1}} \prod_{k > i} \frac{1 - q t^{k-i+1} x_i \cdots x_{k-1}}{1 - t^{k-i} x_i \cdots x_{k-1}} T_{i,q^{-1}}$$

$$\lim_{\alpha \rightarrow \infty} \mathfrak{J}_\alpha(q, t, z) = \prod_{1 \leq i < j \leq N} \frac{(qtz_j/z_i; q)_\infty}{(qz_j/z_i; q)_\infty} \times \left(\frac{(qt; q)_\infty}{(q; q)_\infty} \right)^{N-1} \times \prod_{i=1}^{N-2} \left(\frac{(qt^{i+1}; q)_\infty}{(t^i; q)_\infty} \right)^{N-i-1}.$$

Some remarks about 1.4 are in order. First, the operator D is a version of one of the Macdonald difference operators; it is easy to see that the first assertion of 1.4 implies that \mathfrak{J} is an eigen-function of all the (suitably normalized) Macdonald operators and thus (up to some normalization factor) it is equal to the *Baker-Akhiezer* function for the Macdonald operators in the terminology of [9] or [7]; it is also often called *the Macdonald function*. Moreover, the second assertion can be deduced from the first one and the results of [9], [7], but we are going to give an independent proof of this result.

It should also be noted that some limiting cases of 1.4 have been known before. In particular, the case $t = 0$ is treated in [3] (cf. also [5] for a generalization to arbitrary G). Also, in [16] the $q \rightarrow 1$ version of 1.4 is proved. It should be noted that the proofs in *loc. cit.* are representation-theoretic: they are based on an interpretation of the (localized) equivariant K -theory (resp. localized equivariant cohomology) of all the \mathfrak{Q}^α as the universal Verma module for the quantum group $U_q(\mathfrak{sl}(N))$ (resp. of the lie algebra $\mathfrak{sl}(N)$). On the other hand, the proof of 1.4 given in this paper is purely computational: using Atiyah-Bott-Lefschetz localization formula one can produce a combinatorial expression for the function \mathfrak{J}_α and thus reduce 1.4(1) to a combinatorial identity, which can be proven by an explicit (but fairly long) computation. It would be very interesting to extend the methods of *loc. cit.* to the present situation.

1.5. Geometric interpretation of Macdonald polynomials for $G = \text{SL}(N)$. The Macdonald operators are usually used in order to define the so called Macdonald polynomials. This is a series of W -invariant polynomials $P_{\check{\lambda}}(q, t, z)$ on the torus T (recall that $z \in T$) depending on a dominant weight $\check{\lambda} \in \check{\Lambda}^+$ and on the variables $q, t \in \mathbb{G}_m$. We would like to present a geometric construction of these polynomials. Let us explain how to do it in the $\text{SL}(N)$ -case. The conjectural generalization to arbitrary G is discussed in the next Subsection.

First, for any $\check{\lambda} \in \check{\Lambda}$ one can construct a line bundle $\mathcal{O}(\check{\lambda})$ on \mathcal{QM}^α ; abusing the notation we are going to denote its pull-back to \mathcal{Q}^α also by $\mathcal{O}(\check{\lambda})$. The construction is discussed in [5]. We are not going to recall the construction in the Introduction, but let us just note that it requires a choice of a point $\infty \in \mathbb{P}^1$. Hence, the bundle $\mathcal{O}(\check{\lambda})$ is not $PGL(2)$ -equivariant. However, it is still equivariant with respect to the diagonal torus $\mathbb{G}_m \subset PGL(2)$. In particular, it makes sense to consider the character of $H^\bullet(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^\bullet \otimes \mathcal{O}(\check{\lambda}))$ with respect to the action of $\mathbb{G}_m \times G$, which we shall denote by $[H^\bullet(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^\bullet \otimes \mathcal{O}(\check{\lambda}))]$. By definition this character is W -invariant function on $\mathbb{G}_m \times T$.

THEOREM 1.6. (1) *Assume that $\check{\lambda} \in \check{\Lambda}$ is not dominant. Fix $j, k \in \mathbb{N}$. Then for α sufficiently large we have*

$$H^k(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^j \otimes \mathcal{O}(\check{\lambda})) = 0.$$

(2) *For any $\check{\lambda} \in \check{\Lambda}$ there exists the limit $\lim_{\alpha \rightarrow \infty} [H^\bullet(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^\bullet \otimes \mathcal{O}(\check{\lambda}))]$. We shall denote the above limit by $H_{\check{\lambda}}(q, t, z)$. Note that it follows from the first assertion that $H_{\check{\lambda}} = 0$ when $\check{\lambda}$ is not dominant.*

(3)

$$H_0(q, t, z) = \frac{(1+t)(1+t+t^2)\dots(1+t+\dots+t^{N-1})}{(1-t^{N-1})^2(1-t^{N-2})^4\dots(1-t^3)^{2N-6}(1-t^2)^{2N-4}} \cdot \frac{1}{(1-t^N)(1-t)^{N-2}}.$$

(4) For any $\check{\lambda} = \sum l_i \check{\omega}_i \in \check{\Lambda}^+$ (here $\check{\omega}_i$ denotes the i -th fundamental weight of $\mathrm{SL}(N)$) we have

$$H_{\check{\lambda}} = H_0 \prod_{1 \leq i \leq j \leq N-1} \frac{(t^{j-i+1}; q)_{l_i + \dots + l_j}}{(t^{j-i} q; q)_{l_i + \dots + l_j}} P_{\check{\lambda}}.$$

In other words, $H_{\check{\lambda}}$ is equal to $P_{\check{\lambda}}$ up to an explicit factor.

1.7. The case of arbitrary G . In this subsection we are going to give a conjectural formulation¹ of 1.6 for arbitrary G . The formulation is based on the theory of perverse coherent sheaves developed by D. Arinkin and R. Bezrukavnikov (cf. [1]). For simplicity, in this Introduction we shall assume that G is simply laced (in the general case certain modification of the construction given below is needed; the details are explained in 8).

First let us introduce the infinite type scheme ${}_{\mathfrak{g}}\mathbf{Q}$ (discussed also in [6, Section 2.2]): it is the quotient by the action of the Cartan torus $T \subset G$ of the space of maps from $\mathrm{Spec} R = \mathrm{Spec} \mathbb{C}[[t^{-1}]]$ to the affinization of the base affine space $\overline{G/U_-}$ taking value in G/U_- at the generic point. This scheme is equipped with the action of the proalgebraic group $G(R)$; the open orbit ${}_{\mathfrak{g}}\mathbf{Q}_{\infty} = {}_{\mathfrak{g}}\mathbf{Q}^0$ is nothing but $G(R)/T \cdot U_-(R)$: the maps taking value in G/U_- at the closed point $r \in \mathrm{Spec} R$. We denote by j the open embedding of ${}_{\mathfrak{g}}\mathbf{Q}^0$ into ${}_{\mathfrak{g}}\mathbf{Q}$. All the $G(R)$ -orbits in ${}_{\mathfrak{g}}\mathbf{Q}$ are numbered by the defects at r taking value in the cone of positive coroots Λ_+ of G : ${}_{\mathfrak{g}}\mathbf{Q} = \bigsqcup_{\alpha \in \Lambda_+} {}_{\mathfrak{g}}\mathbf{Q}^{\alpha}$. The codimension of ${}_{\mathfrak{g}}\mathbf{Q}^{\alpha}$ in ${}_{\mathfrak{g}}\mathbf{Q}$ equals $2|\alpha|$.

We introduce the perversity $p({}_{\mathfrak{g}}\mathbf{Q}^{\alpha}) = |\alpha|$; it is immediate that the function p is strictly monotone and comonotone in the sense of [1]. For a locally free $G(R) \rtimes \mathbb{G}_m$ -equivariant sheaf \mathcal{F} on ${}_{\mathfrak{g}}\mathbf{Q}^0$ the construction of [1, Section 4] produces an object $j_{!*}\mathcal{F}$ of $G(R) \rtimes \mathbb{G}_m$ -equivariant quasicohherent derived category on ${}_{\mathfrak{g}}\mathbf{Q}$.

CONJECTURE 1.8. (a) For a nondominant G -weight $\check{\lambda}$ we have $[H^{\bullet}({}_{\mathfrak{g}}\mathbf{Q}, j_{!*}(\Omega_{\mathfrak{g}}^{\bullet}\mathbf{Q}^0) \otimes \mathcal{O}(\check{\lambda}))] = 0$.

(b) For a dominant G -weight $\check{\lambda}$ we have

$$[H^{\bullet}({}_{\mathfrak{g}}\mathbf{Q}, j_{!*}(\Omega_{\mathfrak{g}}^{\bullet}\mathbf{Q}^0) \otimes \mathcal{O}(\check{\lambda}))] = H_0 \prod_{\alpha \in R^+(\check{\mathfrak{g}})} \frac{(t^{|\alpha|}; q)_{\langle \alpha, \check{\lambda} \rangle}}{(t^{|\alpha|-1} q; q)_{\langle \alpha, \check{\lambda} \rangle}} \prod \frac{(t^{|\alpha|-1}; q)_{\infty}}{(qt^{|\alpha|}; q)_{\infty}} P_{\check{\lambda}}$$

where $P_{\check{\lambda}}(q, t, z)$ is the Macdonald polynomial for G , and the second product is taken over all nonsimple positive roots of $R^+(\check{\mathfrak{g}})$.

We explain in 8 why 1.8 is equivalent to 1.6 for $G = \mathrm{SL}(N)$.

1.9. Organization of the paper. In 2 and 3 we gather some combinatorial information about Macdonald polynomials and the ‘‘Macdonald function’’ for root systems of type A. In 4 we prove a generalization of the Sommese vanishing theorem, which in particular implies 1.6(1). In 6 and 7 we prove 1.4 and 1.6. Finally, in 8 we give a careful formulation of 1.8 for arbitrary G and show that for $G = \mathrm{SL}(N)$ it is equivalent to 1.6.

¹The reader should be warned that we do not know how to formulate a version of 1.4 for arbitrary G .

2. Combinatorial notations

2.1. Macdonald polynomials. We follow the notations in [15] (especially, part VI), cf. also [17]. Let N be a positive integer and q, \mathbf{t} be independent indeterminates. Let $\Lambda_{N, \mathbb{F}}$ be the ring of symmetric polynomials in N variables with coefficients in $\mathbb{F} = \mathbb{Q}(q, \mathbf{t})$. Set

$$(2.1) \quad T_{q, y_i} f(y_1, \dots, y_N) = f(y_1, \dots, qy_i, \dots, y_N).$$

For a partition λ , the Macdonald polynomial $P_\lambda(y; q, \mathbf{t}) \in \Lambda_{N, \mathbb{F}}$ is uniquely characterized by the conditions:

$$(2.2) \quad P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu,$$

$$(2.3) \quad D_N^1 P_\lambda = \sum_{i=1}^N q^{\lambda_i} \mathbf{t}^{N-i} \cdot P_\lambda,$$

where m_λ is the monomial symmetric function, and $D_N^1 = D_N^1(q, \mathbf{t})$ is the Macdonald difference operator

$$(2.4) \quad D_N^1 = \sum_{i=1}^N \prod_{j \neq i} \frac{\mathbf{t}y_i - y_j}{y_i - y_j} T_{q, y_i}.$$

2.2. Tableau. Let $\lambda = (\lambda_1, \lambda_2, \dots), \mu = (\mu_1, \mu_2, \dots)$ be partitions satisfying $\mu \subset \lambda$. The necessary and sufficient condition for the skew diagram $\theta = \lambda - \mu$ to be a horizontal strip is

$$(2.5) \quad \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots.$$

This can be written as

$$(2.6) \quad 0 \leq \lambda_i - \mu_i \leq \lambda_i - \lambda_{i+1} \quad (i \geq 1).$$

A (column-strict) tableau T of shape λ is defined to be a sequence of partitions

$$(2.7) \quad \phi = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(N)} = \lambda$$

such that every skew diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)}$ is a horizontal strip. Writing $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots)$, the condition for the T being a tableau reads

$$(2.8) \quad 0 \leq \lambda_i^{(j)} - \lambda_i^{(j-1)} \leq \lambda_i^{(j)} - \lambda_{i+1}^{(j)} \quad (1 \leq i, 1 \leq j \leq N).$$

Note that from $\lambda^{(0)} = \phi$ and the inequality (2.8) we have

$$(2.9) \quad \lambda_i^{(j)} = 0 \quad (i > j).$$

For each skew diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)}$, set

$$(2.10) \quad \theta_{i,j} = \lambda_i^{(j)} - \lambda_i^{(j-1)} \quad (1 \leq i \leq N, 1 \leq j \leq N),$$

for simplicity of display. Then the constraint (2.9) means

$$(2.11) \quad \theta_{i,j} = 0 \quad (i > j),$$

$$(2.12) \quad \lambda_i = \sum_{k=i}^N \theta_{i,k} \quad (1 \leq i \leq N).$$

Hence the tableau T uniquely gives us a set of $N(N-1)/2$ nonnegative integers $\{\theta_{i,j} | 1 \leq i < j \leq N\}$ satisfying (2.8), namely

$$(2.13) \quad 0 \leq \theta_{i,j} \leq \lambda_i - \lambda_{i+1} - \sum_{k=j+1}^N (\theta_{i,k} - \theta_{i+1,k}) \quad (1 \leq i < j \leq N).$$

Conversely, a set of nonnegative integers $\{\theta_{i,j}\}$ satisfying (2.13) uniquely gives us a sequence of partitions $\lambda^{(j)} = (\lambda_1^{(j)}, \lambda_2^{(j)}, \dots)$

$$(2.14) \quad \lambda_i^{(j)} = \sum_{k=1}^j \theta_{i,k},$$

which is a tableau.

It is convenient to consider a set of $N \times N$ upper triangular matrices $M^{(N)}$ having $\{\theta_{i,j}\}$'s as nonzero entries, and zeros on the diagonal:

$$(2.15) \quad M^{(N)} = \{\theta = (\theta_{i,j})_{1 \leq i, j \leq N} | \theta_{i,j} \in \mathbb{Z}_{\geq 0}, \theta_{i,j} = 0 \text{ if } i \geq j\}.$$

We have a natural projection $M^{(N)} \rightarrow M^{(N-1)}$ forgetting the last column.

LEMMA 2.3. *Let $\lambda = (\lambda_1, \dots, \lambda_N)$ be a partition. We have a one to one mapping from the set of (column-strict) tableaux of shape λ to the elements in the polyhedral region $\text{Pol}_\lambda \in M^{(N)}$ defined by*

$$(2.16) \quad \text{Pol}_\lambda = \{\theta \in M^{(N)} | 0 \leq \theta_{i,j} \leq \lambda_i - \lambda_{i+1} - \sum_{k=j+1}^N (\theta_{i,k} - \theta_{i+1,k})\}.$$

LEMMA 2.4. *The size of the skew diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)}$ is written as*

$$(2.17) \quad |\theta^{(i)}| = \lambda_i + \sum_{a=1}^{i-1} \theta_{a,i} - \sum_{b=i+1}^N \theta_{i,b}.$$

2.5. Tableaux sum formula. We recall the tableaux sum formula for the Macdonald polynomials.

The Macdonald polynomial P_λ is written as

$$(2.18) \quad P_\lambda = \sum_T \psi_T(q, \mathbf{t}) y^T.$$

where T runs over the set of tableaux of shape λ , y^T denotes the monomial defined in terms of the weights $\alpha = (|\theta^{(1)}|, |\theta^{(2)}|, \dots, |\theta^{(N)}|)$ of T as

$$(2.19) \quad y^T = y^\alpha = y^\lambda \prod_{1 \leq i < j \leq N} (y_j / y_i)^{\theta_{i,j}},$$

and the coefficient $\psi_T(q, \mathbf{t})$ is given by

$$(2.20) \quad \psi_T(q, \mathbf{t}) = \prod_{i=1}^N \psi_{\lambda^{(i)} / \lambda^{(i-1)}}(q, \mathbf{t}),$$

$$(2.21) \quad \psi_{\lambda / \mu} = \prod_{1 \leq i \leq j \leq \ell(\mu)} \frac{f(q^{\mu_i - \mu_j} \mathbf{t}^{j-i}) f(q^{\lambda_i - \lambda_{j+1}} \mathbf{t}^{j-i})}{f(q^{\lambda_i - \mu_j} \mathbf{t}^{j-i}) f(q^{\mu_i - \lambda_{j+1}} \mathbf{t}^{j-i})},$$

$$(2.22) \quad f(u) = \frac{(\mathbf{t}u; q)_\infty}{(qu; q)_\infty}.$$

For a nonnegative integer $\theta \in \mathbb{Z}_{\geq 0}$, we have

$$\frac{f(u)}{f(q^{-\theta}u)} = \frac{(q^{-\theta+1}u; q)_{\theta}}{(q^{-\theta}tu; q)_{\theta}} = (q/t)^{\theta} \frac{(1/u; q)_{\theta}}{(q/tu; q)_{\theta}}.$$

where $(p; q)_n := (1-p)(1-qp)\dots(1-q^{n-1}p)$. Hence we have

$$\begin{aligned} & \psi_T(q, \mathbf{t}) \\ (2.23) \quad &= \prod_{k=1}^N \prod_{1 \leq i \leq j \leq k-1} \frac{f(q^{\lambda_i^{(k-1)} - \lambda_j^{(k-1)}} \mathbf{t}^{j-i}) f(q^{\lambda_i^{(k)} - \lambda_{j+1}^{(k)}} \mathbf{t}^{j-i})}{f(q^{\lambda_i^{(k)} - \lambda_j^{(k-1)}} \mathbf{t}^{j-i}) f(q^{\lambda_i^{(k-1)} - \lambda_{j+1}^{(k)}} \mathbf{t}^{j-i})} \\ &= \prod_{k=1}^N \prod_{1 \leq i \leq j \leq k-1} \frac{f(q^{-\theta_{i,k} + \lambda_i^{(k)} - \lambda_j^{(k-1)}} \mathbf{t}^{j-i}) f(q^{\lambda_i^{(k)} - \lambda_{j+1}^{(k)}} \mathbf{t}^{j-i})}{f(q^{\lambda_i^{(k)} - \lambda_j^{(k-1)}} \mathbf{t}^{j-i}) f(q^{-\theta_{i,k} + \lambda_i^{(k)} - \lambda_{j+1}^{(k)}} \mathbf{t}^{j-i})} \\ &= \prod_{k=1}^N \prod_{1 \leq i \leq j \leq k-1} \frac{(q^{-\lambda_i^{(k)} + \lambda_j^{(k-1)} + 1} \mathbf{t}^{-j+i-1}; q)_{\theta_{i,k}}}{(q^{-\lambda_i^{(k)} + \lambda_j^{(k-1)}} \mathbf{t}^{-j+i}; q)_{\theta_{i,k}}} \frac{(q^{-\lambda_i^{(k)} + \lambda_{j+1}^{(k)}} \mathbf{t}^{-j+i}; q)_{\theta_{i,k}}}{(q^{-\lambda_i^{(k)} + \lambda_{j+1}^{(k)} + 1} \mathbf{t}^{-j+i-1}; q)_{\theta_{i,k}}}. \end{aligned}$$

3. Macdonald function

3.1. Multiple hypergeometric-type series. Let $q, \mathbf{t}, z_1, z_2, \dots, z_N$ be independent indeterminates. Recall the projection $M^{(N)} \rightarrow M^{(N-1)}$, see the line after (2.15). Define a sequence of rational functions $c_N(\theta; z_1, \dots, z_N; q, \mathbf{t}) \in \mathbb{Q}(q, \mathbf{t}, z_1, \dots, z_N)$ inductively as follows:

$$(3.1) \quad c_1(-; z_1; q, \mathbf{t}) = 1,$$

$$\begin{aligned} (3.2) \quad & c_N(\theta \in M^{(N)}; z_1, \dots, z_N; q, \mathbf{t}) \\ &= c_{N-1}(\theta \in M^{(N-1)}; q^{-\theta_{1,N}} z_1, \dots, q^{-\theta_{N-1,N}} z_{N-1}; q, \mathbf{t}) \\ & \times \prod_{1 \leq i \leq j \leq N-1} \frac{(\mathbf{t} z_{j+1}/z_i; q)_{\theta_{i,N}} (q^{-\theta_{j,N}} q z_j / \mathbf{t} z_i; q)_{\theta_{i,N}}}{(q z_{j+1}/z_i; q)_{\theta_{i,N}} (q^{-\theta_{j,N}} z_j / z_i; q)_{\theta_{i,N}}}. \end{aligned}$$

This can be written explicitly as

$$(3.3)$$

$$\begin{aligned} & c_N(\theta; z_1, \dots, z_N; q, \mathbf{t}) \\ &= \prod_{k=2}^N \prod_{1 \leq i \leq j \leq k-1} \frac{(q^{\sum_{a=k+1}^N (\theta_{i,a} - \theta_{j+1,a})} \mathbf{t} z_{j+1}/z_i; q)_{\theta_{i,k}}}{(q^{\sum_{a=k+1}^N (\theta_{i,a} - \theta_{j+1,a})} q z_{j+1}/z_i; q)_{\theta_{i,k}}} \frac{(q^{-\theta_{j,k} + \sum_{a=k+1}^N (\theta_{i,a} - \theta_{j,a})} q z_j / \mathbf{t} z_i; q)_{\theta_{i,k}}}{(q^{-\theta_{j,k} + \sum_{a=k+1}^N (\theta_{i,a} - \theta_{j,a})} z_j / z_i; q)_{\theta_{i,k}}} \\ &= \prod_{1 \leq i < j \leq N} (q/\mathbf{t})^{\theta_{i,j}} \frac{(\mathbf{t}; q)_{\theta_{i,j}} (q^{\sum_{a=j+1}^N (\theta_{i,a} - \theta_{j,a})} \mathbf{t} z_j / z_i; q)_{\theta_{i,j}}}{(q; q)_{\theta_{i,j}} (q^{1 + \sum_{a=j+1}^N (\theta_{i,a} - \theta_{j,a})} z_j / z_i; q)_{\theta_{i,j}}} \times \\ & \prod_{k=3}^N \prod_{1 \leq l < m < k} (q/\mathbf{t})^{\theta_{l,k}} \frac{(q^{\sum_{b=k+1}^N (\theta_{l,b} - \theta_{m,b})} \mathbf{t} z_m / z_l; q)_{\theta_{l,k}}}{(q^{1 + \sum_{b=k+1}^N (\theta_{l,b} - \theta_{m,b})} z_m / z_l; q)_{\theta_{l,k}}} \frac{(q^{-\theta_{l,k} + \theta_{m,k} - \sum_{b=k+1}^N (\theta_{l,b} - \theta_{m,b})} \mathbf{t} z_l / z_m; q)_{\theta_{l,k}}}{(q^{1 - \theta_{l,k} + \theta_{m,k} - \sum_{b=k+1}^N (\theta_{l,b} - \theta_{m,b})} z_l / z_m; q)_{\theta_{l,k}}} \end{aligned}$$

3.2. Example.

$$(3.4) \quad c_2 = \frac{(tz_2/z_1; q)_{\theta_{1,2}} (q^{-\theta_{1,2}} q/t; q)_{\theta_{1,2}}}{(qz_2/z_1; q)_{\theta_{1,2}} (q^{-\theta_{1,2}}; q)_{\theta_{1,2}}} = \frac{(tz_2/z_1; q)_{\theta_{1,2}} (t; q)_{\theta_{1,2}}}{(qz_2/z_1; q)_{\theta_{1,2}} (q; q)_{\theta_{1,2}}} (q/t)^{\theta_{1,2}},$$

$$(3.5) \quad c_3 = \frac{(q^{\theta_{1,3}-\theta_{2,3}} tz_2/z_1; q)_{\theta_{1,2}} (q^{-\theta_{1,2}} q/t; q)_{\theta_{1,2}}}{(q^{\theta_{1,3}-\theta_{2,3}} qz_2/z_1; q)_{\theta_{1,2}} (q^{-\theta_{1,2}}; q)_{\theta_{1,2}}} \\ \times \frac{(tz_2/z_1; q)_{\theta_{1,3}} (q^{-\theta_{1,3}} q/t; q)_{\theta_{1,3}} (tz_3/z_1; q)_{\theta_{1,3}} (q^{-\theta_{2,3}} qz_1/tz_2; q)_{\theta_{1,3}}}{(qz_2/z_1; q)_{\theta_{1,3}} (q^{-\theta_{1,3}}; q)_{\theta_{1,3}} (qz_3/z_1; q)_{\theta_{1,3}} (q^{-\theta_{2,3}} z_1/z_2; q)_{\theta_{1,3}}} \\ \times \frac{(tz_3/z_2; q)_{\theta_{2,3}} (q^{-\theta_{2,3}} q/t; q)_{\theta_{2,3}}}{(qz_3/z_2; q)_{\theta_{2,3}} (q^{-\theta_{2,3}}; q)_{\theta_{2,3}}}.$$

LEMMA 3.3. *Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition satisfying $\ell(\lambda) \leq N$. The substitution $z_i = t^{N-i} q^{\lambda_i}$ ($1 \leq i \leq N$) in $c_N(\theta; z_1, \dots, z_N; q, t)$ gives us the coefficient ψ_T in the tableau sum formula*

$$(3.6) \quad \psi_T(q, t) = c_N(\theta; t^{N-1} q^{\lambda_1}, \dots, q^{\lambda_N}; q, t).$$

Let $y = (y_1, \dots, y_N), z = (z_1, \dots, z_N)$ be two sets of independent indeterminates. Set

$$(3.7) \quad z_i = t^{N-i} q^{\lambda_i} \quad (1 \leq i \leq N).$$

For simplicity we use the notation

$$(3.8) \quad y^\lambda = \prod_i y_i^{\lambda_i}.$$

Note that we have

$$(3.9) \quad T_{q, y_i} y^\lambda = t^{i-N} z_i \cdot y^\lambda.$$

DEFINITION 3.4. Define a formal power series $f_N(y, z; q, t) \in y^\lambda \mathbb{F}(z)[[y_{i+1}/y_i, (i = 1, \dots, N-1)]]$ by

$$(3.10) \quad f_N(y, z; q, t) = y^\lambda \sum_{\theta \in \mathcal{M}^{(N)}} c_N(\theta; z; q, t) \prod_{1 \leq i < j \leq N} (y_j/y_i)^{\theta_{i,j}}.$$

3.5. Termination of the series $f_N(y, z; q, t)$. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ be a partition, while keeping q, t being generic. Note that we have the following factor in the numerator of $c_N(\theta; z_1, \dots, z_N; q, t)$:

$$(3.11) \quad \prod_{k=1}^{N-1} \prod_{l=1}^k (q^{\sum_{a=k+1}^N (\theta_{i,a} - \theta_{i+1,a})} tz_{i+1}/z_i; q)_{\theta_{i,k}} = \prod_{k=1}^{N-1} \prod_{l=1}^k (q^{\sum_{a=k+1}^N (\theta_{i,a} - \theta_{i+1,a})} q^{\lambda_{i+1} - \lambda_i}; q)_{\theta_{i,k}}.$$

This vanishes unless the following set of inequalities are satisfied:

$$(3.12) \quad 0 \leq \theta_{i,k} \leq \lambda_i - \lambda_{i+1} - \sum_{a=k+1}^N (\theta_{i,a} - \theta_{i+1,a}) \quad (1 \leq i < k \leq N).$$

Namely we have the vanishing of the coefficient $c_N(\theta; z_1, \dots, z_N; q, t)$'s unless $\theta \in \text{Pol}_\lambda \subset \mathcal{M}^{(N)}$. Hence we find that under the specialization in z , the infinite series $f_N(y, z; q, t)$ terminates into a finite one.

PROPOSITION 3.6. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ be a partition, and set $z_i = \mathbf{t}^{N-i} q^{\lambda_i}$. Then we have*

$$(3.13) \quad \begin{aligned} f_N(y, z; q, \mathbf{t}) &= y^\lambda \sum_{\theta \in \text{Pol}_\lambda} c_N(\theta; z; q, \mathbf{t}) \prod_{1 \leq i < j \leq N} (y_j/y_i)^{\theta_{i,j}} \\ &= \sum_T \psi_T(q, \mathbf{t}) y^T = P_\lambda(y, q, \mathbf{t}). \end{aligned}$$

PROPOSITION 3.7. *Let $y = (y_1, \dots, y_N)$ and $z = (z_1, \dots, z_N)$ be generic. We have*

$$(3.14) \quad D_{N,y}^1 f_N(y, z; q, \mathbf{t}) = \sum_{i=1}^N z_i \cdot f_N(y, z; q, \mathbf{t}).$$

LEMMA 3.8. *Let $u(z_1, \dots, z_N) \in \mathbb{F}[z_1, z_2, \dots, z_N]$. If we have $u(\mathbf{t}^{N-1} q^{\lambda_1}, \mathbf{t}^{N-2} q^{\lambda_2}, \dots, q^{\lambda_N}) = 0$ for any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$, then $u(z_1, \dots, z_N) = 0$.*

PROOF. We prove this by the induction on N . When $N = 1$, it is true. Assume it holds for $N - 1$. Expand $u(z_1, \dots, z_N) = \sum_k u_k(z_2, \dots, z_N) z_1^k$. Fix $\lambda_2, \dots, \lambda_N$ and vary $\lambda_1 (\geq \lambda_2)$, then all the coefficients of z_1^k , i.e. $u_k(\mathbf{t}^{N-2} q^{\lambda_2}, \mathbf{t}^{N-3} q^{\lambda_3}, \dots, q^{\lambda_N})$ should vanish. Now we let $\lambda_2, \dots, \lambda_N$ vary and conclude that $u_k(z_2, \dots, z_N) = 0$ by the assumption. \square

PROOF OF 3.7. Set

$$\begin{aligned} & y^{-\lambda} (\text{LHS}(3.14) - \text{RHS}(3.14)) \\ &= \sum_{k_1, \dots, k_{N-1} \geq 0} r_{k_1, \dots, k_{N-1}}(z) \prod_{i=1}^{N-1} (y_{i+1}/y_i)^{k_i} \in \mathbb{F}(z)[[y_{i+1}/y_i, (i = 1, 2, \dots, N-1)]]. \end{aligned}$$

From 3.6 and 3.8, we have $r_{k_1, \dots, k_{N-1}}(z) = 0$ for all $k_1, \dots, k_{N-1} \geq 0$. \square

4. Vanishing

4.1. Sommese vanishing. We need the following version of Sommese vanishing theorem. Let $p : X \rightarrow Y$ be a flat morphism between smooth projective complex varieties. Let \mathcal{L} be a line bundle on X whose restriction to every fiber of p is l -ample [8, Definition 6.5] for certain $l \in \mathbb{N}$. Also, suppose the Iitaka dimension $\kappa(\mathcal{L}_y)$ [8, Definition 5.3] of the restriction of \mathcal{L} to every fiber $X_y = p^{-1}(y)$, $y \in Y$, equals $\dim X_y = \dim X - \dim Y$. Finally, suppose $\dim X - \dim Y - l > M$ for some $M \in \mathbb{N}$.

THEOREM 4.2. *Under the above assumptions, $H^i(X, \Omega_X^j \otimes \mathcal{L}^{-1}) = 0$ for $i + j < M$.*

PROOF. By the Leray spectral sequence, it suffices to prove $R^i p_*(\Omega_X^j \otimes \mathcal{L}^{-1}) = 0$ for $i + j < M$. First, we restrict to a nonempty open $U \subset Y$ over which p is smooth. We set $X_U = p^{-1}(U)$. Then $\Omega_{X_U}^j$ has a filtration whose associated graded bundle is a direct sum of the sheaves $\Omega_{X_U/U}^k \otimes p^* \Omega_U^{j-k}$ over $k \leq j$. Here $\Omega_{X_U/U}^k$ is the bundle of relative k -forms. By the projection formula it suffices to prove $R^i p_*(\Omega_{X_U/U}^k \otimes \mathcal{L}^{-1}) = 0$ for $i + k < M$. By the base change, it suffices to know for any $y \in U$ that $H^i(X_y, \Omega_{X_y}^k \otimes \mathcal{L}^{-1}) = 0$ for $i + k < M$. But this is nothing but Sommese vanishing [8, Corollary 6.6] on the smooth projective variety X_y . So we conclude $R^i p_*(\Omega_X^j \otimes \mathcal{L}^{-1})|_U = 0$ for $i + j < M$.

Now to prove $R^i p_*(\Omega_X^j \otimes \mathcal{L}^{-1}) = 0$ for $i + j < M$ it suffices to know that $R^i p_*(\Omega_X^j \otimes \mathcal{L}^{-1})$ has no torsion for $i + j < M$. We will prove this by induction in $\dim Y$ and the dimension of the support of torsion. Let $Z \subset Y$ be the support of $R^i p_*(\Omega_X^j \otimes \mathcal{L}^{-1})$. Suppose $\dim Z > 0$. Then according to Kleiman's generic transversality theorem [13] there exists a hyperplane section $Y' \subset Y$ intersecting Z transversally at a smooth point $z \in Z$ and such that $X' := p^{-1}(Y')$ is smooth. By the base change, the support of $R^i p_*(\Omega_X^j \otimes \mathcal{L}^{-1}|_{X'})$ contains Z' defined as the irreducible component of $Z \cap Y'$ containing z . However, we have an exact sequence of vector bundles on X' :

$$0 \rightarrow \mathcal{N}_{X'/X}^* \otimes \Omega_{X'}^{j-1} \rightarrow \Omega_{X'}^j|_{X'} \rightarrow \Omega_{X'}^j \rightarrow 0$$

and the conormal bundle $\mathcal{N}_{X'/X}^* = p^* \mathcal{N}_{Y'/Y}^*$. By the projection formula and by the induction (in $\dim Y$) assumption we have $R^i p_*(\mathcal{N}_{X'/X}^* \otimes \Omega_{X'}^{j-1} \otimes \mathcal{L}^{-1}) = 0 = R^i p_*(\Omega_{X'}^j \otimes \mathcal{L}^{-1})$ for $i + j < M$. Hence $R^i p_*(\Omega_{X'}^j|_{X'} \otimes \mathcal{L}^{-1}) = 0$ which contradicts to $Z' \neq \emptyset$.

It remains to establish the base of induction: $\dim Z = 0$. We choose the minimal i_0 among all i such that $R^i p_*(\Omega_X^j \otimes \mathcal{L}^{-1}) \neq 0$ for some j such that $i + j < M$. Let $y \in Z \subset Y$ be a point in the (finite) support of $R^{i_0} p_*(\Omega_X^j \otimes \mathcal{L}^{-1})$. Let us choose a sufficiently ample line bundle \mathcal{M} on Y . Then by the projection formula $R^i p_*(\Omega_X^j \otimes (\mathcal{L} \otimes p^* \mathcal{M})^{-1}) = R^i p_*(\Omega_X^j \otimes \mathcal{L}^{-1}) \otimes \mathcal{M}^{-1}$, and by the Leray spectral sequence $H^{i_0}(X, \Omega_X^j \otimes (\mathcal{L} \otimes p^* \mathcal{M})^{-1}) \neq 0$ (the LHS contains a direct summand $R^{i_0} p_*(\Omega_X^j \otimes \mathcal{L}^{-1})_y \otimes \mathcal{M}_y^{-1}$). However, by the Sommese vanishing [8, Corollary 6.6] applied to the line bundle $\mathcal{L} \otimes p^* \mathcal{M}$ on X (with \mathcal{M} sufficiently ample), we must have $H^{i_0}(X, \Omega_X^j \otimes (\mathcal{L} \otimes p^* \mathcal{M})^{-1}) = 0$. This contradiction proves we cannot have $\dim Z = 0$.

This completes the proof of the theorem. □

5. Parabolic Laumon spaces

Recall the notations of [6]. We denote by \mathcal{QM}^α the Drinfeld moduli space of degree α quasimaps from $\mathbf{C} \simeq \mathbb{P}^1$ to the flag variety $\mathcal{B} = G/B$ of $G = \mathrm{SL}(N)$. Here $\alpha = (d_1, \dots, d_{N-1}) \in \mathbb{N}^{N-1}$. We denote by $\pi_\alpha : \mathcal{Q}^\alpha \rightarrow \mathcal{QM}^\alpha$ the Laumon resolution of \mathcal{QM}^α [14]. Given a subminimal parabolic (with Levi of semisimple rank 1) $\mathrm{SL}(N) \supset P = P_i \supset B$ we consider the corresponding parabolic Laumon space \mathcal{Q}_P^α (see e.g. [4]), and the natural projection $\varpi_\alpha : \mathcal{Q}^\alpha \rightarrow \mathcal{Q}_P^\alpha$. Here $\bar{\alpha} := (d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_{N-1})$.

Recall [6] that $V_{\tilde{\omega}_i} = \Lambda^i \mathbf{C}^N$, $1 \leq i \leq N - 1$, are the fundamental $\mathrm{SL}(N)$ -modules, and \mathcal{QM}^α is equipped with a closed embedding $\psi_\alpha : \mathcal{QM}^\alpha \hookrightarrow \prod_{i \in I} \mathbb{P}\Gamma(\mathbf{C}, V_{\tilde{\omega}_i} \otimes \mathcal{O}(\langle \alpha, \tilde{\omega}_i \rangle))$. Given an $\mathrm{SL}(N)$ -weight $\tilde{\lambda} = \sum_{i \in I} d_i \tilde{\omega}_i \in \Lambda^\vee$ we define a line bundle $\mathcal{O}(\tilde{\lambda})^\alpha$ on $\mathcal{QM}_\alpha^\alpha$ as $\psi_\alpha^* \otimes_{i \in I} \mathcal{O}(d_i)$. Suppose $\tilde{\lambda}$ is *not* dominant, i.e. $l_i < 0$ for some $1 \leq i \leq N - 1$. We fix such an i from now on, and we set $\mathcal{L} := \pi_\alpha^* \mathcal{O}(-\tilde{\lambda})$. For $y \in \mathcal{Q}_P^\alpha$ we denote by X_y the fiber $\varpi_\alpha^{-1}(y)$ with the reduced scheme structure. Our aim is to study the ampleness properties of the line bundle $\mathcal{L}_y := \mathcal{L}|_{X_y}$. They are summarized in the following

PROPOSITION 5.1. *(a) \mathcal{L}_y is generated by the global sections, and gives rise to a morphism $\phi : X_y \rightarrow \mathbb{P}(\Gamma^*(X_y, \mathcal{L}_y))$. We denote by \overline{X}_y the image of ϕ (with the reduced closed subscheme structure).*

(b) The morphism $X_y \xrightarrow{\phi} \overline{X}_y$ equals $X_y \xrightarrow{\pi_\alpha} \pi_\alpha(X_y)$, where $\pi_\alpha(X_y) \subset \mathcal{QM}^\alpha$ is equipped with the reduced closed subscheme structure.

(c) For a fixed $\bar{\alpha}$ and $d_i \gg 0$ we have $\dim X_y = \dim \overline{X}_y = 2d_i - d_{i-1} - d_{i+1} + 1$; in particular, ϖ_α is flat.

(d) Let $l_y := \max\{\dim \phi^{-1}(z), z \in \overline{X}_y\}$. For a fixed $\bar{\alpha}$ and $M \in \mathbb{N}$, there exists D_i such that for $d_i > D_i$ and any $y \in \mathcal{Q}_P^\alpha$ we have $\dim X_y - l_y > M$.

PROOF. (a) and (b) are clear from definitions. A point $y \in \mathcal{Q}_P^\alpha$ is represented by a collection of locally free subsheaves $0 = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \dots \subset \mathcal{W}_{i-1} \subset \mathcal{W}_{i+1} \subset \dots \subset \mathcal{W}_{N-1} \subset \mathcal{W}_N = \mathcal{O}_{\mathbb{C}}^N$ such that $\text{rk } \mathcal{W}_j = j$, and $\text{deg } \mathcal{W}_j = -d_j$. The fiber X_y is the moduli space of subsheaves $\overline{\mathcal{W}}_i \subset \mathcal{W}_{i-1}^{i+1} := \mathcal{W}_{i+1}/\mathcal{W}_{i-1}$ of generic rank 1 and degree $d_{i-1} - d_i$. For such a sheaf $\overline{\mathcal{W}}_i$ we denote by $\overline{\mathcal{W}}_i$ its *saturation* i.e. the maximal subsheaf of \mathcal{W}_{i-1}^{i+1} containing $\overline{\mathcal{W}}_i$, of generic rank 1, and such that $\mathcal{W}_{i-1}^{i+1}/\overline{\mathcal{W}}_i$ has no torsion. We also define the *defect* $\text{def } \overline{\mathcal{W}}_i$ as the cycle of the torsion sheaf $\overline{\mathcal{W}}_i/\overline{\mathcal{W}}_i$. Two points $\overline{\mathcal{W}}_i, \overline{\mathcal{W}}'_i$ are in the same fiber of $\phi = \pi_\alpha|_{X_y}$ iff their saturations and defects coincide. In particular, ϕ is one-to-one when restricted to the open subset $U \subset X_y$ formed by all the saturated $\overline{\mathcal{W}}_i$.

To prove (c) we must check that U is nonempty for $d_i \gg 0$. This is evident. To finish the proof of (c) it remains to compute $\dim U$. Let us decompose $\mathcal{W}_{i-1}^{i+1} \simeq (\mathcal{W}_{i-1}^{i+1})^{\text{tors}} \oplus (\mathcal{W}_{i-1}^{i+1})^{\text{free}}$ into a direct sum of a torsion sheaf and a locally free sheaf. Then a point of U is represented by $\overline{\mathcal{W}}_i \simeq (\mathcal{W}_{i-1}^{i+1})^{\text{tors}} \oplus \overline{\mathcal{W}}_i^{\text{free}}$ where $\overline{\mathcal{W}}_i^{\text{free}} \subset (\mathcal{W}_{i-1}^{i+1})^{\text{free}}$ is a line subbundle of degree $d_{i-1} - d_i - \dim(\mathcal{W}_{i-1}^{i+1})^{\text{tors}}$. Locally around $\overline{\mathcal{W}}_i$, U is isomorphic to $\mathbb{P}\text{Hom}(\overline{\mathcal{W}}_i^{\text{free}}, \mathcal{W}_{i-1}^{i+1})$. For $d_i \gg 0$ the latter space is $\mathbb{P}^{2d_i - d_{i-1} - d_{i+1} + 1}$ which completes the proof of (c).

To prove (d) we fix a saturated subsheaf $\overline{\mathcal{W}}_i = \overline{\mathcal{W}}_i^{\text{tors}} \oplus \overline{\mathcal{W}}_i^{\text{free}}$, and a defect $\delta \in \mathbb{C}^{(d)}$. We have to estimate the dimension of the moduli space $\phi^{-1}(z)$ of subsheaves $\overline{\mathcal{W}}_i \subset \mathcal{W}_{i-1}^{i+1}$ with given saturation $\overline{\mathcal{W}}_i$ and $\text{def } \overline{\mathcal{W}}_i = \delta$. If $\overline{\mathcal{W}}_i^{\text{prfr}}$ stands for the image of projection of $\overline{\mathcal{W}}_i$ to $\overline{\mathcal{W}}_i^{\text{free}}$ along $\overline{\mathcal{W}}_i^{\text{tors}}$, then there are finitely many possible values of $\overline{\mathcal{W}}_i^{\text{prfr}} \subset \overline{\mathcal{W}}_i^{\text{free}}$; more precisely, not more than τ^τ where $\tau = \dim(\mathcal{W}_{i-1}^{i+1})^{\text{tors}}$ (the only ambiguity in the choice of $\overline{\mathcal{W}}_i^{\text{prfr}} \subset \overline{\mathcal{W}}_i^{\text{free}}$ can occur at the support of $(\mathcal{W}_{i-1}^{i+1})^{\text{tors}}$). Now for a fixed value of $\overline{\mathcal{W}}_i^{\text{prfr}} \subset \overline{\mathcal{W}}_i^{\text{free}}$ the dimension of the corresponding stratum of the moduli space in question is independent of d_i . Hence, with d_i growing, $\dim X_y - \dim \phi^{-1}(z)$ grows uniformly in y and z .

The proposition is proved. \square

5.2. Vanishing Theorem. We combine 5.1 and 4.2 setting $Y = \mathcal{Q}_P^\alpha$, $X = \mathcal{Q}^\alpha$, $p = \varpi_\alpha$, $\mathcal{L} = \mathcal{O}(-\check{\lambda})$. We arrive at the following

THEOREM 5.3. *Let $\check{\lambda} = \sum_{i=1}^{N-1} l_i \check{\omega}_i$ be a non-dominant weight, i.e. $l_i < 0$ for certain $1 \leq i \leq N-1$. We fix $j, k \in \mathbb{N}$ and $\bar{\alpha} = (d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_{N-1})$. Then for $d_i \gg 0$ we have $H^k(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^j \otimes \mathcal{O}(\check{\lambda})) = 0$. \square*

6. Euler characteristics of twisted De Rham complexes

6.1. Generating functions. For a weight $\check{\lambda}$ we consider $\chi(H^\bullet(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^\bullet \otimes \mathcal{O}(\check{\lambda})))$ as a virtual graded $\mathbb{G}_m \times T$ -module. Here T is the Cartan torus of $\text{SL}(N)$, and the grading is via De Rham degree $\Omega_{\mathcal{Q}^\alpha}^\bullet$. The generating function of its character is $[H^\bullet(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^\bullet \otimes \mathcal{O}(\check{\lambda}))] := \sum_{i,j} (-1)^{i+j} t^j [H^i(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^j \otimes \mathcal{O}(\check{\lambda}))]$ a function of q, t, z

where q is the coordinate on \mathbb{G}_m , and z are the coordinates on T . We define $H_{\check{\lambda}}(q, t, z)$ as the limit of $[H^\bullet(\mathbb{Q}^\alpha, \Omega_{\mathbb{Q}^\alpha}^\bullet \otimes \mathcal{O}(\check{\lambda}))]$ as $\alpha \rightarrow \infty$. We will see that the limit exists as a formal series in q, t, z converging to a rational function in q, t, z . For instance, if $\check{\lambda}$ is *not* dominant, then according to 5.3, $H_{\check{\lambda}}(q, t, z) = 0$.

6.2. Betti cohomology of Laumon spaces. We start with a computation of $H_0(q, t, z)$.

PROPOSITION 6.3.

$$H_0(q, t, z) = \frac{(1+t)(1+t+t^2)\dots(1+t+\dots+t^{N-1})}{(1-t^{N-1})^2(1-t^{N-2})^4\dots(1-t^3)^{2N-6}(1-t^2)^{2N-4}} \cdot \frac{1}{(1-t^N)(1-t)^{N-2}}.$$

PROOF. According to [11, Theorem 2.9], the Betti cohomology $H^\bullet(\mathbb{Q}^\alpha, \mathbb{C})$ carries a Tate Hodge structure, so $H^i(\mathbb{Q}^\alpha, \Omega_{\mathbb{Q}^\alpha}^j) = 0$ unless $i = j$, while $H^i(\mathbb{Q}^\alpha, \Omega_{\mathbb{Q}^\alpha}^i) = H^{2i}(\mathbb{Q}^\alpha, \mathbb{C})$. The action of $\mathbb{G}_m \times T$ on the latter space is clearly trivial, so $H_0(q, t, z) = H_0(t)$ is the $\alpha \rightarrow \infty$ limit of Poincaré polynomials $P_\alpha(t) := \sum_i t^i \dim H^{2i}(\mathbb{Q}^\alpha, \mathbb{C})$. Now $P_\alpha(t)$ is calculated in [11, Theorem 2.7] (under the perverse normalization). The $\alpha \rightarrow \infty$ limit $P_\infty(t)$ is the product $W(t) \cdot F(t)$ where $W(t)$ is the Poincaré polynomial of the flag variety \mathcal{B} , that is $W(t) = N!_t = (1+t)(1+t+t^2)\dots(1+t+\dots+t^{N-1})$. Furthermore, $F(t) = \sum F_i t^i$ where F_i is the number of unordered collections of positive roots $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m \in R^+(\mathfrak{sl}_N)$ such that none of $\alpha_1, \dots, \alpha_k$ is simple, and $\sum_{j=1}^k (|\alpha_j| - 1) + \sum_{l=1}^m (|\beta_l| + 1) = i$. Here $|\beta| := (\beta, \rho)$.

The proposition follows. □

6.4. Local Laumon spaces. Recall that $\Omega^\alpha \subset \mathbb{Q}^\alpha$ is a locally closed *local* Laumon moduli space of quasiflags based at $\infty \in \mathbb{C}$, see e.g. [10]. Similarly to Section 6.1 we introduce the generating function $\mathfrak{J}_\alpha(q, t, z) = [H^\bullet(\Omega^\alpha, \Omega_{\Omega^\alpha}^\bullet)] := \sum_{i,j} (-1)^{i+j} t^j [H^i(\Omega^\alpha, \Omega_{\Omega^\alpha}^j)]$. We compute the Euler characteristic of $H^\bullet(\Omega^\alpha, \Omega_{\Omega^\alpha}^\bullet)$ via the Atiyah-Bott-Lefschetz localization to the fixed points of $\mathbb{G}_m \times T$ in Ω^α . The characters of $\mathbb{G}_m \times T$ in the tangent spaces of the fixed points are computed in [3, Proposition 2.18a]. To write down the answer we recall the necessary notation. The fixed points are numbered by the collections $\tilde{d} = (d_{ij})_{N-1 \geq i \geq j \geq 1}$ such that $d_{ij} \leq d_{kj}$ for $i \geq k \geq j$, and $d_{i,1} + d_{i,2} + \dots + d_{i,i} = d_i$ (recall that $\alpha = (d_1, \dots, d_{N-1})$). For $1 \leq k < l \leq N$ we set $\theta_{kl} = d_{l-1,k} - d_{lk}$ where $d_{Nk} := 0$. Conversely, $d_{ij} = \theta_{j,i+1} + \theta_{j,i+2} + \dots + \theta_{j,N}$. Recall the set $M^{(N)}$ introduced in (2.15). Let x_i stand for the character of the dual torus \tilde{T} corresponding to the simple coroot α_i . For $\alpha \in \mathbb{N}^{N-1}$ the corresponding character of \tilde{T} is denoted by x^α . We consider the formal generating function

$$J(q, t, z, x) = \sum_{\alpha \in \mathbb{N}^{N-1}} x^\alpha \mathfrak{J}_\alpha(q, t, z).$$

Now

$$(6.1) \quad J(q, t, z, x) = \sum_{(\theta_{ij}) \in M^{(N)}} C_{(\theta_{ij})}(q, t, z) \prod_{1 \leq i < j \leq N} (x_i \cdots x_{j-1})^{\theta_{ij}}$$

where

(6.2)

$$C_{(\theta_{i,j})}(q, t, z) = \prod_{1 \leq i < j \leq N} \frac{(qt; q)_{\theta_{i,j}} (q^{1+\sum_{a=j+1}^N (\theta_{ia} - \theta_{ja})} tz_j/z_i; q)_{\theta_{i,j}}}{(q; q)_{\theta_{i,j}} (q^{1+\sum_{a=j+1}^N (\theta_{ia} - \theta_{ja})} z_j/z_i; q)_{\theta_{i,j}}} \times \\ \prod_{k=3}^N \prod_{1 \leq l < m < k} \frac{(q^{1+\sum_{b=k+1}^N (\theta_{lb} - \theta_{mb})} tz_m/z_l; q)_{\theta_{l,k}} (q^{1-\theta_{lk} + \theta_{mk} - \sum_{b=k+1}^N (\theta_{lb} - \theta_{mb})} tz_l/z_m; q)_{\theta_{l,k}}}{(q^{1+\sum_{b=k+1}^N (\theta_{lb} - \theta_{mb})} z_m/z_l; q)_{\theta_{l,k}} (q^{1-\theta_{lk} + \theta_{mk} - \sum_{b=k+1}^N (\theta_{lb} - \theta_{mb})} z_l/z_m; q)_{\theta_{l,k}}}$$

— this is a restatement of [3, Proposition 2.18a].

6.5. Local stabilization. We will prove that as $\alpha \rightarrow \infty$, the series $\mathfrak{J}_\alpha(q, t, z)$ tends to the limit $\mathfrak{J}_\infty(q, t, z)$. More precisely, for any $n, m \in \mathbb{N}$ the coefficient of $q^n t^m$ in $\mathfrak{J}_\alpha(q, t, z)$ for $\alpha \gg 0$ is independent of α . The resulting series will be denoted by $\mathfrak{J}_\infty(q, t, z)$. The existence of the limit and computation of its value follows from 7.2 below and [7, Proposition 3.11]. For the reader's convenience we present a more elementary computation of the limit.

We introduce a function z_N on the Cartan torus T of $\mathrm{SL}(N)$ defined as $z_N := z_1^{-1} \cdots z_{N-1}^{-1}$.

THEOREM 6.6.

$$\lim_{\alpha \rightarrow \infty} \mathfrak{J}_\alpha(q, t, z) = \prod_{1 \leq i < j \leq N} \frac{(qtz_j/z_i; q)_\infty}{(qz_j/z_i; q)_\infty} \times \left(\frac{(qt; q)_\infty}{(q; q)_\infty} \right)^{N-1} \times \prod_{i=1}^{N-2} \left(\frac{(qt^{i+1}; q)_\infty}{(t^i; q)_\infty} \right)^{N-i-1}.$$

PROOF. Let $\alpha = \sum_{i=1}^{N-1} \ell_i \alpha_i, \ell_i \in \mathbb{N}$. We study the stabilization of $\mathfrak{J}_\alpha(q, t, z)$ in the sector $\ell_1 \gg \ell_2 \gg \cdots \gg \ell_{N-1} \gg 0$. Note that we can recast the coefficient $C_{(\theta_{i,j})}(q, t, z)$ as follows:

$$C_{(\theta_{i,j})}(q, t, z) = \prod_{1 \leq i < j \leq N} \frac{(qt; q)_{\theta_{i,j}} (qtz_j/z_i; q)_{\theta_{i,N}}}{(q; q)_{\theta_{i,j}} (qz_j/z_i; q)_{\theta_{i,N}}} \times \prod_{k=3}^N F_k, \\ F_k = \prod_{1 \leq l < m < k} \frac{(q^{\sigma_{l,k} - \sigma_{m,k}} qtz_m/z_l; q)_{\theta_{l,k-1}} (q^{-\sigma_{l,k} + \sigma_{m,k}} qtz_l/z_m; q)_{\theta_{l,k}}}{(q^{\sigma_{l,k} - \sigma_{m,k}} qz_m/z_l; q)_{\theta_{l,k-1}} (q^{-\sigma_{l,k} + \sigma_{m,k}} qz_l/z_m; q)_{\theta_{l,k}}}, \\ \sigma_{l,k} = \sum_{b=k}^N \theta_{l,b}.$$

Define $F^{(n)}$ ($0 \leq n \leq N-1$) recursively by setting $F^{(0)} = C_{(\theta_{i,j})}(q, t, z)$,

$$F^{(1)} = \lim_{\ell_1 \rightarrow \infty} \sum_{\substack{\theta_{1,2} \geq 0 \\ \ell_1 = \sigma_{1,2}}} F^{(0)}, \quad F^{(2)} = \lim_{\ell_2 \rightarrow \infty} \sum_{\substack{\theta_{1,3}, \theta_{2,3} \geq 0 \\ \ell_2 = \sigma_{1,3} + \sigma_{2,3}}} F^{(1)}, \dots, \\ F^{(n)} = \lim_{\ell_n \rightarrow \infty} \sum_{\substack{\theta_{1,n+1}, \dots, \theta_{n,n+1} \geq 0 \\ \ell_n = \sigma_{1,n+1} + \cdots + \sigma_{n,n+1}}} F^{(n-1)}, \dots, \\ F^{(N-1)} = \lim_{\ell_{N-1} \rightarrow \infty} \sum_{\substack{\theta_{1,N}, \dots, \theta_{N-1,N} \geq 0 \\ \ell_{N-1} = \theta_{1,N} + \cdots + \theta_{N-1,N}}} F^{(N-2)}.$$

Then the coefficient we are interested in is $F^{(N-1)}$. The stabilization of $F^{(N-1)}$ can be studied and stated explicitly as follows. \square

LEMMA 6.7. *For $n = 1, \dots, N - 2$, we have*

$$\begin{aligned}
 F^{(n)} &= \prod_{1 \leq i < j \leq n+1} \frac{(qt; q)_\infty}{(q; q)_\infty} \frac{(qtz_j/z_i; q)_{\theta_{i,N}}}{(qz_j/z_i; q)_{\theta_{i,N}}} \times \\
 &\times \left(\frac{(q; q)_\infty}{(qt; q)_\infty} \right)^{\binom{n}{2}} \prod_{i=1}^{n-1} \left(\frac{(qt^{i+1}; q)_\infty}{(t^i; q)_\infty} \right)^{n-i} \times \\
 &\times \sum_{\substack{(\theta_{i,j})_{j \geq n+2} \\ \ell_i = \sigma_{1,i+1} + \dots + \sigma_{i,i+1} \\ (n+1 \leq i \leq N-1)}} \prod_{\substack{1 \leq i < j \leq N \\ n+2 \leq j}} \frac{(qt; q)_{\theta_{i,j}}}{(q; q)_{\theta_{i,j}}} \frac{(qtz_j/z_i; q)_{\theta_{i,N}}}{(qz_j/z_i; q)_{\theta_{i,N}}} \times \prod_{k=n+2}^N F_k,
 \end{aligned}$$

and

$$F^{(N-1)} = \prod_{1 \leq i < j \leq N} \frac{(qtz_j/z_i; q)_\infty}{(qz_j/z_i; q)_\infty} \times \left(\frac{(qt; q)_\infty}{(q; q)_\infty} \right)^{N-1} \times \prod_{i=1}^{N-2} \left(\frac{(qt^{i+1}; q)_\infty}{(t^i; q)_\infty} \right)^{N-i-1}.$$

PROOF. We prove the statement by the recursive use of the summation formula associated with the root lattice of type A_n [12, the table at p. 136 and references therein]:

$$\begin{aligned}
 \sum_{\chi \in Q} \prod_{\alpha \in R} \frac{(q^{1+\langle \alpha, \chi \rangle} tz_\alpha; q)_\infty}{(q^{1+\langle \alpha, \chi \rangle} z_\alpha; q)_\infty} &= \prod_{\alpha > 0} \frac{(qt^{\langle \rho, \alpha \rangle + 1}; q)_\infty (q^{\delta_\alpha} t^{\langle \rho, \alpha \rangle - 1}; q)_\infty}{(qt^{\langle \rho, \alpha \rangle}; q)_\infty (t^{\langle \rho, \alpha \rangle}; q)_\infty} \\
 &= \left(\frac{(q; q)_\infty}{(qt; q)_\infty} \right)^{n-1} \prod_{i=1}^{n-1} \frac{(qt^{i+1}; q)_\infty}{(t^i; q)_\infty},
 \end{aligned}$$

where $2\rho = \sum_{\alpha > 0} \alpha$, $\delta_\alpha = 1$ if α is a simple root and $\delta_\alpha = 0$ otherwise.

It is clear that we have $F^{(1)}$ by taking the limit $\ell_1 \rightarrow \infty$, namely letting $\theta_{1,2} \rightarrow \infty$ while fixing all the other $\theta_{i,j}$'s. The passage from $F^{(n)}$ to $F^{(n+1)}$ can be studied as follows. We need to take the limit $\ell_n \rightarrow \infty$ and perform the $(n-1)$ -dimensional summation with respect to the variables $\theta_{1,n+1}, \dots, \theta_{n,n+1}$ with the constraint $\ell_n = \sigma_{1,n+1} + \dots + \sigma_{n,n+1}$. It can be easily shown that, the most dominating terms, as a Taylor series in q and t , come from the vicinity of $\sigma_{l,n+1} - \sigma_{m,n+1} \sim 0$ ($1 \leq l < m \leq n$). Therefore the dominating contributions come from $\theta_{1,n+1}, \dots, \theta_{n,n+1} \gg 0$. Hence the $(n-1)$ -dimensional summation can be taken by using the above mentioned summation formula for type A_n .

This completes the proof of the theorem. \square

6.8. From local to global Laumon spaces. The following lemma is very similar to [6, Lemma 4.2]:

LEMMA 6.9.

$$[H^\bullet(\mathcal{Q}^\alpha, \Omega_{\mathcal{Q}^\alpha}^\bullet \otimes \mathcal{O}(\check{\lambda}))] = \sum_{\substack{\gamma + \beta = \alpha \\ w \in W}} z^{w\check{\lambda}} q^{\langle \gamma, \check{\lambda} \rangle} \mathfrak{J}_\gamma(q^{-1}, t, wz) \mathfrak{J}_\beta(q, t, wz) \prod_{\check{\alpha} \in \check{R}^+} \frac{1 - twz^{\check{\alpha}}}{1 - wz^{\check{\alpha}}}.$$

PROOF. Atiyah-Bott-Lefschetz localization to the fixed points of $\mathbb{G}_m \times T$ in \mathcal{Q}^α , see [10, Proof of Theorem 5.8]. \square

6.10. Global stabilization. We consider the formal generating function $\mathfrak{J}(q, t, z, x) = \prod_{i=1}^{N-1} x_i^{\log(\tilde{\omega}_i)/\log q} J(q, t, z, x)$.

Note that if we plug $x = q^\lambda$ into $J(q^{-1}, t, z, x)$ or into $\mathfrak{J}(q^{-1}, t, z, x)$, then for a dominant weight $\tilde{\lambda}$ these formal series converge, and we have $\mathfrak{J}(q^{-1}, t, z, q^\lambda) := \prod_{i=1}^{N-1} (q^{\langle \alpha_i, \tilde{\lambda} \rangle})^{\log(\tilde{\omega}_i)/\log q} J(q^{-1}, t, z, q^\lambda) = z^{\tilde{\lambda}} J(q^{-1}, t, z, q^\lambda)$ (a formal Taylor series in q and t with coefficients in Laurent polynomials in z).

Recall the generating function $H_{\tilde{\lambda}}(q, t, z)$ introduced in Section 6.1. The following proposition is very similar to [6, Proposition 4.4]:

PROPOSITION 6.11.

$$H_{\tilde{\lambda}}(q, t, z) = \sum_{w \in W} \mathfrak{J}(q^{-1}, t, wz, q^\lambda) \mathfrak{J}_\infty(q, t, wz) \prod_{\tilde{\alpha} \in \tilde{R}^+} \frac{1 - twz^{\tilde{\alpha}}}{1 - wz^{\tilde{\alpha}}}.$$

PROOF. As α goes to ∞ , the formula of 6.9 goes to

$$\begin{aligned} & \sum_{\substack{\gamma \in \Lambda_+ \\ w \in W}} z^{w\tilde{\lambda}} q^{\langle \gamma, \tilde{\lambda} \rangle} \mathfrak{J}_\gamma(q^{-1}, t, wz) \mathfrak{J}_\infty(q, t, wz) \prod_{\tilde{\alpha} \in \tilde{R}^+} \frac{1 - twz^{\tilde{\alpha}}}{1 - wz^{\tilde{\alpha}}} = \\ (6.3) \quad & \sum_{w \in W} z^{w\tilde{\lambda}} J(q^{-1}, t, wz, q^\lambda) \mathfrak{J}_\infty(q, t, wz) \prod_{\tilde{\alpha} \in \tilde{R}^+} \frac{1 - twz^{\tilde{\alpha}}}{1 - wz^{\tilde{\alpha}}} = \\ & \sum_{w \in W} \mathfrak{J}(q^{-1}, t, wz, q^\lambda) \mathfrak{J}_\infty(q, t, wz) \prod_{\tilde{\alpha} \in \tilde{R}^+} \frac{1 - twz^{\tilde{\alpha}}}{1 - wz^{\tilde{\alpha}}}. \end{aligned}$$

□

7. Difference equations

7.1. Euler characteristics of De Rham complexes of local Laumon spaces. For $1 \leq i \leq N$, we consider the difference operator $T_{i, q^{\pm 1}}$ on functions of q, t, z, x defined as follows: $T_{i, q^{\pm 1}} F(q, t, z, x_1, \dots, x_{N-1}) := F(q, t, z, x_1, \dots, x_{i-2}, q^{\mp 1} x_{i-1}, q^{\pm 1} x_i, x_{i+1}, \dots, x_{N-1})$. We define

$$D := \sum_{i=1}^N \prod_{j < i} \frac{1 - q^{-1} t^{i-j-1} x_j \cdots x_{i-1}}{1 - t^{i-j} x_j \cdots x_{i-1}} \prod_{k > i} \frac{1 - qt^{k-i+1} x_i \cdots x_{k-1}}{1 - t^{k-i} x_i \cdots x_{k-1}} T_{i, q^{-1}}$$

Recall the function z_N on the Cartan torus T of $\mathrm{SL}(N)$ defined as $z_N := z_1^{-1} \cdots z_{N-1}^{-1}$.

THEOREM 7.2. $D\mathfrak{J}(q, t, z, x) = (z_1 + \dots + z_N)\mathfrak{J}(q, t, z, x)$.

PROOF. We just recall 3.7 and compare 6.1 and (6.2) with formula (3.3) for an eigenfunction $f_N(q, t, z_1, \dots, z_N, y_1, \dots, y_N)$ of the difference operator

$$D_N^1 = \sum_{i=1}^N z_i \prod_{j < i} \frac{1 - t^{-1} y_i / y_j}{1 - y_i / y_j} \prod_{k > i} \frac{1 - ty_k / y_i}{1 - y_k / y_i} T_{q, y_i}$$

where $T_{q, y_i} f(q, t, z_1, \dots, z_N, y_1, \dots, y_N) := f(q, t, z_1, \dots, z_N, y_1, \dots, y_{i-1}, qy_i, y_{i+1}, \dots, y_N)$. It is immediate that after substitution $t = t/q$, $x_i = y_i / y_{i+1}$ we have

$J(q, t, z, t^{-1}x_1^{-1}, \dots, t^{-1}x_{N-1}^{-1}) = f_N(q, \mathbf{t}, z, y)$. It follows that $D'J(q, t, z, x) = (z_1 + \dots + z_N)J(q, t, z, x)$ where

$$D' := \sum_{i=1}^N z_i \prod_{j < i} \frac{1 - q^{-1}t^{i-j-1}x_j \cdots x_{i-1}}{1 - t^{i-j}x_j \cdots x_{i-1}} \prod_{k > i} \frac{1 - qt^{k-i+1}x_i \cdots x_{k-1}}{1 - t^{k-i}x_i \cdots x_{k-1}} T_{i, q^{-1}}$$

The theorem follows. \square

7.3. Difference equation on $H_{\check{\lambda}}$. For a weight $\check{\lambda} = \sum_{i=1}^{N-1} l_i \check{\omega}_i$, and $1 \leq k \leq N$, we define $T_k \check{\lambda}$ as follows: $T_1 \check{\lambda} = (l_1 - 1)\check{\omega}_1 + l_2 \check{\omega}_2 + \dots + l_{N-1} \check{\omega}_{N-1}$, $T_2 \check{\lambda} = (l_1 + 1)\check{\omega}_1 + (l_2 - 1)\check{\omega}_2 + l_3 \check{\omega}_3 + \dots + l_{N-1} \check{\omega}_{N-1}$, \dots , $T_N \check{\lambda} = l_1 \check{\omega}_1 + \dots + l_{N-2} \check{\omega}_{N-2} + (l_{N-1} + 1)\check{\omega}_{N-1}$. We define the operator $\mathfrak{D} := \sum_{r=1}^N K_r(\check{\lambda}) T_r \check{\lambda}$ where

$$K_r(\check{\lambda}) = \frac{(1 - t^2 q^{l_r - 1})(1 - t^3 q^{l_r + l_{r+1} - 1}) \dots (1 - t^{N-r+1} q^{l_r + \dots + l_{N-1}})}{(1 - tq^{l_r})(1 - t^2 q^{l_r + l_{r+1}}) \dots (1 - t^{N-r} q^{l_r + \dots + l_{N-1}})} \times \\ \times \frac{(1 - q^{l_r - 1 + 1})(1 - tq^{l_r - 1 + l_{r-2} + 1}) \dots (1 - t^{r-2} q^{l_r - 1 + \dots + l_1 + 1})}{(1 - tq^{l_r - 1})(1 - t^2 q^{l_r - 1 + l_{r-2}}) \dots (1 - t^{r-1} q^{l_r - 1 + \dots + l_1})}$$

Now 6.11 and 7.2 admit the following

$$\text{COROLLARY 7.4. } \mathfrak{D}H_{\check{\lambda}}(q, t, z) = (z_1 + \dots + z_N)H_{\check{\lambda}}(q, t, z).$$

PROOF. The function $\mathfrak{J}(q^{-1}, t, wz, q^{\check{\lambda}})$ on the weight lattice is an eigenfunction of D (with q inverted) restricted to the weight lattice. According to 6.11, $H_{\check{\lambda}}(q, t, z)$ is a linear combination of the functions $\mathfrak{J}(q^{-1}, t, wz, q^{\check{\lambda}})$ with coefficients independent of $\check{\lambda}$. Hence $H_{\check{\lambda}}(q, t, z)$ is an eigenfunction of D (with q inverted) restricted to the weight lattice (that is \mathfrak{D}) as well. \square

7.5. $H_{\check{\lambda}}$ via Macdonald polynomials. For $\check{\lambda}$ a dominant weight, $P_{\check{\lambda}}$ is the Macdonald polynomial [15]. To avoid a misunderstanding, let us state the relation between our $\check{\lambda} = \sum_{i=1}^{N-1} l_i \check{\omega}_i$, and Macdonald's partitions: we associate to (l_1, \dots, l_{N-1}) the partition $(\check{\lambda}_N \geq \check{\lambda}_{N-1} \geq \dots \geq \check{\lambda}_3 \geq \check{\lambda}_2 \geq 0)$ where $l_1 = \check{\lambda}_2$, $l_2 = \check{\lambda}_3 - \check{\lambda}_2$, \dots , $l_{N-1} = \check{\lambda}_N - \check{\lambda}_{N-1}$.

THEOREM 7.6.

$$H_{\check{\lambda}} = H_0 \prod_{1 \leq i \leq j \leq N-1} \frac{(t^{j-i+1}; q)_{l_i + \dots + l_j}}{(t^{j-i} q; q)_{l_i + \dots + l_j}} P_{\check{\lambda}}$$

PROOF. The Pieri rule [15, Equation (6.24)(iv) at page 341] reads $\mathcal{D}P_{\check{\lambda}} = (z_1 + \dots + z_N)P_{\check{\lambda}}$ where $\mathcal{D} := \sum_{r=1}^N L_r(\check{\lambda}) T_r \check{\lambda}$, and

$$L_r(\check{\lambda}) = \frac{(1 - t^2 q^{l_r - 1})(1 - t^3 q^{l_r + l_{r+1} - 1}) \dots (1 - t^{N+1-r} q^{l_r + \dots + l_{N-1} - 1})}{(1 - tq^{l_r - 1})(1 - t^2 q^{l_r + l_{r+1} - 1}) \dots (1 - t^{N-r} q^{l_r + \dots + l_{N-1} - 1})} \times \\ \times \frac{(1 - q^{l_r})(1 - tq^{l_r + l_{r-1}}) \dots (1 - t^{N-1-r} q^{l_r + \dots + l_{N-1}})}{(1 - tq^{l_r})(1 - t^2 q^{l_r + l_{r-1}}) \dots (1 - t^{N-r} q^{l_r + \dots + l_{N-1}})}$$

Since $H_{\check{\lambda}}(q, t, z)$ is an eigenfunction of \mathfrak{D} , the function

$$P'_{\check{\lambda}}(q, t, z) := H_{\check{\lambda}}(q, t, z) H_0^{-1} \prod_{1 \leq i \leq j \leq N-1} \frac{(t^{j-i+1}; q)_{l_i + \dots + l_j}^{-1}}{(t^{j-i} q; q)_{l_i + \dots + l_j}^{-1}}$$

is an eigenfunction of \mathcal{D} . It vanishes outside the cone of dominant weights according to 5.3, and it equals 1 at $\check{\lambda} = 0$. These properties uniquely characterize the Macdonald polynomials $P_{\check{\lambda}}(q, t, z)$. \square

REMARK 7.7. In view of 7.6, 6.11 expressing the Macdonald polynomials in terms of the Baker-Akhiezer function $\mathfrak{J}(q, t, z, x)$ is nothing but the generalized Weyl formula [9, Proposition 5.3], [7, Theorem 3.9].

8. Speculations for arbitrary simple groups

8.1. Perverse coherent sheaves. Let G be an almost simple simply connected complex group with Lie algebra \mathfrak{g} . We will follow the notations of [5], [6]. Recall the infinite type scheme ${}_{\mathfrak{g}}\mathbf{Q}$ introduced in [6, Section 2.2]: the quotient by the action of the Cartan torus $T \subset G$ of the space of maps from $\text{Spec } R = \text{Spec } \mathbb{C}[[t^{-1}]]$ to the affinization of the base affine space $\overline{G/U_-}$ taking value in G/U_- at the generic point. It is equipped with the action of the proalgebraic group $G(R)$; the open orbit ${}_{\mathfrak{g}}\mathbf{Q}_{\infty} = {}_{\mathfrak{g}}\mathbf{Q}^0$ is nothing but $G(R)/T \cdot U_-(R)$: the maps taking value in G/U_- at the closed point $r \in \text{Spec } R$. We denote by j the open embedding of ${}_{\mathfrak{g}}\mathbf{Q}^0$ into ${}_{\mathfrak{g}}\mathbf{Q}$. All the $G(R)$ -orbits in ${}_{\mathfrak{g}}\mathbf{Q}$ are numbered by the defects at r taking value in the cone of positive coroots Λ_+ of G : ${}_{\mathfrak{g}}\mathbf{Q} = \bigsqcup_{\alpha \in \Lambda_+} {}_{\mathfrak{g}}\mathbf{Q}^{\alpha}$. The codimension of ${}_{\mathfrak{g}}\mathbf{Q}^{\alpha}$ in ${}_{\mathfrak{g}}\mathbf{Q}$ equals $2|\alpha|$.

We introduce the perversity $p({}_{\mathfrak{g}}\mathbf{Q}^{\alpha}) = |\alpha|$; it is immediate that the function p is strictly monotone and comonotone in the sense of [1]. For a locally free $G(R) \rtimes \mathbb{G}_m$ -equivariant sheaf \mathcal{F} on ${}_{\mathfrak{g}}\mathbf{Q}^0$ the construction of [1, Section 4] produces an object $j_{!*}\mathcal{F}$ of $G(R) \rtimes \mathbb{G}_m$ -equivariant quasicoherent derived category on ${}_{\mathfrak{g}}\mathbf{Q}$.

8.2. Laumon resolution. In case $G = \text{SL}(N)$ we denote ${}_{\mathfrak{g}}\mathbf{Q}$ by \mathbf{Q} , and we have a resolution of singularities $\pi : \tilde{\mathbf{Q}} \rightarrow \mathbf{Q}$ where $\tilde{\mathbf{Q}}$ is the moduli space of flags $0 \subset V_1 \subset V_2 \dots \subset V_{N-1} \subset R^N$ of free R -modules, $\text{rk } V_i = i$, along with generators of rank 1 free R -modules $v_i \in \Lambda^i V_i$ defined up to multiplication by a scalar in \mathbb{C} . The smoothness of $\tilde{\mathbf{Q}}$ follows from the equality $\tilde{\mathbf{Q}} \simeq \left(\prod_{i=1}^{N-1} \text{Hom}_{\text{inj}}(R^i, R^{i+1}) \right) / \prod_{i=1}^{N-1} \text{GL}^c(i, R)$ where $\text{Hom}_{\text{inj}}(R^i, R^{i+1})$ stands for the open subscheme in the scheme (pro- finite dimensional vector space) $\text{Hom}_R(R^i, R^{i+1}) \simeq R^{i(i+1)}$ formed by all the injective homomorphisms, while $\text{GL}^c(i, R)$ stands for the group of $i \times i$ matrices with coefficients in R , and with *constant* nonvanishing determinant. For a point $\phi \in \mathbf{Q}$ we have $\dim \pi^{-1}(\phi) \leq |\alpha| - 1$, see [14, Lemma 2.4.6], i.e. the morphism π is *very small*. Hence $j_{!*}(\Omega_{\mathbf{Q}^0}^j) = R\pi_*\Omega_{\tilde{\mathbf{Q}}}^j$.

8.3. Euler characteristics for \mathbf{Q} . Similarly to Section 6.1, we consider the generating function $[H^{\bullet}(\mathbf{Q}, j_{!*}(\Omega_{\mathbf{Q}^0}^{\bullet}) \otimes \mathcal{O}(\check{\lambda}))] := \sum_{i,j} (-1)^{i+j} t^j [H^i(\mathbf{Q}, j_{!*}(\Omega_{\mathbf{Q}^0}^j) \otimes \mathcal{O}(\check{\lambda}))] = \sum_{i,j} (-1)^{i+j} t^j [H^i(\tilde{\mathbf{Q}}, \Omega_{\tilde{\mathbf{Q}}}^j \otimes \pi^*\mathcal{O}(\check{\lambda}))]$.

PROPOSITION 8.4. For $\check{\lambda} = \sum_{i=1}^{N-1} l_i \check{\omega}_i$ we have

$$[H^{\bullet}(\tilde{\mathbf{Q}}, \Omega_{\tilde{\mathbf{Q}}}^{\bullet} \otimes \pi^*\mathcal{O}(\check{\lambda}))] = H_0 \prod_{1 \leq i \leq j \leq N-1} \frac{(t^{j-i+1}; q)_{l_i + \dots + l_j}}{(t^{j-i} q; q)_{l_i + \dots + l_j}} \prod_{i=1}^{N-2} \left(\frac{(t^i; q)_{\infty}}{(qt^{i+1}; q)_{\infty}} \right)^{N-i-1} P_{\check{\lambda}}.$$

PROOF. Applying the Atiyah-Bott-Lefschetz localization to the fixed points of $\mathbb{G}_m \times T$ in $\tilde{\mathbf{Q}}$ we obtain

$$[H^{\bullet}(\tilde{\mathbf{Q}}, \Omega_{\tilde{\mathbf{Q}}}^{\bullet} \otimes \pi^*\mathcal{O}(\check{\lambda}))] = \sum_{w \in W} z^{w\check{\lambda}} \left(\sum_{(\theta_{ij}) \in \mathbb{N}^{M(N)}} C_{(\theta_{ij})}(q^{-1}, t, wz) q^{\sum_{(i,j) \in M(N)} (l_i + \dots + l_{j-1}) \theta_{ij}} \right)$$

$$\times \prod_{1 \leq i < j \leq N} \frac{(qtz_{w(j)}/z_{w(i)}; q)_\infty}{(qz_{w(j)}/z_{w(i)}; q)_\infty} \times \left(\frac{(qt; q)_\infty}{(q; q)_\infty} \right)^{N-1} \times \prod_{\check{\alpha} \in \check{R}^+} \frac{1 - twz^{\check{\alpha}}}{1 - wz^{\check{\alpha}}}$$

It remains to compare 6.1 and the formula 6.3 for $H_{\check{\lambda}}$ with the above formula, taking into account 7.6 and 6.6. \square

8.5. Euler characteristics for ${}_{\mathfrak{g}}\widehat{\mathbf{Q}}$. In case G is of type $BCFG$, following [5, Section 8.2], we consider a simply connected simply laced group G' with Lie algebra \mathfrak{g}' and its outer automorphism σ such that $\mathfrak{g} = (\mathfrak{g}')^\sigma$ (i.e. \mathfrak{g} is obtained by folding of \mathfrak{g}'). We define the scheme ${}_{\mathfrak{g}}\widehat{\mathbf{Q}}$ as a unique irreducible component of the fixed point subscheme of the automorphism ς of ${}_{\mathfrak{g}'}\mathbf{Q}$ having nonempty intersection with ${}_{\mathfrak{g}'}\mathbf{Q}^0$ (notations of *loc. cit.*). The orbits of $(G'[[t^{-1}]])^\varsigma$ on ${}_{\mathfrak{g}}\widehat{\mathbf{Q}}$ are numbered by $\Lambda_+(\mathfrak{g})$, and the minimal extension from ${}_{\mathfrak{g}}\widehat{\mathbf{Q}}^0$ is defined as in 8.1. In order to unify the notation, in the ADE case let us denote ${}_{\mathfrak{g}}\mathbf{Q}$ by ${}_{\mathfrak{g}}\widehat{\mathbf{Q}}$ as well.

Similarly to 6.3, we define $H_0(t)$ as $W(t) \cdot F(t)$ where $W(t)$ is the Poincaré polynomial of the flag variety $\mathcal{B}_{\check{\mathfrak{g}}}$, and $F(t) = \sum F_i t^i$ where F_i is the number of unordered collections of positive roots $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m \in R^+(\check{\mathfrak{g}})$ such that none of $\alpha_1, \dots, \alpha_k$ is simple, and $\sum_{j=1}^k (|\alpha_j| - 1) + \sum_{l=1}^m (|\beta_l| + 1) = i$. Here $|\beta| := \langle \beta, \check{\rho} \rangle$.

CONJECTURE 8.6. (a) For a nondominant G -weight $\check{\lambda}$ we have $[H^\bullet({}_{\mathfrak{g}}\widehat{\mathbf{Q}}, j!_*(\Omega_{\mathfrak{g}\widehat{\mathbf{Q}}^0}^\bullet) \otimes \mathcal{O}(\check{\lambda}))] = 0$.

(b) For a dominant G -weight $\check{\lambda}$ we have

$$[H^\bullet({}_{\mathfrak{g}}\widehat{\mathbf{Q}}, j!_*(\Omega_{\mathfrak{g}\widehat{\mathbf{Q}}^0}^\bullet) \otimes \mathcal{O}(\check{\lambda}))] = H_0 \prod_{\alpha \in R^+(\check{\mathfrak{g}})} \frac{(t^{|\alpha|}; q)_{\langle \alpha, \check{\lambda} \rangle}}{(t^{|\alpha|-1}q; q)_{\langle \alpha, \check{\lambda} \rangle}} \prod \frac{(t^{|\alpha|-1}; q)_\infty}{(qt^{|\alpha|}; q)_\infty} P_{\check{\lambda}}$$

where $P_{\check{\lambda}}(q, t, z)$ is the Macdonald polynomial for G , and the second product is taken over all nonsimple positive roots of $R^+(\check{\mathfrak{g}})$.

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