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Weak Error for Stable Driven Stochastic Differential Equations: Expansion of the Densities

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Abstract Consider a multidimensional stochastic differential equation of the form $X_t = x + \int_0^t b(X_{s-})\,ds + \int_0^t f(X_{s-})\,dZ_s$, where $(Z_s)_{s\ge 0}$ is a symmetric stable process. Under suitable assumptions on the coefficients, the unique strong solution of the above equation admits a density with respect to Lebesgue measure, and so does its Euler scheme. Using a parametrix approach, we derive an error expansion with respect to the time step for the difference of these densities.

Keywords Symmetric stable processes · Parametrix · Euler scheme

Mathematics Subject Classification (2000) 60H30 · 65C30 · 60G52

1 Introduction

Consider the following $\mathbb{R}^d$-valued stochastic differential equation (SDE for short)

$$X_t = x + \int_0^t b(X_{s-})\,ds + \int_0^t f(X_{s-})\,dZ_s, \tag{1.1}$$

where b, f are respectively Lipschitz continuous mappings from $\mathbb{R}^d$ to $\mathbb{R}^d$ and $\mathbb{R}^d$ to $\mathbb{R}^d \otimes \mathbb{R}^d$, and $(Z_s)_{s\ge 0}$ is a general Lévy process. The previous assumptions guarantee the existence of a unique strong solution to (1.1). Also, this solution satisfies the

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strong Markov property, see, e.g., Theorems 7 and 32 in Chap. 5 of Protter [22]. Let $T>0$ be a fixed time horizon, and $(X_t^N)_{t\in\Lambda}$ a given approximation scheme of $(X_t)_{t\in[0,T]}$ associated to the time step $h:=T/N$, $N\in\mathbb{N}^*$ on the grid $\Lambda:=\{t_i:=ih, i\in[\![0,N]\!]\}$. When speaking about weak approximation of (1.1), two kinds of quantities are of interest. The first one writes

$$\mathcal{E}_1(x,T,N):=\mathbb{E}_x\big[g(X_T)\big]-\mathbb{E}_x\big[g\big(X_T^N\big)\big]$$

for a suitable class of test functions g. The second one concerns, when it exists, the approximation of the transition density p of the original SDE (1.1). If the approximation scheme $(X_t^N)_{t\in\Lambda}$ also admits a transition density p^N, the quantity under study becomes

$$\mathcal{E}_2(x,y,T,N):=\big(p-p^N\big)(T,x,y).$$

In both cases, the goal is to give a bound or an error expansion of these quantities in terms of h. The error expansions are particularly useful for practical simulation. For $\mathcal{E}_1$, the expansion allows one to use the Romberg–Richardson extrapolation to improve the convergence of the discretization error, see, e.g., Talay and Tubaro [26]. On the other hand, if p and p^N exist, and a suitable expansion of $\mathcal{E}_2$ holds, it can be useful to estimate the sensitivity of $\mathcal{E}_1$ w.r.t. to the spatial variable x, and it also allows one to get a control on $\mathcal{E}_1$ for a wider class of test functions g than those considered by the direct methods used to control this quantity, see, e.g., Guyon [8]. Indeed, the typical assumptions and techniques associated to the study of $\mathcal{E}_1$ and $\mathcal{E}_2$ are quite of different nature.

In the continuous case, i.e., $Z_s=bs+\sigma W_s$, where $(W_s)_{s\ge 0}$ is a standard d-dimensional Brownian motion, provided that the test function g and the coefficients b,f are sufficiently smooth and g has polynomial growth, without any additional assumption on the generator, Talay and Tubaro [26] derive an error expansion of order 1 for $\mathcal{E}_1(x,T,N)$ when $(X_t^N)_{t\in\Lambda}$ is the Euler approximation. Their proof is based on standard stochastic analysis tools, Itô's expansions and stochastic flows. To obtain the same kind of result for bounded Borel functions g, some nondegeneracy has to be assumed, namely hypoellipticity of the underlying diffusion, and the proof relies on Malliavin calculus techniques, see Bally and Talay [1]. The authors also manage to extend their results to $\mathcal{E}_2(x,y,T,N)$ for a slightly modified Euler scheme [2].

Anyhow, in the uniformly elliptic case, the most natural approach to handle the estimation of the quantity $\mathcal{E}_2(x,y,T,N)$ consists in using the so called "parametrix" technique introduced to obtain existence and controls on the fundamental solutions of PDEs, see, e.g., McKean and Singer [20] or Friedman [7]. Roughly speaking, it consists in expressing the density of X_T in terms of an infinite sum of suitable iterated kernels applied to the density of an SDE with constant coefficients. This has been done successfully by Konakov and Mammen [17]. The main advantage of this approach is that the density of the solution X_T and the Euler approximation X_T^N can be expressed in the same form and therefore quite directly compared. Furthermore, this technique turns out to be quite robust and can be applied as soon as good controls on the densities p,p^N and their derivatives are available, see, e.g., [19] for an extension to a slightly degenerate framework.

For a general Lévy process Z and suitable smooth functions b, f, g, under additional assumptions on the behavior at infinity of the Lévy measure ν of Z, that is, integrability conditions of the large jumps, Protter and Talay [23] manage to get a control of order one or even an error expansion for $\mathcal{E}_1(x,T,N)$ with the same approach as in [26]. In that work the approximation is the Euler scheme, which for a general Lévy measure ν cannot always be exactly simulated on a computer.

The quantity $\mathcal{E}_1(x,T,N)$ for approximations of the Euler scheme that can be simulated has also been studied by Jacod et al. [13], who derived bounds of order 1. Moment conditions are also assumed. We finally refer to the work of Hausenblas and Marchis [10] for approximations of Poisson jump measures that are easy to simulate.

In this work, we consider the case where $(Z_t)_{t\geq 0}$ is an α-stable symmetric process, $\alpha\in(0,2)$. Under suitable nondegeneracy assumptions on its coefficients specified below (see (A-1)–(A-3)), (1.1) is known to have a density p w.r.t. the Lebesgue measure. This can be proved via a Malliavin calculus–Bismut integration-by-parts approach, see, e.g., Bichteller et al. [3]. Also, a direct construction of this density using a parametrix expansion has been obtained by Kolokoltsov [15], who also derived "Aronson's like" bounds with time singularity depending on the index α of the stable process $(Z_t)_{t\geq 0}$.

Analogously to the "diffusion case," the first step of the parametrix is to consider that the density $p(T,x,y)$ of (1.1) can be approximated by the density of the process $\widetilde{X}_t^y = x+b(y)t+f(y)Z_t$ at time T. Namely, we freeze the coefficients in (1.1) at the final spatial point. The next crucial point is to obtain sharp estimates of the stable density $\widetilde{p}^y(T,x,.)$ of $\widetilde{X}_T^y$ and its derivatives in order to solve the parametrix integral equations.

Stable driven SDEs appear in various applicative fields, from mathematical physics to electrical engineering or financial mathematics, see [11, 25], or [14]; therefore their approximation becomes of interest. To approximate (1.1), setting $\phi(t):=\inf\{t_i : t_i\leq t<t_{i+1}\}$, we introduce the Euler scheme

$$X_t^N = x+\int_0^t b\big(X_{\phi(s)}^N\big)\,ds+\int_0^t f\big(X_{\phi(s)}^N\big)\,dZ_s. \tag{1.2}$$

The computation of the above scheme only requires to be able to simulate exactly the increments of $(Z_t)_{t\geq 0}$, which up to a self similarity argument, only amounts to simulate a stable law. This aspect is, for instance, discussed in Samorodnitsky and Taqqu [24], Weron and Weron [27], or Sect. 3 of [23]. Under the same assumptions (A-1)–(A-3), the Euler scheme defined in (1.2) also has a density p^N.

Observe that the results of [23] and [13] cannot be directly applied, even for the study of $\mathcal{E}_1(x,T,N)$, since stable laws have heavy tails. Comparing the parametrix developments of p and p^N, we obtain an expansion with leading term of order 1 in h for $\mathcal{E}_2(x,y,T,N)$. The parametrix expansion of p is discussed in [15], see also Sect. 3 and Appendices B and C in [18], whereas the parametrix expansion of p^N can be related to the ideas developed in [16, 17] for the diffusive case corresponding to an index of stability equal to 2.

This result also emphasizes the robustness of the method that naturally extends to a broad class of processes. Let us mention that, using a Malliavin calculus approach, Hausenblas [9] derived an upper bound of order one w.r.t. h for the quantity

$\mathcal{E}_1(x,T,N),\ g\in L^\infty$ in the scalar case. Concerning functional limit theorems for the approximation of stable driven SDEs, we refer to the work of Jacod [12].

The paper is organized as follows. In Sect. 2 we state our standing assumptions and main results. In Sect. 3 we prove the existence of the densities for both the stable driven equation and its Euler scheme and also give a parametrix representation of these densities. Section 4 is dedicated to the proof of the main results. Eventually, we state in Sect. 5 weaker assumptions under which our main result holds, and we also briefly discuss how to extend it to the case of a stable process perturbed by a compound Poisson process.

2 Assumptions and Main results

2.1 Assumptions and Notation

In the following we consider *symmetric* stable processes, that is, for all $t\ge 0$, $u\in\mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}\big[\exp\big(i\langle u,Z_t\rangle\big)\big]&\\ &=\exp\left(it\langle\gamma,u\rangle+t\int_{S^{d-1}}\int_0^{+\infty}\left(e^{i\rho\langle u,s\rangle}-1-i\frac{\langle u,\rho s\rangle}{1+\rho^2}\right)\frac{d\rho}{\rho^{1+\alpha}}\tilde{\lambda}(ds)\right)\\ &=\exp\left(it\langle\gamma,u\rangle-t\int_{S^{d-1}}\big|\langle s,u\rangle\big|^\alpha\lambda(ds)\right),\end{aligned}\tag{2.1}$$

where $\tilde{\lambda}$ is a symmetric measure on the unit sphere S^{d-1} (i.e., for every A in the Borel σ-field $\mathcal{B}(S^{d-1})$, $\tilde{\lambda}(A)=\tilde{\lambda}(-A)$). The second equality in (2.1) is then obtained by direct integration over ρ and $\lambda=C_\alpha\tilde{\lambda}$ with

$$C_\alpha:=\Gamma(1-\alpha)\alpha^{-1}\cos\left(\frac{\pi\alpha}{2}\right)\mathbb{I}_{\alpha\neq 1}+\frac{\pi}{2}\mathbb{I}_{\alpha=1}.$$

We refer to the proof of Theorem 9.32 in Breiman [4] and Lemma 2, Chap. XVII.4 in Feller [6] for the expression of C_α.

We now introduce our assumptions. Fix an integer $q\ge 2$. We assume that

(A-1) For $d\ge 2$, the spherical measure λ has a $C^q(S^{d-1})$ surface density, and for all $d\ge 1$, there exist constants $0<C_1\le C_2<+\infty$ such that, for all $p\in\mathbb{R}^d$,

$$C_1|p|^\alpha\le\int_{S^{d-1}}\big|\langle p,s\rangle\big|^\alpha\lambda(ds)\le C_2|p|^\alpha.$$

(A-2) The coefficients b and f and their derivatives up to order q are uniformly bounded in x. Thus, for $1<\alpha<2$, $B(x):=b(x)+f(x)\gamma$ is uniformly bounded. We impose for $0<\alpha\le 1$, $B(x)=0$ for all $x\in\mathbb{R}^d$.

(A-3) There exist constants $0<\underline{c}\le\overline{c}<+\infty$ such that, for all $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^d$,

$$\underline{c}|\xi|^2\le\big\langle f(x)\xi,\xi\big\rangle\le\overline{c}|\xi|^2.$$

From now on we assume that Assumptions (A-1)–(A-3) are in force.

Remark 2.1 Note that for $d = 1$, with the convention $S^0 = \{-1, 1\}$, we have $C_1 = C_2$ in (A-1) even without symmetry. The symmetry is actually not needed in that case, see the beginning of Sect. 3 in [15].

Remark 2.2 The zero drift condition in (A-2) comes from the fact that for $\alpha \in (0, 1]$, the addition of a drift of order t does not correspond to a negligible term in small time with respect to the natural scale $t^{1/\alpha}$, see Appendix B in [18] for details.

In the following we denote by C a positive generic constant that can depend on α, d, the bounds appearing in the previous assumptions, but neither on N nor on the spatial points involved. Its value may change from line to line. Other possible dependencies, especially w.r.t. the final time T are explicitly specified. Concerning functional spaces, we denote by $C_b^k(\mathbb{R}^d),\ k \in \mathbb{N}^*$, the Banach space of continuous bounded functions having bounded derivatives up to and including the order k with the norm $\|f\| := \max_{0\le l\le k} \sup_{x\in\mathbb{R}^d} |f^{(l)}(x)|$. Eventually, $C_0^k(\mathbb{R}^d)$ stands for the functions in $C_b^k(\mathbb{R}^d)$ with compact support.

2.2 Generator

From (2.1) and standard computations, see, e.g., (5.11) in [13], we derive that for every smooth function $g \in C_0^2(\mathbb{R}^d)$, the generator of (1.1) writes

$$\Phi g(x) = \big\langle B(x), \nabla_x g(x)\big\rangle + \int_{\mathbb{R}^d} g\big(x + f(x)y\big) - g(x) - \frac{\langle \nabla_x g(x), f(x)y\rangle}{1+|y|^2}\nu(dy),$$

where $B(x) = b(x) + f(x)\gamma$, and ν stands for the Lévy measure of Z. Introduce for all $A \in \mathcal{B}(\mathbb{R}^d)$, $\nu_{f(x)}(A) := \nu(\{y \in \mathbb{R}^d : f(x)y \in A\})$ and denote by $\tilde{\lambda}_{f(x)}$ its spherical part (which is still a symmetric measure). Setting $z = f(x)y$ in the above equation, using the symmetry and the polar coordinates, we derive:

$$\begin{aligned}\Phi g(x) = {}& \big\langle B(x), \nabla_x g(x)\big\rangle \\ & + \int_{S^{d-1}} \int_0^{+\infty} \left(g(x+\rho s) - g(x) - \frac{\rho\langle \nabla_x g(x), s\rangle}{1+\rho^2}\right) \\ & \times \frac{d\rho}{\rho^{1+\alpha}} \tilde{\lambda}_{f(x)}(ds).\end{aligned} \tag{2.2}$$

Remark 2.3 Denote similarly to (2.1), $\lambda_{f(x)} = C_\alpha \tilde{\lambda}_{f(x)}$. The uniform ellipticity condition (A-3) allows one to have good controls on the measure $\lambda_{f(x)}(\cdot)$. As a consequence of (A-1), (A-3), one gets that there exist constants $0 < \underline{C}_1 = \underline{C}_1(\underline{c}, d, \alpha) \le \overline{C}_2 = \overline{C}_2(\overline{c}, d, \alpha) < +\infty$ such that, for all $p \in \mathbb{R}^d$, $x \in \mathbb{R}^d$,

$$\underline{C}_1 |p|^\alpha \le \int_{S^{d-1}} \big|\langle p, s\rangle\big|^\alpha \lambda_{f(x)}(ds) \le \overline{C}_2 |p|^\alpha. \tag{2.3}$$

2.3 Main Results

Proposition 2.1 *For every $t>0$, the solution X_t (resp. X_t^N) of* (1.1) *(resp.* (1.2)*) has a density $p(t,x,\cdot)$ (resp. $p^N(t,x,\cdot)$) w.r.t. the Lebesgue measure. Additionally, as a function of the space variables, the density p is in $C_b^q(\mathbb{R}^d\times\mathbb{R}^d)$ if $\alpha>1$ and in $C_b^{q-1}(\mathbb{R}^d\times\mathbb{R}^d)$ if $\alpha\le 1$.*

To state the theorem we first need some notation. Introduce for all $\xi\in\mathbb{R}^d$ and all smooth functions $\varphi(t,x,y)$, the integro-differential operators

$$\begin{aligned}\widetilde{\Phi}_\xi\varphi(t,x,y) &= \big\langle B(\xi),\nabla_x\varphi(t,x,y)\big\rangle\\ &\quad+\int_{S^{d-1}}\int_0^{+\infty}\left(\varphi(t,x+\rho s,y)-\varphi(t,x,y)-\frac{\rho\langle\nabla_x\varphi(t,x,y),s\rangle}{1+\rho^2}\right)\\ &\quad\times\frac{d\rho}{\rho^{1+\alpha}}\tilde{\lambda}_{f(\xi)}(ds).\end{aligned}\tag{2.4}$$

With this definition we write, for given $(x,y)\in\mathbb{R}^d$,

$$\widetilde{\Phi}^*\varphi(t,x,y)=\widetilde{\Phi}_y\varphi(t,x,y),\quad \forall m\in\mathbb{N}^*,\quad \big(\widetilde{\Phi}_*\big)^m\varphi(t,x,y)=\big(\widetilde{\Phi}_\xi\big)^m\varphi(t,x,y)|_{\xi=x}.\tag{2.5}$$

Note that we have $\widetilde{\Phi}_*\varphi(t,x,y)=\Phi\varphi(t,x,y)$ defined in (2.2), but in general, for $m\ge 2$, $(\widetilde{\Phi}_*)^m\varphi(t,x,y)\ne(\Phi)^m\varphi(t,x,y)$.

Define now, for $t>0$, the kernel

$$H(t,x,y):=\big(\Phi-\widetilde{\Phi}_y\big)\widetilde{p}^{\,y}(t,x,y),\tag{2.6}$$

where $\widetilde{p}^{\,y}(t,x,y)$ denotes the density at point y of $\widetilde{X}_t=x+b(y)t+f(y)Z_t$. Note that the variable y acts here twice: as the argument of the density and as a defining quantity of the process $\widetilde{X}_t(\equiv\widetilde{X}_{t,x,y})$, i.e., the coefficients are frozen in y. Eventually we introduce the continuous and discrete convolution operators

$$\varphi\otimes\psi(t,x,y)=\int_0^t du\int dz\,\varphi(u,x,z)\psi(t-u,z,y),\quad \forall t\in[0,T],$$

$$\varphi\otimes_N\psi(t,x,y)=\int_0^t du\int dz\,\varphi\big(\phi(u),x,z\big)\psi\big(t-\phi(u),z,y\big),\quad \forall t\in\big\{(t_i)_{i\in[\![1,N]\!]}\big\},$$

where $\phi(u)$ is defined just before (1.2) and denotes the largest discretization time lower or equal to u. Also, $\varphi\otimes H^{(0)}=\varphi$, and $\varphi\otimes H^{(r)}=(\varphi\otimes H^{(r-1)})\otimes H$ stands for the r-fold convolution.

Theorem 2.1 *Suppose $q>d+4$. Take $0<M\le q-(d+4)$. There exists a function $R_M(T,x,y)$ with $|R_M(T,x,y)|\le C_M(T)(\frac{1}{1+|y-x|^{d+\alpha}}):=\rho_{\alpha,M}(T,y-x)$ for some positive constant $C_M(T)$ such that*

$$\begin{aligned}\big(p-p^N\big)(T,x,y) &= \sum_{l=1}^{M-1}\frac{h^l}{(l+1)!}\big[p\otimes_N\big(\Phi-\widetilde{\Phi}^*\big)^{l+1}p^d\big](T,x,y)\\ &\quad-\sum_{k=1}^{M-1}\frac{h^k}{(k+1)!}\big[p^d\otimes_N\big(\widetilde{\Phi}_*-\widetilde{\Phi}^*\big)^{k+1}p^N\big](T,x,y)\\ &\quad+h^M R_M(T,x,y)\end{aligned}$$

with $\sum_{l=1}^{0}\cdots=0$ *and* $\forall t\in\{(t_i)_{i\in[\![1,N]\!]}\}$, $p^d(t,x,y):=\sum_{r=0}^{\infty}(\widetilde{p}\otimes_N H^{(r)})(t,x,y)$. *It holds that*

$$\sum_{l=1}^{M-1}\big|\big(p\otimes_N\big(\Phi-\widetilde{\Phi}^*\big)^{l+1}p^d\big)(T,x,y)\big|\le\rho_{\alpha,M}(T,y-x),$$

$$\sum_{k=1}^{M-1}\big|\big(p^d\otimes_N\big(\widetilde{\Phi}_*-\widetilde{\Phi}^*\big)^{k+1}p^N\big)(T,x,y)\big|\le\rho_{\alpha,M}(T,y-x).$$

Remark 2.4 In the above expression, one writes, for all $l\in[\![1,M-1]\!]$,

$$\big(\Phi-\widetilde{\Phi}^*\big)^{l+1}\varphi(t,x,y)=\sum_{k=1}^{l+1}C_{l+1}^k\Phi^k\big(-\widetilde{\Phi}^*\big)^{l+1-k}\varphi(t,x,y),$$

whereas, for all $k\in[\![1,M-1]\!]$,

$$\begin{aligned}\big(\widetilde{\Phi}_*-\widetilde{\Phi}^*\big)^{k+1}\varphi(t,x,y) &= \big[\underbrace{\big(\widetilde{\Phi}_\xi-\widetilde{\Phi}_y\big)\cdots\big(\widetilde{\Phi}_\xi-\widetilde{\Phi}_y\big)}_{(k+1)\text{ times}}\big]\varphi(t,x,y)|_{\xi=x}\\ &= \big(\widetilde{\Phi}_\xi-\widetilde{\Phi}_y\big)^{k+1}\varphi(t,x,y)|_{\xi=x}.\end{aligned}$$

Remark 2.5 The terms in the previous expansion depend on N. Anyhow using iteratively the theorem and controls on $\otimes_N-\otimes$ (see also Lemma 4.1), it is possible to obtain an expansion with terms independent of N. For small M, explicit formulas are thus easily derived, but in all generality the terms become less transparent. For $M=2$, one gets

$$\begin{aligned}\big(p-p^N\big)(T,x,y) &= \frac{h}{2}\big(p\otimes_N\big(\Phi-\widetilde{\Phi}^*\big)^2p^d-p^d\otimes_N\big(\widetilde{\Phi}_*-\widetilde{\Phi}^*\big)^2p^N\big)(T,x,y)\\ &\quad+h^2R_2(T,x,y)\\ &= \frac{h}{2}\big(p\otimes\big(\Phi-\widetilde{\Phi}^*\big)^2p-p\otimes\big(\widetilde{\Phi}_*-\widetilde{\Phi}^*\big)^2p\big)(T,x,y)\\ &\quad+h^2\widetilde{R}_2(T,x,y)=\frac{h}{2}\big(p\otimes\big(\Phi^2-\widetilde{\Phi}_*^2\big)p\big)(T,x,y)\\ &\quad+h^2\widetilde{R}_2(T,x,y),\end{aligned}$$

where $\widetilde{R}_2(T,x,y)\le C(T)\rho_{\alpha,2}(T,y-x)$ for some positive constant $C(T)$.

From the above expansion and the controls on the density and its derivatives, see e.g., Theorems 3.1 and 3.2 and Proposition 3.1 in [15] or Lemma 4.3, we can derive the error expansion for $\mathcal{E}_1(x,T,N)$ for measurable functions g satisfying the growth condition $\exists C>0$, $|g(x)|\le C(1+|x|^\beta)$, $\beta<\alpha$. In particular, we do not need the smoothness assumption on g required in the approach of [23, 26]. We recall that the expansion of $\mathcal{E}_1(x,T,N)$ allows one, from a practical point of view, to improve the convergence rate of the discretization error using the Romberg–Richardson extrapolation. This simply consists in observing that the expansion yields $\mathbb{E}[g(X_T)]-(\mathbb{E}[2g(X_T^{2N})]-\mathbb{E}[g(X_T^N)])=O(h^2)$. The associated Monte Carlo estimator, involving a refined scheme, is then used for simulations, see [26] for details.

Also, the expansion can be used to study the sensitivity of $\mathcal{E}_1(T,x,N)$ w.r.t. x without any additional assumption on g. This is crucial for financial applications (hedging), see, e.g., Guyon [8] for further developments in the diffusive case.

3 Stable Driven Equations and Their Euler Scheme: Existence of the Density and Associated Parametrix Expansion

3.1 Stable Driven Equation

3.1.1 Proof of Proposition 2.1: Existence of the Density for the Solution of (1.1)

For $(X_t)_{t\ge 0}$, the existence of the density derives from Proposition 3.4 in [15], where some properties of the fundamental solution of $\partial_t p(t,x,y)=\Phi p(t,x,y)$, $p(0,x,y)=\delta(y-x)$ are discussed, and a standard identification argument, see, e.g., Dynkin [5], Theorem 2.3, p. 56. The stated smoothness of the density is then a consequence of point (ii) of the same proposition.

Remark 3.1 The existence of the density is discussed in Bichteler et al. [3], where it is proved thanks to the Bismut–Malliavin approach. This technique requires the computation of a tangent equation associated to the gradient flow that involves the derivatives of the coefficients of (1.1). Thus, some additional smoothness of the coefficients is needed, see, e.g., Theorem 6.48 of the above reference. We also mention the result of Picard [21], Theorem 4.1, that gives existence and smoothness of the density for Lévy driven SDEs for very singular Lévy measures, provided that there are sufficiently small jumps. For smooth coefficients b, f, it includes in particular the case of (1.1) where the spherical measure λ can be atomic.

3.1.2 Parametrix Expansion of the Density

For the sake of completeness and also because it is crucial for the discrete model, we briefly recall how to get, through a "parametrix" approach, a series expansion for the density $p(t,x,y)$.

Introduce, for all $x,y\in\mathbb{R}^d$, the following stochastic "frozen" stable driven equation $\widetilde{X}_t\equiv\widetilde{X}_{t,x,y}$ defined for $t\ge 0$ by

$$\widetilde{X}_t = x+\int_0^t b(y)\,du+\int_0^t f(y)\,dZ_u. \tag{3.1}$$

By computation of the Fourier transform of Z_t and Fourier inversion the transition density $\widetilde{p}^y(t,x,z)$ of $\widetilde{X}_t$ at point $z \in \mathbb{R}^d$ explicitly writes

$$\widetilde{p}^y(t,x,z) = \frac{1}{(2\pi)^d} \int e^{-i\langle z-x-tB(y),p\rangle} \exp\left\{-t \int_{S^{d-1}} \left|\langle p,s\rangle\right|^\alpha \lambda_{f(y)}(ds)\right\} dp, \tag{3.2}$$

where $\lambda_{f(y)}$ has been introduced in Sect. 2.2. The densities of the solutions of (3.1) and (1.1) satisfy respectively

$$\begin{aligned}
&\frac{\partial \widetilde{p}^y}{\partial t}(t,x,z) = \widetilde{\Phi}_y \widetilde{p}^y(t,x,z), \quad \text{for } t>0,\ (x,z) \in \left(\mathbb{R}^d\right)^2,\ \widetilde{p}^y(0,x,z) = \delta(z-x),\\
&\frac{\partial p}{\partial t}(t,x,z) = \Phi p(t,x,z), \quad \text{for } t>0,\ (x,z) \in \left(\mathbb{R}^d\right)^2,\ p(0,x,z) = \delta(z-x).
\end{aligned} \tag{3.3}$$

Note carefully that the derivatives in $\widetilde{\Phi}_y$ are taken w.r.t. the x variable.

We will speak about the operators appearing in (3.3) as the "frozen" and "unfrozen" ones. In the following, $\forall (t,x,z) \in \mathbb{R}^{+*} \times (\mathbb{R}^d)^2$, $\widetilde{p}(t,x,z) := \widetilde{p}^z(t,x,z)$. Hence, by (2.6), $\forall (t,z,y) \in \mathbb{R}^{+*} \times (\mathbb{R}^d)^2$, $H(t,z,y) = (\Phi - \widetilde{\Phi}_y)\widetilde{p}(t,z,y) = (\widetilde{\Phi}_z - \widetilde{\Phi}_y)\widetilde{p}(t,z,y)$.

Proposition 3.1 (Parametrix expansion of the density) *With the notation of Sect. 2.2, the following representation holds*:

$$p(t,x,y) = \sum_{r=0}^{\infty} \left(\widetilde{p} \otimes H^{(r)}\right)(t,x,y). \tag{3.4}$$

Proof Equations (3.3) correspond to the forward Kolmogorov equations. Consider now the backward equation for p, namely, $\partial_s p(s,x,z) = {}^t\widetilde{\Phi}_z p(s,x,z)$, where ${}^t\widetilde{\Phi}_z$ stands for the adjoint operator of $\widetilde{\Phi}_z$, and the derivatives are taken w.r.t. z. Differentiating under the integral, from (3.3) we have

$$\begin{aligned}
&(p - \widetilde{p})(t,x,y)\\
&\quad = \int_0^t ds \frac{\partial}{\partial s}\left[\int p(s,x,z)\widetilde{p}(t-s,z,y)\,dz\right]\\
&\quad = \int_0^t ds \int \left[\left({}^t\widetilde{\Phi}_z p\right)(s,x,z)\widetilde{p}(t-s,z,y) - p(s,x,z)\widetilde{\Phi}_y \widetilde{p}(t-s,z,y)\right] dz\\
&\quad = p \otimes H(t,x,y).
\end{aligned}$$

Representation (3.4) then follows by simple iteration. □

Remark 3.2 Note that the previous expansion is "formal." The convergence of the r.h.s. in (3.4) is investigated in the proof of Theorem 3.1 in [15] and can also be derived with the controls of Lemmas A.1 and A.2 below. For the sake of completeness, a short proof of this convergence is also given in Appendix B of [18].

3.2 Euler Scheme

We now consider, for given $N \in \mathbb{N}^*$, the Euler scheme for (1.1) at the discretization times:

$$X_0^N = x, X_{t_{i+1}}^N = X_{t_i}^N + b\big(X_{t_i}^N\big)h + f\big(X_{t_i}^N\big)(Z_{t_{i+1}} - Z_{t_i}),$$

recalling that $h = T/N$.

3.2.1 *Proof of Proposition 2.1 for the Euler Scheme: Existence of the Density*

For each $N \in \mathbb{N}^*$, $(X_{t_i}^N)_{i \in [\![0,N]\!]}$ is a Markov chain. Given the past $\{X_{t_l}^N = x_l,\ l \in [\![0,i]\!]\}$, the conditional distribution of the innovations $b(X_{t_i}^N)h + f(X_{t_i}^N)(Z_{t_{i+1}} - Z_{t_i})$ has conditional density $\widetilde{p}^{x_i}(h,0,\cdot)$ (with the notation of (3.1), (3.2)). This proves the existence of the density for the discretization scheme.

3.2.2 *Parametrix Expansion for the Euler Scheme*

To give for the Euler scheme an expansion similar to (3.4), that will also be the starting point for our error expansion, we need to define, for fixed j,k, $0 \le j < k \le N$, and $x,y \in \mathbb{R}^d$, additional "frozen" Markov chains $(\widetilde{X}_{t_l}^N)_{l \in [\![j,k]\!]} = (\widetilde{X}_{t_l,x,y}^N)_{l \in [\![j,k]\!]}$. Their dynamics is described by

$$\widetilde{X}_{t_j}^N = x, \widetilde{X}_{t_{i+1}}^N = \widetilde{X}_{t_i}^N + b(y)h + f(y)(Z_{t_{i+1}} - Z_{t_i}), \quad i \in [\![j,k-1]\!].$$

Given the past $\{\widetilde{X}_{t_l}^N = x_l, l \in [\![j,i]\!]\}$, the conditional distribution of the innovations $b(y)h + f(y)(Z_{t_{i+1}} - Z_{t_i})$ has conditional density $\widetilde{p}^y(h,0,\cdot)$ and, hence, does not depend on the past. Note that for the grid points $(t_i)_{i \in [\![0,N]\!]}$, the transition densities of the solution $\widetilde{X}_{s,x,y}$ of (3.1) coincide with the transition densities of the chain $\widetilde{X}_{t_j,x,y}^N$ for $N \in \mathbb{N}^*$, $x,y \in \mathbb{R}^d$, and $s = t_j$.

For all $0 \le j < k \le N$, $(x,y) \in (\mathbb{R}^d)^2$, we denote by $p^N(t_k - t_j, x, y)$ and $\widetilde{p}^N(t_k - t_j, x, y)$ the transition probability densities between times t_j and t_k from point x to y of the chains X^N and $\widetilde{X}^N$, respectively. In particular,

$$\widetilde{p}^N(t_k - t_j, x, y) = \widetilde{p}^y(t_k - t_j, x, y) = \widetilde{p}(t_k - t_j, x, y). \tag{3.5}$$

Before stating the parametrix expansion of p^N in terms of $\widetilde{p}^N$, we need to introduce a kernel H_N that is the "discrete" analogue of H defined in (2.6):

$$H_N(t_k - t_j, x, y) = \big\{L_N - \widetilde{L}_N^y\big\}\widetilde{p}^N(t_k - t_j, x, y) \tag{3.6}$$

with

$$L_N\varphi(t_k - t_j, x, y) = h^{-1}\bigg\{\int p^N(h,x,z)\varphi(t_k - t_{j+1}, z, y)\,dz - \varphi(t_k - t_{j+1}, x, y)\bigg\},$$

$$\widetilde{L}_N^y\varphi(t_k - t_j, x, y) = h^{-1}\bigg\{\int \widetilde{p}^y(h,x,z)\varphi(t_k - t_{j+1}, z, y)\,dz - \varphi(t_k - t_{j+1}, x, y)\bigg\}.$$

Note that the previous definitions yield $p^N(h,x,z) = \widetilde{p}^x(h,x,z)$. We also mention that, because of the discretization, there is a slight "shift" in time in the definition of H_N. Namely we have $t_k - t_{j+1}$ instead of the somehow expected $t_k - t_j$.

Lemma 3.1 *For* $0 \le j < k \le N$, *the following formula holds*:

$$p^N(t_k - t_j, x, y) = \sum_{r=0}^{k-j} \big(\widetilde{p}^N \otimes_N H_N^{(r)}\big)(t_k - t_j, x, y), \tag{3.7}$$

where in the calculation of $\widetilde{p}^N \otimes_N H_N^{(r)}$ *(*r*-fold convolution) we define*

$$p^N(0,x,y) = \widetilde{p}^N(0,x,y) = \delta(x-y).$$

The proof of this lemma is given in [16], Lemma 3.6, and does not rely on the specific distribution of the innovations.

Remark 3.3 With the convention that $H_N^{(r)} = 0$ for $r > k - j$, (3.7) also writes $p^N(t_k - t_j, x, y) = \sum_{r=0}^{\infty} (\widetilde{p}^N \otimes_N H_N^{(r)})(t_k - t_j, x, y)$. This expression will often be used in the sequel.

4 Proof of the Main Results

In this section, we state in Sect. 4.1 the various points needed to prove Theorem 2.1. The proofs are postponed to Sect. 4.2. As mentioned earlier, the key idea consists in comparing the parametrix expansions of the densities p and p^N, respectively, given by (3.4) and (3.7). *In the whole section we suppose that the assumptions of Theorem 2.1 hold.*

4.1 Proof of Theorem 2.1

For the previously mentioned comparison to be possible, we first need to estimate a difference between the transition density $p(T,x,y)$ and $p^d(T,x,y) := \sum_{r\ge 0} \widetilde{p} \otimes_N H^{(r)}(T,x,y)$, which is an analogue of (3.4) up to the discrete-time convolution (i.e., $\otimes$ replaced by $\otimes_N$). We refer to (2.2), (2.4), (2.5), and (2.6) for the definition of operators and kernels.

Lemma 4.1 (Time discretization) *One has*:

$$\begin{aligned}\big(p - p^d\big)(T,x,y) = {} & \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} \big(p \otimes_N \big(\varPhi - \widetilde{\varPhi}^*\big)^{l+1} p^d\big)(T,x,y) \\ & + h^M R_{M,1}(T,x,y)\end{aligned}$$

with

$$\sum_{l=1}^{M-1}\left|\left(p\otimes_N\left(\Phi-\widetilde{\Phi}^*\right)^{l+1}p^d\right)(T,x,y)\right|+\left|R_{M,1}(T,x,y)\right|\le\rho_{\alpha,M}(T,y-x).$$

Then the comparison between p^d and p^N is controlled with the following:

Lemma 4.2 (Comparison of the discrete convolutions) *The following expansion holds*:

$$\begin{aligned}\left(p^d-p^N\right)(T,x,y)=&-\sum_{k=1}^{M-1}\frac{h^k}{(k+1)!}\left[p^d\otimes_N\left(\widetilde{\Phi}_*-\widetilde{\Phi}^*\right)^{k+1}p^N\right](T,x,y)\\&+h^MR_{M,2}(T,x,y),\end{aligned}$$

where

$$R_{M,2}(T,x,y)=-\frac{1}{M!}\int_0^1(1-\tau)^M\left[p^d\otimes_N\left(\widetilde{\Phi}_*-\widetilde{\Phi}^*\right)^{M+1}\widetilde{p}_\tau^\Delta\right](T,x,y)\,d\tau,$$

$$\forall t\in\{(t_i)_{i\in[\![1,N]\!]}\},\ \widetilde{p}_\tau^\Delta(t_i,x,y)=\sum_{r=0}^{\infty}\widetilde{p}_\tau\otimes_N H_N^{(r)}(t_i,x,y),\ \ \widetilde{p}_0^\Delta=p^N,\ \textit{and}$$

$$\forall\tau\in[0,1],\ \widetilde{p}_\tau(t,x,y)=\int_{\mathbb{R}^d}\widetilde{p}^x(\tau h,x,z)\widetilde{p}^y(t-\tau h,z,y)\,dz.$$

In particular $\widetilde{p}_0(t,x,y)=\widetilde{p}^y(t,x,y)$. *Also*,

$$\sum_{k=1}^{M-1}\left|\left(p^d\otimes_N\left(\widetilde{\Phi}_*-\widetilde{\Phi}^*\right)^{k+1}p^N\right)(T,x,y)\right|+\left|R_{M,2}(T,x,y)\right|\le\rho_{\alpha,M}(T,y-x).$$

Theorem 2.1 is then a direct consequence of Lemmas 4.1 and 4.2.

4.2 Proofs of the Technical Lemmas

Proof of Lemma 4.1 We start from the recurrence relation for $r\in\mathbb{N}^*$:

$$\begin{aligned}\widetilde{p}\otimes H^{(r)}-\widetilde{p}\otimes_N H^{(r)}=&\left[\left(\widetilde{p}\otimes H^{(r-1)}\right)\otimes H-\left(\widetilde{p}\otimes H^{(r-1)}\right)\otimes_N H\right]\\&+\left[\left(\widetilde{p}\otimes H^{(r-1)}\right)-\left(\widetilde{p}\otimes_N H^{(r-1)}\right)\right]\otimes_N H.\end{aligned}$$

Summing up these terms over $r\in\mathbb{N}^*$ and using the linearity of $\otimes$ and $\otimes_N$, we get $p-p^d=p\otimes H-p\otimes_N H+(p-p^d)\otimes_N H$. An iterative application of this identity yields

$$\left(p-p^d\right)(T,x,y)=\sum_{r=0}^{\infty}[p\otimes H-p\otimes_N H]\otimes_N H^{(r)}(T,x,y).\tag{4.1}$$

By definition, for all $k \in [\![1, N]\!]$,

$$[p \otimes H - p \otimes_N H](t_k, x, y) = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} ds \int \big[p(s, x, z)H(t_k - s, z, y) \\ - p(t_j, x, z)H(t_k - t_j, z, y)\big]\, dz. \tag{4.2}$$

A Taylor expansion of the function $\theta(s, z) := p(s, x, z)H(t_k - s, z, y)$ in the interval $[t_j, s] \subseteq [t_j, t_{j+1}]$ gives

$$\int \big[\theta(s, z) - \theta(t_j, z)\big]\, dz \\ = \sum_{l=1}^{M-1} \frac{(s - t_j)^l}{l!} \int \partial_\tau^l \theta(\tau, z)|_{\tau=t_j}\, dz \\ + \frac{(s - t_j)^M}{(M-1)!} \int_0^1 (1 - \delta)^{M-1} \int \partial_\tau^M \theta(\tau, z)|_{\tau=\tau_j(s,\delta)}\, dz\, d\delta, \tag{4.3}$$

where $\tau_j(s, \delta) = t_j + \delta(s - t_j)$. Note now that $-\partial_s p(t - s, x, z) = \Phi p(t - s, x, z)$, $\partial_t p(t - s, x, z) = {}^t\Phi p(t - s, x, z)$. Here ${}^t\Phi = {}^t\widetilde{\Phi}_z$ is the adjoint operator of Φ, where the derivatives have to be taken w.r.t. z. Hence, $\Phi p(t - s, x, z) = {}^t\Phi p(t - s, x, z)$. The same identity also holds for $\widetilde{p}$ with $\Phi, {}^t\Phi$ respectively replaced by $\widetilde{\Phi}^*, {}^t\widetilde{\Phi}^*$. We therefore derive

$$\int \partial_\tau \theta(\tau, z)|_{\tau=t_j}\, dz = \int \partial_\tau \big[p(\tau, x, z)\big]\big|_{\tau=t_j} H(t_k - t_j, z, y)\, dz \\ + \int p(t_j, x, z)\partial_\tau \big[H(t_k - \tau, z, y)\big]\big|_{\tau=t_j}\, dz \\ = \int {}^t\Phi_z p(t_j, x, z)\big(\Phi - \widetilde{\Phi}^*\big)\widetilde{p}(t_k - t_j, z, y)\, dz \\ - \int p(t_j, x, z)\big(\Phi - \widetilde{\Phi}^*\big)\widetilde{\Phi}^* \widetilde{p}(t_k - t_j, z, y)\, dz \\ = \int p(t_j, x, z)\big(\Phi - \widetilde{\Phi}^*\big)^2 \widetilde{p}(t_k - t_j, z, y)\, dz.$$

Iterating the differentiation, we get

$$\int \partial_\tau^l \theta(\tau, z)|_{\tau=t_j}\, dz = \int p(t_j, x, z)\big(\Phi - \widetilde{\Phi}^*\big)^{l+1} \widetilde{p}(t_k - t_j, z, y)\, dz, \tag{4.4}$$

where we recall that, for two operators A and B, we denote by $(A - B)^k$ the sum $(A - B)^k = \sum_{j=0}^k C_k^j A^{k-j}(-B)^j$.

Plugging (4.3) and (4.4) into (4.2), we get

$$[p \otimes H - p \otimes_N H](t_k, x, y) = \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} p \otimes_N \big(\Phi - \widetilde{\Phi}^*\big)^{l+1} \widetilde{p}(t_k, x, y) + h^M \widetilde{R}_{M,1}(t_k, x, y), \tag{4.5}$$

where

$$\widetilde{R}_{M,1}(t_k, x, y) = \frac{1}{(M-1)!} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \big[h^{-1}(s - t_j)\big]^M \int_0^1 (1-\delta)^{M-1} \times \int \partial_\tau^M \big[p(\tau, x, z) H(t_k - \tau, z, y)\big]\big|_{\tau = \tau_j(s,\delta)} \, ds\, dz\, d\delta. \tag{4.6}$$

Plugging (4.5) and (4.6) into (4.1), we get

$$\big(p - p^d\big)(T, x, y) = \sum_{l=1}^{M-1} \frac{h^l}{(l+1)!} \times \sum_{r=0}^{\infty} p \otimes_N \big(\Phi - \widetilde{\Phi}^*\big)^{l+1} \widetilde{p} \otimes_N H^{(r)}(T, x, y) + h^M R_{M,1}(T, x, y) \tag{4.7}$$

with $R_{M,1}(T, x, y) = \sum_{r=0}^{\infty} (\widetilde{R}_{M,1} \otimes_N H^{(r)})(T, x, y)$.

Now we apply that, for a linear operator S and its adjoint ${}^tS$, we have $p \otimes_N S\widetilde{p} = {}^tSp \otimes_N \widetilde{p}$. This gives

$$\begin{aligned} &\sum_{r=0}^{\infty} p \otimes_N \big(\Phi - \widetilde{\Phi}^*\big)^{l+1} \widetilde{p} \otimes_N H^{(r)}(T, x, y) \\ &\quad = {}^t\big[\big(\Phi - \widetilde{\Phi}^*\big)^{l+1}\big] p \otimes_N \sum_{r=0}^{\infty} \big(\widetilde{p} \otimes_N H^{(r)}\big)(T, x, y) \\ &\quad = p \otimes_N \big(\Phi - \widetilde{\Phi}^*\big)^{l+1} p^d(T, x, y), \end{aligned}$$

which plugged into (4.7) gives the desired expansion. The stated bound follows by application of the estimates given in Lemma 4.3 below. We only give the proof for the first summand; the other terms of the sum over l and the remainder $R_{M,1}(T, x, y)$ can be handled in a similar way. Write

$$\begin{aligned} p \otimes_N \big(\Phi - \widetilde{\Phi}^*\big)^2 p^d(T, x, y) &= \sum_{j=0}^{N-1} h \int p(t_j, x, z) \big(\Phi - \widetilde{\Phi}^*\big)^2 p^d(T - t_j, z, y)\, dz \\ &:= S_1 + S_2, \end{aligned}$$

where in S_1 (resp. S_2) the sum is taken over $I_1 := \{j \in [\![0, \lfloor \frac{N-1}{2} \rfloor]\!]\}$ (resp. $I_2 := \{j \in [\![\lfloor \frac{N-1}{2} \rfloor + 1, N-1]\!]\}$). For S_1 (resp. S_2), $p^d(T - t_j, z, y)$ (resp. $p(t_j, x, z)$) is nonsingular. From (4.12) in Lemma 4.3 below, there exists $C := C(T)$ such that, for all $(x,y,z)\in(\mathbb{R}^d)^3$,

$$\begin{aligned}&p(s,x,z)\le C\widetilde{p}^y(s,x,z),\qquad p^d(s,z,y)\le C\widetilde{p}^y(s,z,y),\quad \forall s\in]0,T],\\&\bigl|\bigl(\Phi-\widetilde{\Phi}^*\bigr)^2p^d(T-t_j,z,y)\bigr|\le C\widetilde{p}^y(T-t_j,z,y),\quad \forall j\in I_1,\\&\bigl|{}^t\bigl[\bigl(\Phi-\widetilde{\Phi}^*\bigr)^2\bigr]p(t_j,x,z)\bigr|\le C\widetilde{p}^y(t_j,x,z),\quad \forall j\in I_2.\end{aligned}\tag{4.8}$$

The semigroup property for $\tilde{p}^y$ yields $|S_1|+|S_2|\le C\widetilde{p}(T,x,y)$. One eventually checks from Proposition B.1 of [18] that $\widetilde{p}(T,x,y):=\widetilde{p}^y(T,x,y)\le \rho_{\alpha,M}(T,y-x)$. □

Proof of Lemma 4.2 Let us denote by $\mathcal{F}[\psi](z)=\int\exp(i\langle z,p\rangle)\psi(p)\,dp$ the Fourier transform of a function ψ. Introduce now for all $u,t,\ u<t,\ u,t\in\{(t_i)_{i\in[\![0,N]\!]}\}$, $p\in\mathbb{R}^d$,

$$\begin{aligned}\psi(p)&=h\bigl(L_N-\widetilde{L}_N^y\bigr)\widetilde{p}^y(t-u,x,p)\\&=\int p^N(h,x,w)\widetilde{p}^y\bigl(t-(u+h),w,p\bigr)\,dw-\widetilde{p}^y(t-u,x,p).\end{aligned}$$

Note that in particular according to (3.6), $\psi(y)=hH_N(t-u,x,y)$. Taking the characteristic functions of the densities involved in the above equation, we obtain from (3.2) and (3.6) that

$$\mathcal{F}[\psi](z):=G_z(1)-G_z(0)$$

with

$$\begin{aligned}G_z(\tau)=\exp\biggl[&i\langle x,z\rangle+i(t-u)\bigl\langle B(y),z\bigr\rangle+i\tau h\bigl\langle\Delta B^{x,y},z\bigr\rangle\\&-\int_{S^{d-1}}\bigl|\langle z,s\rangle\bigr|^\alpha\bigl[(t-u)\lambda_{f(y)}(ds)+\tau h\Delta\lambda^{x,y}(ds)\bigr]\biggr],\end{aligned}$$

where $\Delta B^{x,y}=B(x)-B(y)$, $\Delta\lambda^{x,y}(ds)=\lambda_{f(x)}(ds)-\lambda_{f(y)}(ds)$. Note in particular that for all $\tau\in[0,1]$,

$$G_z(\tau)=G_z(0)\times\exp\biggl(\tau h\biggl[i\bigl\langle\Delta B^{x,y},z\bigr\rangle-\int_{S^{d-1}}\bigl|\langle z,s\rangle\bigr|^\alpha\Delta\lambda^{x,y}(ds)\biggr]\biggr).\tag{4.9}$$

A Taylor expansion yields $\mathcal{F}[\psi](z)=\sum_{k=1}^M\frac{1}{k!}G_z^{(k)}(0)+\frac{1}{M!}\int_0^1(1-\tau)^MG_z^{(M+1)}(\tau)\cdot d\tau$. From (4.9) one derives that for $k\in\mathbb{N}^*$,

$$\frac{1}{k!}G_z^{(k)}(0)=\frac{h^k}{k!}G_z(0)\biggl[i\bigl\langle\Delta B^{x,y},z\bigr\rangle-\int_{S^{d-1}}\bigl|\langle z,s\rangle\bigr|^\alpha\Delta\lambda^{x,y}(ds)\biggr]^k.$$

Observe now that $G_z(0)=\mathcal{F}[\theta](z)$, $\theta(p):=\widetilde{p}^y(t-u,x,p)$. Using the well-known properties of the Fourier transform, one gets, for all $k\in[\![1,M]\!]$,

$$G_z^{(k)}(0)=h^k\mathcal{F}\bigl[\bigl(\widetilde{\Phi}_\xi-\widetilde{\Phi}_y\bigr)^k\theta\bigr]\big|_{\xi=x}(z),$$

where the operators $\widetilde{\Phi}$ are applied w.r.t. the x component, and the Fourier transform is applied w.r.t. the p component of $\widetilde{p}^y(t-u,x,p)$. Also, in the above writing, we compute the Fourier transform for an arbitrary fixed $\xi \in \mathbb{R}^d$, and we then put $\xi = x$.

Hence,

$$\begin{aligned}\mathcal{F}[\psi](z) &= \sum_{k=1}^{M} \frac{1}{k!} G_z^{(k)}(0) + \frac{1}{M!}\int_0^1 (1-\tau)^M G_z^{(M+1)}(\tau)\, d\tau \\ &= \sum_{k=1}^{M} \frac{h^k}{k!} \mathcal{F}\big[\big(\widetilde{\Phi}_\xi - \widetilde{\Phi}_y\big)^k \theta\big]\big|_{\xi=x}(z) \\ &\quad + \frac{h^{M+1}}{M!} \mathcal{F}\bigg[\int_0^1 (1-\tau)^M \big[\big(\widetilde{\Phi}_\xi - \widetilde{\Phi}_y\big)^{M+1}\theta_\tau\big]\big|_{\xi=x}\, d\tau\bigg](z),\end{aligned}$$

where $\forall \tau \in [0,1]$, $\theta_\tau(p) := \int_{\mathbb{R}^d} \widetilde{p}^x(\tau h, x, z)\widetilde{p}^y(t-u-\tau h, z, p)\, dz$. Taking the inverse Fourier transform, putting $p = y$ in the above equation, and observing that $H(t-u,x,y) = (\widetilde{\Phi}_* - \widetilde{\Phi}^*)\widetilde{p}^y(t-u,x,y)$, we obtain

$$\begin{aligned}(H_N - H)(t-u,x,y) &= \sum_{k=1}^{M-1} \frac{h^k}{(k+1)!}\big(\widetilde{\Phi}_* - \widetilde{\Phi}^*\big)^{k+1}\widetilde{p}(t-u,x,y) \\ &\quad + \frac{h^M}{M!}\int_0^1 (1-\tau)^M \big(\widetilde{\Phi}_* - \widetilde{\Phi}^*\big)^{M+1}\widetilde{p}_\tau(t-u,x,y)\, d\tau.\end{aligned} \tag{4.10}$$

Recall now that $(p^d - p^N)(T,x,y) = \sum_{r=0}^{\infty}[(\widetilde{p} \otimes_N H^{(r)}) - (\widetilde{p} \otimes_N H_N^{(r)})](T,x,y)$, where we put $(\widetilde{p} \otimes_N H_N^{(r)})(T,x,y) = 0$ for $hr > T$. Summing over $r \in \mathbb{N}$ in the identity

$$\begin{aligned}&\big(\widetilde{p} \otimes_N H^{(r)} - \widetilde{p} \otimes_N H_N^{(r)}\big)(T,x,y) \\ &\quad = \big(\big(\widetilde{p} \otimes_N H^{(r-1)}\big) \otimes_N (H - H_N)\big)(T,x,y) \\ &\qquad + \big(\big(\widetilde{p} \otimes_N H^{(r-1)} - \widetilde{p} \otimes_N H_N^{(r-1)}\big) \otimes_N H_N\big)(T,x,y),\end{aligned}$$

one gets

$$\big(p^d - p^N\big)(T,x,y) = \big[p^d \otimes_N (H - H_N) + \big(p^d - p^N\big) \otimes_N H_N\big](T,x,y).$$

By iterative application of the last identity we obtain

$$\big(p^d - p^N\big)(T,x,y) = \sum_{r=0}^{\infty}\big[p^d \otimes_N (H - H_N)\big] \otimes_N H_N^{(r)}(T,x,y).$$

We get from (4.10) that for all $t \in \{(t_i)_{i \in [\![1,N]\!]}\}$,

$$
\begin{aligned}
&\big(p^d \otimes_N (H - H_N)\big)(T,x,y) \\
&\quad = -\sum_{k=1}^{M-1} \frac{h^k}{(k+1)!}\big[p^d \otimes_N \big(\widetilde{\Phi}_* - \widetilde{\Phi}^*\big)^{k+1} \widetilde{p}\big](T,x,y) \\
&\qquad - \frac{h^M}{M!} \int_0^1 (1-\tau)^M \big[p^d \otimes_N \big(\widetilde{\Phi}_* - \widetilde{\Phi}^*\big)^{M+1} \widetilde{p}_\tau\big](T,x,y)\, d\tau.
\end{aligned}
$$

Eventually,

$$
\begin{aligned}
\big(p^d - p^N\big)(T,x,y) = &-\sum_{k=1}^{M-1} \frac{h^k}{(k+1)!}\big[p^d \otimes_N \big(\widetilde{\Phi}_* - \widetilde{\Phi}^*\big)^{k+1} p^N\big](T,x,y) \\
&+ h^M R_{M,2}(T,x,y),
\end{aligned}
$$

$$
R_{M,2}(T,x,y) = -\frac{1}{M!}\int_0^1 (1-\tau)^M \big[p^d \otimes_N \big(\widetilde{\Phi}_* - \widetilde{\Phi}^*\big)^{M+1} \widetilde{p}_\tau^\Delta\big](T,x,y)\, d\tau
$$

$$
\forall t \in \big\{(t_i)_{i\in[\![1,N]\!]}\big\},\ \widetilde{p}_\tau^\Delta(t,x,y) = \sum_{r=0}^{\infty} \widetilde{p}_\tau \otimes_N H_N^{(r)}(t,x,y),\ \widetilde{p}_0^\Delta = p^N.
$$

This proves the expansion part of the lemma. The bound follows as in the previous proof from Lemma 4.3. □

We now state Lemma 4.3 that allows us to control the rests appearing in the expansions of Lemmas 4.1 and 4.2. Its proof is postponed to Appendix.

Lemma 4.3 *Let $q > d+4$. For all multiindices a, b s.t. $|a|+|b| < q-(d+4)$, the following inequalities hold*:

$$
\begin{aligned}
&\big|D_y^a D_x^b p^d(t_k,x,y)\big| + \big|D_y^a D_x^b p^N(t_k,x,y)\big| \le C t_k^{-\frac{|a|+|b|}{\alpha}} \widetilde{p}(t_k,x,y), \\
&k \in [\![1,n]\!],
\end{aligned} \tag{4.11}
$$

$$
\big|D_y^a D_x^b p(t,x,y)\big| \le C t^{-\frac{|a|+|b|}{\alpha}} \widetilde{p}(t,x,y), \quad 0 < t \le T.
$$

Also, $\exists C := C(T)$ *s.t. for all* $(x,y,z) \in (\mathbb{R}^d)^3, s \in]0,T]$,

$$
\begin{aligned}
&p(s,x,z) \le C \widetilde{p}^y(s,x,z), \qquad p^d(s,z,y) \le C \widetilde{p}^y(s,z,y), \\
&\big|\big(\Phi - \widetilde{\Phi}^*\big)^k p^d(s,z,y)\big| \le C s^{-\frac{|k|}{\alpha}} \widetilde{p}^y(s,z,y), \\
&\big|{}^t\big[\big(\Phi - \widetilde{\Phi}^*\big)^k\big] p(s,x,z)\big| \le C s^{-\frac{|k|}{\alpha}} \widetilde{p}^y(s,x,z).
\end{aligned} \tag{4.12}
$$

5 Extensions and Conclusion

A careful examination of the proofs in the Appendices of [18] shows that the absolute continuity of λ w.r.t. to the Lebesgue measure of S^{d-1} can be removed in (A-1), provided that the function

$$\zeta(t,x,y) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\int_{S^{d-1}}\left|\langle p,s\rangle\right|^{\alpha}\lambda_{f(x)}(ds)\right)$$
$$\times \exp\left(-t\int_{S^{d-1}}\left|\langle p,s\rangle\right|^{\alpha}\lambda_{f(y)}(ds)\right)\exp\bigl(-i\langle p,x\rangle\bigr)\,dp$$

has bounded derivatives w.r.t. x up to order q (see Appendix B in [18] and Theorem 3.1 in [15]). Also up to a standard perturbative argument, similar controls on the density can be obtained when we consider (1.1) driven by $(Z_s + P_s)_{s\ge 0}$, where $(P_s)_{s\ge 0}$ is a compound Poisson process with Lévy measure $\nu_P(dz) = f(z)\,dz$, and $|f(z)| \le \frac{C}{1+|z|^{d+\beta}}$, $\beta > 0$, see Theorem 4.1 in [15]. In that case our main results remain valid up to a modification of the remainder. Indeed, it is the smallest exponent (or equivalently the largest tail) that leads the asymptotic behavior of $p(t,x,y)$ when $|x-y|$ is large. Thus $\rho_{\alpha,M}(T,y-x)$ has to be replaced by $\rho_{\min(\alpha,\beta),M}(T,y-x)$ in Theorem 2.1. Eventually, good controls have been obtained on p for stable-like processes, i.e., when the stability index in the generator $\Phi\psi(x)$ in (2.2) can depend on the spatial position x, i.e., α turns to $\alpha(x)\in[\underline{\alpha},\overline{\alpha}]$ strictly included in $(0,2]$ (see Sect. 5 in [15]). Anyhow, the processes associated to those generators cannot be approximated by a usual Euler scheme, and the previous analysis breaks down. The approximation of such processes will concern further research.

Appendix: Proof of the Controls on the Derivatives of the Densities (Lemma 4.3)

To conclude the proof, it remains to prove Lemma 4.3. The first step is to get bounds on partial derivatives of the transition densities $\widetilde{p}$ and p. The following estimates generalize the ones obtained in [15], Propositions 2.1–2.3.

Lemma A.1 *Let $q > d+4$. There exists a constant $C > 1$ such that the following estimates hold uniformly for α in any compact subset of the interval* $(0,2)$ *and for all* $0 < t \le T, x, y, z \in \mathbb{R}^d$ *and* $|a| < q-(d+4)$:

$$\left|D_z^a \widetilde{p}^{\,y}(t,x,z)\right| \le \frac{C}{t^{|a|/\alpha}}\widetilde{p}^{\,y}(t,x,z), \tag{A.1}$$

$$\left|D_z^a \widetilde{p}^{\,y}(t,x,z)\right| \le \frac{C}{|z-B(y)t-x|^{|a|}}\widetilde{p}^{\,y}(t,x,z). \tag{A.2}$$

Remark A.1 Equation (A.1) extends to the stable case what is widely known in the Gaussian framework. Namely, each derivation of the density in space remains homogeneous to a stable density up to a multiplicative additional singularity of order $t^{-1/\alpha}$.

Proof From now on we assume w.l.o.g. that $d \ge 3$; the cases $d \in \{1,2\}$ can be addressed more directly. To proceed with the computations, we need to specify a useful change of coordinates. Namely, for a given direction $\zeta \in \mathbb{R}^d\backslash\{0\}$, introduce for $p \in \mathbb{R}^d$ the spherical coordinates $(\rho,\vartheta,\varphi_2,\dots,\varphi_{d-1})$, $\rho = |p|$, with first coordinate or main axis directed along ζ, that is,

$$\begin{aligned}
&p_1=\rho\cos\vartheta,\quad p_2=\rho\sin\vartheta\cos\varphi_2,\quad p_3=\rho\sin\vartheta\sin\varphi_2\cos\varphi_3,\quad \dots,\\
&\quad p_{d-1}=\rho\sin\vartheta\sin\varphi_2\cdots\sin\varphi_{d-2}\cos\varphi_{d-1},\\
&\quad p_d=\rho\sin\vartheta\sin\varphi_2\cdots\sin\varphi_{d-2}\sin\varphi_{d-1},
\end{aligned}\tag{A.3}$$

$\vartheta\in[0,\pi]$, $\varphi_i\in[0,\pi]$, $i\in[\![2,d-2]\!]$, $\varphi_{d-1}\in[0,2\pi]$. Consider then the coordinates (v,τ,ϕ) where $\tau=\cos\vartheta$ and $v=\rho|\zeta|$, with $v\in\mathbb{R}^+$, $\tau\in[-1,1]$, $\phi=(\varphi_2,\dots,\varphi_{d-1})\in[0,\pi]^{d-3}\times[0,2\pi]$. In the following we write $p=p(v,\tau,\phi)$ for the previous r.h.s. in (A.3) written in these new coordinates, that is,

$$\begin{aligned}
&p_1=|\zeta|^{-1}v\tau,\quad p_2=|\zeta|^{-1}v\big(1-\tau^2\big)^{1/2}\cos\varphi_2,\\
&\quad p_3=|\zeta|^{-1}v\big(1-\tau^2\big)^{1/2}\sin\varphi_2\cos\varphi_3,\quad \dots,\\
&\quad p_{d-1}=|\zeta|^{-1}v\big(1-\tau^2\big)^{1/2}\sin\varphi_2\cdots\sin\varphi_{d-2}\cos\varphi_{d-1},\\
&\quad p_d=|\zeta|^{-1}v\big(1-\tau^2\big)^{1/2}\sin\varphi_2\cdots\sin\varphi_{d-2}\sin\varphi_{d-1},
\end{aligned}\tag{A.4}$$

and $\bar p(\tau,\phi)=p(|\zeta|,\tau,\phi)$.

Without loss of generality we suppose $B(y)=0$. The first step consists in differentiating w.r.t. z the inverse Fourier transform for $\widetilde{p}^y(t,x,z)$:

$$\widetilde{p}^y(t,x,z)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\bigg\{-t\int_{S^{d-1}}\big|\langle p,s\rangle\big|^\alpha\lambda_{f(y)}(ds)\bigg\}\exp\big(-i\langle p,z-x\rangle\big)\,dp.\tag{A.5}$$

For $z=x$, (2.3) and standard computations directly give estimate (A.1). Thus, in the following we also assume that $z\neq x$ and use the previous spherical coordinates (v,τ,ϕ) derived from (A.3) setting $\zeta=z-x$ as the main axis. We obtain

$$\begin{aligned}
&D_z^a\widetilde{p}^y(t,x,z)\\
&\quad=\frac{1}{(2\pi)^d|z-x|^{|a|+d}}\int_0^\infty dv\,v^{|a|+d-1}\\
&\qquad\times\int_{-1}^1 d\tau\int_{[0,\pi]^{d-3}\times[0,2\pi]}d\phi\,\Psi\big(v,\tau,|a|\big)\\
&\qquad\times\exp\bigg\{-t\frac{v^\alpha}{|z-x|^\alpha}\int_{S^{d-1}}\big|\langle\overline{p},s\rangle\big|^\alpha\lambda_{f(y)}(ds)\bigg\}\\
&\qquad\times\tau^{a_1}\big(1-\tau^2\big)^{\frac{|a|-a_1+d-3}{2}}h(\phi,a),
\end{aligned}\tag{A.6}$$

where $\overline{p}=p/|p|$, $a=(a_1,\dots,a_d)\in\mathbb{N}^d$, and

$$\begin{aligned}
&\Psi\big(v,\tau,|a|\big)=(-1)^{|a|/2}\cos(v\tau)\mathbb{I}_{|a|\text{ even}}+(-1)^{(|a|+1)/2}\sin(v\tau)\mathbb{I}_{|a|\text{ odd}},\\
&\quad h(\phi,a)=\big\{(\cos\varphi_2)^{a_2}(\sin\varphi_2\cos\varphi_3)^{a_3}\times\cdots\times(\sin\varphi_2\cdots\sin\varphi_{d-2}\cos\varphi_{d-1})^{a_{d-1}}\\
&\qquad\qquad\times(\sin\varphi_2\cdots\sin\varphi_{d-2}\sin\varphi_{d-1})^{a_d}\big\}\times V(\phi),\\
&\quad V(\phi)=(\sin\varphi_2)^{d-3}(\sin\varphi_3)^{d-4}\times\cdots\times(\sin\varphi_{d-3})^2\sin\varphi_{d-2}.
\end{aligned}$$

We first consider the case $|z-x|/t^{1/\alpha} \le \overline{C}$ for a sufficiently small positive constant $\overline{C}$. In this case we expand the trigonometric function $\Psi(v,\tau,|a|)$ in (A.6) in power series and change the variable of integration $\frac{t^{1/\alpha}v}{|z-x|}$ to w in each term. This gives for all $k \in \mathbb{N}$,

$$D_z^a \widetilde{p}^y(t,x,z) = \frac{C_{|a|}}{t^{\frac{|a|+d}{\alpha}}} \left\{ \sum_{m=0}^{k} \frac{(-1)^m}{(2m+\mathbb{I}_{|a| \text{ odd}})!} e_m^{|a|} \left(\frac{|z-x|}{t^{1/\alpha}} \right)^{2m+\mathbb{I}_{|a| \text{ odd}}} + R_{k+1}^{|a|} \right\}, \quad C_{|a|} = \frac{(-1)^{(|a|+\mathbb{I}_{|a| \text{ odd}})/2}}{(2\pi)^d}, \tag{A.7}$$

where for all $m \in [\![1,k]\!]$,

$$e_m^{|a|} = \int_0^\infty dw \int_{-1}^1 d\tau \int_{[0,\pi]^{d-3}\times[0,2\pi]} d\phi \exp\left\{ -w^\alpha \int_{S^{d-1}} \left|\langle \overline{p}, s\rangle\right|^\alpha \lambda_{f(y)}(ds) \right\} \times w^{|a|+2m+d-\mathbb{I}_{|a| \text{ even}}} \times \tau^{a_1+2m+\mathbb{I}_{|a| \text{ odd}}} \left(1-\tau^2\right)^{\frac{|a|-a_1+d-3}{2}} h(\phi,a), \tag{A.8}$$

$$\left|R_{k+1}^{|a|}\right| \le \frac{|e_{k+1}^{|a|}|}{(2(k+1)+\mathbb{I}_{|a| \text{ odd}})!} \left(\frac{|z-x|}{t^{1/\alpha}} \right)^{2(k+1)+\mathbb{I}_{|a| \text{ odd}}}.$$

To simplify the notation, we omit the dependence of the coefficients of our expansions on the direction $\zeta = z - x$. From (A-1), (A-2), and (2.3) one then derives the following bound:

$$\left|e_m^{|a|}\right| \le \frac{A_{d-2}}{\alpha \underline{C}_1^{\frac{|a|+2m+d+\mathbb{I}_{|a| \text{ odd}}}{\alpha}}} \Gamma\left(\frac{|a|+2m+d+\mathbb{I}_{|a| \text{ odd}}}{\alpha} \right) \times B\left(m + \frac{a_1+1+\mathbb{I}_{|a| \text{ odd}}}{2}, \frac{|a|-a_1+d-1}{2} \right). \tag{A.9}$$

Here A_{d-2} denotes the area of the unit sphere S^{d-2}, and B is the β-function. Note that the modulus of each term in expansion (A.7) serves as an estimate of the remainder in a finite Taylor expansion. From (A.7) we have

$$D_z^a \widetilde{p}^y(t,x,z) = \frac{C_{|a|}}{t^{\frac{|a|+d}{\alpha}}} \left(e_0^{|a|} \left(\frac{|z-x|}{t^{1/\alpha}} \right)^{\mathbf{I}_{|a| \text{ odd}}} + R_1^{|a|} \right). \tag{A.10}$$

Recall that we are considering the case $\frac{|z-x|}{t^{1/\alpha}} \le \overline{C}$. By Proposition 3.1(i) from [15] for some $\widetilde{C}$ depending on $\overline{C}$, $\widetilde{C}^{-1} t^{-d/\alpha} \le \widetilde{p}^y(t,x,z) \le \widetilde{C} t^{-d/\alpha}$. Hence, (A.10), (A.9), and (A.8) yield

$$\left|D_z^a \widetilde{p}^y(t,x,z)\right| \le \frac{C}{t^{\frac{|a|}{\alpha}}} \widetilde{p}^y(t,x,z) \le \frac{C\overline{C}^{|a|}}{|z-x|^{|a|}} \widetilde{p}^y(t,x,z). \tag{A.11}$$

To estimate $D_z^a \widetilde{p}^y(t,x,z)$ for $|z-x|/t^{1/\alpha} \ge (\overline{C})^{-1}$, we proceed as in Proposition 2.3 of [15]. This gives the representation $D_z^a \widetilde{p}^y(t,x,z) = [D_z^a \widetilde{p}^y(t,x,z)]_1 + [D_z^a \widetilde{p}^y(t,x,z)]_2$ with

$$\begin{aligned}
&\big[D_z^a \widetilde{p}^y(t,x,z)\big]_j \\
&\quad = \frac{1}{(2\pi)^d}\int_0^\infty d\rho\, \rho^{|a|+d-1}\int_{-1}^1 d\tau\, \Psi\big(\rho|z-x|,\tau,|a|\big) \\
&\qquad \times f_j(\tau)\int_{[0,\pi]^{d-3}\times[0,2\pi]} d\phi \exp\big\{-t\rho^\alpha g_{\lambda_f}(\tau,\phi,y)\big\}h(\phi,a), \qquad j=1,2, \\
&g_{\lambda_f}(\tau,\phi,y) := \int_{S^{d-1}} \big|\big\langle \bar{p}(\tau,\phi),s\big\rangle\big|^\alpha \lambda_{f(y)}(ds),
\end{aligned} \tag{A.12}$$

using the notation introduced after (A.3). Here

$$f_1(\tau) = \tau^{a_1}\big(1-\tau^2\big)^{\frac{|a|-a_1+d-3}{2}}\chi(\tau), \qquad f_2(\tau) = \tau^{a_1}\big(1-\tau^2\big)^{\frac{|a|-a_1+d-3}{2}}\big(1-\chi(\tau)\big),$$

where $\chi(\tau)$ is a C^∞ even truncation function $\mathbb{R}\to[0,1]$ that equals 1 for $|\tau|\le 1-2\varepsilon$ and 0 for $|\tau|\ge 1-\varepsilon$, for some $\varepsilon\in(0,\frac{1}{2})$. Because of the symmetry in τ, it is easy to see that the integral in (A.12) is nonzero only if a_1 and $|a|$ are both even or odd. Expanding the exponential at order 2 in (A.12) and making the change of variables $\rho|z-x| = v$, we get

$$\big[D_z^a \widetilde{p}^y(t,x,z)\big]_1 = \frac{C_{|a|}}{|z-x|^{|a|+d}}\sum_{m=0}^2 \frac{1}{m!} b_m^{|a|}\bigg(\frac{t}{|z-x|^\alpha}\bigg)^m, \tag{A.13}$$

where $C_{|a|}$ is defined in (A.7), and for $m\in[\![0,1]\!]$,

$$\begin{aligned}
b_m^{|a|} &= (-1)^m \int_0^\infty F_m^{|a|}(v) v^{|a|+m\alpha+d-1}\, dv, \\
F_m^{|a|}(v) &= [\mathbb{I}_{|a|\ \mathrm{even}}\mathrm{Re} - \mathbb{I}_{|a|\ \mathrm{odd}}\mathrm{Im}]\bigg[\int_{-\infty}^\infty \exp(-iv\tau)\varphi_m(\tau)\, d\tau\bigg], \\
\varphi_m(\tau) &= f_1(\tau)\int_{[0,\pi]^{d-3}\times[0,2\pi]} g_{\lambda_f}^m(\tau,\phi,y) h(\phi,a)\, d\phi,
\end{aligned}$$

and

$$\begin{aligned}
b_2^{|a|} &= 2\int_0^1 (1-\delta)\int_0^\infty F_{2,\delta}^{|a|}(v) v^{|a|+2\alpha+d-1}\, dv\, d\delta, \\
F_{2,\delta}^{|a|}(v) &= [\mathbb{I}_{|a|\ \mathrm{even}}\mathrm{Re} - \mathbb{I}_{|a|\ \mathrm{odd}}\mathrm{Im}]\bigg[\int_{-\infty}^\infty \exp(-iv\tau)\varphi_{2,\delta}(\tau)\, d\tau\bigg],
\end{aligned}$$

$$\begin{aligned}
\varphi_{2,\delta}(\tau) = f_1(\tau)&\int_{[0,\pi]^{d-3}\times[0,2\pi]} g_{\lambda_f}^2(\tau,\phi,y)\exp\bigg\{-\delta t\bigg(\frac{v}{|z-x|}\bigg)^\alpha g_{\lambda_f}(\tau,\phi,y)\bigg\} \\
&\times h(\phi,a)\, d\phi.
\end{aligned}$$

To extend the integration to $\mathbb{R}$ in the definition of $(F_m^{|a|}(v))_{m\in[\![0,1]\!]}$, $F_{2,\delta}^{|a|}(v)$, we simply use that the functions $(\varphi_m)_{m\in[\![0,1]\!]}$, $\varphi_{2,\delta}$ have compact support in τ. However, to check that the coefficients $(b_m^{|a|})_{m\in[\![0,2]\!]}$ are well defined, we have to equilibrate at infinity the term in $(v^{|a|+m\alpha+d-1})_{m\in[\![0,2]\!]}$. This can be done computing iterated integration by parts in τ in the definition of $(F_m^{|a|}(v))_{m\in[\![0,1]\!]}$, $F_{2,\delta}^{|a|}(v)$. Namely, if $\varphi_m(\tau), m=0,1,$ and $\varphi_{2,\delta}(\tau)$ are C^q functions of τ with compact support and $q>|a|+4+d>|a|+2\alpha+d$, performing q integrations by parts w.r.t. τ, one derives that the coefficients $(b_m^{|a|})_{m\in[\![0,2]\!]}$ are well defined. Let us now check that assumption (A-1) implies that $\varphi_m(\tau), m=0,1,$ and $\varphi_{2,\delta}(\tau)$ are C^q functions of τ with compact support. Indeed, for the unit vectors $\overline{p}(\tau+\Delta\tau,\phi)$ and $\overline{p}(\tau,\phi)$, from elementary algebra there exists an orthogonal matrix $A:=A(\Delta\tau)$ such that $\overline{p}(\tau+\Delta\tau,\phi)=A\overline{p}(\tau,\phi)$. Hence, if $\lambda_{f(x)}(ds)=\Theta(x,s)\,ds$, where Θ has the previous smoothness, one can show that

$$\begin{aligned}
&\lim_{\Delta\tau\to 0}\frac{g_{\lambda_f}(\tau+\Delta\tau,\phi,x)-g_{\lambda_f}(\tau,\phi,x)}{\Delta\tau}\\
&\quad=\lim_{\Delta\tau\to 0}\frac{\int_{S^{d-1}}\{|\langle\overline{p}(\tau,\phi),A^*s\rangle|^\alpha-|\langle\overline{p}(\tau,\phi),s\rangle|^\alpha\}\lambda_{f(x)}(ds)}{\Delta\tau}\\
&\quad=\int_{S^{d-1}}\left|\langle\overline{p}(\tau,\phi),s\rangle\right|^\alpha\lim_{\Delta\tau\to 0}\frac{[\Theta(x,As)-\Theta(x,s)]}{\Delta\tau}\,ds\\
&\quad=\int_{S^{d-1}}\left|\langle\overline{p}(\tau,\phi),s\rangle\right|^\alpha\Theta_s'(x,s)\beta(\tau,\phi,s)\,ds,
\end{aligned}$$

where $\beta(\tau,\phi,s)$ is C^∞ function in τ uniformly bounded in (τ,ϕ,s) in our region. The process can then be iterated other $q-1$ times.

Thus all coefficients $(b_m^{|a|})_{m\in[\![0,2]\!]}$ are well defined.

Next, analogously to Proposition 2.3 in [15] (where the case $|a|=0$ was considered) and with the same rotations of the integration contours for $\alpha\in(0,1]$, $\alpha\in(1,2)$, we obtain for all $k\in\mathbb{N}$,

$$\begin{aligned}
\left[D_z^a\widetilde{p}^y(t,x,z)\right]_2&=\frac{C_{|a|}}{|z-x|^{|a|+d}}\left\{\sum_{m=0}^k\frac{1}{m!}c_m^{|a|}\left(\frac{t}{|z-x|^\alpha}\right)^m+R_{2,k+1}^{|a|}\right\},\\
c_m^{|a|}&=2[\mathbb{I}_{|a|\ \mathrm{even}}\mathrm{Re}-\mathbb{I}_{|a|\ \mathrm{odd}}\mathrm{Im}]\left[\int_{1-2\varepsilon}^1 d\tau\int_{[0,\pi]^{d-3}\times[0,2\pi]}d\phi\,h(\phi,a)\right.\\
&\quad\times\left(-g_{\lambda_f}(\tau,\phi)\right)^m\exp\left(-\frac{i\pi\alpha m}{2}\right)(-i)^{|a|+d}\tau^{-(\alpha m+d+|a|)}\\
&\quad\left.\times\Gamma\left(\alpha m+d+|a|\right)f_2(\tau)\right],
\end{aligned}\tag{A.14}$$

and $|R_{2,k+1}^{|a|}|\le\frac{|c_{k+1}^{|a|}|}{(k+1)!}(\frac{t}{|x-z|^\alpha})^{k+1}$. Note that the coefficients $c_m^{|a|}$ are also well defined because τ does not approach zero (recall that $1-\chi(\tau)\neq 0\Leftrightarrow|\tau|>1-2\varepsilon$). Precisely, $|c_m^{|a|}|\le 2A_{d-2}C_2^m(1-2\varepsilon)^{-\alpha m+d+|a|}\Gamma(\alpha m+d+|a|)$.

Now the sum of expansions (A.13) and (A.14) gives the expansion for $D_z^a\widetilde{p}^y(t,x,z)$. Note that by construction, the first coefficient $b_0^{|a|}+c_0^{|a|}$ does not depend on the spectral measure $\lambda_{f(y)}(\cdot)$, and it vanishes when the spectral measure is uniform (that is, $C_1=C_2=1$ in (2.3)). This can be shown by means of representations involving Bessel and Whittaker functions and the same rotations of the integration contours as in Proposition 2.2 of [15], see Appendix C in [18]. Thus, for all $k\in\mathbb{N}^*$, we get the representation

$$D_z^a\widetilde{p}^y(t,x,z)=\frac{C_{|a|}}{|z-x|^{|a|+d}}\left\{\sum_{m=1}^{k}\frac{1}{m!}d_m^{|a|}\left(\frac{t}{|z-x|^\alpha}\right)^m+R_{k+1}^{|a|}\right\},\tag{A.15}$$

where $d_m^{|a|}=b_m^{|a|}+c_m^{|a|}$ with $b_m=0$ for $m\geq 3$, and $|R_{k+1}^{|a|}|\leq\frac{|d_{k+1}^{|a|}|}{(k+1)!}(\frac{t}{|x-z|^\alpha})^{k+1}$. Now, by Proposition 3.1(ii) in [15], $d_1^0>0$. Equation (A.15) yields

$$D_z^a\widetilde{p}^y(t,x,z)=\frac{C_{|a|}d_1^0t}{|z-x|^{|a|+d+\alpha}}\left(\frac{d_1^{|a|}}{d_1^0}+\widetilde{R}_2^{|a|}\right),\quad \left|\widetilde{R}_2^{|a|}\right|\leq\frac{|d_2^{|a|}|}{2d_1^0}\frac{t}{|x-z|^\alpha},$$

$$\widetilde{p}^y(t,x,z)=\frac{C_0}{|z-x|^d}\left(\frac{d_1^0t}{|z-x|^\alpha}+R_2^0\right)\geq\frac{C_0d_1^0t}{2|z-x|^{d+\alpha}}$$

for sufficiently small $\overline{C}$. Hence, we have

$$\begin{aligned}\left|D_z^a\widetilde{p}^y(t,x,z)\right|&\leq\frac{CC_{|a|}}{|z-x|^{|a|}}\frac{d_1^0t}{|z-x|^{d+\alpha}}\leq\frac{C}{|z-x|^{|a|}}\widetilde{p}^y(t,x,z)\\&\leq\frac{C\overline{C}^{|a|}}{t^{|a|/\alpha}}\widetilde{p}^y(t,x,z),\end{aligned}\tag{A.16}$$

recalling that $\frac{t^{1/\alpha}}{|z-x|}\leq\overline{C}$ for the last inequality. Without loss of generality, we can assume that $\overline{C}<1$. It remains to consider the case $|x-z|/t^{1/\alpha}\in]\overline{C},\overline{C}^{-1}[:=I(\overline{C})$. It follows from (A.6) that $|z-x|^d\widetilde{p}^y(t,x,z)$ and $|z-x|^{d+|a|}D_z^a\widetilde{p}^y(t,x,z)$ are continuous functions of $|x-z|/t^{1/\alpha}$. Since the stable density is also strictly positive, we deduce that there exists $\widetilde{C}$ such that on $I(\overline{C})$, $|D_z^a\widetilde{p}^y(t,x,z)|\leq\frac{\widetilde{C}}{|z-x|^{|a|+d}}\leq\frac{C}{|z-x|^{|a|}}\widetilde{p}^y(t,x,z)\leq\frac{C\overline{C}^{-|a|}}{t^{|a|/\alpha}}\widetilde{p}^y(t,x,z)$, which concludes the proof. □

Lemma A.2 *Let $q>d+4$. There exists a constant $C>1$ such that the following estimates hold uniformly for α in any compact subset of the interval $(0,2)$ and for all $0<t\leq T,\ x,y,v\in\mathbb{R}^d$ and $|a|+|b|<q-(d+4)$:*

$$\left|D_y^aD_x^bH(t,x,y)\right|\leq\frac{C}{t^{\frac{|a|+|b|}{\alpha}}}\widetilde{p}(t,x,y)\left(1+\frac{\min(1,|y-x|)}{t}\right),\tag{A.17}$$

$$\left|D_x^bH(t,x,x+v)\right|\leq C\widetilde{p}(t,x,x+v)\left(1+\frac{\min(1,|v|)}{t}\right),\tag{A.18}$$

$$\left|D_y^a D_x^b \widetilde{p}(t,x,y)\right| \le \frac{C}{|y-B(y)t-x|^{|a|+|b|}} \widetilde{p}(t,x,y). \tag{A.19}$$

Proof Inequalities (A.17) and (A.18) follow from the representation

$$\begin{aligned} H(t,x,y) = {} & \left\langle B(x)-B(y), \nabla_x \widetilde{p}(t,x,y)\right\rangle + \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |p|^\alpha \int_{S^{d-1}} \left|\langle \overline{p}, s\rangle\right|^\alpha \\ & \times \left(\lambda_{f(y)}(ds) - \lambda_{f(x)}(ds)\right) \exp\left\{-t|p|^\alpha \int_{S^{d-1}} \left|\langle \overline{p}, s\rangle\right|^\alpha \lambda_{f(y)}(ds)\right\} \\ & \times \exp\left\{-i\left\langle p, y-B(y)t-x\right\rangle\right\} dp, \end{aligned} \tag{A.20}$$

analogously to the proof of Proposition 2.3 in [15], see also Appendix B in [18], where (A.17) is proved for $|a|=|b|=0$. Inequality (A.18) is contained in (3.23′), p. 748, of that reference. Inequality (A.19) can be derived following the proof of Lemma A.1. □

The proof of Lemma 4.3 can then be achieved from Lemmas A.1 and A.2 adapting the arguments in Appendix B of [18] concerning the control in terms of the frozen density for the "formal" series appearing in (3.4). See also the proof of Theorem 2.3 in [17] or Theorem 3.1 in [15].

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