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**IDIOSYNCRATIC SHOCKS,
AGGREGATION AND WEALTH
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One of the fundamental sources of heterogeneity in real-life economies lies in stochastic nature of individual income. However to incorporate this heterogeneity into a tractable general equilibrium model presents a difficult problem since the dynamics of such economies is the type-distribution of agents. For that reason research has been concentrated for the last 20 years on numerical methods. This paper offers an analytical framework that helps to characterize the behavior of economies with idiosyncratic income shocks in close analogy to the standard representative agent model. The first part deals with aggregation issue. The second part develops special models, the so called constant prudence economies, that yield a remarkably high level of approximation (0,05% precision in the calibrated model).

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Один из фундаментальных источников неоднородности в экономике связан со стохастическим характером индивидуального дохода. Включение данного фактора в подающуюся анализу модель общего равновесия представляет сложную задачу, так как динамика такой экономики зависит от специфики распределения агентов. По этой причине в исследованиях последних 20 лет использовался численный подход. В настоящей работе развивается аналитический подход, который позволяет охарактеризовать поведение экономики с идиосинкразическими шоками дохода в тесной аналогии со стандартной моделью репрезентативного агента. Первая часть посвящена проблеме агрегирования. Во второй части строятся специальные модели, так называемые экономики с постоянной предусмотрительностью, обеспечивающие высокую степень приближения (в калиброванной модели точность составляет 0,05%).

Классификация JEL: D58, C68, C62, E21

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I. Introduction

Most of the dynamic general equilibrium models are developed in the framework of representative agent paradigm. In this setting all individuals are identical and face the same choices so their decisions coincide and therefore the aggregation is trivial. However, real economies contain various sources of heterogeneity and so the question arises is there a way to incorporate heterogeneity in an analytically tractable general equilibrium framework, or it makes the economy too complex to be expressed in relatively simple and intuitively appealing equations similar to the representative agent approach.

One fundamental example of heterogeneity is due to the stochastic nature of individual income. Idiosyncratic shocks of income deal with situations where individuals face uncertainty about their future income but due to the law of large numbers uncertainty on macro level may be absent. This means that although individual consumption and asset holdings are stochastic, aggregated counterparts and prices may be not.

As has already been stressed by Modigliani (1966) heterogeneity of this type makes aggregation in general case impossible, i.e. there is no translation of equations from micro level to macro level. Nakajima (2007) argues that to solve a model with idiosyncratic shocks is in general a daunting task. This is because of the fact that aggregate variables become the type distribution of agents, which in general is an infinite object.

It is for this reason that idiosyncratic income shocks are so difficult to incorporate into general equilibrium frameworks. The model of Aiyagari (1994) with uninsurable productivity shocks that will further be analyzed illustrates this point well. In this model each of the infinitely lived agents goes through unique realization of stochastic process that governs the shifts in his employment status. Due to uncertainty which the individual faces, consumption-saving decisions are likely to depend in a complex manner on assets he accumulated throughout his unique life path. Thus aggregate dynamics are likely to depend not only on the initial distribution of agents but are also realization sensitive. How to deal with such an immense object?

One way to proceed is to resort to numerical methods. Aiyagari (1994) made an important contribution by developing the numerical approach for solving for the steady state of the model with uninsurable labor income shocks when there is no uncertainty on aggregate level. Krussel and Smith (1998) generalized his approach and constructed a numerical algorithm for obtaining an approximate solution for stochastic steady state in the presence of aggregated productivity shocks. They proposed that a very high precision can be obtained by approximating the distribution of agents using some moments of the distribution. In particular, they

demonstrated numerically that, the mean asset holding is sufficient statistics to obtain a reasonable level of precision.

Since the breakthrough results of Krussel and Smith research has been mainly focused on exploring the effects of different settings by applying essentially the same computational technique. The main advantage of the numerical approach is that it offers a solution. The main drawback is that it represents a black box. Leaving aside the complexity of the algorithms involved, results obtained are hard to interpret. How robust is the constructed solution? Is it unique? Is there room for understanding the reported characteristics or are they as mysterious as those which we observe in real-life economies? In short, without knowing how to analyze the behavior of such economies numerical methods alone leave us in the void. Although sophisticated numerical models help us to approach reality, we still need simpler models that would help us to approach the virtual reality generated by numerical models.

The main goal of this paper is to offer an analytical framework which attempts to close this gap. We develop a method that throws light on the dynamics and steady state properties of an economy with idiosyncratic shocks of income and allows us to explore such economies in close analogy with the standard analytical tools used for representative agent models. Although the exact solution is not feasible (we explain why), this method gives a remarkably high level of approximation (in calibrated model it is 0,05%). The approach developed in this paper should be regarded as a useful complementary tool for numerical methods of analysis. On the other hand if our analysis is correct it means that computational techniques to solve these models can be immensely simplified.

An important feature of an economy with idiosyncratic shocks of income as opposed to representative agent framework is that equilibrium wealth distribution is purely endogenous. This was stressed by earlier contributors and becomes even more explicit in the context of the present analysis. Thus some interesting questions arise. How do the characteristics of idiosyncratic shocks translate into the characteristics of equilibrium wealth distribution? Can the presence of idiosyncratic income shocks explain the scale of wealth inequality that we observe in real economies? We address these questions in what follows. One important conclusion we come to replicates the numerical results of earlier research. We find that small idiosyncratic shocks lead to large variations in wealth very close to what one observes in the data. The main advantage of the analytical approach is that it shows explicitly why this is so.

The paper is organized as follows. Section II contains the general setting and deals with the aggregation issue. It formulates the necessary and sufficient conditions for the existence of equations in aggregates, the so called aggregation theorem. This framework includes most of the important macroeconomic models with idiosyncratic shocks (not necessarily income-shocks)

known in the literature. The main goal is to offer a formal criterion for an economy to be captured in aggregates.

Section III builds on the results of previous section and turns to idiosyncratic shocks of income in the framework close to Aiyagari (1994). It begins with individual consumption-saving decisions and demonstrates that an exact solution is not feasible. The main focus is on the analytical approach that permits to explore the dynamic behavior of such an economy with a satisfactory level of precision. By constructing special boundary economies (the so called, constant prudence economies) we form a narrow channel in which the dynamics of original economy is trapped.

Section IV contains calibration and numerical results. The model calibrated to capture unemployment shocks demonstrates that important characteristics can be extracted with high a level of precision and without resorting to complicated numerical methods. This includes pinning down interest rate, consumption behavior and reaction to the shift in parameters. A separate point of interest is the evolution and stationary level of wealth inequality.

Section V concludes. For clarity of exposition all proofs are given in Appendix I. Appendix II revisits Blanchard (1985) finite horizon model. The reason to include this model is that it can be nicely reinterpreted in terms of idiosyncratic income (wealth) shocks and is straightforwardly analyzed by application of general approach developed in Section II.

II. General Setting and Aggregation Theorem

Consider an economy populated by a 1-continuum of agents. Each individual in period t finds himself in one of two possible states of nature that will be referred as good, denoted G , and bad, denoted B . These states may take different content depending on specific model we are focusing on. Further we will present some examples of the models with idiosyncratic shocks in which good state may be interpreted as the state in which individual is employed, earns high salary, is alive or resets her price whereas bad state means unemployed, low salary, dead or not resetting price.

Assume that transition between states follows a two-state Markov chain. Specifically, let p_I be the probability of retaining the state I ($I=G,B$) in the next period, correspondingly, $1 - p_I$, be the probability of switching to the other state. In what follows I^* denotes *other than* I . Since there is a continuum of agents, the fraction of population at the state I in any period, denoted by α_I , is constant and equals $(1 - p_{I^*})/(2 - p_I - p_{I^*})$ with $\alpha_I + \alpha_{I^*} = 1$.

This scheme is quite flexible to include most frameworks used in macroeconomics. If $p_I = 1 - p_{I^*}$ than the probability of going to the state I does not depend on the current state and we have Poisson process. If $p_I = 1 - p_{I^*} = 1$ we have certainty case.

We now introduce a useful concept to deal with aggregation issue. An economy is said to exhibit a *dichotomy of mean values and higher moments*, or, in short, a *dichotomy of moments* if relations between first moments of distributions for economic variables form a closed set of equations, i.e. a set of equations in which higher moments of distributions are absent. Although it is required that higher moments do not enter this set of equations, evolution of higher moments is allowed to depend on lower moments.

Let $a_{I,t}^j$ denote an individual state variable of agent j being at time t at state I . As we will see in further examples this state variable can take on different interpretations ranging from asset holdings to price of individually produced good. Generally, by solving individual optimization problem, it is possible to write the next period value of the state variable a in terms of its current period value, current state I (state variable), and a vector of economy wide parameters \mathbf{z} :

$$a_{I,t+1}^j = T_I(a_{I,t}^j, \mathbf{z}).$$

We now present one of the central results of this paper, the aggregation theorem. It states that an economy exhibits the dichotomy of moments if and only if transition functions $T_I(\cdot)$ are linear in respect to individual state variable. For clarity of exposition we break down this claim into necessary and sufficient condition.

The necessary condition for an economy to exhibit a dichotomy of moments is given by the following proposition.

Proposition 1

Let $a_{I,t+1}^j = T_I(a_{I,t}^j, \mathbf{z})$ for $I=G,B$ be analytic functions in $a_{I,t}^j$ defined on the domain A . An economy is characterized by a dichotomy of moments only if transition functions T_I are linear in respect to $a_{I,t}^j$:

$$T_I(a_{I,t}^j, \mathbf{z}) = \lambda_t(\mathbf{z}) \cdot a_{I,t}^j + \mu_{I,t}(\mathbf{z})$$

This statement provides a clear condition that must be satisfied in order for any economy to be captured by equations in aggregates. If this condition is not met then economy is not tractable in aggregates. Further we give some examples of important macroeconomic models that satisfy that condition and allow for aggregation.

Sufficiency results are summarized in the following claim.

Proposition 2

If the law of motion for individual state variable satisfies a linear form $\mathbf{a}_{t+1}^j = \lambda_t \cdot \mathbf{a}_t^j + \mu_{I,t}$ then:

Mean value $\mathbb{E}(\mathbf{a}_t)$ follows

$$\mathbb{E}(\mathbf{a}_{t+1}) = \lambda_t \cdot \mathbb{E}(\mathbf{a}_t) + \mathbb{E}(\mu_{I,t})$$

where $\mathbb{E}(\mu_{I,t}) = \alpha_G \mu_{G,t} + \alpha_B \mu_{B,t}$

Mean square $\mathbb{E}(\mathbf{a}_{t+1}^2)$ follows

$$\mathbb{E}(\mathbf{a}_{t+1}^2) = \lambda_t^2 \cdot \mathbb{E}(\mathbf{a}_t^2) + 2 \cdot \lambda_t \cdot \mathbb{E}(\mu_{I,t} \cdot \mathbb{E}_I(\mathbf{a}_t)) + \mathbb{E}(\mu_{I,t}^2)$$

where $\mathbb{E}(\mu_{I,t} \cdot \mathbb{E}_I(\mathbf{a}_t)) = \alpha_G \cdot \mu_{G,t} \cdot \mathbb{E}_G(\mathbf{a}_t) + \alpha_B \cdot \mu_{B,t} \cdot \mathbb{E}_B(\mathbf{a}_t)$ and

$$\mathbb{E}(\mu_{I,t}^2) = \alpha_G \cdot \mu_{G,t}^2 + \alpha_B \cdot \mu_{B,t}^2$$

Higher moments are related by the chain rule of the form

$$\mathbb{E}(\mathbf{a}_{t+1}^n) = \lambda_t^n \cdot \mathbb{E}(\mathbf{a}_t^n) + F_n(\mathbb{E}_I(\mathbf{a}_t^{n-i}), \mu_{I,t}, \lambda_t)$$

where $F_n(\mathbb{E}_I(\mathbf{a}_t^{n-i}), \mu_{B,t}, \lambda_t)$ represents a function of moments less than n .

We obtain these equations in Appendix I by deriving dynamic equations for conditional density functions for both continuous and discrete type distributions. The results stated reveal the dichotomy of moments when linearity condition is satisfied. Note that dynamic equation for mean value of the state variable does not include higher moments. Thus dynamics can be analyzed in aggregates. Equilibrium path of an economy with dichotomy of moments (if such path exists) determines the paths for mean values and the economy wide parameters z (equilibrium interest rates, wages et t.c.) and thus determines the paths for λ_t and $\mu_{I,t}$. Dynamics of higher moments is then obtained by applying the chain rule: evolution of higher moment depends only on its initial level and lower moments. In the further analysis we are interested primarily in the second moments.

Some examples

We present some examples of macroeconomic general equilibrium models with idiosyncratic shocks that can be captured in aggregates. The purpose is to demonstrate that our framework can serve as a useful unification principle.

The New-Keynsian model with Calvo (1983) process for price setting offers a good example of such model. Each period the firms that adjust their price are randomly selected, and a fraction w adjust while remaining fraction $1-w$ do not adjust. Those firms that do adjust their price do so to maximize the expected discounted value of current and future profits. It is easy to

verify that our general setting applies to this case. To see this note that in this case we can define as G the state in which individual firm receives a signal and adjusts, whereas B refers to the state with no adjustment. Then $p_G = 1 - p_B = w$ and we have the Poisson process. State variable $a_{l,t}^j$ represents individual price. Transition equations are straightforward: $a_{t+1}^j = a_t^*$ for firms at G at time t , where a_t^* is the optimal price chosen by all adjusting firms; $a_{t+1}^j = a_t^j$ for firms at B at time t . This conforms with Proposition 1 and, thus, the model allows for exact aggregation.

In the agency costs models of Carlstrom and Fuerst (1997) or Bernanke, Gertler and Gilshrist (1999) entrepreneurs borrow external funds to invest in a project that is subject to idiosyncratic productivity shock. It is noteworthy that solution in aggregates is obtained because utility is assumed to be linear in consumption. We see again that some form of linearity requirement is indispensable. However *ad hoc* assumption of linear utility is not plausible. In Sections III and IV we show how aggregated dynamics can be extracted without reverting to such strong assumptions.

Next example refers to Blanchard (1985) eternal youth model. Each period and regardless his age an individual faces constant probability of death, p . In this setting good state quite naturally refers to staying alive while bad state occurs when individual passes away. Alternatively one can interpret that in the bad state individual stays alive but is stripped of all his wealth. In our terms $p_G = 1 - p$, $p_B = 0$. The Euler condition with logarithmic utility is linear in individual consumption. This in turn implies that individual asset holdings at the good state follow linear dynamic equation. At the bad state asset transition equation is just $a_{t+1}^j = 0$. We analyze this model more thoroughly in Appendix II. Now it suffices to point that aggregation theorem clearly applies to this model and it is tractable in aggregates.

The last but not least example is the standard neo-classical growth model without uncertainty ($p_G = 1 - p_B = 1$). In this case the basic Euler equation that links current and next-period individual consumptions is linear in consumption. Thus there is a linear relation between the next period and current period individual assets and the aggregation theorem applies. Two last examples demonstrate that what really matters for aggregation is linearity of decision rules, not linearity of utility.

III. Economy with labor income shocks

Individual consumption

The basic model is a version of Aiyagari (1994) model with uninsurable idiosyncratic labor productivity shocks. An agent can have either high wage $w_{G,t}$ or low wage $w_{B,t}$ in each period t . The productivity status follows a Markov process. One particular interpretation for

agent's productivity status is his employment: he can either be employed or unemployed. In this case following Krussel and Smith (1998) $w_{B,t}$ is set to be zero and $w_{G,t}$ equals market clearing wage.

The problem for the agent j being at the state I at time t and holding assets a_t^j is to maximize

$$u(c_{I,t}^j) + E_{I,t} \left[\sum_{i=1}^{\infty} \beta^i \cdot u(c_{t+i}^j) \right] \quad (1)$$

subject to the budget constraint

$$a_{t+1}^j = R_{t+1} \cdot (a_t^j + w_{I,t} - c_{I,t}^j) \quad (2)$$

First order necessary conditions condition is given by Euler equations that in the present case takes the form

$$u'(c_{I,t}^j) = \beta \cdot R_{t+1} \cdot [p_I \cdot u'(c_{I,t+1}^j) + (1 - p_I) \cdot u'(c_{I^*,t+1}^j)] \text{ for } I=G,B$$

Suppose that preferences are given by

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}$$

where θ is coefficient of risk averseness ($\theta \geq 1$). With these preferences, the Euler conditions become

$$\beta \cdot R_{t+1} \cdot \left[p_I \cdot \left(\frac{c_{I,t+1}^j}{c_{I,t}^j} \right)^{-\theta} + (1 - p_I) \cdot \left(\frac{c_{I^*,t+1}^j}{c_{I,t}^j} \right)^{-\theta} \right] = 1 \text{ for } I = G, B \quad (3)$$

Equation (3) is nonlinear in ratios of consumption, hence, these ratios are likely to depend in non-trivial manner on other state variables, specifically, on the level of assets individual is holding. The non-linearity of basic consumption equation makes impossible to get its counterpart in terms of aggregated consumption.

In order to proceed, it proves helpful to express consumption equations in terms of propensity to consume. As we will demonstrate shortly consumption and saving rules expressed in this form have transparent and easy to interpret structure.

Let $\Omega_{I,t}^j$ denote the propensity to consume out of wealth for agent j in state I and time t . Using this notation propensity to consume are allowed to depend on time t , individual state I and individual asset holdings a_t^j . By definition

$$c_{I,t}^j \equiv \Omega_{I,t+1}^j \cdot (a_t^j + h_{I,t}) \quad (4)$$

so that $\Omega_{I,t}^j$ relates individual consumption to his wealth, the sum of net asset holdings and human wealth (the agent's superscript is dropped out in the expression for human wealth as it

depends only on state I at time t). Human wealth is expressed as the current value of the present and future expected labor income

$$h_{I,t} = w_{I,t} + E_{I,t} \left[\sum_{i=1}^{\infty} (R_{t+i})^{-i} \cdot w_{t+i} \right] \quad (5)$$

where $w_{I,t}$ is labor income at state I at time t .

Denote $\Delta h_{I,t} = h_{I,t} - h_{I^*,t}$ the difference in human wealth between two states ($\Delta h_{I,t} + \Delta h_{I^*,t} = 0$). The following proposition sheds light on the optimal individual decision rule for consumption.

Proposition 3

Suppose $\left| \frac{\Delta h_{I,t+1}}{a_t^j + h_{I,t}} \right|$ is small. Then the optimal consumption strategy for agent j at state I at time t can be expressed as follows

$$c_{I,t}^j = \left[\Omega_t - \varphi_{I,t} \cdot \left(\frac{\Delta h_{I,t+1}}{a_t^j + h_{I,t}} \right)^2 + \mathcal{O} \left(\left(\frac{\Delta h_{I,t+1}}{a_t^j + h_{I,t}} \right)^3 \right) \right] \cdot (a_t^j + h_{I,t}) \quad (6)$$

where function Ω_t satisfies equation

$$\beta \cdot R_{t+1} \cdot \left(R_{t+1} \Omega_{t+1} \cdot \frac{1 - \Omega_t}{\Omega_t} \right)^{-\theta} = 1 \quad (7)$$

and precautionary saving functions $\varphi_{I,t}$ ($I=G,B$) satisfy the system of two equations

$$\varphi_{I,t} = \frac{\Omega_t(1-\Omega_t)}{\Omega_{t+1}} [p_I \varphi_{I,t+1} + (1-p_I) \varphi_{I^*,t+1}] + \frac{(\theta+1)}{2} \frac{\Omega_t}{(1-\Omega_t)R_{t+1}^2} p_I (1-p_I) \quad (8)$$

These results are obtained in Appendix by application of perturbation theory. To see what they imply for propensity to consume first note that dynamic equation for Ω_t is identical to the propensity to consume equation in an economy without idiosyncratic shocks. Thus Ω_t can be treated as the unperturbed part of propensity to consume. Expressing Ω_t from equation (7) and re-denoting one obtains the following forward looking equation

$$x_t = 1 + s_{t+1} \cdot x_{t+1}$$

where $x_t = 1/\Omega_t$ and $s_{t+1} = (\beta \cdot R_{t+1})^{1/\theta} / R_{t+1}$. Solving this equation forward and imposing no-bubble condition, yields

$$\Omega_t = \frac{1}{x_t} = \left(1 + \sum_{i=1}^{\infty} \prod_{k=1}^i s_{t+k} \right)^{-1}$$

With logarithmic utility ($\theta = 1$) this gives constant propensity to consume $s_{t+k} = \beta$ and $\Omega_t = 1 - \beta$. In general case this equation determines Ω_t given future interest rate path that is assumed to be known to rational consumers.

Individual consumption rules given in Proposition 3 provide a close approximation to the optimal consumption strategy as long as $\left| \Delta h_{I,t+1} / (a_t^j + h_{I,t}) \right|$ stays small. Suppose for the

moment that this is the case. What does this imply? Rewrite consumption rule ignoring the terms of order of smallness greater than two.

$$c_{I,t}^j = \Omega_t \cdot (a_t^j + h_{I,t}) - \varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{a_t^j + h_{I,t}} \quad I = G, B \quad (9)$$

The second term on the right side of equation (9) represents the precautionary saving motive. It shows that there are two factors at play here. First is measured by precautionary saving functions $\varphi_{I,t}$ that are only time and state dependent. Second factor measured by the term $\frac{\Delta h_{I,t+1}}{a_t^j + h_{I,t}}$ depends on individual assets as well. It represents the ratio of human wealth differential in both states to the current individual wealth.

Intuitively $\varphi_{I,t}$ at both states are positive functions so that uncertainty about the future income forces consumers to save more and consume less than in the certainty case. It can be verified that $\varphi_{I,t}$ are indeed positive. Note that these functions depend on the state I thus precautionary behavior at the good state can slightly differ from that at the bad state. For further analysis it will be helpful to write a single equation for the state-average precautionary saving function. Define $\varphi_t \equiv \alpha_G \varphi_{G,t} + \alpha_B \varphi_{B,t}$. From equations (8) it follows (see Appendix)

$$\varphi_t = \frac{\Omega_t(1 - \Omega_t)}{\Omega_{t+1}} \cdot \varphi_{t+1} + \frac{(\theta + 1)}{2} \frac{\Omega_t}{(1 - \Omega_t)R_{t+1}^2} \frac{(1 - p_G)(1 - p_B)(p_G + p_B)}{2 - p_G - p_B} \quad (10)$$

Equation (10) represents a forward looking linear equation in respect to φ_t . Note that both coefficient on the next period value of endogenous variable and the free term are positive. Solving this equation forward and ruling out explosive paths one obtains the fundamental solution for φ_t that is positive at any t for any interest rate path.

As individual accumulates more assets she reacts less to future income uncertainty and moves closer to certainty equivalent behavior. Whatever interesting is this non-linearity of consumption in respect to asset holdings it creates the major problem for analysis. For if individual consumption at t is non-linear in his assets at t so are his assets at $t+1$. Thus by Proposition 1 this economy does not exhibit a dichotomy of moments and there is no translation of individual equations into aggregated counterparts. Put differently, dynamics of economy populated by prudent consumers is of the type distribution.

Boundary economies

We now develop an analytical approach that offers a solution to the aggregation problem described above.

The idea is to characterize the (infinitely) complex behavior of the original economy by exploring simpler limiting cases that allow for aggregation and whose dynamics serve as the upper and lower bounds for the economy of interest. Moreover, if stationary solutions for these simpler economies can be shown to be close to each other, the behavior of the original economy would be pinned down. We proceed by defining what is required from these boundary economies.

Upper bound asset accumulation economy is an economy which is characterized by 1) dichotomy of moments 2) average assets in any period are higher than in the original economy provided the same for two economies initial mean asset holdings a_0 and interest rate, wage and capital stock paths: $\mathbb{E}\bar{a}_t \geq \mathbb{E}a_t$ for $t = 1, \dots, \infty$.

Lower bound asset accumulation economy is an economy which is characterized by 1) dichotomy of moments 2) average assets in any period are lower than in the original economy provided the same for two economies initial mean asset holdings a_0 and interest rate, wage and capital stock paths: $\mathbb{E}\underline{a}_t \leq \mathbb{E}a_t$ for $t = 1, \dots, \infty$.

The main issue is to construct such economies. It turns out that it is possible to find economies with desired properties by considering the so called *constant prudence economy* (CPE). By this we mean a hypothetical economy in which precautionary saving behavior does not depend on individual asset level. Formally, a constant prudence economy differs from the original economy only in that individuals consume according to the rule

$$c_{I,t}^j(CPE) = \Omega_t \cdot (a_t^j + h_{I,t}) - \varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{a_{I,t}^* + h_{I,t}} \quad I = G, B \quad (11)$$

The only difference between consumption rules (9) and (11) is that in the later agent j asset holdings $a_{I,t}^j$ are replaced by economy-wide level $a_{I,t}^*$.

It is straightforward to demonstrate that any constant prudence economy exhibits the dichotomy of moments. To see this substitute (11) in the asset accumulation equation (2). This yields

$$a_{t+1}^j = R_{t+1} \cdot (1 - \Omega_t) \cdot a_t^j + \left[R_{t+1} \left(w_{I,t} - \Omega_t h_{I,t} + \varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{a_{I,t}^* + h_{I,t}} \right) \right] \quad \text{for } I = G, B$$

This equation clearly satisfies Proposition 2.

A constant prudence economy captures in most simple way both heterogeneous and homogenous aspects of consumption. All individuals behave alike in respect to precautionary savings. Nevertheless the time heterogeneity is presented by the first term in (11) since

consumption depends on current individual wealth. It is due to the linearity of this term in respect to wealth that economy is tractable in aggregates.

We can use constant prudence economies to construct boundary cases. One limiting case is obtained when $a_{I,t}^*$ is very large or goes to infinity. Then individuals consume according to

$$c_{I,t}^j(\text{MinPE}) = \Omega_t \cdot (a_t^j + h_{I,t}) \quad (12)$$

Thus they consume as in the certainty case. It is natural to call economy in which individual consumption is given by (12) *the minimal prudence economy*. Further it will be shown that minimal prudence economy presents a lower bound asset accumulation economy.

The other side of the specter is not so obvious. As we see from (11) precautionary savings rises when $a_{I,t}^*$ decreases. This term goes to infinity when wealth, $a_{I,t}^* + h_{I,t}$, approaches zero. Recall however that (9) is a good approximation for optimal consumption only if $\frac{\Delta h_{I,t+1}}{a_t^j + h_{I,t}}$ is sufficiently small in absolute value. In other words it serves as approximation for sufficiently high levels of wealth. With decreasing wealth optimal consumption rule diverts further from (11).

Relation between optimal consumption rule and second order approximation given by (11) is summarized in Fig.1. It can be seen that there is such value of a at state I and time t for which approximation (11) calls for zero consumption. Denote this value $a_{I,t}^\#$. By definition

$$a_{I,t}^\# \equiv \left(\frac{\varphi_{I,t}}{\Omega_t}\right)^{1/2} \cdot \Delta h_{I,t+1} - h_{I,t+1} \quad (13)$$

We call economy in which individual consumption is given by (11) with $a_{I,t}^\#$ from (13) *the maximum prudence economy*. It follows directly from Fig.1 that individual optimal consumption can be characterized by the following inequality

$$c_{I,t}^j > c_{I,t}^j(\text{MaxPE}) \equiv \Omega_t \cdot (a_t^j + h_{I,t}) - \varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{a_{I,t}^{\min} + h_{I,t}} \quad (14)$$

It turns out that maximum prudence economy is an upper bound asset accumulation economy.

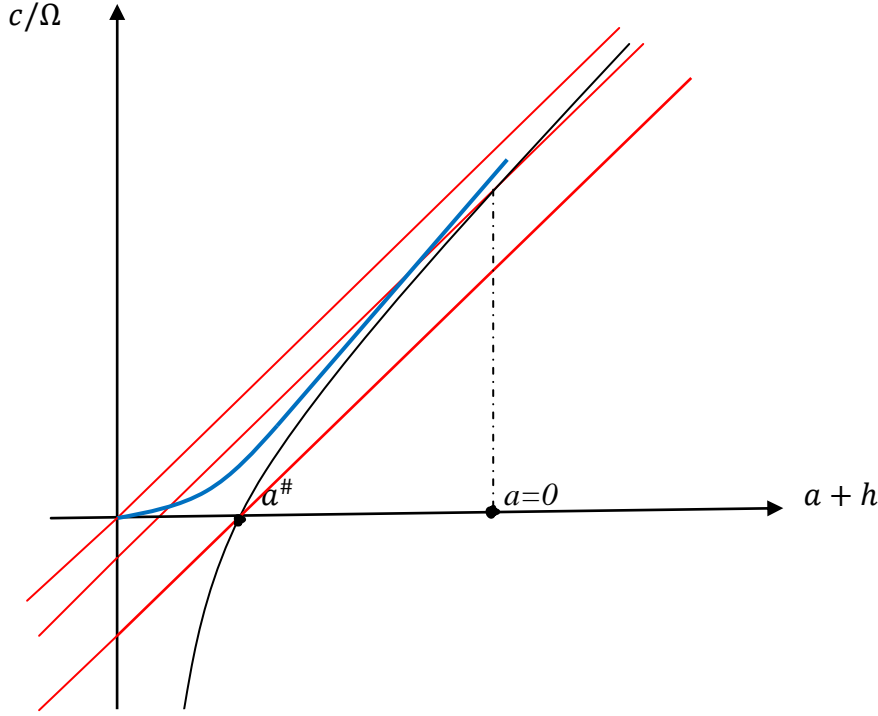


Fig. 1. Consumption rules

- Blue line: optimal consumption rule
- Black line: second order approximation of optimal consumption rule
- Higher red line: consumption rule for Minimum Prudence Economy
- Middle red line: consumption rule for Constant Prudence Economy with $a_{l,t}^* = 0$
- Lower red line: consumption rule for Maximum Prudence Economy

It should be stressed that consumption rules described for constant prudence economies should not be regarded as providing optimal solutions for some specific consumer decision taking problem. Moreover, we do not even require that “consumption” given by (11) to be non-negative as is shown in Fig. 1. One should treat these economies as a useful formal instrument to evaluate the behavior of the original economy. The main advantage is that these economies allow for solution in aggregates and set bounds for original economy. This is formulated in the following proposition.

Proposition 4

An economy in which individual j consumes according to the rules

$$\underline{c}_{G,t}^j = \Omega_t \cdot (a_t^j + h_{G,t}) - \sqrt{\Omega_t \varphi_t} \cdot \Delta h_{t+1} \quad (15)$$

$$\underline{c}_{B,t}^j = \Omega_t \cdot (a_t^j + h_{B,t}) - \sqrt{\Omega_t \varphi_t} \cdot \Delta h_{t+1} \quad (16)$$

is an upper bound asset accumulation economy.

An economy in which individual j consumes according to the rules

$$\underline{c}_{G,t}^j = \Omega_t \cdot (a_t^j + h_{G,t}) \quad (17)$$

$$\underline{c}_{B,t}^j = \Omega_t \cdot (a_t^j + h_{B,t}) \quad (18)$$

is a lower bound asset accumulation economy.

This claim is quite strong. Recall that dynamics of the original economy is influenced not only by initial probability distribution but also by specific realizations of individual agent life paths, i.e. individual transitions between states. Nevertheless (15)–(16) and (17)–(18) are proved to be the bounds regardless of initial distribution or realization paths.

It should be stressed that this proposition deals with partial equilibrium aspect. It shows that with price paths set exogenously there is an economy with dichotomy of moments in which assets accumulate faster than in the original economy (upper bound) and an economy with slower asset accumulation (lower bound). This result provides the basis for general equilibrium analysis.

What can be said about dynamic properties of upper and lower bound economies presented in Proposition 4? In the next subsection we will analyze this question in more detail and show that these two economies are characterized by dynamics similar to the standard Ramsey model. Both economies are saddle path stable. Moreover the lower bound economy coincides with the standard representative agent economy. It settles in a steady state with modified golden rule interest rate $\underline{R}_{SS} = 1/\beta$. The upper bound is also characterized by saddle path with slightly lower steady state interest rate: $\overline{R}_{SS} < \underline{R}_{SS}$.

As should be already clear from preceding analysis the original economy has no dichotomy of moments and is a complex infinite-dimension object that cannot be expressed by some closed system of dynamic equations. Thus, it is not possible to make general judgments about the existence of the steady state or even the existence of equilibrium path in this economy. It is possible however to identify dynamic properties of this model presuming there exists a steady state. This is given by the following proposition.

Proposition 5

Assume that there is a steady state in the original economy. Then in capital-consumption space this steady state lies between the steady states of lower and upper bound economies:

$$\underline{k}_{SS} < k_{SS} < \overline{k}_{SS} \text{ and } \underline{c}_{SS} < c_{SS} < \overline{c}_{SS}$$

This is the key result. Let us outlay the main argument (for detailed proof see Appendix). Suppose the contrary. Specifically, imagine that in the steady state original economy accumulates more capital than the upper bound economy: $k_{SS} > \overline{k}_{SS}$. Since capital is a decreasing function of interest rate, we have $R_{SS} < \overline{R}_{SS}$. It can be shown that stationary level of assets accumulated in the upper bound economy increase with stationary interest rate. Thus, $\overline{a}_{SS}(R_{SS}) < \overline{a}_{SS}(\overline{R}_{SS})$. We see that R_{SS} level of interest rate is associated with a capital deficit in

the upper bound economy: demand exceeds supply. But we know by definition that upper bound economy accumulates more assets than the original economy. By consequence there should be even more severe misbalance of capital and assets in the original economy at R_{ss} and we run into contradiction. Similar argument using the lower bound economy can be made to show that $k_{ss} < \underline{k_{ss}}$ cannot occur in equilibrium.

Generally speaking nothing can be said about uniqueness of the steady state. Proposition 5 set bounds for stationary equilibria. If the steady states for upper and lower bounds economies are very close to each other then steady state(s) for the original economy (assuming it exists) can be pinned down precisely.

Upper and Lower Bounds: Dynamics

Now we are well positioned to formulate the system of equations that rules the dynamics of the upper and lower bound economies. We begin with the upper bound that represents an economy with constant (independent of individual asset holdings) precautionary savings.

In order to simplify slightly analysis we assume logarithmic utility ($\theta = 1$). In this case undisturbed propensity to consume Ω_t does not depend on interest rates: $\Omega_t = 1 - \beta$. This is without loss of generality and lowers the number of dimensions by one.

Consumption rules for this economy are given in Proposition 4. Integrating these equations in respect to the states I and individuals j we obtain the following equation for the mean value (aggregate) of consumption

$$\mathbb{E}c_t = (1 - \beta) \cdot (\mathbb{E}a_t + \mathbb{E}h_t) - \sqrt{(1 - \beta)\varphi_t} \cdot \Delta h_{t+1} \quad (19)$$

It is convenient to write down the consumption equation in the form that relates mean values of consumptions in periods t and $t+1$. This can be done by writing (19) for the period $t+1$, using mean asset accumulation equation $\mathbb{E}a_{t+1} = R_{t+1}(\mathbb{E}a_t + w_t - \mathbb{E}c_t)$ to substitute $\mathbb{E}a_{t+1}$ in it and expressing $\mathbb{E}a_t$ from (19) to substitute $\mathbb{E}a_t$ in it. After some rearrangement we arrive at

$$\mathbb{E}c_{t+1} = \beta R_{t+1} \cdot \mathbb{E}c_t + \sqrt{(1 - \beta)} [R_{t+1} \sqrt{\varphi_t} \cdot \Delta h_{t+1} - \sqrt{\varphi_{t+1}} \cdot \Delta h_{t+2}] \quad (20)$$

This equation deserves short comment. It can be interpreted as generalized Euler equation that incorporates precautionary savings motive. Relation $\mathbb{E}c_{t+1} = \beta R_{t+1} \cdot \mathbb{E}c_t$ is the standard Euler equation for an economy without uncertainty. Second term on the right of (20) represents distortion of the standard form due to uncertainty about future income. We see that the impact of uncertainty on consumption depends positively on human wealth differential between two states and precautionary saving function φ_t .

Equation for state-average precautionary savings function φ_t was derived earlier and is given by (10). Using expression for constant Ω_t it simplifies to

$$\varphi_t = \beta \cdot \varphi_{t+1} + \frac{1}{R_{t+1}^2} \frac{(1-\beta)(1-p_G)(1-p_B)(p_G+p_B)}{\beta(2-p_G-p_B)} \quad (21)$$

Human wealth dynamics can be extracted from the definition of $h_{I,t}$ (5). Using this definition one can relate current human wealth to it next period values

$$h_{I,t} = w_{I,t} + \frac{1}{R_{t+1}} \cdot [p_I \cdot h_{I,t+1} + (1-p_I) \cdot h_{I^*,t+1}] \text{ for } I=G,B$$

From this it is easy to obtain an equation for human wealth differential between states, $\Delta h_t = h_{G,t} - h_{B,t}$, the term that is present in consumption equation (19)

$$\Delta h_t = \Delta w_t + \frac{1}{R_{t+1}} \cdot (p_G + p_B - 1) \cdot \Delta h_{t+1} \quad (22)$$

where $\Delta w_t = w_{G,t} - w_{B,t}$ is wage differential between two states.

We assume the Cobb-Douglas production function $y_t = l_t^{1-\mu} \cdot k_t^\mu$ with inelastic labor supply l_t that equals α_G (the fraction of population that is employed). This implies standard relations for production, interest rate and market-clearing wage

$$y_t = \alpha_G^{1-\gamma} \cdot k_t^\gamma \quad (23)$$

$$\mu \cdot \left(\frac{\alpha_G}{k_t}\right)^{1-\gamma} + 1 - \delta = R_t \quad (24)$$

$$(1-\gamma) \cdot \left(\frac{\alpha_G}{k_t}\right)^{-\gamma} = w_{G,t} \quad (25)$$

$$w_{B,t} = 0$$

Using last two equations wage differential is

$$\Delta w_t = (1-\gamma) \cdot \left(\frac{\alpha_G}{k_t}\right)^{-\gamma}$$

Substituting this wage differential in (22)

$$\Delta h_t = \frac{1}{R_{t+1}} \cdot (p_G + p_B - 1) \cdot \Delta h_{t+1} + (1-\mu) \cdot \left(\frac{\alpha_G}{k_t}\right)^{-\gamma} \quad (26)$$

Finally we have the following capital accumulation equation

$$k_{t+1} = \alpha_G^{1-\gamma} \cdot k_t^\gamma + (1-\delta) \cdot k_t - \mathbb{E}c_t \quad (27)$$

There are four dynamic equations. Aggregate consumption, $\mathbb{E}c_t$, dynamics is given by (20); dynamic equation for precautionary savings function, φ_t , is represented by (21); (22) governs the dynamics of human wealth differential, Δh_t ; finally, equation for capital stock k_t dynamics is (27). Together with relation between capital stock and interest rate (24) and transversality condition these equations determine equilibrium path for this economy.

We now turn to the steady state of the upper bound economy and show that it is unique. In the steady state all endogenous variables take constant values that we denote by subscript *ss*.

Begin with equation for consumption (20). In the steady state it becomes

$$\mathbb{E}c_{ss} = \frac{R_{ss} - 1}{1 - \beta R_{ss}} \sqrt{(1 - \beta) \cdot \varphi_{ss} \cdot \Delta h_{ss}} \quad (29)$$

Firstly, this relation reveals that equilibrium interest rate should be lower than $1/\beta$, the level of interest rate in the certainty case. Secondly, it shows that it stays very close to that level as long as human wealth differential is small. Both results are intuitive as prudent behavior forces individuals to save more and thus lowers equilibrium interest rate.

The steady state level of precautionary saving function φ_{ss} is obtained from (21). This yields

$$\varphi_{ss} = \frac{1}{\beta R_{ss}^2} \frac{(1 - p_G)(1 - p_B)(p_G + p_B)}{2 - p_G - p_B} \quad (30)$$

Equation (26) provides the steady state level of human wealth differential

$$\Delta h_{ss} = \left(1 - \frac{1}{R_{ss}} \cdot (p_G + p_B - 1) \right)^{-1} (1 - \gamma) \cdot \left(\frac{\alpha_G}{k_{ss}} \right)^{-\gamma} \quad (31)$$

Finally, equations (24) and (27) become in the steady state

$$\gamma \cdot \left(\frac{\alpha_G}{k_{ss}} \right)^{1-\gamma} + 1 - \delta = R_{ss} \quad (32)$$

$$k_{ss} = \frac{1}{\delta} (y_{ss} - \mathbb{E}c_{ss}) \quad (33)$$

Equations (29)–(33) simultaneously determine the steady state values of R_{ss} , $\mathbb{E}c_{ss}$, k_{ss} , Δh_{ss} , φ_{ss} . It can be shown that this steady state exists and is unique. The easy way to verify this is to note that equation (32) defines the steady state capital stock as decreasing function of the steady state interest rate. This relation can be expressed as follows

$$k_{ss}^{(1)}(R_{ss}) = \frac{\alpha_G}{\delta^{1/(1-\gamma)}} \left[\frac{\gamma}{1 + \frac{(R_{ss} - 1)}{\delta}} \right]^{1/1-\gamma} \quad (34)$$

On the other hand, equations (29)-(31) and (33) yield the following relation between capital stock and interest rate

$$k_{ss}^{(2)}(R_{ss}) = \frac{\alpha_G}{\delta^{1/(1-\gamma)}} \left[1 - A \cdot \frac{R_{ss} - 1}{1 - \beta R_{ss}} \cdot \frac{1}{R_{ss}} \cdot \left(1 - \frac{1}{R_{ss}} \cdot (p_G + p_B - 1) \right)^{-1} \right]^{1/1-\gamma} \quad (35)$$

where $A \equiv \frac{1-\gamma}{\alpha_G} \sqrt{\frac{1-\beta}{\beta} \frac{(1-p_G)(1-p_B)(p_G+p_B)}{2-p_G-p_B}}$

Functions $k_{ss}^{(1)}(R_{ss})$ and $k_{ss}^{(2)}(R_{ss})$ from (34) and (35) are plotted in the Fig.2. Their unique intersection is close to $1/\beta$ from below and determines the equilibrium interest rate

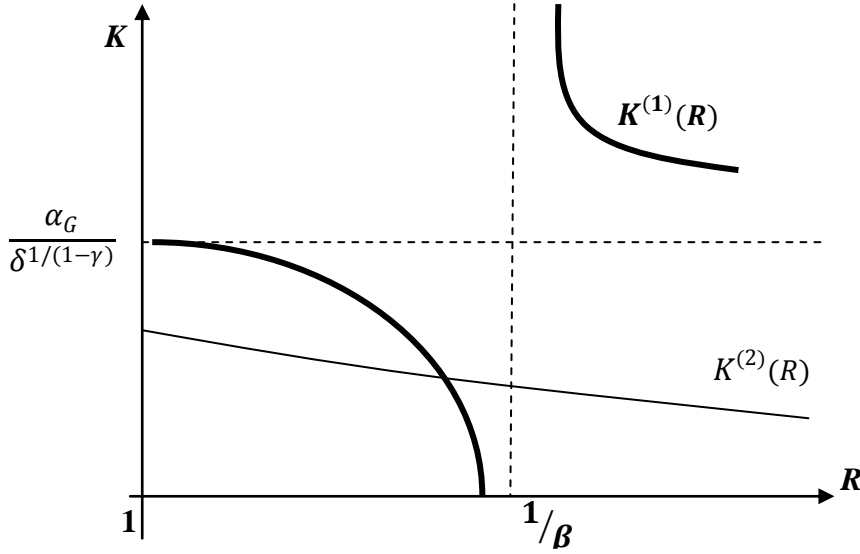


Fig. 2. Steady state in the upper bound economy

It can be shown that the equilibrium has a saddle-point structure. Behavior of the upper and lower bound economies is very similar. As was pointed earlier dynamic equations for lower bound economies for average consumption and capital stock coincides with the standard Ramsey model. Dynamics of the upper bound economy is a little trickier because it contains two additional dimensions, Δh_t , and φ_t . Nevertheless a stable arm converging to the steady state can be constructed. Steady state for the upper bound is slightly shifted relative to the lower bound economy due to precautionary savings. Stable arms for both economies can be viewed as a channel for the path of the original economy (presuming it exists). This is summarized in Fig.3. Since equilibrium paths of the original economy are of the type distribution there must be an infinite number of paths that converges to the steady state C that lie in the channel formed by saddle paths of lower and upper bounds economies.

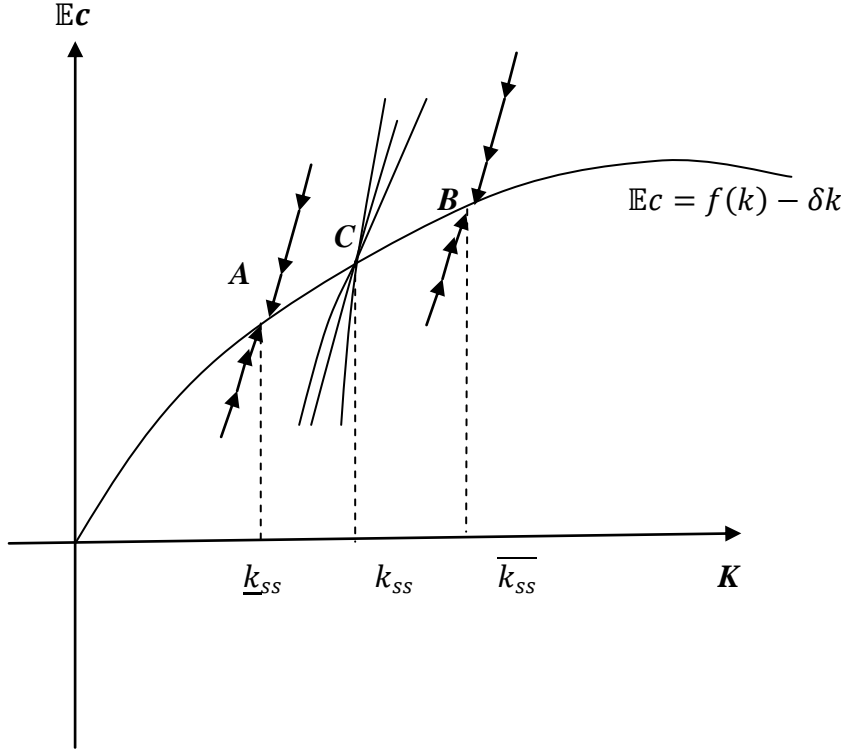


Fig. 3. Boundary economies

Evolution of Higher Moments

We now focus on the implications of idiosyncratic income shocks for wealth inequality. There are various measures of inequality used by economists. In the context of our exposition the most relevant is coefficient of variation (the ratio of standard deviation to mean value).

The derivation of dynamic equation that governs unconditional square assets $\mathbb{E}(a_t^2)$ in the upper bound economy follows directly from Proposition 1 and equations of preceding subsection. We obtain

$$\mathbb{E}(a_{t+1}^2) = \lambda_t^2 \cdot \mathbb{E}(a_t^2) + 2 \cdot \lambda_t \cdot \mathbb{E}(\mu_{I,t} \cdot \mathbb{E}_I(a_t)) + \mathbb{E}(\mu_{I,t}^2) \quad (36)$$

where $\lambda_t = \beta R_t$ $\mathbb{E}(\mu_{I,t} \cdot \mathbb{E}_I(a_t)) = \sum_{I=G,B} \alpha_I \cdot \mu_{I,t} \cdot \mathbb{E}_I a_t = \sum_{I=G,B} \alpha_I \cdot [R_{t+1} \cdot (w_{I,t} - (1 - \beta) \cdot h_{I,t} - (1 - \beta)\varphi_t \cdot \Delta h_t + 1 \cdot \mathbb{E}_I a_t)$ and

$$\mathbb{E}(\mu_{I,t}^2) = \sum_{I=G,B} \alpha_I \cdot (\mu_{I,t})^2 = \sum_{I=G,B} \alpha_I \cdot [R_{t+1} \cdot (w_{I,t} - (1 - \beta) \cdot h_{I,t} - \sqrt{(1 - \beta)\varphi_t} \cdot \Delta h_{t+1})]^2$$

To gain some insight recall that in any economy characterized by the dichotomy of moments mean values of variables form a closed system of dynamic equations and determine the paths for λ_t , $\mu_{I,t}$, $\mathbb{E}_I(a_t)$, $\mathbb{E}a_t$. Given these paths and initial condition equation (36) determines

dynamics of $\mathbb{E}(a_t^2)$. Economy in aggregates follows the saddle path and converges to the steady state. So if aggregates are settled in the steady state, (36) becomes

$$\mathbb{E}(a_{t+1}^2) = \lambda_{ss}^2 \cdot \mathbb{E}(a_t^2) + 2 \cdot \lambda_{ss} \cdot \mathbb{E}(\mu_{I,ss} \cdot \mathbb{E}_I(a_{ss})) + \mathbb{E}(\mu_{I,ss}^2) \quad (37)$$

Equation (37) shows that even when first moments are already settled in the steady state, second and higher moments need not to be so. Recall that $\lambda_{ss} = R_{ss}/\beta$ and $R_{ss} < \frac{1}{\beta}$. Thus $\lambda_{ss} < 1$ and (37) ultimately converges to the stationary level

$$\mathbb{E}a_{ss}^2 = \frac{1}{1 - \lambda_{ss}^2} \left(2 \cdot \lambda_{ss} \cdot \mathbb{E}(\mu_{I,ss} \cdot \mathbb{E}_I(a_{ss})) + \mathbb{E}(\mu_{I,ss}^2) \right)$$

As $\sigma_{ss}^2(a) = \mathbb{E}a_{ss}^2 - (\mathbb{E}a_{ss})^2$, stationary dispersion is

$$\sigma_{ss}^2(a) = \frac{2 \cdot \lambda_{ss} \cdot \mathbb{E}(\mu_{I,ss} \cdot \mathbb{E}_I(a_{ss})) + \mathbb{E}(\mu_{I,ss}^2)}{1 - \lambda_{ss}^2} - (\mathbb{E}a_{ss})^2 \quad (38)$$

Using (38) one obtains the stationary coefficient of variation.

Equation (38) also shows that equilibrium asset distribution can be characterized by high inequality if equilibrium λ_{ss} is close to unity. This exercise demonstrates that asset structure becomes purely endogenous. Evolution of higher moments can be described by the chain rule. Once first $n-1$ moments are determined, there is a dynamic equation that determines the evolution of n -s moment given $n-1$ moments. In Appendix I we derive explicitly dynamic equations that govern evolution of density functions. In view of this it is possible not only recover higher moments separately but get the whole picture of distribution dynamics. In Appendix I after the proof of Proposition 2 we provide an illustrative numerical example how density distribution functions evolves.

IV. Calibration and Results

One natural way to apply analytical framework developed above and calibrate the model consists in interpreting good and bad states as referring to employed-unemployed status. Two-state Markov chain might be a relevant approximation for transition between employed and unemployed since data suggests there is substantial difference in probabilities of keeping a job and finding one.

There are five constants to be calibrated: probabilities p_G and p_B , utility factors θ and β , capital share γ , depreciation rate δ . At quarter frequency we set: $\beta = 0.99$, $\gamma = 0.36$, $\delta = 0.025$. These values are close to commonly used in simulation experiments. We choose $p_B = 0.6$ so that average duration of an unemployment spell, $p_B/(1 - p_B)$, equals 1.5 quarters which corresponds to US data. We set $p_G = 0.98$ to match the unemployment rate,

$(1 - p_G)/(2 - p_G - p_B)$, of 5%. It is commonly accepted that realistic values for coefficient of risk aversion are somewhere between 1 and 5. We set $\theta = 1$ for the benchmark case and then explore how higher risk averseness affects the results.

In the previous subsection we analyzed two boundary economies whose stable arms form the channel for original economy. Consumption rules for lower bound (15)–(16) and upper bound (17)–(18) are constructed to provide the simplest possible dynamic systems. Indeed, lower bound economy so chosen coincides with the standard neo-classical model and upper bound economy contains just two additional dimensions. For the purpose of analytical simplicity we also used logarithmic utility.

However for the purpose of simulation experiments we can construct much more refined bounds by choosing from the class of constant prudence economies introduced by (11).

All that is required for an economy to present a lower bound asset accumulation economy is that precautionary savings in it are less than *average* precautionary savings in the original economy (given same price paths). Recall that precautionary savings motive is captured by the second term in (9). Thus we need to choose $a_{I,t}^*$ so that

$$\int \varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{a_t^j + h_{I,t}} dF_I(a_t^j) > \varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{a_{I,t}^* + h_{I,t}} \quad \text{for } I = G, B \quad (39)$$

If we set $a_{I,t}^* = \mathbb{E}_I a_t$ then (39) is clearly satisfied. Put simply average precautionary savings are higher than precautionary savings of “representative” agent with average asset holdings. There is one caveat however. As Fig.1 illustrates, optimal consumption rule diverts from the second order approximation as individual wealth approaches zero. Thus there is a fraction of population in the original economy with wealth close to zero whose extra savings (compared to certainty case) are less than $\varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{\mathbb{E}_I a_t + h_{I,t}}$. However this fraction with lowest possible assets is miserable and it is highly improbable that it changes the inequality sign. To rule out completely such possibility we generously set $a_{I,t}^* = 2\mathbb{E}_I a_t$ for lower bound estimation.

On the opposite, if we choose $a_{I,t}^* = 0$ we produce an economy with relatively high average precautionary saving. All individuals in such economy behave with much prudence as if they had zero assets. Again Fig.1 shows that there are some individuals with negative assets whose extra savings are greater than $\varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{h_{I,t}}$. The fraction of such individuals is likely to be very small and it is highly improbable that they can make average saving in the original economy greater than $\varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{h_{I,t}}$. To rule out such possibility completely we set $a_{I,t}^* = -\mathbb{E}_I a_t$ for the upper bound economy.

Note that with this new setting lower and upper bound economies become more complicated objects compared to (15)–(16) and (17)–(18). Now there are two precautionary saving functions $\varphi_{l,t}$ one for each state instead of just one state-average φ_t , two conditional means $\mathbb{E}_l a_t$ instead of one $\mathbb{E} a_t$, three human wealth functions $h_{l,t}$, Δh_{t+1} instead of one Δh_{t+1} and finally propensity to consume term Ω_t is no longer constant with $\theta > 1$. Thus, number of equations that characterizes these economies is increased by five. This is the price one has to pay to reach higher level of precision than by using simpler bounds of the previous section.

How this setting translates into steady state values of interest rate, capital stock, processes for consumption, asset accumulation and asset inequality?

The key finding is that the steady states for aggregates in the boundary economies are extremely close to each other. The difference between steady state values for capital stock in two economies is 0,05% (0,02% for average consumption). This means that the original economy can be pinned down with astonishing precision. If we apply simpler bounds presented in (15)–(16) and (17)–(18) the difference between capital stocks widens to 1.5% (0,54% between average consumptions). Thus even in this case we gain very good approximation.

Further we report basic equilibrium relations in one of the two refined boundary economies. Since they form extremely narrow channel there is no observable difference between the two.

We find that equilibrium interest rate is lower but very close to $1/\beta$, the steady state value for an economy without uncertainty. The difference between the two is 0.00005 quarterly. This corresponds to $\lambda = 0.99996$, compared to unity in certainty case. Thus idiosyncratic uncertainty implied by this model leads to 0.2% shift of capital stock and 0.07% shift of average consumption from the certainty case. This shows that effects of these shocks are considerably smaller than effects of aggregate productivity shocks in the standard RBC models.

Individual consumption clearly depends on employment status. Equilibrium consumption strategies for both states are as follows

$$c_G^j = 0.01 \cdot (a_t^j + 64.2) - 0.002 \quad (40)$$

$$c_B^j = 0.01 \cdot (a_t^j + 62.7) - 0.003 \quad (41)$$

In the certainty case individual consumption would be

$$c^j = 0.01 \cdot (a_t^j + 64.1)$$

Figures in parentheses represent human wealth in corresponding state measured in quarterly output. Figures on the right of (40), (41) capture precautionary savings in both states. We see that uncertainty about future income increases savings of employed by 0.2% of output and by 0.3% of unemployed. These figures might give impression that precautionary savings are

minor. However they should be put in perspective. Recall that implied income uncertainty is very small as well: employed faces a 2% chance that his human wealth falls by 2% in the next period. Thus, in relative terms prudent behavior is not negligible as will become clear in further example with increased income uncertainty.

Consumption is extremely smooth. When individual becomes unemployed he reduces his consumption by just 1.6% . A slight decrease is due mainly to reduction in human wealth. Since current labor income falls sharply for unemployed and consumption is stable, assets are melting in bad times. Specifically, we find that assets evolve

$$a_{t+1}^j = 0.99996 \cdot a_t^j + 0.03 \text{ for employed}$$

$$a_{t+1}^j = 0.99996 \cdot a_t^j - 0.63 \text{ for unemployed}$$

Thus, individual asset holding are continuously changing whereas average assets for two states and whole economy do not.

One of the most intriguing results concerns equilibrium measure of inequality. We find that even small labor income uncertainty builds into large variations of assets and income. Coefficient of asset variation, $\sigma_a/\mathbb{E}A$, is as high as 4.2. Interestingly this value corresponds nicely to what one finds in US data.

Now we show how shifts in parameters affect the results. First let us look at the effects of growing uncertainty.

In the following example probability p_B is increased to 0.94. This translates in that average duration of bad times goes to 16 quarters and difference in human wealth between the states goes to 17% from just 1.5 quarters and 2% in the benchmark case. We report

$$\lambda = 0.9991$$

$$R = 1/\beta - 0.001$$

$$\sigma_a/\mathbb{E}A = 4.7$$

$$c_G^j = 0.01 \cdot (a_t^j + 71.4) - 0.08$$

$$c_B^j = 0.01 \cdot (a_t^j + 61.5) - 0.09$$

We see that increased income uncertainty lowers interest rate. Steady state shifts much further than in the benchmark case. Capital stock moves by 4,2% and average consumption by 1,5% from certainty case. Measure of asset inequality stays virtually unaffected. However, precautionary savings grow substantially: each period individual in both states saves additional 8% and 9% of output compared to the certainty case.

In order to analyze the impact of risk averseness we set $\theta = 5$ with other parameter values as in the benchmark case. We report the following results

$$\lambda = 0.99991$$

$$R = 1/\beta - 0.0001$$

$$\sigma_a/\mathbb{E}A = 2.9$$

$$c_G^j = 0.01 \cdot (a_t^j + 68.9) - 0.0066$$

$$c_G^j = 0.01 \cdot (a_t^j + 67.4) - 0.0073$$

This shows that measure of risk averseness has strong effect on equilibrium asset deviation. Growing risk averseness also increases precautionary savings.

V. Conclusion

The purpose of this paper was to characterize rigorously the dynamics of economies with idiosyncratic income shocks. To this end we developed an approach that helps to fulfill the general equilibrium analysis of such economies. The main finding is that it is possible to construct an analytically tractable economy, so called constant prudence economy, that replicates the behavior of original economy with great precision. We believe that this approach can prove helpful in other settings that include idiosyncratic shocks of different nature.

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Appendix I

Proposition 1

Let $T_I(\alpha_{I,t}^j, \mathbf{z})$ for $I=G,B$ be analytic functions in $\alpha_{I,t}^j$ defined on the domain A . An economy is characterized by a dichotomy of means and higher moments only if transition functions T_I are linear in $\alpha_{I,t}^j$ and take the form:

$$T_I(\alpha_{I,t}^j, \mathbf{z}) = \lambda_t(\mathbf{z}) \cdot \alpha_{I,t}^j + \mu_{I,t}(\mathbf{z})$$

Proof

To simplify the notation let a' denote the next period value of a . Then an individual agent that holds a in the current period will hold in the next period $a' = T_G(a, \mathbf{z})$ if she is currently at the good state and $a' = T_B(a, \mathbf{z})$ if she is at the bad state.

Consider the two delta-function type distributions of a :

$f_G^{(1)}(a) = \delta(a - a_0)$, $f_B^{(1)}(a) = \delta(a - a_0)$ and $f_G^{(2)}(a) = \delta(a - (a_0 + \Delta))$, $f_B^{(2)}(a) = \delta(a - (a_0 - \frac{\alpha_G}{\alpha_B} \cdot \Delta))$ where α_G, α_B are the fraction of population in good and bad state. Note that both distributions are constructed to produce the same unconditional mean a_0 . Using the transition functions for two states it is easy to verify that next period mean values of a produced by two distributions are:

$$\mathbb{E}a'(1) = \alpha_G \cdot T_G(a_0, \mathbf{z}) + \alpha_B \cdot T_B(a_0, \mathbf{z}) \quad (1)$$

$$\mathbb{E}a'(2) = \alpha_G \cdot T_G(a_0 + \Delta, \mathbf{z}) + \alpha_B \cdot T_B(a_0 - \frac{\alpha_G}{\alpha_B} \cdot \Delta, \mathbf{z}) \quad (2)$$

By assumption of the dichotomy next period mean does not depend on the characteristics of current distribution other than its current mean value. As both distributions has same mean, next period means are equal to each other: $\mathbb{E}a'(1) = \mathbb{E}a'(2)$. Subtracting (1) from (2) and rearranging yields the following equation:

$$T_G(a_0, \mathbf{z}) - T_G(a_0 + \Delta, \mathbf{z}) = \frac{\alpha_B}{\alpha_G} \cdot \left(T_B\left(a_0 - \frac{\alpha_G}{\alpha_B} \cdot \Delta, \mathbf{z}\right) - T_B(a_0, \mathbf{z}) \right)$$

This equation must hold for any a_0 and Δ provided that arguments belong to the domain. Dividing both sides by Δ and rewriting the equation:

$$\frac{T_G(a_0 + \Delta, \mathbf{z}) - T_G(a_0, \mathbf{z})}{\Delta} = \frac{T_B\left(a_0 - \frac{\alpha_G}{\alpha_B} \cdot \Delta, \mathbf{z}\right) - T_B(a_0, \mathbf{z})}{\left(-\frac{\alpha_G}{\alpha_B} \cdot \Delta\right)} \quad (3)$$

Letting Δ go to zero we see that two sides of this equation represent partial derivatives of transition functions in respect to a . We can rewrite it in the form

$$\frac{\partial T_G(a, \mathbf{z})}{\partial a} = \frac{\partial T_B(a, \mathbf{z})}{\partial a} \quad (4)$$

This holds for any a in the domain. Thus, transition functions can be related

$$T_G(a, \mathbf{z}) = T_B(a, \mathbf{z}) + c(\mathbf{z})$$

Use this relation to substitute T_B in the equation (3) and rewrite it as:

$$T_G(a_0, \mathbf{z}) - T_G(a_0 + \Delta, \mathbf{z}) = \frac{\alpha_B}{\alpha_G} \cdot \left(T_G\left(a_0 - \frac{\alpha_G}{\alpha_B} \cdot \Delta, \mathbf{z}\right) - T_G(a_0, \mathbf{z}) \right) \quad (5)$$

Since transition functions are analytic in a they can be represented by convergent Taylor series for small enough Δ :

$$T_G(a_0 + \Delta, \mathbf{z}) = T_G(a_0, \mathbf{z}) + \sum_{n=1}^{\infty} C_n \cdot \Delta^n \quad (6)$$

$$T_G\left(a_0 - \frac{\alpha_G}{\alpha_B} \cdot \Delta, \mathbf{z}\right) = T_G(a_0, \mathbf{z}) + \sum_{n=1}^{\infty} C_n \cdot \left(-\frac{\alpha_G}{\alpha_B} \cdot \Delta\right)^n \quad (7)$$

where $C_n = \frac{1}{n!} \cdot T_G^{(n)}(a_0, \mathbf{z})$

Substituting (6) and (7) into (5) and isolating the first term in the series yields:

$$-C_1 \cdot \Delta - \sum_{n=2}^{\infty} C_n \cdot \Delta^n = -C_1 \cdot \Delta + \frac{\alpha_B}{\alpha_G} \cdot \sum_{n=2}^{\infty} C_n \cdot \left(-\frac{\alpha_G}{\alpha_B} \cdot \Delta\right)^n$$

or

$$\sum_{n=2}^{\infty} C_n \cdot \left[1 - \left(-\frac{\alpha_G}{\alpha_B}\right)^{n-1}\right] \cdot \Delta^n = 0.$$

The last equation must hold for any (small enough) Δ . This is true only if $C_n = 0$ for $n = 2, \dots, \infty$. Recalling that $C_n = \frac{1}{n!} \cdot T_G^{(n)}(a_0, \mathbf{z})$ and that the value a_0 was picked arbitrary from the domain we conclude that $T_G^{(n)}(a, \mathbf{z}) = 0$ for $n = 2, \dots, \infty$ for any a .

Thus, $T_G(a, \mathbf{z})$ is linear in respect to a : $T_G(a, \mathbf{z}) = \lambda(\mathbf{z}) \cdot a + \mu_G(\mathbf{z})$. Since it was shown that both transition functions differ by $c(\mathbf{z})$, $T_B(a, \mathbf{z}) = \lambda(\mathbf{z}) \cdot a + \mu_B(\mathbf{z})$.

Proposition 2

If the law of motion for individual state variable satisfies a linear form $\mathbf{a}_{t+1}^j = \lambda_t \cdot \mathbf{a}_{I,t}^j + \mu_{I,t}$ then:

Mean $\mathbb{E}(\mathbf{a}_t)$ follows:

$$\mathbb{E}(\mathbf{a}_{t+1}) = \lambda_t \cdot \mathbb{E}(\mathbf{a}_t) + \mathbb{E}(\mu_{I,t})$$

where $\mathbb{E}(\mu_{I,t}) = \alpha_G \mu_{G,t} + \alpha_B \mu_{B,t}$

Mean square $\mathbb{E}(\mathbf{a}_{t+1}^2)$ follows:

$$\mathbb{E}(\mathbf{a}_{t+1}^2) = \lambda_t^2 \cdot \mathbb{E}(\mathbf{a}_t^2) + 2 \cdot \lambda_t \cdot \mathbb{E}(\mu_{I,t} \cdot \mathbb{E}_I(\mathbf{a}_t)) + \mathbb{E}(\mu_{I,t}^2)$$

where $\mathbb{E}(\mu_{I,t} \cdot \mathbb{E}_I(\mathbf{a}_t)) = \alpha_G \cdot \mu_{G,t} \cdot \mathbb{E}_G(\mathbf{a}_t) + \alpha_B \cdot \mu_{B,t} \cdot \mathbb{E}_B(\mathbf{a}_t)$ and

$$\mathbb{E}(\mu_{I,t}^2) = \alpha_G \cdot \mu_{G,t}^2 + \alpha_B \cdot \mu_{B,t}^2$$

Higher moments are related by the chain rule of the form:

$$\mathbb{E}(\mathbf{a}_{t+1}^n) = \lambda_t^n \cdot \mathbb{E}(\mathbf{a}_t^n) + F_n(\mathbb{E}_I(\mathbf{a}_t^{n-i}), \mu_{I,t}, \lambda_t)$$

where $F_n(\mathbb{E}_I(\mathbf{a}_t^{n-i}), \mu_{B,t}, \lambda_t)$ represents a function of moments less than n .

Proof

To obtain these equations in the case of continuous probability distributions we first derive an evolution equation for density functions for both states. Consider a small neighborhood of a_0 : $(a_0 - \varepsilon, a_0 + \varepsilon)$. Then $f_{G,t+1}(a_0)$ is probability density of population at the point a_0 in the good state in period $t+1$ and $f_{G,t+1}(a_0) \cdot 2\varepsilon$ is the fraction of population in the neighborhood of a_0 of the population in the good state in period $t+1$. There are two ways an individual agent finds himself in this fraction. Firstly she could arrive from the good state in the previous period if her assets were in the domain $(\frac{a_0 - \mu_G}{\lambda} - \frac{\varepsilon}{\lambda}, \frac{a_0 - \mu_G}{\lambda} + \frac{\varepsilon}{\lambda})$. The mass of such agents was $f_{G,t}(\frac{a_0 - \mu_G}{\lambda}) \cdot 2\frac{\varepsilon}{\lambda}$ in period t . Since only part p_G of that mass retained the state there is mass $p_G \cdot f_{G,t}(\frac{a_0 - \mu_G}{\lambda}) \cdot 2\frac{\varepsilon}{\lambda}$ of individuals who are in the good state in $t+1$ in the given domain and retained their state from the previous period. Alternatively she could arrive to the given neighborhood in the good state in $t+1$ from the bad state in period t if her assets in t lied in the domain $(\frac{a_0 - \mu_B}{\lambda} - \frac{\varepsilon}{\lambda}, \frac{a_0 - \mu_B}{\lambda} + \frac{\varepsilon}{\lambda})$. Since probability of changing states from B to G is $1 - p_B$ and $\frac{\alpha_G}{\alpha_B}$ is the ratio of population sizes in both states, there is mass $\frac{\alpha_G}{\alpha_B} \cdot (1 - p_B) \cdot f_{B,t}(\frac{a_0 - \mu_B}{\lambda}) \cdot 2\frac{\varepsilon}{\lambda}$ of individuals who are in the good state in $t+1$ in the given domain and who switched the state from B to G . Collecting the terms one has the following mass conservation equation

$$f_{G,t+1}(a_0) \cdot 2\varepsilon = p_G \cdot f_{G,t} \left(\frac{a_0 - \mu_G}{\lambda} \right) \cdot 2 \frac{\varepsilon}{\lambda} + \frac{\alpha_G}{\alpha_B} \cdot (1 - p_B) \cdot f_{B,t} \left(\frac{a_0 - \mu_B}{\lambda} \right) \cdot 2 \frac{\varepsilon}{\lambda}$$

Dividing by 2ε , noting that $\frac{\alpha_G}{\alpha_B} = \frac{1-p_G}{1-p_B}$ and recalling that a_0 was chosen arbitrary

$$f_{G,t+1}(a) = p_G \cdot \frac{f_{G,t} \left(\frac{a - \mu_{G,t}}{\lambda_t} \right)}{\lambda_t} + (1 - p_G) \cdot \frac{f_{B,t} \left(\frac{a - \mu_{B,t}}{\lambda_t} \right)}{\lambda_t} \quad (1)$$

Evolution equation for conditional density function for bad state is derived in complete analogy

$$f_{B,t+1}(a) = p_B \cdot \frac{f_{B,t} \left(\frac{a - \mu_{B,t}}{\lambda_t} \right)}{\lambda_t} + (1 - p_B) \cdot \frac{f_{G,t} \left(\frac{a - \mu_{G,t}}{\lambda_t} \right)}{\lambda_t} \quad (2)$$

Using evolution equation for density functions one can obtain evolution equations for the moments. Convenient way to proceed is to move from density functions to moment generating function representation.

By definition

$$m_{I,t}(x) \equiv \int_{-\infty}^{\infty} f_{I,t}(a) e^{xa} da$$

Multiplying equations (1) and (2) by e^{xa} , integrating them in respect to a and using replacement of variables in order to carry through the integration of the right hand side of (1) and (2) yield

$$m_{G,t+1}(x) = p_G \cdot e^{\mu_{G,t} \cdot x} \cdot m_{G,t}(\lambda_t x) + (1 - p_G) \cdot e^{\mu_{B,t} \cdot x} \cdot m_{B,t}(\lambda_t x) \quad (3)$$

$$m_{B,t+1}(x) = p_B \cdot e^{\mu_{B,t} \cdot x} \cdot m_{B,t}(\lambda_t x) + (1 - p_B) \cdot e^{\mu_{G,t} \cdot x} \cdot m_{G,t}(\lambda_t x) \quad (4)$$

Denoting by a dote derivative in respect to x and differentiating equations (3) and (4)

$$\begin{aligned} \dot{m}_{G,t+1}(x) &= p_G \cdot [\mu_{G,t} \cdot m_{G,t}(\lambda_t x) + \dot{m}_{G,t}(\lambda_t x)] \cdot e^{\mu_{G,t} \cdot x} + (1 - p_G) \\ &\cdot [\mu_{B,t} \cdot m_{B,t}(\lambda_t x) + \dot{m}_{B,t}(\lambda_t x)] \cdot e^{\mu_{B,t} \cdot x} \quad (5) \end{aligned}$$

$$\begin{aligned} \dot{m}_{B,t+1}(x) &= p_B \cdot [\mu_{B,t} \cdot m_{B,t}(\lambda_t x) + \dot{m}_{B,t}(\lambda_t x)] \cdot e^{\mu_{B,t} \cdot x} + (1 - p_B) \\ &\cdot [\mu_{G,t} \cdot m_{G,t}(\lambda_t x) + \dot{m}_{G,t}(\lambda_t x)] \cdot e^{\mu_{G,t} \cdot x} \quad (6) \end{aligned}$$

By properties of moment generating functions

$$m_{I,t}(0) = 1$$

$\dot{m}_{I,t}(0) = \mathbb{E}_I(a_t)$, where $\mathbb{E}_I(a_t)$ denotes conditional mean of a in states $I=G,B$ in period t .

Using these facts in equations (5) and (6) one obtains

$$\mathbb{E}_G(a_{t+1}) = p_G \cdot (\mu_{G,t} + \lambda_t \cdot \mathbb{E}_G(a_t)) + (1 - p_G) \cdot (\mu_{B,t} + \lambda_t \cdot \mathbb{E}_B(a_t))$$

$$\mathbb{E}_B(a_{t+1}) = p_B \cdot (\mu_{B,t} + \lambda_t \cdot \mathbb{E}_B(a_t)) + (1 - p_B) \cdot (\mu_{G,t} + \lambda_t \cdot \mathbb{E}_G(a_t))$$

Evolution equation for unconditional mean follows from these equations using relation $\mathbb{E}(a_t) = \alpha_G \mathbb{E}_G(a_t) + \alpha_B \mathbb{E}_B(a_t)$ and definition of α_G and α_B . It takes the form

$$\mathbb{E}(\mathbf{a}_{t+1}) = \lambda_t \cdot \mathbb{E}(\mathbf{a}_t) + \mathbb{E}(\boldsymbol{\mu}_{I,t})$$

where $\mathbb{E}(\boldsymbol{\mu}_{I,t}) = \alpha_G \boldsymbol{\mu}_{G,t} + \alpha_B \boldsymbol{\mu}_{B,t}$

To derive equations for conditional mean squares $\mathbb{E}_I(a_t^2)$ differentiate equations (5) and (6) in respect to x and use the following properties of moment producing functions: $m_{I,t}(0) = 1$, $\dot{m}_{I,t}(0) = \mathbb{E}_I(a_t)$, $\ddot{m}_{I,t}(0) = \mathbb{E}_I(a_t^2)$. This gives

$$\begin{aligned} \mathbb{E}_G(a_{t+1}^2) &= p_G \cdot \left(\lambda_t^2 \cdot \mathbb{E}_G(a_t^2) + 2 \cdot \lambda_t \cdot \mu_{G,t} \cdot \mathbb{E}_G(a_t) + (\mu_{G,t})^2 \right) + \\ &(1 - p_G) \cdot \left(\lambda_t^2 \cdot \mathbb{E}_B(a_t^2) + 2 \cdot \lambda_t \cdot \mu_{B,t} \cdot \mathbb{E}_B(a_t) + (\mu_{B,t})^2 \right) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}_B(a_{t+1}^2) &= p_B \cdot \left(\lambda_t^2 \cdot \mathbb{E}_B(a_t^2) + 2 \cdot \lambda_t \cdot \mu_{B,t} \cdot \mathbb{E}_B(a_t) + (\mu_{B,t})^2 \right) + \\ &(1 - p_B) \cdot \left(\lambda_t^2 \cdot \mathbb{E}_G(a_t^2) + 2 \cdot \lambda_t \cdot \mu_{G,t} \cdot \mathbb{E}_G(a_t) + (\mu_{G,t})^2 \right) \end{aligned}$$

Equation for unconditional mean square follows from these two equations using the relation $\mathbb{E}(a_t^2) = \alpha_G \mathbb{E}_G(a_t^2) + \alpha_B \mathbb{E}_B(a_t^2)$. It can be written in the form

$$\mathbb{E}(a_{t+1}^2) = \lambda_t^2 \cdot \mathbb{E}(a_t^2) + 2 \cdot \lambda_t \cdot \mathbb{E}(\boldsymbol{\mu}_{I,t} \cdot \mathbb{E}_I(a_t)) + \mathbb{E}(\boldsymbol{\mu}_{I,t}^2)$$

where $\mathbb{E}(\boldsymbol{\mu}_{I,t} \cdot \mathbb{E}_I(a_t)) = \alpha_G \cdot \mu_{G,t} \cdot \mathbb{E}_G(a_t) + \alpha_B \cdot \mu_{B,t} \cdot \mathbb{E}_B(a_t)$ and

$$\mathbb{E}(\boldsymbol{\mu}_{I,t}^2) = \alpha_G \cdot \mu_{G,t}^2 + \alpha_B \cdot \mu_{B,t}^2$$

Same method can be used to derive equations for expected values of powers of a higher than two. For instance, third moments can be expressed by twice differentiating equations (5) and (6) and using the fact $\ddot{m}_{I,t}(0) = \mathbb{E}_I(a_t^3)$. This procedure leads to the chain equations for higher powers of a of the form

$$\mathbb{E}(\mathbf{a}_{t+1}^n) = \lambda_t^n \cdot \mathbb{E}(\mathbf{a}_t^n) + F_n(\mathbb{E}_I(\mathbf{a}_t^{n-i}), \boldsymbol{\mu}_{I,t}, \lambda_t)$$

where $F_n(\mathbb{E}_I(\mathbf{a}_t^{n-i}), \boldsymbol{\mu}_{I,t}, \lambda_t)$ represents a function of moments less than n .

If distribution is of discrete type than probability density can be represented by generalized δ -functions. Assume that current period conditional distributions are given by $f_G(a) = \sum_{i=1}^{n_G} \omega_G^i \cdot \delta(a - a_G^i)$ and $f_B(a) = \sum_{i=1}^{n_B} \omega_B^i \cdot \delta(a - a_B^i)$ with $\sum_{i=1}^{n_I} \omega_I^i = 1$ ($I=G,B$). Transition equations for individual a in the two states are $a'_I = \lambda \cdot a_I + \mu_I$ ($I=G,B$).

It is easy to verify that next period conditional density functions can be written as

$$f'_G(a) = p_G \cdot \sum_{i=1}^{n_G} \omega_G^i \cdot \delta\left(a - (\lambda \cdot a_G^i + \mu_G)\right) + (1 - p_G) \cdot \sum_{i=1}^{n_B} \omega_B^i \cdot \delta\left(a - (\lambda \cdot a_B^i + \mu_B)\right)$$

and similarly

$$f'_B(a) = p_B \cdot \sum_{i=1}^{n_B} \omega_B^i \cdot \delta\left(a - (\lambda \cdot a_B^i + \mu_B)\right) + (1 - p_B) \cdot \sum_{i=1}^{n_G} \omega_G^i \cdot \delta\left(a - (\lambda \cdot a_G^i + \mu_G)\right)$$

These density function evolution equations imply the above stated dynamic equations for expected values of a . We confine ourselves by demonstration of equation for means. This is obtained by multiplying equations () and () by a integrating them in respect to a . Recalling the property of δ -functions $\int_{-\infty}^{\infty} a \cdot \delta(a - a_0) da = a_0$ we have

$$\begin{aligned} \mathbb{E}_G(a') &= p_G \cdot \sum_{i=1}^{n_G} \omega_G^i \cdot (\lambda \cdot a_G^i + \mu_G) + (1 - p_G) \cdot \sum_{i=1}^{n_B} \omega_B^i \cdot (\lambda \cdot a_B^i + \mu_B) \\ \mathbb{E}_B(a') &= p_B \cdot \sum_{i=1}^{n_B} \omega_B^i \cdot (\lambda \cdot a_B^i + \mu_B) + (1 - p_B) \cdot \sum_{i=1}^{n_G} \omega_G^i \cdot (\lambda \cdot a_G^i + \mu_G) \end{aligned}$$

Noting that $\sum_{i=1}^{n_I} \omega_I^i \cdot a_I^i = \mathbb{E}_I(a)$ for $I=G,B$ these equations become

$$\begin{aligned} \mathbb{E}_G(a') &= p_G \cdot (\mathbb{E}_G(a) + \mu_G) + (1 - p_G) \cdot (\mathbb{E}_B(a) + \mu_B) \\ \mathbb{E}_B(a') &= p_B \cdot (\mathbb{E}_B(a) + \mu_B) + (1 - p_B) \cdot (\mathbb{E}_G(a) + \mu_G) \end{aligned}$$

Since $\mathbb{E}(a) = \alpha_G \mathbb{E}_G(a) + \alpha_B \mathbb{E}_B(a)$, combining these equations gives

$$\mathbb{E}(a') = \lambda \cdot \mathbb{E}(a) + \mathbb{E}(\mu_I)$$

where $\mathbb{E}(\mu_I) = \alpha_G \mu_G + \alpha_B \mu_B$

Numerical example of density function evolution

We set $p_G = p_B = 0.5$. Transition equations for individual state variable x are symmetrical

$$\begin{aligned} x_{t+1} &= 0.9x_t + 1 \text{ for } I = G \\ x_{t+1} &= 0.9x_t - 1 \text{ for } I = B \end{aligned}$$

Initial distribution is uniform with support $[0,5]$ for both states. Equations (1) and (2) show that in this case conditional density functions for both states coincide: $f_{G,t}(x) = f_{B,t}(x) \equiv f(t, x)$. The following figures present how the density function evolves in time. The limit distribution function is symmetrical around zero. In this example after just 12 periods density function looks very smooth but its microstructure reveals extremely complex fractal-like pattern.

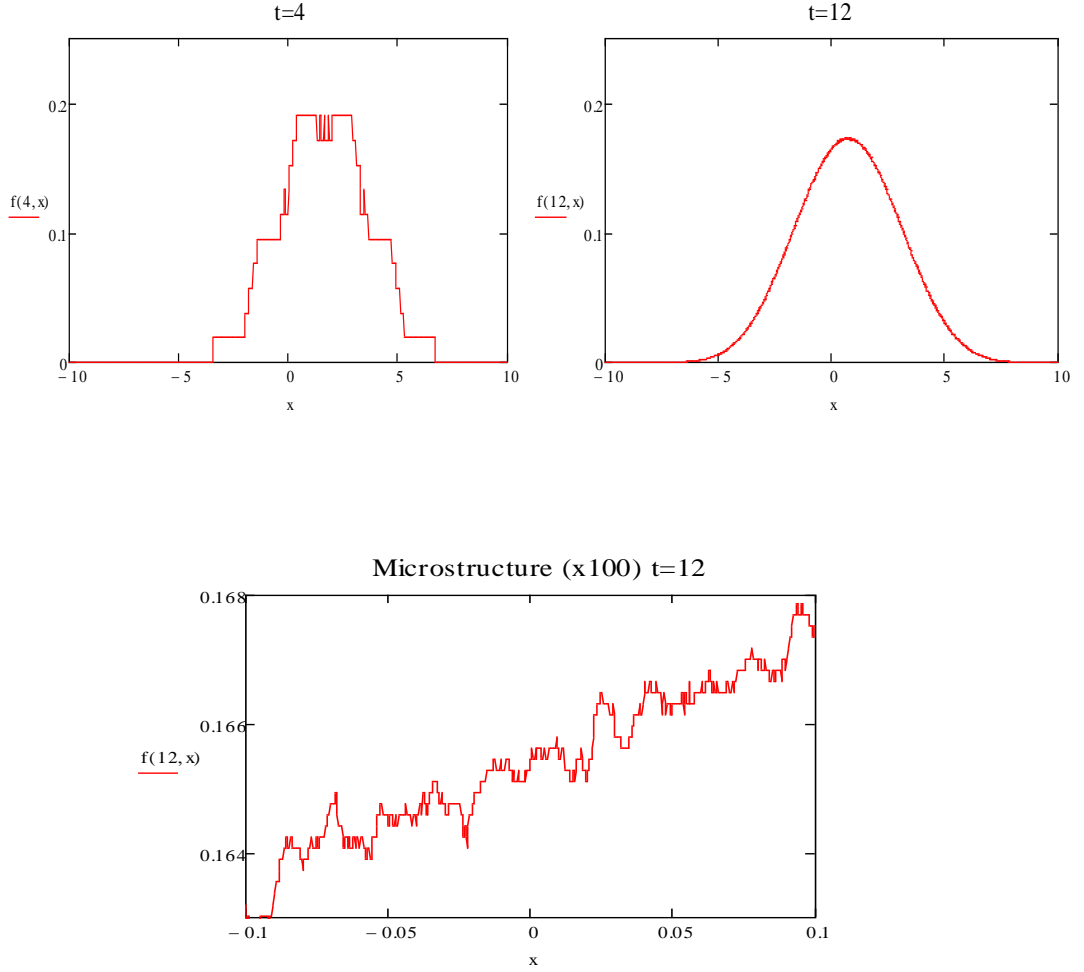


Fig. 4. Numerical example of density function evolution

Proposition 3

Suppose $\left| \frac{\Delta h_{I,t+1}}{a_t^j + h_{I,t}} \right|$ is small. Then the optimal consumption strategy for the agent j at state I at time t can be expressed as follows

$$c_{I,t}^j = \left[\Omega_t - \varphi_{I,t} \cdot \left(\frac{\Delta h_{I,t+1}}{a_t^j + h_{I,t}} \right)^2 + \mathcal{O} \left(\left(\frac{\Delta h_{I,t+1}}{a_t^j + h_{I,t}} \right)^3 \right) \right] \cdot (a_t^j + h_{I,t})$$

where function Ω_t satisfies equation

$$\beta \cdot R_{t+1} \cdot \left(R_{t+1} \Omega_{t+1} \cdot \frac{1 - \Omega_t}{\Omega_t} \right)^{-\theta} = 1$$

and precautionary saving functions $\varphi_{I,t}$ ($I=G,B$) satisfy the system of two equations

$$\varphi_{I,t} = \frac{\Omega_t(1-\Omega_t)}{\Omega_{t+1}} [p_I \varphi_{I,t+1} + (1-p_I) \varphi_{I^*,t+1}] + \frac{(\theta+1)}{2} \frac{\Omega_t}{(1-\Omega_t)R_{t+1}^2} p_I (1-p_I)$$

Proof

Take as a starting point the Euler equation for an individual agent who is at state G in period t :

$$u'(c_{G,t}) = \beta R_{t+1} [p_G \cdot u'(c_{G,t+1}) + (1-p_G) \cdot u'(c_{B,t+1})] \quad (1)$$

The identity relations that defines propensity to consume at good state $\Omega_{G,t}$ and bad state $\Omega_{B,t}$:

$$c_{G,t} = \Omega_{G,t} \cdot (a_t + h_{G,t}) \quad (2)$$

$$c_{G,t+1} = \Omega_{G,t+1} \cdot (a_{t+1} + h_{G,t+1}) \quad (3)$$

$$c_{B,t+1} = \Omega_{B,t+1} \cdot (a_{t+1} + h_{G,t+1}) \quad (4)$$

The budget constraint is

$$a_{t+1} = R_{t+1} \cdot (a_t + w_{G,t} - c_{G,t}) \quad (5)$$

Substituting (2) into (5)

$$a_{t+1} = R_{t+1} [(1 - \Omega_{G,t})(a_t + h_{G,t}) + w_{G,t} - h_{G,t}] \quad (6)$$

Equation for human wealth at good state is given by:

$$h_{G,t} = w_{G,t} + \frac{1}{R_{t+1}} (p \cdot h_{G,t+1} + (1-p_G) \cdot h_{B,t+1}) \quad (7)$$

Substituting (6) into (3) and using (7) one obtains

$$c_{G,t+1} = R_{t+1} \Omega_{G,t+1} (1 - \Omega_{G,t}) \cdot (a_t + h_{G,t}) + (1-p_G) \Omega_{G,t+1} \cdot \Delta h_{t+1} \quad (8)$$

$$\text{where } \Delta h_{t+1} \equiv h_{G,t+1} - h_{B,t+1} \quad (9)$$

Similar equation holds in respect to $c_{B,t+1}$, and is derived in the same way as (8)

$$c_{B,t+1} = R_{t+1} \Omega_{B,t+1} (1 - \Omega_{G,t}) \cdot (a_t + h_{G,t}) - p_G \Omega_{B,t+1} \cdot \Delta h_{t+1} \quad (10)$$

Inserting (2) (9) and (10) into (1) and dividing both sides by $(a_t + h_{G,t})^{-\theta}$:

$$\Omega_{G,t}^{-\theta} = \beta R_{t+1} p_G [R_{t+1} \Omega_{G,t+1} (1 - \Omega_{G,t}) + (1-p_G) \Omega_{G,t+1} \cdot \varepsilon_{G,t}]^{-\theta} + \beta R_{t+1} (1-p_G) [R_{t+1} \Omega_{B,t+1} (1 - \Omega_{G,t}) - p_G \cdot \Omega_{B,t+1} \cdot \varepsilon_{G,t}]^{-\theta} \quad (11)$$

where

$$\varepsilon_{G,t} \equiv \frac{\Delta h_{t+1}}{a_t + h_{G,t}}$$

In complete analogy to derivation of (11) started with Euler equation (1) it can be repeated for bad-state Euler equation

$$u'(c_{B,t}) = \beta R_{t+1} [p_B \cdot u'(c_{B,t+1}) + (1-p_B) \cdot u'(c_{G,t+1})]$$

This provides us with equation in respect to $\Omega_{B,t}$ analogous to (11):

$$\Omega_{B,t}^{-\theta} = \beta R_{t+1} \cdot p_B \cdot [R_{t+1} \Omega_{B,t+1} (1 - \Omega_{B,t}) - (1 - p_B) \Omega_{G,t+1} \cdot \varepsilon_{B,t}]^{-\theta} + \beta R_{t+1} (1 - p_B) \cdot [R_{t+1} \Omega_{G,t+1} (1 - \Omega_{B,t}) + p_B \cdot \Omega_{G,t+1} \cdot \varepsilon_{B,t}]^{-\theta} \quad (12)$$

where $\varepsilon_{B,t} \equiv \frac{\Delta h_{t+1}}{a_t + h_{B,t}}$

$\varepsilon_{G,t}$ and $\varepsilon_{B,t}$ defines perturbation terms in (11) and (12).

We will now use the method of perturbations to determine approximate solution for propensities to consume $\Omega_{G,t}$ and $\Omega_{B,t}$. First note that unperturbed equations (11) and (12) coincide. Let Ω_t be the solution of unperturbed equation

$$\Omega_t^{-\theta} = \beta R_{t+1} [R_{t+1} \Omega_{t+1} (1 - \Omega_t)]^{-\theta} \quad (13)$$

To find first order approximation assume:

$$\Omega_{G,t} = \Omega_t + \gamma_{G,t} \cdot \varepsilon_{G,t} + \mathcal{O}(\varepsilon^2) \quad (14)$$

$$\Omega_{B,t} = \Omega_t + \gamma_{B,t} \cdot \varepsilon_{B,t} + \mathcal{O}(\varepsilon^2) \quad (15)$$

Using (14) and (15) and taking linear expansion of (11) and (12) in respect to ε it is easy to verify that $\gamma_{G,t} = 0$ and $\gamma_{B,t} = 0$. Thus the first non-zero perturbation term is of second order:

$$\Omega_{G,t} = \Omega_t - \varphi_{G,t} \cdot \varepsilon_{G,t}^2 + \mathcal{O}(\varepsilon^3) \quad (16)$$

$$\Omega_{B,t} = \Omega_t - \varphi_{B,t} \cdot \varepsilon_{B,t}^2 + \mathcal{O}(\varepsilon^3) \quad (17)$$

Now we can use the form of solution given by (16) and (17) to determine second order perturbation functions $\varphi_{G,t}$ and $\varphi_{B,t}$. Use (16) and (17) to write Taylor expansion of (11) and (12). After isolating terms of order ε^2 we have the following equation

$$\begin{aligned} \frac{1}{\beta R_{t+1}} \theta \cdot \Omega_t^{-(\theta+1)} \varphi_{G,t} &= \theta p_G [R_{t+1} \Omega_{t+1} (1 - \Omega_t)]^{-(\theta+1)} \cdot R_{t+1} [(1 - \Omega_t) \varphi_{G,t+1} - \\ &\Omega_{t+1} \varphi_{G,t} + \theta (1 - p_G) R_{t+1} \Omega_{t+1} (1 - \Omega_t) - \theta + 1 \cdot R_{t+1} (1 - \Omega_t) \varphi_{B,t+1} - \Omega_{t+1} \varphi_{G,t} + \theta \theta + 12 p_G R_{t+1} \\ &+ 1 \Omega_{t+1} (1 - \Omega_t) - \theta + 21 - p_G 2 \Omega_{t+1} + 12 + \theta \theta + 12 (1 - p_G) R_{t+1} \Omega_{t+1} (1 - \Omega_t) - (\theta + 2) p_G 2 \Omega_{t+1} \\ &+ 12] \end{aligned} \quad (16)$$

Similar equation can be obtained from (12) for the second state. It has mirror form to (16) with indexes G and B interchanged. Dividing both sides of (16) by $\theta \cdot \Omega_t^{-(\theta+1)}$ and using (13)

$$\frac{1}{\beta R_{t+1}} \varphi_{G,t} = (\beta R_{t+1})^{-\frac{\theta+1}{\theta}}$$

$$\cdot R_{t+1} [p_G(1 - \Omega_t) \varphi_{G,t+1} - p \Omega_{t+1} \varphi_{G,t} + (1 - p_G)(1 - \Omega_t) \varphi_{B,t+1} - (1 - p_G) \Omega_{t+1} \varphi_{G,t}] + \frac{(\theta + 1)}{2} (\beta R_{t+1})^{-\frac{\theta+2}{\theta}} \frac{\Omega_{t+1}^2}{\Omega_t} [p_G \cdot (1 - p_G)^2 + (1 - p_G) \cdot p_G^2]$$

After some arrangement it simplifies to

$$\varphi_{G,t} = \frac{\Omega_t(1-\Omega_t)}{\Omega_{t+1}} [p_G \varphi_{G,t+1} + (1 - p_G) \varphi_{B,t+1}] + \frac{(\theta+1)}{2} \frac{\Omega_t}{(1-\Omega_t)R_{t+1}^2} p_G(1 - p_G) \quad (17)$$

And mirror equation for $\varphi_{B,t}$

$$\varphi_{B,t} = \frac{\Omega_t(1-\Omega_t)}{\Omega_{t+1}} [p_B \varphi_{B,t+1} + (1 - p_B) \varphi_{G,t+1}] + \frac{(\theta+1)}{2} \frac{\Omega_t}{(1-\Omega_t)R_{t+1}^2} p_B(1 - p_B) \quad (18)$$

For state average function $\varphi_t \equiv \alpha_G \varphi_{G,t} + \alpha_B \varphi_{B,t}$ using (17), (18) and definition of α_G and α_B one obtains

$$\varphi_t = \frac{\Omega_t(1 - \Omega_t)}{\Omega_{t+1}} \cdot \varphi_{t+1} + \frac{(\theta + 1)}{2} \frac{\Omega_t}{(1 - \Omega_t)R_{t+1}^2} \frac{(1 - p_G)(1 - p_B)(p_G + p_B)}{2 - p_G - p_B}$$

Proposition 4

An economy in which individual j consumes according to the rules

$$\overline{c}_{G,t}^j = \Omega_t \cdot (a_t^j + h_{G,t}) - \sqrt{\Omega_t \varphi_t} \cdot \Delta h_{t+1}$$

$$\overline{c}_{B,t}^j = \Omega_t \cdot (a_t^j + h_{B,t}) - \sqrt{\Omega_t \varphi_t} \cdot \Delta h_{t+1}$$

is an upper bound asset accumulation economy.

An economy in which individual j consumes according to the rules

$$\underline{c}_{G,t}^j = \Omega_t \cdot (a_t^j + h_{G,t})$$

$$\underline{c}_{B,t}^j = \Omega_t \cdot (a_t^j + h_{B,t})$$

is a lower bound asset accumulation economy.

Proof

Consider an upper bound case first. Note that consumption rules given in the proposition imply that individual assets change according to:

$$a_{t+1} = \lambda_t \cdot a_{I,t}^j + \mu_{I,t}$$

where $\lambda_t = R_{t+1}(1 - \Omega_t)$ and $\mu_{I,t} = R_{t+1}(w_{I,t} - \Omega_t h_{I,t} + \sqrt{\Omega_t \varphi_t} \cdot \Delta h_{t+1})$ for $I=G,B$. Thus by the proposition 2 this economy is characterized by dichotomy of moments.

Relation between individual optimal consumption rule and second order approximation given by (9) from the text is shown in Fig.1 from the text. It can be seen that there is such value of a at time t and state I that the rule () prescribes zero consumption. Denote this value $a_{I,t}^\#$. Then, by definition

$$a_{I,t}^\# + h_{I,t} \equiv \left(\frac{\varphi_{I,t}}{\Omega_t^*} \right)^{1/2} \cdot \Delta h_{I,t+1} \quad (1)$$

Consider a hypothetical economy in which individual consumption rules are given by

$$c_{I,t}^{MaxP} = \Omega_t \cdot (a_t^j + h_{I,t}) - \varphi_{I,t} \cdot \frac{(\Delta h_{I,t+1})^2}{a_{I,t}^\# + h_{I,t}} \text{ for } I=G,B \quad (2)$$

We call this economy a maximum prudence economy (MP). Substitute (1) into (2)

$$c_{I,t}^{MaxPj} = \Omega_t \cdot (a_t^j + h_{I,t}) - \sqrt{\Omega_t \varphi_{I,t}} \cdot \Delta h_{t+1} \text{ for } I=G,B \quad (3)$$

Maximum prudence economy satisfies conditions of proposition 0 and, thus, is also dichotomic in moments.

It is clear from Fig. 1 that given the same interest rate and wage paths an individual's consumption in maximum prudence economy is less than that in the original economy provided they are at the same state and with same assets:

$$c_{I,t}^{MaxP} (a^j) < c_{I,t} (a^j) \quad (4)$$

Next step is to compare mean assets accumulation paths in the two economies. In order to do so assume the following set identities of parameters for the two economies:

- 1) Identical initial distribution of assets with mean a_0
- 2) Identical paths for capital stock, wages, interest rates
- 3) Every individual J in the MaxPE economy with initial assets a_0^J has his twin in the original economy with the same initial assets a_0^J and the same sequence of states $I_t(J) \ t = 1, \dots, \infty$.

It can be shown that asset accumulation paths for twins J in the two economies are such that

$$a_{I,t}^{MaxP} (J) > a_{I,t} (J) \text{ for any } J,I \text{ and } t = 1, \dots, \infty \quad (5)$$

To demonstrate this, write individual asset accumulation equations for twins J

$$a_{I,t+1}^{MaxP} (J) = R_{t+1} [a_{I,t}^{MaxP} (J) + w_{I,t} - c_{I,t}^{MaxP} (J)] \quad (6)$$

$$a_{I,t+1} (J) = R_{t+1} [a_{I,t} (J) + w_{I,t} - c_{I,t} (J)] \quad (7)$$

Consumption rules in the two economies are such that we can write them in the form

$$c_{I,t}^{MaxP}(J) = \Omega_t \cdot (a_{I,t}^{MaxP}(J) + h_{I,t}) - \Delta_t^{MaxP} \quad (8)$$

$$c_{I,t}(J) = \Omega_t \cdot (a_{I,t}(J) + h_{I,t}) - \Delta_t \quad (9)$$

where precautionary saving functions Δ_t^{MaxP} and Δ_t satisfy

$$\Delta_t^{MaxP} > \Delta_t \text{ for } t = 0, \dots, \infty \quad (10)$$

The last inequality is satisfied by construction of MaxPE economy (see Fig. 1)

Inserting (8) and (9) into (6) and (7) and then subtracting (7) from (6)

$$a_{I,t+1}^{MaxP}(J) - a_{I,t+1}(J) = R_{t+1}(1 - \Omega_t) \left(a_{I,t}^{MaxP}(J) - a_{I,t}(J) \right) + R_{t+1}(\Delta_t^{MaxP} - \Delta_t) \quad (11)$$

Re-denoting (11)

$$x_{t+1} = \lambda_t x_t + u_t \quad (12)$$

$$\text{with } \lambda_t > 0 \text{ and } u_t > 0 \text{ for } t = 0, \dots, \infty \quad (13)$$

Last inequality follows from (10).

Equation (12) can be solved backwards relative to u_t . Solution is given by

$$x_t = \sum_{k=1}^t u_{k-1} \prod_{i=0}^{k-1} \lambda_i \text{ for } t = 1, \dots, \infty \text{ and } x_0 = 0 \quad (14)$$

It follows from (14) and (13) that

$$x_t \equiv a_{I,t}^{MaxP}(J) - a_{I,t}(J) > 0 \text{ or } a_{I,t}^{MaxP}(J) > a_{I,t}(J) \text{ for } t = 1, \dots, \infty \text{ and all } J \quad (14)$$

Integrating in respect to J

$$\mathbb{E}a_t^{MaxP} > \mathbb{E}a \text{ for } t = 1, \dots, \infty \quad (15)$$

Since MaxPE economy has dichotomy of moments, $\mathbb{E}a_t^{MP}$ dynamics does not depend on type distribution but only on identities 1) and 2) (same initial mean assets and price paths). Thus, inequality (15) holds even if initial asset distribution in the two economies differ (but have the same mean) and there is no twins, i.e. it holds in the absence of identity 3).

We proved that MaxPE is an upper bound asset accumulation economy.

Note that MaxPE slightly differs from the consumption rules presented in the proposition

$$\overline{c_{I,t}^j} = \Omega_t \cdot (a_t^j + h_{G,t}) - \sqrt{\Omega_t \varphi_t} \cdot \Delta h_{t+1} \text{ for } I=G,B$$

instead of (3). To see what this implies integrate them on j . This gives

$$\mathbb{E}c_t^{MaxP} = \Omega_t \cdot (\mathbb{E}a_t^{MaxP} + \mathbb{E}h_t) - (\alpha_G \sqrt{\varphi_{G,t}} + \alpha_B \sqrt{\varphi_{B,t}}) \cdot \sqrt{\Omega_t} \Delta h_{t+1} \quad (16)$$

$$\mathbb{E}\overline{c}_t = \Omega_t \cdot (\mathbb{E}a_t + \mathbb{E}h_t) - \sqrt{\Omega_t \varphi_t} \cdot \Delta h_{t+1} \quad (17)$$

By Jensen inequality: $\alpha_G\sqrt{\varphi_{G,t}} + \alpha_B\sqrt{\varphi_{B,t}} < \sqrt{\alpha_G\varphi_{G,t} + \alpha_B\varphi_{B,t}} = \sqrt{\varphi_t}$. Using this fact together with economy-wide asset accumulation equations

$\mathbb{E}a_{t+1}^{MaxP} = R_{t+1}(\mathbb{E}a_t^{MaxP} + w_t - \mathbb{E}c_t^{MaxP})$ and $\mathbb{E}a_{t+1} = R_{t+1}(\mathbb{E}a_t + w_t - \mathbb{E}c_t)$ it is easy to show that (16) and (17) imply

$$\mathbb{E}\bar{a}_t > \mathbb{E}a_t^{MaxP} \text{ for } t = 1, \dots, \infty \quad (18)$$

Finally, from (15) and (18)

$$\mathbb{E}\bar{a}_t > \mathbb{E}a_t \text{ for } t = 1, \dots, \infty$$

This completes the proof for the upper bound case.

The second part of the proposition concerning the lower bound economy can be proven in complete analogy.

Proposition 5

Assume that there is a steady state in the original economy. Then in capital-consumption space this steady state lies between the steady states of lower and upper bound economies $\underline{k}_{ss} < k_{ss} < \bar{k}_{ss}$ and $\underline{c}_{ss} < c_{ss} < \bar{c}_{ss}$

We outlined the proof of this claim in the text. The only element that lacks in this proof is showing that stationary level of asset holdings in the upper bound economy is increasing function of stationary interest rate. To see this plug (15) and (16) in the individual asset accumulation equation (2). This yields

$$a_{t+1}^j = R_{t+1} \cdot (1 - \Omega_t) \cdot a_t^j + [R_{t+1}(w_{I,t} - \Omega_t h_{I,t} + \sqrt{\Omega_t \varphi_t} \cdot \Delta h_{t+1})] \quad (1)$$

Integrating (1) in respect to j and I and dropping time index for stationary levels it becomes

$$a = R \cdot (1 - \Omega) \cdot a + [R(\mathbb{E}w - \Omega \mathbb{E}h + \sqrt{\Omega \varphi} \cdot \mathbb{E}\Delta h)] \quad (2)$$

From equation (7) in the text

$$R(1 - \Omega) = \beta R^{1/\theta} \quad (3)$$

From (2) and (3)

$$a = \frac{R(\mathbb{E}w - \Omega \mathbb{E}h)}{1 - \beta R^{1/\theta}} + \frac{\sqrt{\Omega \varphi} \cdot \mathbb{E}\Delta h}{1 - \beta R^{1/\theta}} \quad (4)$$

Let us inspect the terms that appear on the right of (4).

As $\mathbb{E}h = \frac{R}{R-1} \mathbb{E}w$, $\mathbb{E}w - \Omega \mathbb{E}h = \left(\frac{R(1-\Omega)-1}{R-1}\right) \mathbb{E}w = \frac{\beta R^{1/\theta}-1}{R-1} \mathbb{E}w$ and the first term collapses to $-\frac{R}{R-1} \mathbb{E}w$. From (25) and (32) average wage is a decreasing function of interest rate

$$\mathbb{E}w = \frac{(1-\gamma)a_G^{1-\gamma}}{\delta^{1/(1-\gamma)}} \left[\frac{\gamma}{1+\frac{(R-1)}{\delta}} \right]^{\gamma/1-\gamma}. \text{ From two last expressions it follows that the first term in}$$

(4) increases in R for $R > 1$.

Now we analyze the second term. From (31), (30), (34) in the text this term can be expressed as

$$\frac{\sqrt{\Omega\varphi} \cdot \mathbb{E}\Delta h}{1 - \beta R^{1/\theta}} = c \left[\left(1 - \beta R^{\frac{1}{\theta}}\right) (R - (1 - p_G - p_B)) \left(1 + \frac{(R-1)}{\delta}\right)^{\gamma/1-\gamma} \right]^{-1}$$

where c is a positive constant. It can be seen that this is an increases in R for $R \in [1, 1/\beta]$.

Appendix II

Finite Horizons

Another potential source for wealth inequality arises when individuals face finite life horizons. Assets different individuals are holding may vary for two main reasons 1) individual's propensity to consume and to save is likely to depend on his/her age. Elder with less life expectancy are likely to consume more and save less or dissave 2) individual's asset holding may differ with age, the longer individual is in place the more assets he/she had time to accumulate.

Blanchard (1985) provides a useful framework for exploring the later possibility. The original paper concentrates on dynamic and steady state properties of the model and effects of government spending. Our focus is somewhat different: we are interested in the implications of finite horizons for equilibrium asset structure. We do so by applying density functions approach developed earlier to this case. In this model individuals face the same probability of death irrespective of how long they have been alive so that life expectancy is constant throughout life. With this assumption propensity to consume does not depend on individual's age and, by consequence, results of Proposition 1 can be used.

In discrete time version of the model an individual j alive at time t maximizes

$$u(c_{I,t}^j) + \sum_{i=1}^{\infty} (\beta')^i \cdot u(c_{t+i}^j)$$

subject to the budget constraint

$$a_{t+1}^j = R'_{t+1} \cdot (a_t^j + w_{I,t} - c_{I,t}^j)$$

where $\beta' = p \cdot \beta$ and $R'_{t+1} = R_{t+1} + (1 - p)$.

In these expressions p represents the probability of survival into the next period. Asset accumulation equation differs from the standard version in that effective interest on assets, R'_{t+1} , is standard gross interest, R_t , plus $1-p$. This additional interest is interpreted as payment received by an individual from an insurance company in exchange for his assets in case of death (so that expected profit for company is zero). We see that probability of death, $1-p$, increases the value of current consumption relative to the future consumption and provides additional source of income from asset holdings.

The Euler equation for this problem takes the form

$$u'(c_t^j) = \beta' \cdot R'_{t+1} \cdot u'(c_{t+1}^j)$$

Assume CRRA utility so that Euler equation becomes:

$$c_{t+1}^j = (\beta' \cdot R'_{t+1})^{1/\theta} \cdot c_t^j$$

Next period consumption is linear in current t period consumption. This implies that propensity to consume does not depend on individual's wealth. Specifically one can show that in the present case propensity to consume, Ω_t , obey the following equation

$$\beta' \cdot R'_{t+1} \cdot \left(R'_{t+1} \Omega_{t+1} \cdot \frac{1 - \Omega_t}{\Omega_t} \right)^{-\theta} = 1 \quad (41)$$

This equation is identical to (7) with effective discount factor and effective interest rate.

Using this linear relation between consumption and wealth in one obtains

$$a_{t+1}^j = \lambda_t \cdot a_t^j + \mu_t \quad (42)$$

where $\lambda_t = R'_{t+1} \cdot (1 - \Omega_t)$ and $\mu_t = R'_{t+1} \cdot (w_t - \Omega_t \cdot h_t)$

In last expression labor income, w_t , and, by consequence, human wealth, h_t , is assumed to not vary with age.

Human wealth is present value of expected labor income

$$h_t = \sum_{i=0}^{\infty} (R'_{t+1})^{-i} \cdot w_{t+i} \quad (43)$$

or, equivalently:

$$h_t = w_t + \frac{h_{t+1}}{R'_{t+1}} \quad (44)$$

Now in order to apply a two-state scheme developed earlier one can interpret these two states as “being alive” and “being dead”. We can think that “being alive” means that individual goes on with his assets in which case his assets are transformed according to (40) whereas “being

dead” means that he is just stripped of all his wealth and $a_{t+1}^j = 0$ in this case. Thus we can apply aggregation theorem of Section I.

Mean value of assets follows the law of motion

$$\mathbb{E}a_{t+1} = p \cdot (\lambda_t \cdot \mathbb{E}a_t + \mu_t) \quad (45)$$

Mean squared assets follow the law of motion:

$$\mathbb{E}a_{t+1}^2 = p \cdot (\lambda_t^2 \cdot \mathbb{E}a_t^2 + 2 \cdot \lambda_t \cdot \mathbb{E}a_t \cdot \mu_t + (\mu_t)^2) \quad (46)$$

The Cobb-Douglas production function $y_t = l_t^{1-\mu} \cdot k_t^\mu$ with inelastic labor supply l_t that equals 1 implies that production, interest rate and wage

$$y_t = k_t^\mu \quad (47)$$

$$\mu \cdot (k_t)^{\mu-1} + 1 - \delta = R_t \quad (48)$$

$$(1 - \mu) \cdot (k_t)^\mu = w_t \quad (49)$$

The model is closed by adding a capital market equilibrium condition which takes the form

$$\mathbb{E}a_t = R_t \cdot k_t \quad (50)$$

Six equations form a closed system for six endogenous variables. These equations and variables are: equation for propensity to consume term Ω_t (41), equation for aggregated assets $\mathbb{E}a_t$ (45), equation for human wealth h_t (44), equation for wage w_t (49), equation for gross interest rate R_t (48), capital market equilibrium condition for k_t (50). Together with initial condition for capital stock and transversality condition these equations determine the equilibrium path for this economy.

Finally, equation (46) together with initial condition for mean square assets determines how this variable change over time.

The dynamic system can be characterized by saddle point structure. Stable arm converges to unique steady state to which we turn to.

Determination of the steady state is straightforward. In the steady state propensity to consume

$$\Omega_{ss} = 1 - (p \cdot \beta)^{1/\theta} \cdot (R_{ss} + 1 - p)^{(1-\theta)/\theta} \quad (51)$$

We have

$$\lambda_{ss} = R'_{ss} \cdot (1 - \Omega_{ss}) = [p \cdot \beta \cdot (R_{ss} + 1 - p)]^{1/\theta} \quad (52)$$

Then from (45)

$$\mathbb{E}a_{ss} = \frac{p \cdot \mu_{ss}}{1 - p \cdot \lambda_{ss}}$$

Using the fact that $w_{ss} = \frac{R'_{ss}}{R_{ss}-1} \cdot h_{ss}$ last expression takes form

$$\mathbb{E}a_{ss} = \frac{p \cdot (\lambda_{ss} - 1)}{1 - p \cdot \lambda_{ss}} \cdot h_{ss} \quad (53)$$

Note from (52) that there is a positive relation between R_{ss} and λ_{ss} . Expressing human wealth and capital as functions of lambda, capital market equilibrium condition can be written

$$\frac{\lambda_{ss} - 1}{1 - p \cdot \lambda_{ss}} \cdot h_{ss}(\lambda_{ss}) = R_{ss}(\lambda_{ss}) \cdot k_{ss}(\lambda_{ss}) \quad (54)$$

From this equation equilibrium value of λ_{ss} is determined which is then used to solve for interest rate and other steady state values.

It can be shown that equation (54) has a unique solution. This solution lies in a well defined interval. Note that in order for the left side of (54) to be positive λ_{ss} should belong to the interval: $(1; 1/p)$. If the left side is associated with supply of assets and right side with demand for assets one can determine equilibrium as intersection of supply and demand functions.

Now we can focus on implications for equilibrium asset structure. For this we turn to equation (46). The fact that steady state value of λ_{ss} is less than $1/p$ ensures that this equation is stationary and $\mathbb{E}a_t^2$ converges to the steady state value

$$\mathbb{E}a_{ss}^2 = \frac{p \cdot (2 \cdot \lambda_{ss} \cdot \mathbb{E}a_{ss} \cdot \mu_{ss} + (\mu_{ss})^2)}{1 - p \cdot \lambda_{ss}^2} \quad (55)$$

Thus, using relation $\delta_{ss}^2 = \mathbb{E}a_{ss}^2 - (\mathbb{E}a_{ss})^2$ we have

$$\delta_{ss}^2(a) = \frac{\frac{1}{p} - 1}{1 - p \cdot \lambda_{ss}^2} \cdot (\mathbb{E}a_{ss})^2$$

and coefficient of variation is

$$\frac{\delta_{ss}}{A_{ss}} = \left(\frac{\frac{1}{p} - 1}{1 - p \cdot \lambda_{ss}^2} \right)^{1/2} \quad (56)$$

How does inequality of assets measured by this ratio change when life horizon increases. One might expect that as individuals live longer they accumulate more assets through time and thus inequality between old and young rises. Expression (55) shows that this need not to be so. When life expectancy that is given by $1/(1-p)$ rises p moves closer to unity and this implies λ_{ss} goes to unity as well. Thus, nominator and denominator in (56) both go to zero and result is ambiguous. In fact in calibrated model inequality slightly decreases with increasing life expectancy.

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