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On Families of Diffeomorphisms with Bifurcations of Attractive and Repelling Sets

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In the survey, we consider bifurcations of attracting (or repelling) invariant sets of some classical dynamical systems with a discrete time.

Keywords: Attractor; repeller; bifurcation.

1. Introduction

We dedicate this paper to the memory of Leonid Pavlovich Shilnikov, who was the founding father of the theory of nonlocal bifurcations of multidimensional dynamical systems.

In it, we discuss bifurcations of attracting (and repelling) invariant sets of some classical dynamical systems with a discrete time (i.e. dynamical systems generated by diffeomorphisms). In many instances, such an invariant set defines complete invariants of topological conjugacy (see, for example, [Bonatti & Grines, 2000; Franks, 1970; Grines & Zhuzhoma, 2005]).

In the early 1960's, mathematicians realized that structurally stable dynamical systems (flows and diffeomorphisms of dimensions three and two, and higher, resp.) could admit the occurrence of rather complex limit sets; in particular, attracting limit sets, which could not be locally homeomorphic to a direct product of a disk and Cantor set. As follows from works of Shilnikov ([Shilnikov *et al.*, 1998, 2001] and references therein), attracting sets of dynamical systems, which describe real physical processes (for example, geometric models of the Lorenz attractor [Afraimovich *et al.*, 1977, 1983]) can be even more complex.

In 1960, Smale [1960] introduced a new class of dynamical systems (flows and diffeomorphisms), which were later called Morse–Smale systems. It was proved that Morse–Smale systems are structurally stable, and have zero entropy [Palis, 1969; Palis & Smale, 1970; Robinson, 1999]. Hence, Morse–Smale systems are viewed as the simplest structurally stable system. A Morse–Smale system can be defined as a structurally stable one with a nonwandering set consisting of a finite number of limit orbits. There is a tight connection between dynamics and the topological structure of manifolds, on which dynamical systems are defined [Bonatti *et al.*, 2001, 2002; Grines *et al.*, 2003; Medvedev & Zhuzhoma, 2008; Smale, 1960]. On a closed manifold, any Morse–Smale system has at least one attracting orbit and at least one repelling orbit. Thus, the simplest Morse–Smale system has the nonwandering set consisting of two points: a sink and source. For 3D Morse–Smale diffeomorphisms, Bonatti *et al.* [2004] introduced the special attracting and repelling sets to represent the dynamics in the sink-source spirit. In the series of papers [Bonatti *et al.*, 2004, 2005, 2006; Pochinka, 2011] Bonatti, Grines, Medvedev, Pecou, and Pochinka obtained the complete topological classification of Morse–Smale diffeomorphisms on closed 3D-manifolds. There is the high-dimensional analog of the sink-source representation (see [Grines *et al.*, 2010]), which can be used in a future topological classification. In what follows, we will study bifurcations of special attracting sets of 3D Morse–Smale diffeomorphisms.

A well-known example of hyperbolic attractors is the Smale solenoid [Smale, 1967]. The generalization of this Smale construction was proposed in [Isaenkova & Zhuzhoma, 2009, 2011; Robinson & Williams, 1976; Williams, 1967, 1974]. Shilnikov and Turaev made use of the Smale construction to study the onset of the so-called blue sky catastrophes [Shilnikov & Turaev, 1995]. Below, we will also introduce Smale–Vietoris diffeomorphisms that contain the generalizations of the Smale example, and represent some bifurcations between different types of solenoidal basic sets, which can lean to the breakdown (or the birth) of the Smale solenoid.

Anosov diffeomorphisms on a compact manifold are well known to be structurally stable. Several papers have investigated the relationship between the topology of underlying manifolds and the dynamics of codimension one Anosov diffeomorphisms [Anosov, 1967]. Franks [1970], Manning [1974] and Newhouse [1970] have shown that any codimension one Anosov diffeomorphism is conjugated to a hyperbolic torus automorphism (as a consequence, a manifold admitting such diffeomorphisms is homeomorphic to the torus T^n, $n \geq 2$). An important class of dynamical systems is comprised of the diffeomorphisms satisfying Smale's Axiom A [Smale, 1967], the so-called A-diffeomorphisms. Given such a diffeomorphism, its recurrent behavior is captured in its nonwandering set, which can be decomposed into invariant topologically transitive pieces (basic sets). One remarkable type of basic sets is an expanding attractor, introduced by Williams [1967, 1974]. A structurally stable A-diffeomorphism of T^n with an orientable codimension one expanding attractor can be obtained using Smale's surgery [Smale, 1967], from a codimension one Anosov diffeomorphism. Such diffeomorphisms are called *DA-diffeomorphisms*. Below, we will consider bifurcations of DA-diffeomorphisms as well.

2. An Array of Attracting Set

We consider dynamical systems with the discrete time $\mathbb{Z}$. This means that a dynamical system on an n-manifold M^n is the family of iterations f^i, $i \in \mathbb{Z}$, of some diffeomorphism $f : M^n \to M^n$. An *invariant set* of f is a subset $\Lambda \subset M^n$ such that $f(\Lambda) = \Lambda$. An invariant set Λ is called *attractive* if there is a closed neighborhood N of Λ such that

$$f(N) \subset \operatorname{int} N, \quad \Lambda = \bigcap_{i \geq 0} f^i(N).$$

The neighborhood N is called an *isolated* (*attractive*) *neighborhood*.

2.1. *Attractive sets of Morse–Smale diffeomorphisms*

Let $f : M^n \to M^n$ be an orientation-preserving Morse–Smale diffeomorphism. Denote by Σ_f, Δ_f^s and Δ_f^u all saddles, sinks, and sources of the diffeomorphism f, respectively. Let us divide the saddle points Σ_f into two disjoint subsets Σ_f^A and Σ_f^R such that the sets

$$A_f = \Delta_f^s \cup W^u_{\Sigma_f^A} \quad \text{and} \quad R_f = \Delta_f^u \cup W^s_{\Sigma_f^R}$$

are closed and invariant. Such a partition of the set of saddle points is always possible. Moreover, the sets A_f and R_f are disjoint and their union contains all periodic points of the diffeomorphism f. The maximal dimension of the unstable (stable) manifolds of the points in Σ_f^A (Σ_f^R) will be referred to as the *dimension of* A_f (R_f). The following result was proved in [Grines *et al.*, 2010].

Theorem 1. *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism. Then the set A_f (resp., R_f) is an attractor (repeller) of the diffeomorphism f. Moreover, if the dimension of the attractor A_f (repeller R_f) is at most $(n-2)$, then the repeller R_f (attractor A_f) is connected.*

We will call A_f and R_f *global attractor and a global repeller*, *respectively, of the Morse–Smale diffeomorphism* $f : M^n \to M^n$.

2.2. *Attractive sets of Smale–Vietoris diffeomorphisms*

The most important attractive set is an attractor. There are several types of attractors: hyperbolic, expanding, etc. An invariant set $\Lambda \subset M$ is a *hyperbolic attractor* if the following conditions hold: (1) there is an isolated neighborhood N of Λ; (2) any point $x \in \Lambda$ is nonwandering; (3) there is a continuous df-invariant splitting of the tangent bundle TM_Λ into *stable* and *unstable bundles* $E^s_\Lambda \oplus E^u_\Lambda$, $\dim E^s_x + \dim E^u_x = \dim M$ $(x \in \Lambda)$, with

$$\|df^i(v)\| \le C_s\lambda^i\|v\|, \quad v \in E^s_\Lambda,$$
$$\|df^{-i}(v)\| \le C_u\lambda^i\|v\|, \quad v \in E^u_\Lambda, \quad i > 0$$

for some fixed $C_s > 0$, $C_u > 0$, $0 < \lambda < 1$.

The hyperbolic structure implies the existence of stable and unstable manifolds $W^s(x)$, $W^u(x)$ respectively through any point $x \in \Lambda$:

$$W^s(x) = \left\{y \in M : \lim_{j\to\infty} \rho(f^j(x), f^j(y)) \to 0\right\},$$
$$W^u(x) = \left\{y \in M : \lim_{j\to\infty} \rho(f^{-j}(x), f^{-j}(y)) \to 0\right\},$$

which are smooth, injective immersions of E^s_x and E^u_x into M. Moreover, $W^s(x)$, $W^u(x)$ are tangent to E^s_x and E^u_x at x, respectively.

A nontrivial hyperbolic attractor Λ is *expanding* if the topological dimension of Λ equals the dimension of a fiber E^u_Λ, $\dim E^u_x = \dim \Lambda$ $(x \in \Lambda)$. The most familiar expanding attractors are: (1) Smale solenoid; (2) DA-attractor (a nontrivial attractor of a DA-diffeomorphism); (3) Plykin attractor [Plykin, 1974].

Recall that a topological solenoid was introduced by Vietoris [1927] (independently, a solenoid was introduced by Van Danzig [1930]). It can be presented as the intersection of a nested sequence of solid tori $T_1 \supset T_1 \supset \ldots T_i \supset \ldots$, where T_{i+1} is wrapped around inside T_i longitudinally p_i times in a smooth fashion without folding back.

The solenoids were introduced into dynamics by Smale in his celebrated paper [Smale, 1967] as DE-maps which arise from expanding maps (the abbreviation DE is formed by first letters of *Derived* from *Expanding* map). Omitting details, one can say that a DE map is the skew map

$$f : T \times N \to T \times N, \quad (x; y) \mapsto (g_1(x); g_2(x, y)),$$

where $g_1 : T \to T$ is an expanding map of degree $d \ge 2$, and $g_2|_{\{x\}\times N} : \{x\} \times N \to \{g_1(x)\} \times N$ is an uniformly attracting map of n-disk $\{x\} \times N$ into n-disk $\{g_1(x)\} \times N$ for every $x \in T$. In addition, f must be a diffeomorphism onto its image $T \times N \to f(T \times N)$. In the spirit of Smale construction of DE-maps, one can introduce diffeomorphisms, called Smale–Vietoris ones, which arise from nonsingular endomorphisms. A skew-mapping:

$$F : \mathbb{T}^k \times N \to \mathbb{T}^k \times N, \quad (t, z) \mapsto (g(t); \omega(t, z))$$

is called a Smale skew-mapping if the following conditions hold: (1) $F : \mathbb{T}^k \times N \to F(\mathbb{T}^k \times N)$ is a diffeomorphism on its image; (2) $g : \mathbb{T}^k \to \mathbb{T}^k$ is a d-cover, $d \ge 2$; (3) given any $t \in \mathbb{T}^k$, the restriction $w|_{\{t\}\times N} : \{t\} \times N \to \mathbb{T}^k \times N$ is the uniformly attracting embedding $\{t\} \times N \to \mathrm{int}(\{g(t)\} \times N)$ i.e. there are $0 < \lambda < 1$, $C > 0$ such that

$$\begin{aligned}&\mathrm{diam}(F^n(\{t\} \times N))\\ &\qquad \le C\lambda^n \mathrm{diam}(\{t\} \times N), \quad \forall\, n \in \mathbb{N}.\end{aligned}$$

When $g = E_d$, Smale skew-mapping is a DE-mapping introduced by Smale [1967].

A diffeomorphism $f : M^n \to M^n$ is called a *Smale–Vietoris* diffeomorphism if there is the n-submanifold $\mathbb{T}^k \times N \subset M^n$ such that the restriction $f|_{\mathbb{T}^k\times N} \overset{\mathrm{def}}{=} F$ is a Smale skew-mapping. The submanifold $\mathbb{T}^k \times N \subset M^n$ is called a *support of Smale skew-mapping*.

By definition,

$$\bigcap_{l\ge 0} F^l(\mathbb{T}^k \times N) \overset{\mathrm{def}}{=} \mathfrak{S}(f).$$

It is easy to see that the set $\mathfrak{S}(f) = \mathfrak{S}$ is attractive, invariant, and closed, so that the restriction $f|_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S}$ is a homeomorphism. The following theorem describes the symbolic model of the restriction $f|_{\mathfrak{S}}$. Following the classical results by Williams

[1967, 1974], one can prove that the restriction $f|_{\mathfrak{S}}$ is conjugate to the inverse limit of the mapping $g : \mathbb{T}^k \to \mathbb{T}^k$, where $\mathfrak{S} = \cap_{l\geq 0} F^l(\mathbb{T}^k \times N)$. Using this result, one can prove the following statement [Isaenkova & Zhuzhoma, 2011].

Theorem 2. *Let $f : M^n \to M^n$ be a Smale–Vietoris A-diffeomorphism of closed n-manifold M^n and $\mathbb{T}^1 \times N = \mathfrak{B} \subset M^n$ the support of Smale skew-mapping $f|_{\mathfrak{B}} = F$. Then the nonwandering set $NW(F)$ of F belongs to $\mathfrak{S} = \bigcap_{l\geq 0} F^l(\mathbb{T}^1 \times N)$, and $NW(F)$ contains a unique nontrivial basic set $\Lambda(f)$ that is either*

- *a one-dimensional expanding attractor, and $\Lambda(f) = \mathfrak{S}$; or*
- *a zero-dimensional basic set, and $NW(F)$ consists of $\Lambda(f)$ and finitely many (nonzero) isolated attracting periodic points plus finitely many (possibly, zero) saddle type isolated periodic points of codimension one stable Morse index.*

Both possibilities hold.

2.3. *DA-attractor*

A diffeomorphism with a DA-attractor is obtained by a so-called Smale's surgery performed on a codimension one Anosov automorphism of the n-torus T^n. For this reason, it is called the *Derived from Anosov diffeomorphism* or *DA-diffeomorphism*. It was first introduced by Smale [1967].

Take a codimension one Anosov automorphism $A : T^n \to T^n$ with one-dimensional stable splitting and codimension one unstable splitting. Let p_0 be a fixed point of A. One can carefully insert at point p_0 a tiny ball with a source P_0 and two saddle type fixed points, see Fig. 1, in such a way that we get a diffeomorphism, say $f : T^n \to T^n$, with a codimension one expanding attractor $\Lambda = T^n \backslash W^u(P_0)$ which is called a *DA-attractor*. Smale's surgery is similar to the Poincare–Denjoy blowing up operation producing a Denjoy foliation from a minimal foliation on 2-torus.

In 1979, Grines and Zhuzhoma considered the following problem. Let $f_1, f_2 : T^n \to T^n$ be two diffeomorphisms of the n-torus T^n, $n \geq 3$, having orientable expanding attractors of codimension one Λ_1 and Λ_2, respectively. When there exists a homeomorphism φ: $T^n \to T^n$ such that

$$\varphi(\Lambda_1) = \Lambda_2 \quad \text{and} \quad f_2|_{\Lambda_2} = \varphi f_1 \varphi^{-1}|_{\Lambda_2}.$$

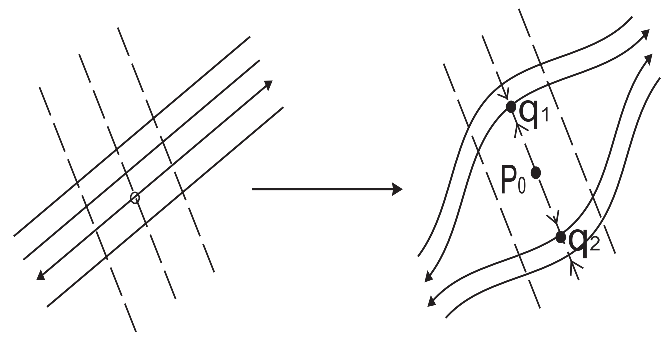

Fig. 1. Smale's surgery.

Following ideas of Newhouse [1970], one proved that given any orientable expanding attractor Λ of a diffeomorphism $f : T^n \to T^n$, there is a neighborhood $U(\Lambda)$ of Λ such that: (1) $U(\Lambda) \subset W^s(\Lambda)$; (2) $T^n \backslash U(\Lambda)$ consists of a finitely many n-balls $B_1, \ldots, B_m$; and (3) $f^{-j}(B_i) \subset B_i$ for some $j \in \mathbb{N}$ and any $1 \leq i \leq m$. It means that f looks like a DA-diffeomorphism up to dynamics in balls B_i where it globally remains the dynamics of a periodic repelling point. Moreover, with the additional restriction for f being structurally stable, Grines and Zhuzhoma [2005] proved that f must be almost DA-diffeomorphism of the n-torus:

Theorem 3. *Suppose f is a structurally stable diffeomorphism of a closed n-manifold M^n $(n \geq 3)$ and Λ is a codimension one orientable expanding attractor of f. Then*:

(1) *M^n is homeomorphic to T^n.*
(2) *The spectral decomposition of f consists of Λ, and a finite nonzero number of repelling*

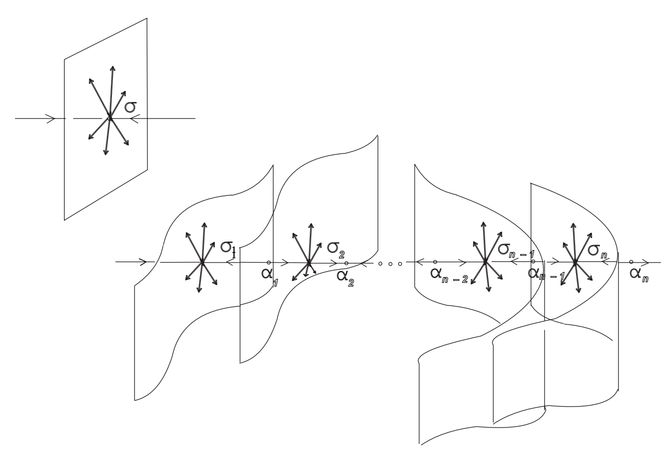

Fig. 2. Structurally stable diffeomorphism of $\mathbb{T}^3$ with a 2D orientable expanding attractor.

periodic orbits of index[1] n, *and a finite number* (*maybe zero*) *of periodic saddle orbits of index* $n-1$.

Figure 2 sketches a phase portrait of a structurally stable diffeomorphism on $\mathbb{T}^3$ with a two-dimensional orientable expanding attractor.

3. Bifurcations of Attractive and Repelling Sets

In the given paper we consider special pathways (or one-parameter families) in a space of diffeomorphisms with a nonempty set of bifurcation parameters.

3.1. *On a simple isotopy class of sink-source diffeomorphism on a 3-sphere*

This section deals with the Palis–Pugh problem on the existence of an arc with finitely or countably many bifurcations joining two Morse–Smale systems on a closed smooth manifold [Palis & Pugh, 1975]. In [Newhouse & Peixoto, 1976], the authors proved that any Morse–Smale vector fields are joined by a simple arc. Simplicity means that the entire arc, except finitely many points, consists of Morse–Smale systems, and at the exceptional points, a minimal (in a certain sense) deviation of the vector field from a Morse–Smale system occurs. The situation with discrete dynamical systems is different. Two orientation-preserving Morse–Smale diffeomorphisms on the circle can be joined by a simple arc if and only if they have the same rotation number. As follows from results of Matsumoto [1979] and Blanchard [1980], any orientable closed surface admits isotopic Morse–Smale diffeomorphisms which cannot be joined by a simple arc. We say that two isotopic Morse–Smale diffeomorphisms belong to the same simple isotopy class if they can be joined by a simple arc. According to the paper [Blanchard, 1980], there exist infinitely many simple isotopy classes of Morse–Smale diffeomorphisms on any orientable surface inside an isotopy class admitting Morse–Smale diffeomorphisms.

The problem of the existence of a simple arc in dimension three is complicated by the presence of Morse–Smale diffeomorphisms whose saddle periodic points have separatrices *wildly embedded* in the underlying manifold. The first "wild" example on the 3-sphere S^3 was constructed by Pixton [1977]. This diffeomorphism belongs to the class (which we called the Pixton class in [Grines & Pochinka, 2011]) formed by those three-dimensional Morse–Smale diffeomorphisms whose nonwandering set consists of precisely four points, namely, two sinks, a source, and a saddle. According to [Bonatti *et al.*, 2007], any Pixton diffeomorphism is joined by a simple arc to a source-sink diffeomorphism. This is caused by the fact that, for any diffeomorphism from the Pixton class, at least a one-dimensional separatrix of its saddle point is tame [Bonatti & Grines, 2000]. By using the connected sum of two 3-spheres, it is easy to construct a diffeomorphism for which all separatrices of all saddles are wildly embedded (see Fig. 3, in which the 3-balls used to obtain the connected sum are shaded); one can prove that such a diffeomorphism is not joined by a simple arc to any source-sink diffeomorphism.

Let $\mathrm{Diff}(M^n)$ be the space of diffeomorphisms on a closed manifold M^n endowed with the C^1-topology. A *smooth arc* in $\mathrm{Diff}(M^n)$ is defined as a smooth map $\xi : M^n \times [0,1] \to M^n$ that gives a family of diffeomorphisms $\{\xi_t \in \mathrm{Diff}(M^n), t \in [0,1]\}$. Let $KS(M^n)$ be the set of all *Kupka–Smale* diffeomorphisms, i.e. diffeomorphisms whose periodic orbits are hyperbolic and have transversal stable and unstable manifolds. The Kupka–Smale diffeomorphisms with finite nonwandering set form the set $MS(M^n)$ of Morse–Smale diffeomorphisms. For a smooth arc ξ, the set $B(\xi) = \{b \in [0,1], \xi_b \notin KS(M^n)\}$ is called the *bifurcation set*. According

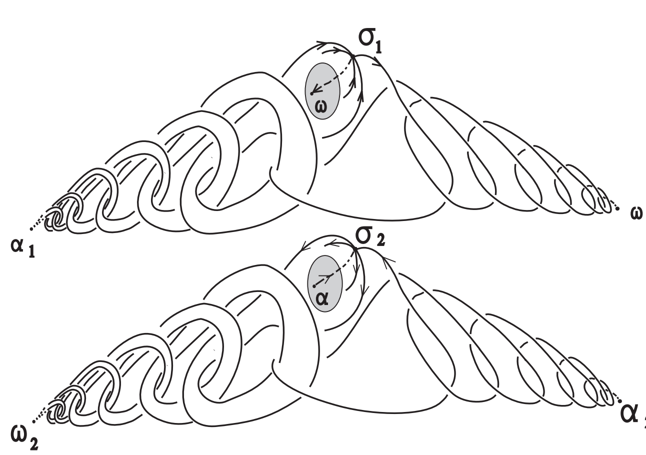

Fig. 3. Connected sums of Pixton's examples.

[1]Recall that if x is a periodic point, then the *index* at x is defined to be the dimension of the unstable manifold $W^u(f^k(x))$, where k is the period of x. The index of a periodic orbit equals the index of any point of the orbit.

to [Palis, 1975], for a generic set of arcs (which is the intersection of open dense subsets in the space of smooth arcs), the bifurcation set is countable, and each diffeomorphism $\xi_b, b \in B(\xi)$ experiences one of the following bifurcations up to the direction of motion along the arc: a saddle-node bifurcation, a period-doubling, a Hopf bifurcation, and a heteroclinic tangency. An arc ξ is called *simple* if the bifurcation set $B(\xi)$ is finite, ξ_t is structurally stable for any $t \in ([0,1]\backslash B(\xi))$ and the bifurcations are one of the following types: saddle-node, period-doubling, heteroclinic tangency.

The simplest Morse–Smale diffeomorphism is a source-sink diffeomorphism. The nonwandering set of such a diffeomorphism consists of two points, a source and a sink, and the ambient manifold is homeomorphic to the sphere S^n. In [Bonatti *et al.*, 2007], it was proved that all source-sink diffeomorphisms on $\mathbb{S}^3$ belong to the same simple isotopy class, which we denote by I_{NS}. Let $f \in MS(M^3)$ be a diffeomorphism with a saddle point σ and let ℓ^u_σ be unstable *separatrix* (that is, a connected component of the set $W^u_\sigma\backslash\sigma$). If the separatrix ℓ^u_σ does not participate in heteroclinic intersections (that is, no intersections of stable and unstable manifolds of different saddle points), then $cl(\ell^u_\sigma)\backslash(\ell^u_\sigma \cup \sigma) = \{\omega\}$, where ω is a sink periodic point. Moreover, if $\dim W^u_\sigma = 1$ then $cl(\ell^u_\sigma)$ is a topologically embedded arc in M^3. The set $\ell^u_\sigma \cup \sigma$ is a smooth submanifold of M^3. But $cl(\ell^u_\sigma)$ may be wild at the point ω; in this case, the separatrix ℓ^u_σ is said to be *wild*, and otherwise, it is said to be *tame*. The tameness and the wildness of a stable one-dimensional separatrix are defined in a similar way.

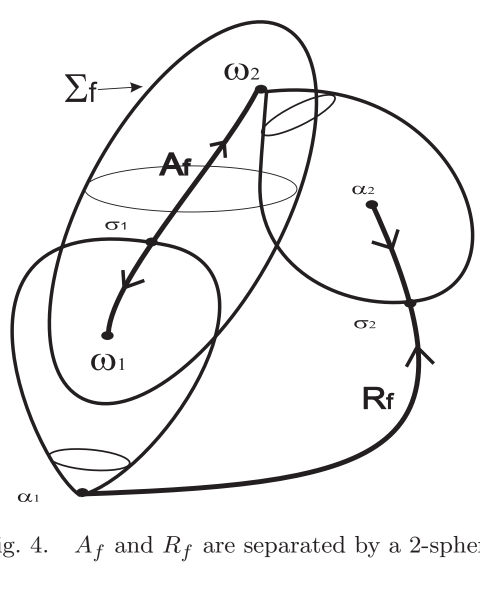

Fig. 4. A_f and R_f are separated by a 2-sphere.

Recall that the dynamics of any cascade $f \in MS(M^3)$ can be represented as follows. Let $\Omega^q_f, q = 0,1,2,3$ denote the set of periodic points p for which we have $\dim W^u_p = q$. Then $A_f = W^u_{\Omega^0_f\cup\Omega^1_f}$ is a connected attractor, and $R_f = W^s_{\Omega^3_f\cup\Omega^2_f}$ is a connected repeller with topological dimension at most 1. The sets A_f and R_f do not intersect, and each point from the set $V_f = M^3\backslash(A_f \cup R_f)$ is wandering and moves from R_f to A_f under the action of f. We say that A_f and R_f are *separated by a 2-sphere* if there exists a smooth 2-sphere $\Sigma_f \subset V_f$ such that A_f and R_f belong to different connected components of $M^3\backslash\Sigma_f$. Let $MS_0(M^3)$ denote the class of Morse–Smale diffeomorphisms without heteroclinic intersections on a 3-manifold M^3. Grines and Pochinka proved the following criterion for the existence of a simple arc joining a Morse–Smale diffeomorphism without heteroclinic intersections to a source-sink diffeomorphism. The next theorem was proved in [Grines & Pochinka, 2013].

Theorem 4. *A diffeomorphism $f \in MS_0(\mathbb{S}^3)$ belongs to the class I_{NS} if and only if the attractor A_f and the repeller R_f are separated by a 2-sphere.*

3.2. *On a breakdown of Smale solenoids*

After Theorem 2, it is natural to consider bifurcations from one type of dynamics to another which can be thought of as a destruction (or, a birth) of Smale solenoid. For simplicity, we represent two such bifurcations for $M^3 = S^3$ a 3-sphere.

Theorem 5. *There is the family of Ω-stable Smale–Vietoris diffeomorphisms $f_\mu : S^3 \to S^3$, $0 \le \mu \le 1$, continuously depending on the parameter μ such that the nonwandering set $NW(f_\mu)$ of f_μ is the following*:

- *$NW(f_0)$ consists of a one-dimensional expanding attractor (Smale solenoid attractor) and one-dimensional contracting repeller (Smale solenoid repeller)*;
- *for $\mu > 0$, $NW(f_\mu)$ consists of two nontrivial zero-dimensional basic sets and finitely many isolated periodic orbits.*

Recall that a diffeomorphism $f : M \to M$ is Ω-*stable* if there is a neighborhood $U(f)$ of f in the space of C^1 diffeomorphisms $\mathrm{Diff}^1(M)$ such that $f|_{\mathrm{NW}(f)}$ conjugate to every $g|_{\mathrm{NW}(g)}$ provided

$g \in U(f)$. Therefore, there is no similar family of Ω-stable diffeomorphisms which smoothly (at least C^1) depends on parameter. Actually, we have to break an Ω-stability (hence, hyperbolicity) to pass smoothly from an Ω-stable diffeomorphism to another one. The following result was obtained in collaboration with Gonchenko.

Theorem 6. *There is the family of diffeomorphisms* $f_\mu : S^3 \to S^3$, $-1 \le \mu \le 1$, *smoothly depending on the parameter* μ *such that*

- *given any* $-1 \le \mu < 0$, f_μ *is* Ω*-stable, and the nonwandering set* $NW(f_\mu)$ *of* f_μ *consists of a* (*hyperbolic*) *one-dimensional expanding attractor and one-dimensional contracting repeller* (*Smale solenoid attractor and repeller*);
- f_0 *is partially hyperbolic on the nonwandering set* $NW(f_0)$, *and* $NW(f_0)$ *consisting of* (*nonhyperbolic*) *Smale solenoid attractor and Smale solenoid repeller*;
- *given any* $0 < \mu \le 1$, f_μ *is* Ω*-stable, and the nonwandering set* $NW(f_\mu)$ *of* f_μ *consists of two* (*hyperbolic*) *nontrivial zero-dimensional basic sets and finitely many isolated periodic orbit.*

3.3. *Bifurcations of DA-diffeomorphisms*

Taking in mind Theorem 3, one can prove the following result [Medvedev & Zhuzhoma, 2003].

Theorem 7. *Any structurally stable diffeomorphism* $f \in \mathrm{Diff}^r(M^n)$, $n \ge 4$, *with a codimension one orientable expanding attractor can be connected by a simple arc with a DA-diffeomorphism.*

Theorem 7 does not hold true for $n = 2$. The proof works for $n = 3$ because it is possible to show that there are no wildly embedded separatrices of saddle periodic points in this case.

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