Poincaré and Brunn–Minkowski inequalities on weighted Riemannian manifolds with boundary

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Abstract

It is known that by dualizing the Bochner–Lichnerowicz–Weitzenböck formula, one obtains Poincaré-type inequalities on Riemannian manifolds equipped with a density, which satisfy the Bakry–Émery Curvature-Dimension condition (combining the Ricci curvature with the "curvature" of the density). When the manifold has a boundary, an appropriate generalization of the Reilly formula may be used instead. By systematically dualizing this formula for various combinations of boundary conditions of the domain (convex, mean-convex) and the function (Neumann, Dirichlet), we obtain new Poincaré-type inequalities - not only on the manifold, but more interestingly, also on its boundary. For instance, we may handle Neumann conditions on a mean-convex domain, and obtain generalizations to the weighted-manifold setting of a purely Euclidean Poincaré-type inequality of Colesanti on the boundary of a convex domain. All other previously known Poincaré-type inequalities of Lichnerowicz, Brascamp–Lieb, Bobkov–Ledoux and Veysseire are recovered, extended to the Riemannian setting and generalized into a single unified formulation, and their appropriate versions in the presence of a boundary are obtained, as well as new spectral-gap estimates. Finally, a novel geometric evolution equation is proposed, which extends to the Riemannian setting the Minkowski addition operation of convex domains, a notion previously confined to the purely linear setting. This geometric flow is intimately related to a homogeneous Monge-Ampère equation on the exterior of the convex domain, and yields a novel Brunn–Minkowski inequality in the weighted-Riemannian setting. Our framework allows to encompass the entire class of Borell's convex measures, including heavy-tailed measures, and extends the latter class to weighted-manifolds having negative generalized dimension.

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1 Introduction

Throughout the paper we consider a compact *weighted-manifold* (M, g, μ), namely a compact smooth complete connected and oriented n-dimensional Riemannian manifold (M, g) with boundary ∂M, equipped with a measure:

$$\mu = \exp(-V) d\mathrm{Vol}_M \ ,$$

where Vol_M is the Riemannian volume form on M and $V \in C^2(M)$ is twice continuously differentiable. The boundary ∂M is assumed to be a C^2 manifold with outer unit-normal $\nu = \nu_{\partial M}$. The corresponding symmetric diffusion operator with invariant measure μ, which is called the weighted-Laplacian, is given by:

$$L = L_{(M,g,\mu)} := \exp(V)\mathrm{div}(\exp(-V)\nabla) = \Delta - \langle \nabla V, \nabla \rangle \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric g, $\nabla = \nabla_g$ denotes the Levi-Civita connection, $\mathrm{div} = \mathrm{div}_g = tr(\nabla \cdot)$ denotes the Riemannian divergence operator, and $\Delta = \mathrm{div}\nabla$ is the Laplace-Beltrami operator. Indeed, note that with these generalized notions, the usual integration by part formula is satisfied for $f, g \in C^2(M)$:

$$\int_M L(f) g d\mu = \int_{\partial M} f_\nu g d\mu - \int_M \langle \nabla f, \nabla g \rangle \, d\mu = \int_{\partial M} (f_\nu g - g_\nu f) d\mu + \int_M L(g) f d\mu \ ,$$

where $u_\nu = \nu \cdot u$, and integration on ∂M with respect to μ means with respect to $\exp(-V) d\mathrm{Vol}_{\partial M}$.

The second fundamental form $\mathrm{II} = \mathrm{II}_{\partial M}$ of $\partial M \subset M$ at $x \in \partial M$ is as usual (up to sign) defined by $\mathrm{II}_x(X, Y) = \langle \nabla_X \nu, Y \rangle$, $X, Y \in T\partial M$. The quantities

$$H_g(x) := tr(\mathrm{II}_x) \ , \ H_\mu(x) := H_g(x) - \langle \nabla V(x), \nu(x) \rangle \ ,$$

are called the Riemannian mean-curvature and *generalized* mean-curvature of ∂M at $x \in \partial M$, respectively. It is well-known that H_g governs the first variation of $\mathrm{Vol}_{\partial M}$ under the normal-map $t \mapsto \exp(t\nu)$, and similarly H_μ governs the first variation of $\exp(-V) d\mathrm{Vol}_{\partial M}$ in the weighted-manifold setting, see e.g. [45] or Subsection 8.3.

In the purely Riemannian setting, it is classical that positive lower bounds on the Ricci curvature tensor Ric_g and upper bounds on the topological dimension n play a fundamental role in governing various Sobolev-type inequalities on (M, g), see e.g. [15, 22, 23, 37, 73] and the references therein. In the weighted-manifold setting, the pertinent information on *generalized* curvature and *generalized* dimension may be incorporated into a single tensor, which was put forth by Bakry and Émery [4, 3] following Lichnerowicz [39, 40]. The N-dimensional Bakry–Émery Curvature tensor ($N \in (-\infty, \infty]$) is defined as (setting $\Psi = \exp(-V)$):

$$\mathrm{Ric}_{\mu,N} := \mathrm{Ric_g} + \nabla^2 \mathrm{V} - \frac{1}{\mathrm{N-n}} \mathrm{dV} \otimes \mathrm{dV} = \mathrm{Ric_g} - (\mathrm{N-n}) \frac{\nabla^2 \Psi^{1/(\mathrm{N-n})}}{\Psi^{1/(\mathrm{N-n})}} ,$$

and the Bakry–Émery Curvature-Dimension condition $CD(\rho, N)$, $\rho \in \mathbb{R}$, is the requirement that as 2-tensors on M:

$$\mathrm{Ric}_{\mu,N} \geq \rho g .$$

Here $\nabla^2 V$ denotes the Riemannian Hessian of V. Note that the case $N = n$ is only defined when V is constant, i.e. in the classical non-weighted Riemannian setting where μ is proportional to Vol_M, in which case $\mathrm{Ric}_{\mu,n}$ boils down to the usual Ricci curvature tensor. When $N = \infty$ we set:

$$\mathrm{Ric}_\mu := \mathrm{Ric}_{\mu,\infty} = \mathrm{Ric}_g + \nabla^2 V .$$

It is customary to only treat the case when $N \in [n, \infty]$, with the interpretation that N is an upper bound on the "generalized dimension" of the weighted-manifold (M, g, μ); however, our method also applies with no extra effort to the case when $N \in (-\infty, 0]$, and so our results are treated in this greater generality, which in the Euclidean setting encompasses the entire class of Borell's convex (or "$1/N$-concave") measures [11] (cf.[12, 8]). It will be apparent that the more natural parameter is actually $1/N$, with $N = 0$ interpreted as $1/N = -\infty$, and so our results hold in the range $1/N \in [-\infty, 1/n]$. Clearly, the $CD(\rho, N)$ condition is monotone in $1/N$ in that range, so for all $N_+ \in [n, \infty], N_- \in (-\infty, 0]$:

$$CD(\rho, n) \Rightarrow CD(\rho, N_+) \Rightarrow CD(\rho, \infty) \Rightarrow CD(\rho, N_-) \Rightarrow CD(\rho, 0) ;$$

note that $CD(\rho, 0)$ is the weakest condition in this hierarchy. It seems that outside the Euclidean setting, this extension of the Curvature-Dimension condition to the range $N \leq 0$, has not attracted much attention in the weighted-Riemannian and more general metric-measure space setting (cf. [65, 41]); an exception is the work of Ohta and Takatsu [55, 56]. We expect this gap in the literature to be quickly filled (in fact, concurrently to posting our work on the arXiv, Ohta [54] has posted a first attempt of a systematic treatise of the range $N \leq 0$).

A convenient equivalent form of the $CD(\rho, N)$ condition may be formulated as follows. Let Γ_2 denote the iterated carré-du-champ operator of Bakry–Émery:

$$\Gamma_2(u) := \left\| \nabla^2 u \right\|^2 + \langle \mathrm{Ric}_\mu \nabla u, \nabla u \rangle ,$$

where $\left\|\nabla^2 u\right\|$ denotes the Hilbert-Schmidt norm of $\nabla^2 u$. Then the $CD(\rho, N)$ condition is equivalent when $1/N \in (-\infty, 1/n]$ (see [3, Section 6] for the case $N \in [n, \infty]$ or Lemma 2.3 in the general case) to the requirement that:

$$\Gamma_2(u) \geq \rho \left|\nabla u\right|^2 + \frac{1}{N}(Lu)^2 \quad \forall u \in C^2(M) \ . \tag{1.1}$$

Denote by $\mathcal{S}_0(M)$ the class of functions u on M which are C^2 smooth in the interior of M and C^1 smooth on the entire compact M. Denote by $\mathcal{S}_N(M)$ the subclass of functions which in addition satisfy that u_ν is C^1 smooth on ∂M. The main tool we employ in this work is the following:

Theorem 1.1 (Generalized Reilly Formula)**.** *For any function $u \in \mathcal{S}_N(M)$:*

$$\int_M (Lu)^2 d\mu = \int_M \left\|\nabla^2 u\right\|^2 d\mu + \int_M \langle \mathrm{Ric}_\mu \ \nabla u, \nabla u \rangle \, d\mu +$$
$$\int_{\partial M} H_\mu (u_\nu)^2 d\mu + \int_{\partial M} \langle II_{\partial M} \ \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, d\mu - 2 \int_{\partial M} \langle \nabla_{\partial M} u_\nu, \nabla_{\partial M} u \rangle \, d\mu \ . \tag{1.2}$$

Here $\nabla_{\partial M}$ denotes the Levi-Civita connection on ∂M with its induced Riemannian metric.

This natural generalization of the (integrated) Bochner–Lichnerowicz–Weitzenböck formula for manifolds with boundary was first obtained by R.C. Reilly [59] in the classical Riemannian setting ($\mu = \mathrm{Vol}_M$). The version above is a modification (obtained by integrating by parts on ∂M) of a previous version due to M. Li and S.-H. Du [42]. For completeness, we sketch in Section 2 the proof of the version (1.2) which we require for deriving our results.

1.1 Poincaré-type inequalities on M

It is known that by dualizing the Bochner–Lichnerowicz–Weitzenböck formula, various Poincaré-type inequalities such as the Lichnerowicz [38], Brascamp–Lieb [12, 35] and Veysseire [68] inequalities, may be obtained under appropriate bounds on curvature and dimension. Recently, heavy-tailed versions of the Brascamp–Lieb inequalities have been obtained in the Euclidean setting by Bobkov–Ledoux [8] and sharpened by Nguyen [52]. By employing the generalized Reilly formula, we begin this work by unifying, extending and generalizing many of these previously known results to various new combinations of boundary conditions on the domain (locally convex, mean-convex) and the function (Neumann, Dirichlet) in the weighted-Riemannian setting. We mention in passing another celebrated application of the latter duality argument in the Complex setting, namely Hörmander's L^2 estimate [27], but we refrain from attempting to generalize it here; further more recent applications may be found in [26, 35, 32, 5, 33].

Given a finite measure ν on a measurable space Ω, and a ν-integrable function f on Ω, we denote:

$$\fint_\Omega f d\nu := \frac{1}{\nu(\Omega)} \int_\Omega f d\nu \; , \; Var_\nu(f) := \int_\Omega \left(f - \fint_\Omega f d\nu \right)^2 d\nu \; .$$

The following result is proved in Section 3.

Theorem 1.2 (Generalized Dimensional Brascamp–Lieb With Boundary)**.** *Assume that* $\mathrm{Ric}_{\mu,N} > 0$ *on* M *with* $1/N \in (-\infty, 1/n]$*. The generalized Reilly formula implies all of the inequalities below for any* $f \in C^1(M)$*:*

(1) (Neumann Dimensional Brascamp–Lieb inequality on locally convex domain)

Assume that $II_{\partial M} \geq 0$ *(*M *is locally convex). Then:*

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu \; .$$

(2) (Dirichlet Dimensional Brascamp–Lieb inequality on generalized mean-convex domain)

Assume that $H_\mu \geq 0$ *(*M *is generalized mean-convex),* $f \equiv 0$ *on* $\partial M \neq \emptyset$*. Then:*

$$\frac{N}{N-1} \int_M f^2 d\mu \leq \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu \; .$$

(3) (Neumann Dimensional Brascamp–Lieb inequality on strictly generalized mean-convex domain)

Assume that $H_\mu > 0$ *(*M *is strictly generalized mean-convex). Then for any* $C \in \mathbb{R}$*:*

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu + \int_{\partial M} \frac{1}{H_\mu} \Big(f - C \Big)^2 d\mu \; .$$

In particular, if $\int_{\partial M} \frac{1}{H_\mu} d\mu < \infty$*, we have:*

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu + Var_{\mu/H_\mu}(f|_{\partial M}) \; .$$

In the Euclidean setting with $1/N = 0$, recall that $\mathrm{Ric}_{\mu,\infty} = \nabla^2 V$, reducing Case (1) to an inequality obtained by H. J. Brascamp and E. H. Lieb [12] as an infinitesimal version of the Prekopá–Leindler inequality, a functional infinite-dimensional version of the Brunn–Minkowski inequality (see Section 8). When $\mathrm{Ric}_{\mu,N} \geq \rho g$ with $\rho > 0$ (i.e. (M, g, μ) satisfies the $CD(\rho, N)$ condition), by replacing the $\int_M \langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \rangle d\mu$ term with the larger $\frac{1}{\rho} \int_M |\nabla f|^2 \, d\mu$ one in all occurrences above, we obtain a generalization of the classical Lichnerowicz estimate [38] on the spectral-gap of the weighted-Laplacian $-L$. When $N \leq -1$, Case (1) was obtained in the *Euclidean setting* (and under the stronger assumption that $\mathrm{Ric}_{\mu,\infty} = \nabla^2 V > 0$) with a constant better than $\frac{N}{N-1}$ on the left-hand-side above by V. H. Nguyen [52], improving a previous estimate of S. Bobkov and M. Ledoux [8] valid when $N \leq 0$. However, on a general *weighted Riemannian manifold*, our constant $\frac{N}{N-1}$ is best possible in Case (1) for the entire range $N \in (-\infty, -1] \cup [n, \infty]$, see Subsection 3.2. We refer to Subsection 3.1 for a long exposition on some of the previously known generalizations in these directions; with few exceptions, Cases (2) and (3) and also Case (1) when $1/N \neq 0$ seem new.

When $\mathrm{Ric}_\mu \geq \rho g$ for a function $\rho : M \to \mathbb{R}_+$ which is not necessarily bounded away from zero, we also extend in Section 4 a result of L. Veysseire [68] who obtained a spectral-gap estimate of $1/⨍_M (1/\rho) d\mu$, to the case of Neumann boundary conditions when M is locally convex.

Remark 1.3. Although all of our results are formulated for compact weighted-manifolds with boundary, the results easily extend to the non-compact case, if the manifold M can be exhausted by compact submanifolds $\{M_k\}$ so that each $(M_k, g|_{M_k}, \mu|_{M_k})$ has an appropriate boundary (locally-convex or generalized mean-convex, in accordance with the desired result). In the Dirichlet case, the asserted inequalities then extend to all functions in $C_0^1(M)$ having compact support and vanishing on the boundary ∂M. In the Neumann cases, the asserted inequalities extend to all functions $f \in C^1_{loc}(M) \cap L^2(M, \mu)$ when μ is a finite measure.

1.2 Poincaré-type inequalities on ∂M

Next, we obtain various Poincaré-type inequalities on the boundary of (M, g, μ).

Theorem 1.4 (Generalized Colesanti Inequality)**.** *Assume that (M, g, μ) satisfies the $CD(0, N)$ condition ($1/N \in (-\infty, 1/n]$) and that $II_{\partial M} > 0$ (M is locally strictly-convex). Then the following inequality holds for any $f \in C^1(\partial M)$:*

$$\int_{\partial M} H_\mu f^2 d\mu - \frac{N-1}{N} \frac{\left(\int_{\partial M} f d\mu\right)^2}{\mu(M)} \leq \int_{\partial M} \left\langle II_{\partial M}^{-1} \nabla_{\partial M} f, \nabla_{\partial M} f \right\rangle d\mu \; . \tag{1.3}$$

Theorem 1.4 was obtained by A. Colesanti in [16] with $N = n$ for a compact subset M of Euclidean space $\mathbb{R}^n$ endowed with the Lebesgue measure ($V = 0$) and having a C^2 strictly convex boundary. Colesanti derived this inequality as an infinitesimal version of the celebrated Brunn-Minkowski inequality, and so his method is naturally confined to the Euclidean setting. In contrast, we derive Theorem 1.4 in Section 5 directly from the generalized Reilly formula, and thus obtain in the Euclidean setting another proof of the Brunn–Minkowski inequality for convex domains (see more on this in the next subsection).

We also obtain a dual-version of Theorem 1.4, which in fact applies to mean-convex domains (a slightly more general version is given in Theorem 5.4):

Theorem 1.5 (Dual Generalized Colesanti Inequality)**.** *Assume that (M, g, μ) satisfies the $CD(\rho, 0)$ condition, $\rho \in \mathbb{R}$, and that $H_\mu > 0$ on ∂M (M is strictly generalized mean-convex). Then for any $f \in C^{2,\alpha}(\partial M)$ and $C \in \mathbb{R}$:*

$$\int_{\partial M} \langle II_{\partial M} \, \nabla_{\partial M} f, \nabla_{\partial M} f \rangle \, d\mu \leq \int_{\partial M} \frac{1}{H_\mu} \Big(L_{\partial M} f + \frac{\rho(f - C)}{2} \Big)^2 d\mu \ .$$

Here $L_{\partial M} = L_{(\partial M, g|_{\partial M}, \exp(-V) d Vol_{\partial M})}$ is the weighted-Laplacian on the boundary.

By specializing to the constant function $f \equiv 1$, various mean-curvature inequalities for convex and mean-convex boundaries of $CD(0, N)$ weighted-manifolds are obtained in Section 6, immediately recovering (when $N \in [n, \infty]$) and extending (when $N \leq 0$) recent results of Huang–Ruan [29]. Under various combinations of non-negative lower bounds on H_μ, $\mathrm{II}_{\partial M}$ and ρ, spectral-gap estimates on convex boundaries of $CD(\rho, 0)$ weighted-manifolds are deduced in Sections 6 and 7. For instance, we show:

Theorem 1.6. *Assume that (M, g, μ) satisfies $CD(\rho, 0)$, $\rho \geq 0$, and that $II_{\partial M} \geq \sigma g|_{\partial M}$, $H_\mu \geq \xi$ on ∂M with $\sigma, \xi > 0$. Then:*

$$\lambda_1 Var_\mu(f) \leq \int_{\partial M} |\nabla_{\partial M} f|^2 d\mu \ , \ \forall f \in C^1(\partial M) \ ,$$

with:

$$\lambda_1 \geq \frac{\rho + a + \sqrt{2a\rho + a^2}}{2} \geq \max\Big(a, \frac{\rho}{2}\Big) \ \ , \ a := \sigma\xi \ .$$

This extends and refines the estimate $\lambda_1 \geq (n-1)\sigma^2$ of Xia [72] in the classical unweighted Riemannian setting ($V \equiv 0$) when $Ric_g \geq 0$ ($\rho = 0$), since in that case $\xi \geq (n-1)\sigma$. Other estimates where σ and ξ are allowed to vary on ∂M are obtained in Section 7. To this end, we show that the boundary $(\partial M, g|_{\partial M}, \exp(-V) d\mathrm{Vol}_{\partial M})$ satisfies the $CD(\rho_0, N-1)$ condition for an appropriate ρ_0.

1.3 Connections to the Brunn–Minkowski Theory

Recall that the classical Brunn–Minkowski inequality in Euclidean space [63, 24] asserts that:

$$Vol((1-t)K+tL)^{1/n} \geq (1-t)Vol(K)^{1/n} + tVol(L)^{1/n} \;,\; \forall t \in [0,1] \;, \tag{1.4}$$

for all convex $K, L \subset \mathbb{R}^n$; it was extended to arbitrary Borel sets by Lyusternik. Here Vol denotes Lebesgue measure and $A + B := \{a + b \; ; \; a \in A, b \in B\}$ denotes Minkowski addition. We refer to the excellent survey by R. Gardner [24] for additional details and references.

By homogeneity of Vol, (1.4) is equivalent to the concavity of the function $t \mapsto Vol(K+tL)^{1/n}$. By Minkowski's theorem, extending Steiner's observation for the case that L is the Euclidean ball, $Vol(K+tL)$ is an n-degree polynomial $\sum_{i=0}^n \binom{n}{i} W_{n-i}(K,L) t^i$, whose coefficients

$$W_{n-i}(K,L) := \frac{(n-i)!}{n!} \left(\frac{d}{dt}\right)^i Vol(K+tL)|_{t=0} \;, \tag{1.5}$$

are called mixed-volumes. The above concavity thus amounts to the following "Minkowski's second inequality", which is a particular case of the Alexandrov–Fenchel inequalities:

$$W_{n-1}(K,L)^2 \geq W_{n-2}(K,L) W_n(K,L) = W_{n-2}(K,L) Vol(K) \;. \tag{1.6}$$

It was shown by Colesanti [16] that (1.6) is equivalent to (1.3) in the Euclidean setting. In fact, a Poincaré-type inequality on the sphere, which is a reformulation of (1.3) obtained via the Gauss-map, was established already by Hilbert (see [13, 28]) in his proof of (1.6) and thus the Brunn–Minkowski inequality for convex sets. Going in the other direction, the Brunn–Minkowski inequality was used by Colesanti to establish (1.3). See e.g. [12, 7, 34] for further related connections.

In view of our generalization of (1.3) to the weighted-Riemannian setting, it is all but natural to wonder whether there is a Riemannian Brunn–Minkowski theory lurking in the background. Note that when L is the Euclidean unit-ball D, then $K + tD$ coincides with $K_t := \{x \in \mathbb{R}^n \; ; \; d(x,K) \leq t\}$, where d is the Euclidean distance. The corresponding distinguished mixed-volumes $W_{n-i}(K) = W_{n-i}(K,D)$, which are called intrinsic-volumes or quermassintegrals, are obtained (up to normalization factors) as the i-th variation of $t \mapsto Vol(K_t)$. Analogously, we may define K_t on a general Riemannian manifold with d denoting the geodesic distance, and given $1/N \in (-\infty, 1/n]$, define the following *generalized* quermassintegrals of K as the i-th variations of $t \mapsto \mu(K_t)$, $i = 0, 1, 2$ (up to normalization):

$$W_N(K) := \mu(K) \;,\; W_{N-1}(K) := \frac{1}{N} \int_{\partial K} d\mu \;,\; W_{N-2}(K) := \frac{1}{N(N-1)} \int_{\partial K} H_\mu d\mu \;.$$

Applying (1.3) to the constant function $f \equiv 1$, we obtain in Section 8 the following interpretation of the resulting inequality:

Theorem 1.7. *(Riemannian Geodesic Brunn-Minkowski for Convex Sets) Let K^n denote a submanifold of (M^n, g, μ) having C^2 boundary and bounded away from ∂M. Assume that $(K, g|_K, \mu|_K)$ satisfies the $CD(0, N)$ condition ($1/N \in (-\infty, 1/n]$) and that $II_{\partial K} > 0$ (K is locally strictly-convex). Then the following generalized Minkowski's second inequality for geodesic extensions holds:*

$$W_{N-1}(K)^2 \geq W_N(K) W_{N-2}(K) \; .$$

Equivalently, $(d/dt)^2 N\mu(K_t)^{1/N}|_{t=0} \leq 0$, so that the function $t \mapsto N\mu(K_t)^{1/N}$ is concave on any interval $[0, T]$ so that for all $t \in [0, T)$, K_t is C^2 smooth, locally strictly-convex, bounded away from ∂M, and $(K_t, g|_{K_t}, \mu|_{K_t})$ satisfies $CD(0, N)$.

A greater challenge is to find an extension of the Minkowski sum $K + tL$ beyond the linear setting for a general convex L. Observe that due to lack of homogeneity, this is *not* the same as extending the operation of Minkowski interpolation $(1-t)K+tL$, a trivial task on any geodesic metric space by using geodesic interpolation. Motivated by the equivalence between (1.3) and (1.6) which should persist in the weighted-Riemannian setting, we propose in Section 8 a generalization of $K + tL$ based on a seemingly novel geometric-flow we dub "Parallel Normal Flow", which is characterized by having parallel normals to the evolving surface along the trajectory, and which is related to an appropriate homogeneous Monge-Ampère equation. Given a locally strictly-convex K and $\varphi \in C^2(\partial K)$, this flow produces a set denoted by $K_{\varphi,t} := K + t\varphi$ (which coincides in the Euclidean setting with $K + tL$ when φ is the support function of L). We do not go here into justifications for the existence of such a flow on an interval $[0, T]$ (except when all of the data is analytic, in which case the short-time existence it is easy to justify), but rather observe in Theorem 8.6 that in such a case and under the $CD(0, N)$ condition, $t \mapsto N\mu(K_{\varphi,t})^{1/N}$ is concave on $[0, T]$; indeed, the latter concavity turns out to be precisely equivalent to our generalized Colesanti inequality (1.3). In view of the remarks above, this observation should be interpreted as a version of the Brunn–Minkowski inequality in the weighted Riemannian setting. Furthermore, this leads to a natural way of defining the mixed-volumes of K and φ in this setting, namely as variations of $t \mapsto \mu(K + t\varphi)$. Yet another natural flow producing the aforementioned concavity is also suggested in Section 8; however, this flow does not seem to produce Minkowski summation in the Euclidean setting. See Remark 8.7 for a comparison with other known extensions of the Brunn–Minkowski inequality to the metric-measure space setting.

To conclude this work, we provide in Section 9 some further applications of our results to the study of isoperimetric inequalities on weighted Riemannian manifolds. Additional applications will be developed in a subsequent work.

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2 Generalized Reilly Formula and Other Preliminaries

2.1 Notation

We denote by $int(M)$ the interior of M. Given a compact differentiable manifold Σ (which is at least C^k smooth), we denote by $C^k(\Sigma)$ the space of real-valued functions on Σ with continuous derivatives $\left(\frac{\partial}{\partial x}\right)^a f$, for every multi-index a of order $|a| \leq k$ in a given coordinate system. Similarly, the space $C^{k,\alpha}(\Sigma)$ denotes the subspace of functions whose k-th order derivatives are Hölder continuous of order α on the $C^{k,\alpha}$ smooth manifold Σ. When Σ is non-compact, we may use $C^{k,\alpha}_{loc}(\Sigma)$ to denote the class of functions u on M so that $u|_{\Sigma_0} \in C^{k,\alpha}(\Sigma_0)$ for all compact subsets $\Sigma_0 \subset \Sigma$. These spaces are equipped with their usual corresponding topologies.

Throughout this work we employ Einstein summation convention. By abuse of notation, we denote different covariant and contravariant versions of a tensor in the same manner. So for instance, Ric_μ may denote the 2-covariant tensor $(Ric_\mu)_{\alpha,\beta}$, but also may denote its 1-covariant 1-contravariant version $(Ric_\mu)^\alpha_\beta$, as in:

$$\langle \mathrm{Ric}_\mu \nabla f, \nabla f \rangle = g_{i,j} (\mathrm{Ric}_\mu)^i_k \nabla^k f \nabla^j f = (\mathrm{Ric}_\mu)_{i,j} \nabla^i f \nabla^j f = \mathrm{Ric}_\mu(\nabla f, \nabla f) \ .$$

Similarly, reciprocal tensors are interpreted according to the appropriate context. For instance, the 2-contravariant tensor $(\mathrm{II}^{-1})^{\alpha,\beta}$ is defined by:

$$(\mathrm{II}^{-1})^{i,j} \mathrm{II}_{j,k} = \delta^i_k \ .$$

We freely raise and lower indices by contracting with the metric when there is no ambiguity regarding which underlying metric is being used; this is indeed the case throughout this work, with the exception of Subsection 8.5. Since we mostly deal with 2-tensors, the only possible contraction is often denoted by using the trace notation tr.

In addition to the already mentioned notation in the weighted-Riemannian setting, we will also make use of $\mathrm{div}_{g,\mu} = \mathrm{div}_{(M,g,\mu)}$ to denote the weighted-divergence operator on the weighted-manifold (M, g, μ), so that if $\mu = \exp(-V) d\mathrm{Vol}_M$ then:

$$\mathrm{div}_{g,\mu}(X) := \exp(V) \mathrm{div}_g(\exp(-V)X) = \mathrm{div}_g(X) - g(\nabla_g V, X) \ , \ \forall X \in TM \ ;$$

this is the natural notion of divergence in the weighted-manifold setting, satisfying the usual integration by parts formula (say if M is closed):

$$\int_M f \cdot \mathrm{div}_{g,\mu}(X) d\mu = - \int_M g(\nabla_g f, X) d\mu \ , \ \forall X \in TM \ .$$

Finally, when studying consequences of the $CD(\rho, N)$ condition, the various expressions in which N appears are interpreted in the limiting sense when $1/N = 0$. For instance, $N/(N-1)$ is interpreted as 1, and $N f^{1/N}$ is interpreted as $\log f$ (since $\lim_{1/N \to 0} N(x^{1/N} - 1) = \log(x)$; the constant -1 in the latter limit does not influence our application of this convention).

2.2 Proof of the Generalized Reilly Formula

For completeness, we sketch the proof of our main tool, Theorem 1.1 from the Introduction, following the proof given in [42].

Proof of Theorem 1.1. The generalized Bochner–Lichnerowicz–Weitzenböck formula [39, 4] states that for any $u \in C^3_{loc}(int(M))$, we have:

$$\frac{1}{2} L |\nabla u|^2 = \left\|\nabla^2 u\right\|^2 + \langle \nabla L u, \nabla u \rangle + \langle \mathrm{Ric}_\mu \nabla u, \nabla u \rangle \ . \tag{2.1}$$

We introduce an orthonormal frame of vector fields $e_1, \ldots, e_n$ so that that $e_n = \nu$ on ∂M, and denote $u_i = du(e_i)$, $u_{i,j} = \nabla^2 u(e_i, e_j)$. Assuming in addition that $u \in C^2(M)$, we may integrate by parts:

$$\int_M \frac{1}{2} L |\nabla u|^2 d\mu = \int_{\partial M} \sum_{i=1}^n u_i u_{i,n} d\mu \ , \ \int_M \langle \nabla L u, \nabla u \rangle d\mu = \int_{\partial M} u_n (Lu) d\mu - \int_M (Lu)^2 d\mu \ .$$

Consequently, integrating (2.1) over M, we obtain:

$$\int_M \left((Lu)^2 - \left\|\nabla^2 u\right\|^2 - \langle \mathrm{Ric}_\mu \nabla u, \nabla u \rangle \right) d\mu = \int_{\partial M} \left(u_n (Lu) - \sum_{i=1}^n u_i u_{i,n} \right) d\mu \ .$$

Now:

$$u_n(Lu) - \sum_{i=1}^n u_i u_{i,n} = \sum_{i=1}^{n-1} (u_n u_{i,i} - u_i u_{i,n}) - u_n \langle \nabla u, \nabla V \rangle \ .$$

Computing the different terms:

$$\begin{aligned} \sum_{i=1}^{n-1} u_{i,i} &= \sum_{i=1}^{n-1} (e_i(e_i u) - (\nabla_{e_i} e_i) u) = \sum_{i=1}^{n-1} (e_i(e_i u) - ((\nabla_{\partial M})_{e_i} e_i) u) + \left(\sum_{i=1}^{n-1} (\nabla_{\partial M})_{e_i} e_i - \nabla_{e_i} e_i \right) u \\ &= \Delta_{\partial M} u + \left(\sum_{i=1}^{n-1} \mathrm{II}_{i,i} \right) e_n u = \Delta_{\partial M} u + tr(\mathrm{II}) u_n \ ; \end{aligned}$$

$$\sum_{i=1}^{n-1} u_i u_{i,n} = \sum_{i=1}^{n-1} u_i \left(e_i(e_n u) - (\nabla_{e_i} e_n) u \right) = \langle \nabla_{\partial M} u, \nabla_{\partial M} u_n \rangle - \langle \mathrm{II} \ \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \ .$$

Putting everything together:

$$\int_M \left((Lu)^2 - \left\|\nabla^2 u\right\|^2 - \langle \mathrm{Ric}_\mu \nabla u, \nabla u\rangle\right) d\mu = \int_{\partial M} \left(u_n(\Delta_{\partial M} u - \langle \nabla u, \nabla V\rangle) + \mathrm{tr}(\mathrm{II})(u_n)^2\right) d\mu$$
$$- \int_{\partial M} \langle \nabla_{\partial M} u, \nabla_{\partial M} u_n\rangle \, d\mu + \int_{\partial M} \langle \mathrm{II}\ \nabla_{\partial M} u, \nabla_{\partial M} u\rangle \, d\mu \ .$$

This is the formula obtained in [42] for smooth functions. To conclude the proof, simply note that:

$$\langle \nabla u, \nabla V\rangle = \langle \nabla_{\partial M} u, \nabla_{\partial M} V\rangle + u_n V_n \ , \ L_{\partial M} = \Delta_{\partial M} - \langle \nabla_{\partial M} V, \nabla_{\partial M}\rangle \ , \ H_\mu = tr(\mathrm{II}) - V_n \ ,$$

and thus:

$$\int_{\partial M} \left(u_n(\Delta_{\partial M} u - \langle \nabla u, \nabla V\rangle) + tr(\mathrm{II})(u_n)^2\right) d\mu = \int_{\partial M} \left(u_n L_{\partial M} u + H_\mu u_n^2\right) d\mu \ .$$

Integrating by parts one last time, this time on ∂M, we obtain:

$$\int_{\partial M} u_n L_{\partial M} u \, d\mu = -\int_{\partial M} \langle \nabla_{\partial M} u_n, \nabla_{\partial M} u\rangle \, d\mu \ .$$

Finally, plugging everything back, we obtain the asserted formula for u as above:

$$\int_M \left((Lu)^2 - \left\|\nabla^2 u\right\|^2 - \langle \mathrm{Ric}_\mu\ \nabla u, \nabla u\rangle\right) d\mu$$
$$= \int_{\partial M} H_\mu u_n^2 d\mu - 2\int_{\partial M} \langle \nabla_{\partial M} u_n, \nabla_{\partial M} u\rangle \, d\mu + \int_{\partial M} \langle \mathrm{II}\ \nabla_{\partial M} u, \nabla_{\partial M} u\rangle \, d\mu \ .$$

To conclude that the assertion in fact holds for $u \in \mathcal{S}_N(M)$, we employ a standard approximation argument using a partition of unity and mollification. Since the metric is assumed at least C^3 and ∂M is C^2, we may approximate any $u \in \mathcal{S}_N(M)$ by functions $u_k \in C^3_{loc}(int(M)) \cap C^2(M)$, so that $u_k \to u$ in $C^2_{loc}(int(M))$ and $C^1(M)$, and $(u_k)_\nu \to u_\nu$ in $C^1(\partial M)$. The assertion then follows by passing to the limit. □

Remark 2.1. For minor technical reasons, it will be useful to record the following variants of the generalized Reilly formula, which are obtained by analogous approximation arguments to the one given above:

- If u_ν or u are constant on ∂M and $u \in \mathcal{S}_0(M)$ (recall $\mathcal{S}_0(M) := C^2_{loc}(int(M)) \cap C^1(M)$), then:

$$\int_M (Lu)^2 d\mu = \int_M \left\|\nabla^2 u\right\|^2 d\mu + \int_M \langle \mathrm{Ric}_\mu\ \nabla u, \nabla u\rangle \, d\mu + \int_{\partial M} H_\mu (u_\nu)^2 d\mu + \int_{\partial M} \langle \mathrm{II}_{\partial M}\ \nabla_{\partial M} u, \nabla_{\partial M} u\rangle \, d\mu \ . \quad (2.2)$$

- If $u \in \mathcal{S}_D(M) := \mathcal{S}_0(M) \cap C^2(\partial M)$, then integration by parts yields:

$$\int_M (Lu)^2 d\mu = \int_M \left\|\nabla^2 u\right\|^2 d\mu + \int_M \langle \mathrm{Ric}_\mu \; \nabla u, \nabla u \rangle \, d\mu +$$
$$\int_{\partial M} H_\mu (u_\nu)^2 d\mu + \int_{\partial M} \langle \mathrm{II}_{\partial M} \; \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, d\mu + 2 \int_{\partial M} \langle u_\nu, L_{\partial M} u \rangle \, d\mu \; . \quad (2.3)$$

Remark 2.2. Throughout this work, when integrating by parts, we employ a slightly more general version of the textbook Stokes Theorem $\int_M d\omega = \int_{\partial M} \omega$, in which one only assumes that ω is a continuous differential $(n-1)$-form on M which is differentiable on $int(M)$ (and so that $d\omega$ is integrable there); a justification may be found in [43]. This permits us to work with the classes $C^k_{loc}(int(M))$ occurring throughout this work.

2.3 The $CD(\rho, N)$ condition for $1/N \in [-\infty, 1/n]$

The results in this subsection for $1/N \in [0, 1/n]$ are due to Bakry (e.g. [3, Section 6]).

Lemma 2.3. *For any $u \in C^2_{loc}(M)$ and $1/N \in [-\infty, 1/n]$:*

$$\Gamma_2(u) = \langle \mathrm{Ric}_\mu \; \nabla u, \nabla u \rangle + \left\|\nabla^2 u\right\|^2 \geq \langle \mathrm{Ric}_{\mu,N} \; \nabla u, \nabla u \rangle + \frac{1}{N}(Lu)^2 \; . \quad (2.4)$$

Our convention throughout this work is that $-\infty \cdot 0 = 0$, and so if $Lu = 0$ at a point $p \in M$, the assertion when $\frac{1}{N} = -\infty$ is that:

$$\Gamma_2(u) \geq \langle \mathrm{Ric}_{\mu,0} \; \nabla u, \nabla u \rangle \;\; ,$$

at that point.

Proof. Recalling the definitions, this is equivalent to showing that:

$$\left\|\nabla^2 u\right\|^2 + \frac{1}{N-n} \langle \nabla u, \nabla V \rangle^2 \geq \frac{1}{N}(Lu)^2 \; .$$

Clearly the case that $1/N = 0$ ($N = \infty$) follows. But by Cauchy–Schwartz:

$$\left\|\nabla^2 u\right\|^2 \geq \frac{1}{n}(\Delta u)^2 \; ,$$

and so the case $N = n$, which corresponds to a constant function V so that $\mathrm{Ric}_\mu = \mathrm{Ric}_{\mu,n} = \mathrm{Ric}_g$ and $L = \Delta$, also follows. It remains to show that:

$$\frac{1}{n}(\Delta u)^2 + \frac{1}{N-n} \langle \nabla u, \nabla V \rangle^2 \geq \frac{1}{N}(Lu)^2 \; .$$

The case $1/N = -\infty$ ($N = 0$) follows since when $0 = Lu = \Delta u - \langle \nabla u, \nabla V \rangle$ then:

$$\frac{1}{n}(\Delta u)^2 - \frac{1}{n} \langle \nabla u, \nabla V \rangle^2 = \frac{1}{n}(\Delta u + \langle \nabla u, \nabla V \rangle)(\Delta u - \langle \nabla u, \nabla V \rangle) = 0 \ .$$

In all other cases, the assertion follows from another application of Cauchy–Schwartz:

$$\frac{1}{\alpha} A^2 + \frac{1}{\beta} B^2 \geq \frac{1}{\alpha + \beta}(A + B)^2 \quad \forall A, B \in \mathbb{R} \ ,$$

valid as soon as (α, β) lay in either the set $\{\alpha, \beta > 0\}$ or the set $\{\alpha + \beta < 0$ and $\alpha\beta < 0\}$.

□

Remark 2.4. It is immediate to deduce from Lemma 2.3 that for $1/N \in (-\infty, 1/n]$, $\mathrm{Ric}_{\mu,N} \geq \rho g$ on M, $\rho \in \mathbb{R}$, if and only if:

$$\Gamma_2(u) \geq \rho \, |\nabla u|^2 + \frac{1}{N}(Lu)^2 \ , \ \forall u \in C^2_{loc}(M) \ .$$

Indeed, the necessity follows from Lemma 2.3. The sufficiency follows by locally constructing given $p \in M$ and $X \in T_p M$ a function u so that $\nabla u = X$ at p and equality holds in both applications of the Cauchy–Schwartz inequality in the proof above, as this implies that $\mathrm{Ric}_{\mu,N}(X, X) \geq \rho \, |X|^2$. Indeed, equality in the first application implies that $\nabla^2 u$ is a multiple of g at p, whereas the equality in the second implies when $1/N \notin \{0, 1/n\}$ that $\langle \nabla u, \nabla V \rangle$ and Δu are appropriately proportional at p; clearly all three requirements can be simultaneously met. The cases $1/N \in \{0, 1/n\}$ follow by approximation.

2.4 Solution to Poisson Equation on Weighted Riemannian Manifolds

As our manifold is smooth, connected, compact, with C^2 smooth boundary and strictly positive C^2-density all the way up to the boundary, all of the classical elliptic existence, uniqueness and regularity results immediately extend from the Euclidean setting to our weighted-manifold one (see e.g. [66, Chapter 5] and [51]); for more general situations (weaker regularity of metric, Lipschitz domains, etc.) see e.g. [49] and the references therein. We summarize the results we require in the following:

Theorem 2.5. *Given a weighted-manifold (M, g, μ) , $\mu = \exp(-V) d\,Vol_M$, we assume that ∂M is C^2 smooth. Let $\alpha \in (0, 1)$, and assume that g is $C^{2,\alpha}$ smooth and $V \in C^{1,\alpha}(M)$. Let $f \in C^\alpha(M)$, $\varphi_D \in C^{2,\alpha}(\partial M)$ and $\varphi_N \in C^{1,\alpha}(\partial M)$. Then there exists a function $u \in C^{2,\alpha}_{loc}(int(M)) \cap C^{1,\beta}(M)$ for all $\beta \in (0, 1)$, which solves:*

$$Lu = f \ \text{ on } M \ ,$$

with either of the following boundary conditions on ∂M:

(1) Dirichlet: $u|_{\partial M} = \varphi_D$, assuming $\partial M \neq \emptyset$.

(2) Neumann: $u_\nu|_{\partial M} = \varphi_N$, assuming the following compatibility condition is satisfied:

$$\int_M f d\mu = \int_{\partial M} \varphi_N d\mu \ .$$

In particular, $u \in S_0(M)$ in either case. Moreover, $u \in S_N(M)$ in the Neumann case and $u \in \mathcal{S}_D(M)$ in the Dirichlet case.

Remark 2.6. Since ∂M is only assumed C^2, when writing $\varphi_D \in C^{2,\alpha}(\partial M)$ we mean that φ_D may be extended to a $C^{2,\alpha}$ function on the entire M. If we were to assume that ∂M is $C^{2,\alpha}$ smooth instead of merely C^2, we could conclude that $u \in C^{2,\alpha}(M)$ and hence $u \in \mathcal{S}_N(M)$ for both Neumann and Dirichlet boundary conditions. But we make the extra effort to stay with the C^2 assumption.

Remark 2.7. For future reference, we remark that it is enough to only assume in the proof of the generalized Reilly formula (including the final approximation argument) that the metric g is C^3 smooth, so in particular the above regularity results apply.

We will not require the uniqueness of u above, but for completeness we mention that this is indeed the case for Dirichlet boundary conditions, and up to an additive constant in the Neumann case.

2.5 Spectral-gap on Weighted Riemannian Manifolds

Let λ_1^N denote the best constant in the Neumann Poincaré inequality:

$$\lambda_1^N Var_\mu(f) \leq \int_M |\nabla f|^2 \, d\mu \ , \ \forall f \in H^1(M) \ ,$$

and let λ_1^D denote the best constant in the Dirichlet Poincaré inequality:

$$\lambda_1^D \int_M f^2 d\mu \leq \int_M |\nabla f|^2 \, d\mu \ , \ \forall f \in H_0^1(M) \ .$$

Here $H^1(M)$ and $H_0^1(M)$ denote the Sobolev spaces obtained by completing $C^1(M)$ and $C_0^1(M)$ in the H^1-norm $\sqrt{\int_M f^2 d\text{Vol} + \int_M |\nabla f|^2 \, d\text{Vol}}$. It is well-known that λ_1^N and λ_1^D coincide with the spectral-gaps of the self-adjoint positive semi-definite extensions of $-L$ to the appropriate dense subspaces of $L^2(M)$; furthermore, since M is assumed compact, both instances have an orthonormal complete basis of eigenfunctions with corresponding discrete non-negative spectra. In the first case, λ_1^N is the first positive eigenvalue of $-L$ with zero Neumann boundary conditions:

$$-Lu = \lambda_1^N u \text{ on } M \ \ , \ u_\nu \equiv 0 \text{ on } \partial M \ ;$$

the zero eigenvalue corresponds to the eigenspace of constant functions, and so only functions u orthogonal to constants are considered. In the second case, λ_1^D is the first (positive) eigenvalue of $-L$ with zero Dirichlet boundary conditions:

$$-Lu = \lambda_1^D u \text{ on } M \;\;,\; u \equiv 0 \text{ on } \partial M \;.$$

Our assumptions on the smoothness of M, its boundary, and the density $\exp(-V)$, guarantee by elliptic regularity theory that in either case, all eigenfunctions are in $\mathcal{S}_0(M)$ (in fact, in $\mathcal{S}_N(M)$ in the Neumann case and in $\mathcal{S}_D(M)$ in the Dirichlet case).

3 Generalized Poincaré-type inequalities on M

In this section we provide a proof of Theorem 1.2 from the Introduction, which we repeat here for convenience:

Theorem 3.1 (Generalized Dimensional Brascamp–Lieb With Boundary)**.** *Assume that* $\mathrm{Ric}_{\mu,N} > 0$ *on* M *with* $1/N \in (-\infty, 1/n]$*. The generalized Reilly formula implies all of the inequalities below for any* $f \in C^1(M)$*:*

(1) (Neumann Dimensional Brascamp–Lieb inequality on locally convex domain)
Assume that $\mathrm{II}_{\partial M} \geq 0$ *(*M *is locally convex). Then:*

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu \;.$$

(2) (Dirichlet Dimensional Brascamp–Lieb inequality on generalized mean-convex domain)
Assume that $H_\mu \geq 0$ *(*M *is generalized mean-convex),* $f \equiv 0$ *on* $\partial M \neq \emptyset$*. Then:*

$$\frac{N}{N-1} \int_M f^2 d\mu \leq \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu \;.$$

(3) (Neumann Dimensional Brascamp–Lieb inequality on strictly generalized mean-convex domain)
Assume that $H_\mu > 0$ *(*M *is strictly generalized mean-convex). Then for any* $C \in \mathbb{R}$*:*

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu + \int_{\partial M} \frac{1}{H_\mu} \Big(f - C\Big)^2 d\mu \;.$$

In particular, if $\int_{\partial M} \frac{1}{H_\mu} d\mu < \infty$*, we have:*

$$\frac{N}{N-1} Var_\mu(f) \leq \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu + Var_{\mu/H_\mu}(f|_{\partial M}) \;.$$

3.1 Previously Known Partial Cases

3.1.1 $1/N = 0$ - Generalized Brascamp–Lieb Inequalities

Recall that when $1/N = 0$, $\mathrm{Ric}_{\mu,N} = \mathrm{Ric}_\mu$, and $\frac{N}{N-1} = 1$. When (M, g) is Euclidean space $\mathbb{R}^n$ and $\mu = \exp(-V)dx$ is a finite measure, the Brascamp–Lieb inequality [12] asserts that:

$$Var_\mu(f) \leq \int_{\mathbb{R}^n} \left\langle (\nabla^2 V)^{-1} \, \nabla f, \nabla f \right\rangle d\mu \ , \ \forall f \in C^1(\mathbb{R}^n) \ .$$

Observe that in this case, $\mathrm{Ric}_\mu = \nabla^2 V$, and so taking into account Remark 1.3, we see that the Brascamp–Lieb inequality follows from Case (1). The latter is easily seen to be sharp, as witnessed by testing the Gaussian measure in Euclidean space.

The extension to the weighted-Riemannian setting for $1/N = 0$, at least when (M, g) has no boundary, is well-known to experts, although we do not know who to accredit this to (see e.g. the Witten Laplacian method of Helffer–Sjöstrand [26] as exposed by Ledoux [35]). The case of a locally-convex boundary with Neumann boundary conditions (Case 1 above) can easily be justified in Euclidean space by a standard approximation argument, but this is less clear in the Riemannian setting; probably this can be achieved by employing the Bakry–Émery semi-group formalism (see Qian [58] and Wang [69, 70]). To the best of our knowledge, the other two Cases (2) and (3) are new even for $1/N = 0$.

3.1.2 $\mathrm{Ric}_{\mu,N} \geq \rho g$ with $\rho > 0$ - Generalized Lichnerowicz Inequalities

Assume that $\mathrm{Ric}_{\mu,N} \geq \rho g$ with $\rho > 0$, so that (M, g, μ) satisfies the $CD(\rho, N)$ condition. It follows that:

$$\int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \, \nabla f, \nabla f \right\rangle d\mu \leq \frac{1}{\rho} \int_M |\nabla f|^2 \, d\mu \ , \tag{3.1}$$

and so we may replace in all three cases of Theorem 3.1 every occurrence of the left-hand term in (3.1) by the right-hand one. So for instance, Case (1) implies that:

$$\frac{N}{N-1} \mathrm{Var}_\mu(f) \leq \frac{1}{\rho} \int_M |\nabla f|^2 \, d\mu \ , \tag{3.2}$$

and similarly for the other two cases; we refer to the resulting inequalities as Cases (1'), (2') and (3'). Clearly, Cases (1') and (2') are spectral-gap estimates for $-L$ with Neumann and Dirichlet boundary conditions, respectively.

Recall that in the non-weighted Riemannian setting ($\mu = \mathrm{Vol}_M$ and $N = n$), $\mathrm{Ric}_{\mu,N} = \mathrm{Ric}_g$. In this classical setting, the above spectral-gap estimates are due to the following authors: when $\partial M = \emptyset$, Cases (1') and (3') degenerate to a single statement, due to Lichnerowicz [38], and by Obata's theorem [53] equality is attained if and only if M is the n-sphere. When $\partial M \neq \emptyset$, Case (1') is due to Escobar [21] and independently Xia [71] ; Case (2') is due to Reilly [59] ; in both cases, one has equality if and only if M is the n-hemisphere ; Case (3') seems new even in the classical case.

On weighted-manifolds with $N \in [n, \infty]$, Case (1') is certainly known, see e.g. [36] (in fact, a stronger log-Sobolev inequality goes back to Bakry and Émery [4]); Case (2') was recently obtained under a slightly stronger assumption by Ma and Du [42, Theorem 2]; for an adaptation to the $CD(\rho, N)$ condition see Li and Wei [36, Theorem 3], who also showed that in both cases one has equality if and only if $N = n$ and M is the n-sphere or n-hemisphere endowed with its Riemannian volume form, corresponding to whether ∂M is empty or non-empty, respectively. As already mentioned, Case (3') seems new.

To the best of our knowledge, the case of $N < 0$ has not been previously treated in the Riemannian setting. Concurrently to posting our work on the arXiv, Ohta [54] has also obtained Case (1') for $N < 0$ when $\partial M = \emptyset$.

3.1.3 Generalized Bobkov–Ledoux–Nguyen Inequalities

In the *Euclidean setting* with $N \leq -1$ (and under the stronger assumption that $\mathrm{Ric}_\mu = \nabla^2 V > 0$), Case (1) with a better constant of $\frac{n-N-1}{n-N}$ instead of our $\frac{N}{N-1} = \frac{-N}{-N+1}$ is due to Nguyen [52, Proposition 10], who generalized and sharpened a previous version valid for $N \leq 0$ by Bobkov–Ledoux [8]. However, on a general *weighted Riemannian manifold*, our constant $\frac{N}{N-1}$ is best possible in the range $N \in (-\infty, -1] \cup [n, \infty]$, see Subsection 3.2 below.

Note that in the Euclidean case, $CD(0, N)$ condition with $N \in \mathbb{R}$ corresponds to Borell's class of convex measures [11], also known as "$1/N$-concave measures" (cf. [48]). When $N < 0$, these measures are heavy-tailed, having tails decaying to zero only polynomially fast, and consequently the corresponding generator $-L$ may not have a strictly positive spectral-gap. This is compensated by the weight $\mathrm{Ric}_{\mu,N}^{-1}$ in the resulting Poincaré-type inequality. A prime example is given by the Cauchy measure in $\mathbb{R}^n$, which satisfies $CD(0, 0)$ (it is $-\infty$-concave). See [8, 52] for more information.

Still in the Euclidean setting with $N \geq n$ (in fact $N > n - 1$), a dimensional version of the Brascamp–Lieb inequality which is reminiscent of Case (1) was obtain by Nguyen [52, Theorem 9]. The Bobkov–Ledoux results were obtained as an infinitesimal version of the Borell–Brascamp–Lieb inequality [11, 12] (see Subsection 8.4) - a generalization of the Brunn-Minkowski inequality, which is strictly confined to the Euclidean setting. Nguyen's approach is already more similar to our own, dualizing an ad-hoc Bochner formula obtained for a non-stationary diffusion operator.

In any case, our unified formulation (and treatment) of both regimes $N \leq 0$ and $N \in [n, \infty]$, the weaker assumption that $\mathrm{Ric}_{\mu,N} > 0$, the extension to the Riemannian setting with sharp constant $\frac{N}{N-1}$ and the treatment of the different boundary conditions in Cases (1), (2) and (3) seem new.

3.2 Sharpness of the $\frac{N}{N-1}$ constant in the Riemannian setting

We briefly comment on the sharpness of the constant $\frac{N}{N-1}$ for the range $N \in (-\infty, -1] \cup [n, \infty]$ in the more traditional setting of Case (1); the sharpness of Case (2) is also shown for $N \geq n$. This constant is no longer sharp in Case (1) for $N < 0$ with $|N| \ll 1$, since under the $CD(\rho, N)$ condition with $\rho > 0$, the spectral-gap remains bounded below as $N < 0$ increases to 0, see [46].

As described in Subsection 3.1.2, it is classical that equality in the Lichnerowicz estimate (3.2) is attained by the n-sphere and n-hemisphere in Cases (1) (and (3)) and by the n-hemisphere in Case (2), both endowed with the usual Riemannian volume. This demonstrates the sharpness of the constant $\frac{N}{N-1}$ when $N = n$.

For general $N \in (-\infty, -1] \cup (n, \infty]$, the sharpness may be shown as follows. Given $\rho > 0$, set $\delta = \frac{\rho}{N-1}$ and:

$$\beta := \begin{cases} \frac{\pi}{2\sqrt{\delta}} & \delta > 0 \\ \infty & \delta < 0 \end{cases} , \; \alpha := \begin{cases} -\beta & \text{Case (1)} \\ 0 & \text{Case (2)} \end{cases} .$$

Define the following functions of $t \in [\alpha, \beta]$:

$$R(t) := \begin{cases} \cos(\sqrt{\delta} t) & \delta > 0 \\ \cosh(\sqrt{-\delta} t) & \delta < 0 \end{cases} , \; \Psi_{N-1}(t) := R^{N-1}(t) \; .$$

If we extend our setup to include the case of one-dimensional ($n = 1$) weighted manifolds, namely the case of the real line endowed with a density, then it is immediate to check that $([\alpha, \beta], |\cdot|, \mu = \Psi_{N-1}(t)dt)$ satisfies the $CD(\rho, N)$ condition, since:

$$\mathrm{Ric}_{\mu,N} = -(N-1)\frac{(\Psi_{N-1}^{\frac{1}{N-1}})''}{\Psi_{N-1}^{\frac{1}{N-1}}} = -(N-1)\frac{R''}{R} = (N-1)\delta = \rho \; .$$

Note that when $n = 1$, our constant $\frac{N}{N-1}$ and Nguyen's one $\frac{n-N-1}{n-N}$ coincide. As we have learned from Nguyen, his constant is sharp in the Euclidean setting for any $n \geq 1$. One consequently verifies the sharpness for $n = 1$ by using the same test function used by Nguyen in [52], namely $f(t) = \frac{d}{dt} R(t)$. Indeed, when $N < -1$ or $N > 1$ (to ensure convergence of the integrals below) we have:

$$\int f(t) d\mu = \int_{-\beta}^{\beta} R'(t) R^{N-1}(t) dt = \frac{1}{N} \int_{-\beta}^{\beta} (R^N(t))' dt = 0 \; ,$$

since $\lim_{t\to\beta} R^N(t) = 0$, and since also $f(0) = R'(0) = 0$ (so that the Dirichlet boundary condition at $t = 0$ is satisfied in Case (2)), we may integrate by parts:

$$\int f^2(t)d\mu = \frac{1}{N}\int_\alpha^\beta R'(t)(R^N(t))'dt = -\frac{1}{N}\int_\alpha^\beta R''(t)R^N(t)dt = \frac{\rho}{N(N-1)}\int_\alpha^\beta R^{N+1}(t)dt\ .$$

On the other hand:

$$\int \mathrm{Ric}_{\mu,N}^{-1} f'(t)^2 d\mu = \frac{1}{\rho}\int_\alpha^\beta (R''(t))^2 R^{N-1}(t)dt = \frac{\rho}{(N-1)^2}\int_\alpha^\beta R^{N+1}(t)dt\ .$$

Comparing the last two expressions, we conclude the sharpness of the constant $\frac{N}{N-1}$ for $n = 1$ in Case (1) when $|N| > 1$ and in Case (2) when $N > 1$ (the function $f(t)$ does not vanish at infinity when $N < 0$ so this range is excluded in Case (2)). When $N = -1$, one uses an appropriately truncated version of the above test function. In any case, to assert sharpness for a *compact* weighted manifold with strictly positive density, we truncate the above construction at a finite $\beta_\epsilon \in (0, \beta)$, and let β_ϵ tend to β.

To see the sharpness for $n \geq 2$, we proceed by repeating the construction from [47], which emulates the above 1-dimensional model space on a thin weighted n-dimensional manifold of revolution. For $n \geq 3$, define:

$$\Psi_{N-n}(t) := R^{N-n}(t),$$

and given $\epsilon > 0$, consider the n-dimensional manifold $M := [\alpha, \beta] \times S^{n-1}$ endowed with the metric g_ϵ and measure μ_ϵ given by:

$$g_\epsilon := dt^2 + \epsilon^2 R(t)^2 g_{S^{n-1}}\ ;$$
$$\mu_\epsilon := \Psi(t,\theta)dvol_{g_\epsilon}(t,\theta)\ ,\ \Psi(t,\theta) = \Psi_{N-n}(t)\ ,\ (t,\theta) \in [\alpha,\beta] \times S^{n-1}\ .$$

The intuition behind this construction is that when $\epsilon > 0$ is small enough, the geometry of (M, g_ϵ) will contribute (at least) $(n-1)\delta$ to the generalized Ricci curvature tensor $\mathrm{Ric}_{g,\mu,N}$, and a factor of $R^{n-1}(t)$ to the density $d\mu_\epsilon\left((-\infty, t] \times S^{n-1}\right)/dt$, whereas the measure μ_ϵ will contribute $(N-n)\delta g_\epsilon$ to the former and a factor of $\Psi_{N-n}(t) = R^{N-n}(t)$ to the latter, totaling $(N-1)\delta = \rho$ and $R^{N-1}(t) = \Psi_{N-1}(t)$, respectively. Consequently $(M, g_\epsilon, \mu_\epsilon)$ satisfies the $CD(\rho, N)$ condition for small enough $\epsilon > 0$, and its measure projection onto the axis of revolution is $c_\epsilon \Psi_{N-1}(t)$; the sharpness of the constant then follows from our previous one-dimensional analysis. Note that in Case (2), the boundary component $\{0\} \times S^{n-1}$ is totally geodesic and hence satisfies our boundary curvature assumptions. In practice, when $N \geq n$ (and thus $\beta < \infty$), we need to ensure that the resulting compact weighted manifold is smooth at its vertices (at $t \in \{-\beta, \beta\}$ in Case (1) and $t = \beta$ in Case (2)), and this is achieved as in

[47] by gluing appropriate caps. When $N \leq -1$ (and thus $\beta = \infty$), in order to obtain a compact manifold as in the formulation of Theorem 3.1, we also need to truncate the above construction at a finite $\beta_\epsilon > 0$; the resulting boundary $\{-\beta_\epsilon, \beta_\epsilon\} \times S^{n-1}$ turns out to indeed be locally convex since $R'(\beta_\epsilon) = -R'(-\beta_\epsilon) > 0$, according to the calculation in [47]. The construction is even more complicated for the case $n = 2$; we refer to [47] for further precise details and rigorous justifications.

3.3 Proof of Theorem 3.1

Proof of Theorem 3.1. Plugging (2.4) into the generalized Reilly formula, we obtain for any $u \in \mathcal{S}_N(M)$:

$$\frac{N-1}{N} \int_M (Lu)^2 d\mu \geq \int_M \langle \mathrm{Ric}_{\mu,N} \nabla u, \nabla u \rangle \, d\mu +$$
$$\int_{\partial M} H_\mu (u_\nu)^2 d\mu + \int_{\partial M} \langle \mathrm{II}_{\partial M} \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \, d\mu - 2 \int_{\partial M} \langle \nabla_{\partial M} u_\nu, \nabla_{\partial M} u \rangle \, d\mu \ . \quad (3.3)$$

Recall that this remains valid for $u \in \mathcal{S}_0(M)$ if u or u_ν are constant on ∂M. Lastly, note that if $Lu = f$ in M with $f \in C^1(M)$ and $u \in \mathcal{S}_0(M)$, then:

$$\int_M f^2 d\mu = \int_M (Lu)^2 d\mu = \int_M f Lu \, d\mu = -\int_M \langle \nabla f, \nabla u \rangle \, d\mu + \int_{\partial M} f u_\nu d\mu \ . \quad (3.4)$$

Consequently, by Cauchy–Schwartz:

$$\int_M f^2 d\mu \leq \left(\int_M \langle \mathrm{Ric}_{\mu,N} \nabla u, \nabla u \rangle \, d\mu \right)^{1/2} \left(\int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \nabla f, \nabla f \right\rangle d\mu \right)^{1/2} + \int_{\partial M} f u_\nu d\mu \ . \quad (3.5)$$

We now proceed to treat the individual three cases.

(1) Assume that $\int_M f d\mu = 0$ and solve the Neumann Poisson problem for $u \in \mathcal{S}_0(M)$:

$$Lu = f \text{ on } M \ \ , \ u_\nu \equiv 0 \text{ on } \partial M \ ;$$

note that the compatibility condition $\int_{\partial M} u_\nu d\mu = \int_M f d\mu = 0$ is indeed satisfied, so a solution exists. Since $u_\nu|_{\partial M} \equiv 0$ and $\mathrm{II}_{\partial M} \geq 0$, we obtain from (3.3):

$$\frac{N}{N-1} \int_M \langle \mathrm{Ric}_{\mu,N} \nabla u, \nabla u \rangle \, d\mu \leq \int_M (Lu)^2 d\mu = \int_M f^2 d\mu \ . \quad (3.6)$$

Plugging this back into (3.5) and using that $u_\nu \equiv 0$ yields the assertion of Case (1).

(2) Assume that $f|_{\partial M} \equiv 0$ and solve the Dirichlet Poisson problem for $u \in \mathcal{S}_0(M)$:

$$Lu = f \text{ on } M \quad , \; u \equiv 0 \text{ on } \partial M \; .$$

Observe that (3.6) still holds since $u|_{\partial M} \equiv 0$ and $H_\mu \geq 0$. Plugging (3.6) back into (3.5) and using that $f|_{\partial M} \equiv 0$ yields the assertion of Case (2).

(3) Assume that $\int_M f d\mu = 0$ and solve the Dirichlet Poisson problem:

$$Lu = f \text{ on } M \quad , \; u \equiv 0 \text{ on } \partial M \; .$$

The difference with the previous case is that the $\int f u_\nu d\mu$ term in (3.4) does not vanish since we do not assume that $f|_{\partial M} \equiv 0$. Consequently, we cannot afford to omit the positive contribution of $\int_{\partial M} H_\mu (u_\nu)^2 d\mu$ in (3.3):

$$\frac{N-1}{N} \int_M f^2 d\mu \geq \int_M \langle \mathrm{Ric}_{\mu,N} \, \nabla u, \nabla u \rangle \, d\mu + \int_{\partial M} H_\mu u_\nu^2 d\mu \; .$$

Applying the duality argument, this time in additive form, we obtain for any $\lambda > 0$:

$$\begin{aligned} \int_M f^2 d\mu &= -\int_M \langle \nabla f, \nabla u \rangle \, d\mu + \int_{\partial M} f u_\nu d\mu \\ &\leq \frac{1}{2\lambda} \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \, \nabla f, \nabla f \right\rangle d\mu + \frac{\lambda}{2} \int_M \langle \mathrm{Ric}_{\mu,N} \, \nabla u, \nabla u \rangle \, d\mu + \int_{\partial M} f u_\nu d\mu \; . \end{aligned}$$

Since $\int_{\partial M} u_\nu d\mu = \int_M f d\mu = 0$, we may as well replace the last term by $\int_{\partial M} (f - C) u_\nu d\mu$. Plugging in the previous estimate and applying the Cauchy–Schwartz inequality again to eliminate u_ν, we obtain:

$$\begin{aligned} \left(1 - \frac{\lambda}{2} \frac{N-1}{N}\right) \int_M f^2 d\mu &\leq \frac{1}{2\lambda} \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \, \nabla f, \nabla f \right\rangle d\mu + \int_{\partial M} (f - C) u_\nu d\mu - \frac{\lambda}{2} \int_{\partial M} H_\mu u_\nu^2 d\mu \\ &\leq \frac{1}{2\lambda} \int_M \left\langle \mathrm{Ric}_{\mu,N}^{-1} \, \nabla f, \nabla f \right\rangle d\mu + \frac{1}{2\lambda} \int_{\partial M} \frac{1}{H_\mu} (f - C)^2 d\mu \; . \end{aligned}$$

Multiplying by 2λ and using the optimal $\lambda = \frac{N}{N-1}$, we obtain the assertion of Case (3).

□

4 Generalized Veysseire Spectral-gap inequality on convex M

The next result was recently obtained by L. Veysseire [68] for compact weighted-manifolds without boundary. It may be thought of as a spectral-gap version of the Generalized Brascamp–Lieb inequality. We provide an extension in the case that M is locally convex.

Theorem 4.1 (Veysseire Spectral-Gap inequality with locally-convex boundary)**.** *Assume that as* 2*-tensors on* M*:*

$$\mathrm{Ric}_\mu \geq \rho g \ ,$$

for some measurable function $\rho : M \to \mathbb{R}_+$*. The generalized Reilly formula implies that for any* $f \in C^1(M)$*:*

(1) (Neumann Veysseire inequality on locally convex domain)

Assume that $II_{\partial M} \geq 0$ *(*M *is locally convex). Then:*

$$Var_\mu(f) \leq \fint_M \frac{1}{\rho} d\mu \int_M |\nabla f|^2 \, d\mu.$$

Remark 4.2. We do not know whether the analogous results for Dirichlet or Neumann boundary conditions (Cases (2) and (3) in the previous section) hold on a generalized mean-convex domain, as the proof given below breaks down in those cases.

Remark 4.3. As in Veysseire's work [68], further refinements are possible. For instance, if in addition the $CD(\rho_0, N)$ condition is satisfied for $\rho_0 > 0$ and $1/N \in [-\infty, 1/n]$, then one may obtain an estimate on the corresponding spectral-gap λ_1^N of the form:

$$\lambda_1^N \geq \frac{N}{N-1}\rho_0 + \frac{1}{\fint_M \frac{1}{\rho - \rho_0} d\mu} \ .$$

As explained in [68], this may be obtained by using an appropriate convex combination of the Lichnerowicz estimate (Case (1) of Theorem 1.2 after replacing $Ric_{\mu,N}^{-1}$ with $1/\rho_0$) and the estimates obtained in this section, with a final application of the Cauchy–Schwartz inequality. Similarly, it is possible to interpolate between the Lichnerowicz estimates and the Dimensional Brascamp–Lieb ones of Theorem 1.2. We leave this to the interested reader.

Veysseire's proof in [68] is based on the Bochner formula and the following observation, valid for any $u \in C^2(M)$ at any point so that $\nabla u \neq 0$:

$$\left\|D^2 u\right\| \geq |\nabla |\nabla u|| \ . \tag{4.1}$$

At a point where $\nabla u = 0$, we define $|\nabla |\nabla u|| := 0$.

Proof of Theorem 4.1. Plugging (4.1) into the generalized Reilly formula and integrating the $\int_M (Lu)^2 d\mu$ term by parts, we obtain for any $u \in \mathcal{S}_N(M)$ so that $Lu \in C^1(M)$:

$$\int_{\partial M} u_\nu L u d\mu - \int_M \langle \nabla u, \nabla L u \rangle d\mu \geq \int_M |\nabla |\nabla u||^2 d\mu + \int_M \langle \mathrm{Ric}_\mu \nabla u, \nabla u \rangle d\mu +$$
$$\int_{\partial M} H_\mu (u_\nu)^2 d\mu + \int_{\partial M} \langle \mathrm{II}_{\partial M} \nabla_{\partial M} u, \nabla_{\partial M} u \rangle d\mu - 2 \int_{\partial M} \langle \nabla_{\partial M} u_\nu, \nabla_{\partial M} u \rangle d\mu \ . \tag{4.2}$$

(1) Let $u \in \mathcal{S}_N(M)$ denote an eigenfunction of $-L$ with zero Neumann boundary conditions corresponding to λ_1^N, so that in particular $Lu = -\lambda_1^N u \in C^1(M)$, and denote $h = |\nabla u| \in H^1(M)$. Applying (4.2) to u, using that $\mathrm{II}_{\partial M} \geq 0$, and that $\int_{\{h=0\}} |\nabla h|^2 d\mathrm{Vol}_M = 0$ for any $h \in H^1(M)$, we obtain:

$$\lambda_1^N \int_M h^2 d\mu \geq \int_M |\nabla h|^2 d\mu + \int_M \rho h^2 d\mu \ .$$

Applying the Neumann Poincaré inequality to the function h, we obtain:

$$\lambda_1^N \int_M h^2 d\mu \geq \lambda_1^N \left(\int_M h^2 d\mu - \frac{1}{\mu(M)} (\int_M h d\mu)^2 \right) + \int_M \rho h^2 d\mu \ .$$

It follows by Cauchy–Schwartz that:

$$\lambda_1^N \geq \frac{\mu(M) \int_M \rho h^2 d\mu}{(\int_M h d\mu)^2} \geq \frac{\mu(M)}{\int_M \frac{1}{\rho} d\mu} \ ,$$

concluding the proof.

□

Remark 4.4. The proof above actually yields a meaningful estimate on the spectral-gap λ_1^N even when $\mathrm{II}_{\partial M}$ is negatively bounded from below. However, this estimate depends on upper bounds on $|\nabla u|$, where u is the first non-trivial Neumann eigenfunction, both in M and on its boundary.

5 Poincaré-type inequalities on ∂M

5.1 Generalized Colesanti Inequality

Theorem 5.1 (Generalized Colesanti Inequality)**.** *Assume that (M, g, μ) satisfies the $CD(0,N)$ condition ($1/N \in [-\infty, 1/n]$) and that $II_{\partial M} > 0$ (M is locally strictly-convex). Then the following inequality holds for any $f \in C^1(\partial M)$:*

$$\int_{\partial M} H_\mu f^2 d\mu - \frac{N-1}{N} \frac{\left(\int_{\partial M} f d\mu\right)^2}{\mu(M)} \leq \int_{\partial M} \left\langle II_{\partial M}^{-1} \nabla_{\partial M} f, \nabla_{\partial M} f \right\rangle d\mu \ . \tag{5.1}$$

Remark 5.2. Theorem 5.1 was obtained by A. Colesanti in [16] with $N = n$ for a compact subset M of Euclidean space $\mathbb{R}^n$ having a C^2 strictly convex boundary and endowed with the Lebesgue measure ($V = 0$). Colesanti was mainly interested in the case that f has zero mean $\int_{\partial M} f \ d\mu = 0$, but his proof yields the additional second term above. Colesanti derived this inequality as an infinitesimal version of the Brunn-Minkowski inequality, and so his method is naturally confined to the Euclidean setting; see [17] for further possible extensions in the Euclidean setting. As observed in [16], Theorem 5.1 yields a sharp Poincaré inequality on S^{n-1} when M is a Euclidean ball in $\mathbb{R}^n$.

Proof. By applying the Cauchy–Schwartz inequality to the last-term in the generalized Reilly formula:

$$2 \langle \nabla_{\partial M} u_\nu, \nabla_{\partial M} u \rangle \leq \langle \mathrm{II}_{\partial M} \ \nabla_{\partial M} u, \nabla_{\partial M} u \rangle + \left\langle \mathrm{II}_{\partial M}^{-1} \nabla_{\partial M} u_\nu, \nabla_{\partial M} u_\nu \right\rangle \ ,$$

we obtain for any $u \in \mathcal{S}_N(M)$:

$$\int_M (Lu)^2 d\mu \geq \int_M \left(\left\| \nabla^2 u \right\|^2 + \langle \mathrm{Ric}_\mu \ \nabla u, \nabla u \rangle \right) d\mu + \int_{\partial M} H_\mu (u_\nu)^2 d\mu - \int_{\partial M} \left\langle \mathrm{II}_{\partial M}^{-1} \ \nabla_{\partial M} u_\nu, \nabla_{\partial M} u_\nu \right\rangle d\mu \ .$$

Using the $CD(0,N)$ condition as in Lemma 2.3 with the convention that $-\infty \cdot 0 = 0$, we conclude:

$$\frac{N-1}{N} \int_M (Lu)^2 d\mu \geq \int_{\partial M} H_\mu (u_\nu)^2 d\mu - \int_{\partial M} \left\langle \mathrm{II}_{\partial M}^{-1} \ \nabla_{\partial M} u_\nu, \nabla_{\partial M} u_\nu \right\rangle d\mu \ .$$

Now assume $f \in C^{1,\alpha}(\partial M)$ and solve the following Neumann Laplace problem for $u \in \mathcal{S}_N(M)$ satisfying:

$$Lu \equiv \frac{1}{\mu(M)} \int_{\partial M} f d\mu \text{ on } M \quad , \ u_\nu = f \text{ on } \partial M \ ;$$

note that the compatibility condition $\int_{\partial M} u_\nu d\mu = \int_M (Lu) d\mu$ is indeed satisfied, so that a solution exists. Plugging this back into the previous estimate, the generalized

Colesanti inequality follows for $f \in C^{1,\alpha}(\partial M)$. The result for $f \in C^1(\partial M)$ follows by approximating f in $C^1(\partial M)$ by $C^{1,\alpha}$ functions using a standard partition of unity and mollification argument (this is possible since ∂M is assumed C^2 smooth). $\square$

Remark 5.3. Peculiarly, it is possible to strengthen this inequality by using it for $f + z$ and optimizing over $z \in \mathbb{R}$; alternatively and equivalently, we may solve in the last step above:

$$Lu \equiv z \text{ on } M \quad , \; u_\nu = f - \fint_{\partial M} f d\mu + z \frac{\mu(M)}{\mu(\partial M)} \text{ on } \partial M \; .$$

This results in the following stronger inequality:

$$\int_{\partial M} H_\mu f^2 d\mu - \frac{N-1}{N} \frac{\left(\int_{\partial M} f d\mu\right)^2}{\mu(M)} + \frac{\left(\int_{\partial M} f \beta d\mu\right)^2}{\int_{\partial M} \beta d\mu} \leq \int_{\partial M} \left\langle \mathrm{II}_{\partial M}^{-1} \, \nabla_{\partial M} f, \nabla_{\partial M} f \right\rangle d\mu \; ,$$

where:

$$\beta(x) := \frac{N-1}{N} \frac{\mu(\partial M)}{\mu(M)} - H_\mu(x) \; .$$

Note that $\int_{\partial M} \beta d\mu \geq 0$ by testing (5.1) on the constant function $f \equiv 1$. It may be shown that this integral is in fact strictly positive, unless M is isometric to a Euclidean ball and V is constant - see Remark 6.5; so in all other cases, this yields a strict improvement over (5.1).

By Colesanti's argument in the Euclidean setting, the weaker (5.1) inequality constitutes an infinitesimal version of the (sharp) Brunn–Minkowski inequality (for convex sets), and so one cannot hope to improve (5.1) in the corresponding cases where Brunn–Minkowski is sharp.

On the other hand, it would be interesting to integrate back the stronger inequality and obtain a refined version of Brunn–Minkowski, which would perhaps be better suited for obtaining delicate stability results.

5.2 A Dual Version

Next, we establish a dual version of Theorem 5.1, which in fact applies whenever M is only assumed *mean-convex* and under a general $CD(\rho, N)$ condition; however, the price we pay is that we do not witness the dependence on N in the resulting inequality, so we might as well assume $CD(\rho, 0)$.

Theorem 5.4 (Dual Generalized Colesanti Inequality)**.** *Assume that (M, g, μ) satisfies the $CD(\rho, 0)$ condition, $\rho \in \mathbb{R}$, and that $H_\mu > 0$ on ∂M (M is strictly generalized mean-convex). Then for any $f \in C^{2,\alpha}(\partial M)$ and $C \in \mathbb{R}$:*

$$\int_{\partial M} \langle II_{\partial M} \, \nabla_{\partial M} f, \nabla_{\partial M} f \rangle \, d\mu \leq \int_{\partial M} \frac{1}{H_\mu} \Big(L_{\partial M} f + \frac{\rho(f - C)}{2} \Big)^2 d\mu \; .$$

Moreover, if we assume that ∂M is $C^{2,\alpha}$ smooth, then the result holds for $f \in C^2(\partial M)$.

Proof. First, assume that $f \in C^{2,\alpha}(\partial M)$. This time, we solve the Dirichlet Laplace problem for $u \in \mathcal{S}_D(M)$ satisfying:

$$Lu \equiv 0 \text{ on } M \quad , \; u = f \text{ on } \partial M \; .$$

By the generalized Reilly formula as in (2.3) and the $CD(\rho, 0)$ condition:

$$0 \geq \rho \int_M |\nabla u|^2 \, d\mu + \int_{\partial M} H_\mu (u_\nu)^2 d\mu + \int_{\partial M} \langle \mathrm{II}_{\partial M} \, \nabla_{\partial M} f, \nabla_{\partial M} f \rangle \, d\mu + 2 \int_{\partial M} L_{\partial M} f \cdot u_\nu d\mu \; .$$

Integrating by parts we obtain:

$$0 \geq \rho \int_{\partial M} f u_\nu d\mu + \int_{\partial M} H_\mu (u_\nu)^2 d\mu + \int_{\partial M} \langle \mathrm{II}_{\partial M} \, \nabla_{\partial M} f, \nabla_{\partial M} f \rangle \, d\mu + 2 \int_{\partial M} L_{\partial M} f \cdot u_\nu d\mu \; .$$

Since $\int_{\partial M} u_\nu d\mu = \int_M (Lu) d\mu = 0$, we may as well replace the first term above by $\int_{\partial M} (f - C) u_\nu d\mu$. The asserted inequality for $f \in C^{2,\alpha}(\partial M)$ is obtained following an application of the Cauchy–Schwartz inequality:

$$H_\mu u_\nu^2 + 2 u_\nu \Big(L_{\partial M} f + \frac{\rho (f - C)}{2} \Big) \geq - \frac{1}{H_\mu} \Big(L_{\partial M} f + \frac{\rho (f - C)}{2} \Big)^2 \; .$$

When ∂M is $C^{2,\alpha}$ smooth, the result remains valid for $f \in C^2(\partial M)$ by approximating f in $C^2(\partial M)$ by functions in $C^{2,\alpha}(\partial M)$ by the usual partition of unity and mollification argument. □

Remark 5.5. When $\mathrm{II}_{\partial M} > 0$ and $\rho = 0$, Theorem 5.4 for the $CD(0, \infty)$ condition may be heuristically obtained from Theorem 5.1 by a non-rigorous duality argument:

$$\begin{aligned} \int_{\partial M} \langle \mathrm{II}_{\partial M} \, \nabla_{\partial M} f, \nabla_{\partial M} f \rangle \, d\mu \quad &=^? \quad \sup_g \frac{\big(\int_{\partial M} \langle \nabla_{\partial M} f, \nabla_{\partial M} g \rangle \, d\mu \big)^2}{\int_{\partial M} \langle \mathrm{II}_{\partial M}^{-1} \, \nabla_{\partial M} g, \nabla_{\partial M} g \rangle \, d\mu} \\ &\leq \quad \sup_g \frac{\big(\int_{\partial M} g L_{\partial M} f d\mu \big)^2}{\int_{\partial M} H_\mu g^2 d\mu} \leq \int_{\partial M} \frac{1}{H_\mu} (L_{\partial M} f)^2 d\mu \; , \end{aligned}$$

where the supremum above is over all functions $g \in C^1(\partial M)$ with $\int_{\partial M} g d\mu = 0$. The delicate point is justifying the equality in question above: Cauchy–Schwartz implies the $\geq$ direction, and so given $f \in C^1(\partial M)$ it remains to find a function $g \in C^1(\partial M)$ so that $\nabla_{\partial M} g = \mathrm{II}_{\partial M} \, \nabla_{\partial M} f$ on ∂M. It is well known that on a simply-connected manifold (and more generally, with vanishing first homology), a vector field is a gradient field if and only if its covariant derivative is a symmetric tensor, but this does not seem to be the case for us.

6 Applications of Generalized Colesanti Inequalities

6.1 Topological Consequences

Theorem 6.1. *Assume that (M, g, μ) satisfies the $CD(0,0)$ condition and that $II_{\partial M} > 0$ (M is locally strictly-convex). Then ∂M is connected.*

Proof. Otherwise, ∂M has at least two connected components. By constructing a function $f \in C^1(\partial M)$ which is equal to an appropriate constant on each of the components so that $\int_{\partial M} f d\mu = 0$, we obtain a contradiction to (5.1). □

Remark 6.2. Observe that one cannot relax most of the conditions of the theorem. For instance, taking M to be $[0,1] \times T^{n-1}$ with the product metric, where T^{n-1} is the flat $n-1$-dimensional torus, we see that the strict convexity condition cannot be relaxed to $\mathrm{II}_{\partial M} \geq 0$. In addition, taking M to be the submanifold of Hyperbolic space H, which in the Poincaré model in the open unit-disc in $\mathbb{R}^n$ is represented by:

$$M := \{x \in \mathbb{R}^n \; ; \; |x| < 1 \; , \; |x + 10e_n| < 10.5 \; , \; |x - 10e_n| < 10.5\} \; ,$$

since M is strictly convex as a subset of Euclidean space, the same holds in H, but ∂M has two connected components. Consequently, we see that the $CD(0,0)$ condition cannot be relaxed to $CD(-1,0)$ and hence (by scaling the metric) neither to $CD(-\epsilon, 0)$.

6.2 Mean-Curvature Inequalities

Setting $f \equiv 1$ in Theorem 5.1, we recover and generalize to the entire range $1/N \in [-\infty, 1/n]$ the following recent result of Huang and Ruan [29, Theorem 1.3] for $N \in [n, \infty]$, who generalized the same result obtained by Reilly [60] in the classical Riemannian volume case ($V = 0$ and $N = n$).

Corollary 6.3 (Extending Reilly and Huang–Ruan)**.** *Assume that (M, g, μ) satisfies the $CD(0,N)$ condition ($1/N \in (-\infty, 1/n]$) and that $II_{\partial M} > 0$ (M is locally strictly-convex). Then:*

$$\int_{\partial M} H_\mu d\mu \leq \frac{N-1}{N} \frac{\mu(\partial M)^2}{\mu(M)} \; . \tag{6.1}$$

Applying Cauchy–Schwartz, it immediately follows that in above setting:

$$\int_{\partial M} \frac{1}{H_\mu} d\mu \geq \frac{\mu(\partial M)^2}{\int_{\partial M} H_\mu d\mu} \geq \frac{N}{N-1} \mu(M) \; . \tag{6.2}$$

Interestingly, it was shown by A. Ros [61] in the classical non-weighted case, and generalized by Huang and Ruan [29, Theorem 1.1] to the weighted-Riemannian setting for $N \in [n, \infty]$, that it is enough to assume that M is strictly (generalized) mean-convex for the inequality between first and last terms in (6.2) to hold. We extend this to the entire range $1/N \in (-\infty, 1/n]$:

Theorem 6.4 (Extending Ros and Huang–Ruan)**.** *Assume that (M, g, μ) satisfies the $CD(0, N)$ condition ($1/N \in (-\infty, 1/n]$) and that $H_\mu > 0$ (M is strictly generalized mean-convex). Then:*

$$\int_{\partial M} \frac{1}{H_\mu} d\mu \geq \frac{N}{N-1} \mu(M) \ . \tag{6.3}$$

This is very much related to our dual version of the generalized Colesanti inequality (Theorem 5.4), and in fact both inequalities may be obtained simultaneously from the generalized Reilly formula by invoking the Cauchy–Schwartz inequality in two different ways. In a sense, this explains why we lost the dependence on N in Theorem 5.4 and why we lose the dependence on ρ in Theorem 6.4. The idea for proving Theorem 6.4 is the same as in [29], but our argument is somewhat more direct.

Proof. Let us solve for $u \in \mathcal{S}_0(M)$ the following Dirichlet Poisson equation:

$$Lu \equiv 1 \text{ on } M \ \ , \ u \equiv 0 \text{ on } \partial M \ .$$

By the generalized Reilly formula and the $CD(0, N)$ condition:

$$\begin{aligned} \mu(M) = \int_M (Lu)^2 d\mu \ &= \ \int_M \left\|\nabla^2 u\right\|^2 d\mu + \int_M \langle \mathrm{Ric}_\mu \, \nabla u, \nabla u \rangle \, d\mu + \int_{\partial M} H_\mu (u_\nu)^2 d\mu \\ &\geq \ \frac{1}{N} \int_M (Lu)^2 d\mu + \int_{\partial M} H_\mu (u_\nu)^2 d\mu \ . \end{aligned}$$

Coupled with an application of the Cauchy–Schwartz inequality, this yields:

$$\mu(M)^2 = (\int_M (Lu) d\mu)^2 = (\int_{\partial M} u_\nu d\mu)^2 \leq \int_{\partial M} H_\mu (u_\nu)^2 d\mu \int_{\partial M} \frac{1}{H_\mu} d\mu \leq \frac{N-1}{N} \mu(M) \int_{\partial M} \frac{1}{H_\mu} d\mu \ ,$$

and the assertion follows. □

Remark 6.5. It may be shown by analyzing the cases of equality in all of the above used inequalities, that when $N \in [n, \infty]$, equality occurs in (6.1) or (6.3) if and only if M is isometric to a Euclidean ball and V is constant. See [60, 61, 29] for more details.

6.3 Spectral-Gap Estimates on ∂M

Next, we recall a result of Xia [72] in the classical non-weighted Riemannian setting ($V = 0$), stating that when $Ric_g \geq 0$ on M and $\mathrm{II}_{\partial M} \geq \sigma g|_{\partial M}$ on ∂M with $\sigma > 0$, then:

$$Var_{Vol_{\partial M}}(f) \leq \frac{1}{(n-1)\sigma^2} \int_{\partial M} |\nabla_{\partial M} f|^2 dVol_{\partial M} \ , \ \forall f \in C^1(\partial M) \ . \tag{6.4}$$

In other words, the spectral-gap of $-L_{\partial M}$ on $(\partial M, g|_{\partial M}, Vol_{\partial M})$ away from the trivial zero eigenvalue is at least $(n-1)\sigma^2$. Since in that case we have $H_g = tr(\mathrm{II}_{\partial M}) \geq (n-1)\sigma$, our next result, which is an immediate corollary of Theorem 5.1 applied to f with $\int_{\partial M} f \, d\mu = 0$, is both a refinement and an extension of Xia's estimate to the more general $CD(0,0)$ condition in the weighted Riemannian setting:

Corollary 6.6. *Assume that (M, g, μ) satisfies $CD(0,0)$, and that $II_{\partial M} \geq \sigma g|_{\partial M}$, $H_\mu \geq \xi$ on ∂M with $\sigma, \xi > 0$. Then:*

$$Var_\mu(f) \leq \frac{1}{\sigma \xi} \int_{\partial M} |\nabla_{\partial M} f|^2 d\mu \ , \ \forall f \in C^1(\partial M) \ .$$

In the Euclidean setting, and more generally when all sectional curvatures are non-negative, an improved bound will be obtained in the next section. The next result extends the previous one to the $CD(\rho, 0)$ setting:

Corollary 6.7. *Assume that (M, g, μ) satisfies $CD(\rho, 0)$, $\rho \geq 0$, and that $II_{\partial M} \geq \sigma g|_{\partial M}$, $H_\mu \geq \xi$ on ∂M with $\sigma, \xi > 0$. Then:*

$$\lambda_1 Var_\mu(f) \leq \int_{\partial M} |\nabla_{\partial M} f|^2 d\mu \ , \ \forall f \in C^1(\partial M) \ ,$$

with:

$$\lambda_1 \geq \frac{\rho + a + \sqrt{2a\rho + a^2}}{2} \geq \max\left(a, \frac{\rho}{2}\right) \ , \ a := \sigma \xi \ .$$

Proof. Let u denote the first non-trivial eigenfunction of $-L_{\partial M}$, satisfying $-L_{\partial M} u = \lambda_1 u$ with $\lambda_1 > 0$ the spectral-gap (we already know it is positive by Corollary 6.6). Note that since $g|_{\partial M}$ is C^2 smooth, $\nabla_{\partial M} V \in C^1(\partial M)$ and ∂M has no boundary, then $u \in C^{2,\beta}(\partial M)$ for all $\beta \in (0,1)$. Plugging the estimates $\mathrm{II}_{\partial M} \geq \sigma g|_{\partial M}$, $H_\mu \geq \xi$ into the dual generalized Colesanti inequality (Theorem 5.4) and applying it to the function u, we obtain:

$$\sigma \lambda_1 \int_{\partial M} u^2 d\mu \leq \frac{1}{\xi} \int_{\partial M} (-\lambda_1 u + \frac{\rho_1}{2} u)^2 d\mu \ , \ \forall \rho_1 \in [0, \rho] \ .$$

Opening the brackets, this yields:

$$\lambda_1^2 - (\rho_1 + \xi\sigma)\lambda_1 + \frac{\rho_1^2}{4} \geq 0 \ , \ \forall \rho_1 \in [0,\rho] \ .$$

The assertion then follows by using all values of $\rho_1 \in [0,\rho]$.

□

In the next section, we extend our spectral-gap estimates on $(\partial M, g|_{\partial M}, \mu)$ for the case of varying lower bounds σ and ξ.

7 Boundaries of $CD(\rho, N)$ weighted-manifolds

7.1 Curvature-Dimension of the Boundary

Denote the full Riemann curvature 4-tensor on (M, g) by R_g^M, and let $\mathrm{Ric}_\mu^{\partial M}$ denote the weighted Ricci tensor on $(\partial M, g|_{\partial M}, \exp(-V)d\mathrm{Vol}_{\partial M})$.

Lemma 7.1. *Set $g_0 := g|_{\partial M}$ the induced metric on ∂M. Then:*

$$Ric_\mu^{\partial M} = (Ric_\mu^M - R_g^M(\cdot,\nu,\cdot,\nu))|_{T\partial M} + (H_\mu g_0 - II_{\partial M}) II_{\partial M} \ .$$

Proof. Let $e_1, \ldots, e_n$ denote an orthonormal frame of vector fields in M so that e_n coincides on ∂M with the outer normal ν. The Gauss formula asserts that for any $i,j,k,l \in \{1,\ldots,n-1\}$:

$$R_{g_0}^{\partial M}(e_i,e_j,e_k,e_l) = R_g^M(e_i,e_j,e_k,e_l) + \mathrm{II}_{\partial M}(e_i,e_k)\mathrm{II}_{\partial M}(e_j,e_l) - \mathrm{II}_{\partial M}(e_j,e_k)\mathrm{II}_{\partial M}(e_i,e_l) \ .$$

Contracting by applying $g_0^{j,l}$ and using the orthogonality, we obtain:

$$Ric_{g_0}^{\partial M} = (Ric_g^M - R_g^M(\cdot,\nu,\cdot,\nu))|_{T\partial M} + (H_g g_0 - \mathrm{II}_{\partial M})\mathrm{II}_{\partial M} \ . \tag{7.1}$$

In addition we have:

$$\begin{aligned} \nabla_{g_0}^2 V(e_i,e_j) &= e_i(e_j(V)) - ((\nabla_{\partial M})_{e_i} e_j)(V) \\ &= e_i(e_j(V)) - ((\nabla_M)_{e_i} e_j)(V) - \mathrm{II}_{\partial M}(e_i,e_j) e_n(V) \\ &= \nabla_g^2 V(e_i,e_j) - \mathrm{II}_{\partial M}(e_i,e_j)\ g(\nabla V,\nu) \ . \end{aligned}$$

In other words:

$$\nabla_{g_0}^2 V = \nabla_g^2 V|_{T\partial M} - \mathrm{II}_{\partial M} g(\nabla V,\nu) \ . \tag{7.2}$$

Adding (7.1) and (7.2) and using that $H_\mu = H_g - g(\nabla V,\nu)$, the assertion follows. □

Remark 7.2. The induced metric g_0 on ∂M is only as smooth as the boundary, namely C^2. Observe that in order to apply our previously described results to $(\partial M, g_0, \exp(-V)d\mathrm{Vol}_{\partial M})$, we would need to assume that g_0 and hence ∂M are C^3 smooth, as noted in Section 2. We continue to denote the measure $\exp(-V)d\mathrm{Vol}_{\partial M}$ on ∂M by μ.

Corollary 7.3. *Assume that $0 \leq II_{\partial M} \leq H_\mu g_0$ and $R^M_g(\cdot,\nu,\cdot,\nu) \leq \kappa g_0$ as 2-tensors on ∂M. If (M,g,μ) satisfies $CD(\rho,N)$ then $(\partial M, g_0, \mu)$ satisfies $CD(\rho-\kappa, N-1)$.*

Proof. The first assumption ensures that:

$$(H_\mu g_0 - \mathrm{II}_{\partial M})\mathrm{II}_{\partial M} \geq 0 \ ,$$

since the product of two commuting positive semi-definite matrices is itself positive semi-definite. It follows by Lemma 7.1 that:

$$\begin{aligned} Ric^{\partial M}_{\mu,N-1} &= Ric^{\partial M}_\mu - \frac{1}{N-1-(n-1)} dV \otimes dV|_{T\partial M} \\ \geq \ & (Ric^M_\mu - R^M_g(\cdot,\nu,\cdot,\nu) - \frac{1}{N-n} dV \otimes dV)|_{T\partial M} = (Ric^M_{\mu,N} - R^M_g(\cdot,\nu,\cdot,\nu))|_{T\partial M} \ . \end{aligned}$$

The assertion follows from our assumption that $Ric^M_{\mu,N} \geq \rho g$ and $R^M_g(\cdot,\nu,\cdot,\nu)|_{T\partial M} \leq \kappa g_0$. □

An immediate modification of the above argument yields:

Corollary 7.4. *Assume that (M,g,μ) satisfies $CD(\rho,N)$ and that $R^M_g(\cdot,\nu,\cdot,\nu) \leq \kappa g_0$ as 2-tensors on ∂M. If $\sigma_1 g_0 \leq II_{\partial M} \leq \sigma_2 g_0$, for some functions $\sigma_1,\sigma_2 : \partial M \to \mathbb{R}$, then:*

$$Ric^{\partial M}_{\mu,N-1} \geq (\rho - \kappa + \min(\sigma_1(H_\mu - \sigma_1), \sigma_2(H_\mu - \sigma_2)))g_0 \ .$$

In particular, if $H_\mu \geq \xi$ and $\sigma g_0 \leq II_{\partial M} \leq (H_\mu - \sigma)g_0$ for some constants $\xi, \sigma \in \mathbb{R}$, then:

$$Ric^{\partial M}_{\mu,N-1} \geq (\rho - \kappa + \sigma(\xi - \sigma))g_0 \ .$$

When $\mathrm{II}_{\partial M} \geq \sigma g_0$ with $\sigma \geq 0$, we obviously have $\mathrm{II}_{\partial M} \leq (H_g - (n-2)\sigma)g_0$ and $H_g \geq (n-1)\sigma$. In addition if $\langle \nabla V, \nu \rangle \leq 0$ on ∂M, we have $H_g \leq H_\mu$. Consequently, if $n \geq 3$ we obtain:

$$(H_\mu g_0 - \mathrm{II}_{\partial M})\mathrm{II}_{\partial M} \geq \sigma(H_g - \sigma)g_0 \geq (n-2)\sigma^2 g_0 \ . \tag{7.3}$$

Putting all of this together, we obtain:

Proposition 7.5. *Assume that $n \geq 3$, (M^n,g,μ) satisfies $CD(\rho,N)$, $II_{\partial M} \geq \sigma g_0$ with $\sigma \geq 0$ and $\langle \nabla V, \nu \rangle \leq 0$ and $R^M_g(\cdot,\nu,\cdot,\nu) \leq \kappa g_0$ on $T\partial M$. Then $(\partial M, g_0, \mu)$ satisfies $CD(\rho - \kappa + (n-2)\sigma^2, N-1)$.*

Observe that this is sharp for the sphere S^{n-1}, both as a hypersurface of Euclidean space $\mathbb{R}^n$, and as a hypersurface in a sphere RS^n with radius $R \geq 1$.

7.2 Spectral-Gap Estimates on ∂M

We can now apply the known results for weighted-manifolds (without boundary!) satisfying the $CD(\rho_0, N-1)$ condition to $(\partial M, g_0, \mu)$.

The first estimate is an immediate consequence of the Bakry–Émery criterion [4] for log-Sobolev inequalities, see [34] for definitions and more details:

Corollary 7.6. *With the same assumptions as in Proposition 7.5 and for $N \in [n, \infty]$, $(\partial M, g_0, \mu)$ satisfies a log-Sobolev inequality with constant $\lambda_{LS} := \left(\rho - \kappa + (n-2)\sigma^2\right)\frac{N-1}{N-2}$, assuming that the latter is positive. In particular, the spectral-gap is at least λ_{LS}.*

The latter yields an improvement over Xia's spectral-gap estimate (6.4) for the boundary of a strictly convex manifold of non-negative sectional curvature. For concreteness, we illustrate this below for geodesic balls:

Example 7.7. *Assume that $\partial M = \emptyset$, $n \geq 3$, and that (M^n, g) has sectional curvatures in the interval $[\kappa_0, \kappa_1]$. Let B_r denote a geodesic ball around $p \in M$ of radius $0 < r \leq inj_p$, where inj_p denotes the radius of injectivity at p, and consider $(\partial B_r, g_0, Vol_{\partial B_r})$ where $g_0 = g|_{\partial B_r}$. By [57, Chapter 6, Theorem 27], $II_{\partial B_r} \geq \sqrt{\kappa_1}\cot(\sqrt{\kappa_1}r)g_0$. Consequently, by Lemma 7.1:*

$$\mathrm{Ric}_{g_0}^{\partial B_r} \geq (n-2)\kappa_0 g_0 + (H_g g_0 - II_{\partial B_r})II_{\partial B_r} \geq \rho_0 g_0 \ , \ \rho_0 := (n-2)\left(\kappa_0 + \kappa_1 \cot^2(\sqrt{\kappa_1}r)\right) \ .$$

It follows that $(\partial B_r, g_0, Vol_{\partial B_r})$ satisfies $CD(\rho_0, n-1)$, and hence by the Bakry–Émery criterion as above this manifold satisfies a log-Sobolev inequality with constant $\lambda_{LS} \geq \frac{n-1}{n-2}\rho_0 = (n-1)(\kappa_0 + \kappa_1\cot^2(\sqrt{\kappa_1}r))$ whenever the latter is positive, strengthening Xia's result for the spectral-gap (6.4) in the case of non-negative sectional curvature ($\kappa_0 = 0$).

Furthermore, if we replace the lower bound assumption on the sectional curvatures by the assumption that $\mathrm{Ric}_g^{B_r} \geq \rho g$, we obtain by Corollary 7.6 that $\lambda_{LS} \geq (n-1)(\frac{\rho - \kappa_1}{n-2} + \kappa_1 \cot^2(\sqrt{\kappa_1}r))$ whenever the latter is positive.

7.3 Spectral-Gap Estimates on ∂M involving varying curvature

Proceeding onward, we formulate our next results in Euclidean space with constant density, since then the assumptions of the previous subsection are the easiest to enforce. We continue to denote by g_0 the induced Euclidean metric on ∂M. By Lemma 7.1 we know that in this case:

$$Ric_{g_0}^{\partial M} = (H_g g_0 - \mathrm{II}_{\partial M})\mathrm{II}_{\partial M} \ , \tag{7.4}$$

and so if $\mathrm{II}_{\partial M} \geq 0$, we verify as in Corollary 7.3 that $(\partial M, g_0, \mathrm{Vol}_{\partial M})$ satisfies $CD(0, n-1)$. Consequently, the following spectral-gap estimates immediately follow from (7.4), (7.3) and the Lichnerowicz and Veysseire Theorems (see Theorems 3.1 and 4.1), at least for C^3 boundaries. The general case of a C^2 boundary follows by a standard Euclidean approximation argument. The first estimate below improves (in the Euclidean setting) the spectral-gap estimate given by Corollary 6.6:

Theorem 7.8 (Lichnerowicz Estimate on ∂M)**.** *Let $n \geq 3$ and let M denote a compact subset of Euclidean space $(\mathbb{R}^n, g)$ with C^2-smooth boundary. Assume that $II_{\partial M} \geq \sigma g_0$ and $H = tr(II_{\partial M}) \geq \xi$ for some $\sigma, \xi > 0$. Then:*

$$Var_{Vol_{\partial M}}(f) \leq \frac{n-2}{n-1} \frac{1}{(\xi - \sigma)\sigma} \int_{\partial M} |\nabla_{\partial M} f|^2 \, dVol_{\partial M} \ , \ \forall f \in C^1(\partial M) \ .$$

Theorem 7.9 (Veysseire Estimate on ∂M)**.** *Let $n \geq 3$ and let M denote a compact subset of Euclidean space $(\mathbb{R}^n, g)$ with C^2-smooth boundary. Assume that:*

$$II_{\partial M} \geq \sigma g_0$$

for some positive measurable function $\sigma : \partial M \to \mathbb{R}_+$, and set $H = tr(II_{\partial M})$. Then:

$$Var_{Vol_{\partial M}}(f) \leq \fint_{\partial M} \frac{1}{(H - \sigma)\sigma} dVol_{\partial M} \int_{\partial M} |\nabla_{\partial M} f|^2 \, dVol_{\partial M} \ , \ \forall f \in C^1(\partial M) \ .$$

We conclude this section with a similar estimate to the one above, by employing the generalized Colesanti inequality:

Theorem 7.10 (Colesanti Estimate on ∂M)**.** *With the same assumptions as in the previous theorem, we have:*

$$Var_{Vol_{\partial M}}(f) \leq C \left(\fint_{\partial M} \frac{1}{H} dVol_{\partial M} \fint_{\partial M} \frac{1}{\sigma} dVol_{\partial M} \right) \int_{\partial M} |\nabla_{\partial M} f|^2 \, dVol_{\partial M} \ , \ \forall f \in C^1(\partial M) \ ,$$

where $C > 1$ is some universal (dimension-independent) numeric constant.

Proof. Given a 1-Lipschitz function $f : \partial M \to \mathbb{R}$ with $\int_{\partial M} f dVol_{\partial M} = 0$, we may estimate using Cauchy–Schwartz and Theorem 5.1:

$$\begin{aligned} \left(\fint_{\partial M} |f| \, dVol_{\partial M} \right)^2 &\leq \fint_{\partial M} \frac{1}{H} dVol_{\partial M} \fint_{\partial M} H f^2 dVol_{\partial M} \\ &\leq \fint_{\partial M} \frac{1}{H} dVol_{\partial M} \fint_{\partial M} \left\langle \mathrm{II}_{\partial M}^{-1} \, \nabla_{\partial M} f, \nabla_{\partial M} f \right\rangle dVol_{\partial M} \\ &\leq \fint_{\partial M} \frac{1}{H} dVol_{\partial M} \fint_{\partial M} \frac{1}{\sigma} dVol_{\partial M} \ . \end{aligned} \tag{7.5}$$

It follows by a general result of the second-named author [44], which applies to any weighted-manifold satisfying the $CD(0,\infty)$ condition, and in particular to $(\partial M, g_0, Vol_{\partial M})$, that up to a universal constant, the same estimate as in (7.5) holds for the variance of any function $f \in C^1(\partial M)$ with $\fint_{\partial M} |\nabla_{\partial M} f|^2 \, dVol_{\partial M} \leq 1$, and so the assertion follows. We remark that the results in [44] were proved assuming the metric in question is C^∞ smooth, but an inspection of the proof, which builds upon the regularity results in [50], verifies that it is enough to have a C^2 metric, so the results are indeed applicable to $(\partial M, g_0, Vol_{\partial M})$. □

Note that the estimate given by Theorem 7.8 is sharp and that the ones in Theorems 7.9 and 7.10 are sharp up to constants, as witnessed by $S^{n-1} \subset \mathbb{R}^n$.

Remark 7.11. The Euclidean setting was only used to easily establish that $(\partial M, g_0, Vol_{\partial M})$ satisfies $CD(0,\infty)$. The above results remain valid whenever both $(M, g, \exp(-V)Vol_M)$ and $(\partial M, g_0, \exp(-V)Vol_{\partial M})$ satisfy the $CD(0,\infty)$ condition, if we assume that ∂M is C^3-smooth.

8 Connections to the Brunn–Minkowski Theory

It was shown by Colesanti [16] that in the Euclidean case, the inequality (5.1) is *equivalent* to the statement that the function $t \mapsto Vol(K+tL)^{1/n}$ is concave at $t=0$ when K, L are strictly convex and C^2 smooth. Here $A+B := \{a+b \; ; \; a \in A, b \in B\}$ denotes Minkowski addition. Using homogeneity of the volume and a standard approximation procedure of arbitrary convex sets by ones as above, this is in turn equivalent to:

$$Vol((1-t)K + tL)^{1/n} \geq (1-t)Vol(K)^{1/n} + tVol(L)^{1/n} \;\; , \;\; \forall t \in [0,1] \; ,$$

for all convex $K, L \subset \mathbb{R}^n$. This is precisely the content of the celebrated Brunn–Minkowski inequality in Euclidean space (e.g. [63, 24]), at least for convex domains. Consequently, Theorem 5.1 provides yet another proof of the Brunn–Minkowski inequality in Euclidean space via the generalized Reilly formula. Conceptually, this is not surprising since the Bochner formula is a dual version of the Brascamp–Lieb inequality (see Section 3), and the latter is known to be an infinitesimal form of the Prekopá–Leindler inequality, which in turn is a functional (essentially equivalent) form of the Brunn-Minkowski inequality. So all of these inequalities are intimately intertwined and essentially equivalent to one another; see [12, 7, 34] for more on these interconnections. We also mention that as a by-product, we may obtain all the well-known consequences of the Brunn–Minkowski inequality (see [24]); for instance, by taking the first derivative in t above, we may deduce the (anisotropic) isoperimetric inequality (for convex sets K).

Since our generalization of Colesanti's Theorem holds on any weighted-manifold satisfying the $CD(0,N)$ condition, it is then natural to similarly try and obtain a Brunn–Minkowski or isoperimetric-type inequality in the latter setting. The main difficulties arising with such an attempt in this generality are the lack of homogeneity, the lack of a previously known generalization of Minkowski addition, and the fact that enlargements of convex sets are in general non-convex (consider geodesic balls on the sphere which are extended past the equator). At least some of these issues are addressed in what follows.

8.1 Riemannian Brunn–Minkowski for Geodesic Extensions

Let K denote a compact subset of (M^n, g) with C^2 smooth boundary ($n \geq 2$). Denote:

$$\delta^0(K) := \mu(K) \; , \; \delta^1(K) := \mu(\partial K) := \int_{\partial K} d\mu \; , \; \delta^2(K) := \int_{\partial K} H_\mu d\mu \; .$$

It is well-known (see e.g. [45] or Subsection 8.3) that δ^i, $i = 0, 1, 2$, are the i-th variations of $\mu(K_t)$, where K_t is the t-neighborhood of K, i.e. $K_t := \{x \in M; d(x,K) \leq t\}$ with d denoting the geodesic distance on (M, g). Given $1/N \in (-\infty, 1/n]$, denote in analogy to the Euclidean case the "generalized quermassintegrals" by:

$$W_N(K) = \delta^0(K) \; , \; W_{N-1}(K) = \frac{1}{N}\delta^1(K) \; , \; W_{N-2}(K) = \frac{1}{N(N-1)}\delta^2(K) \; .$$

Observe that when $\mu = \mathrm{Vol}_M$ and $N = n$, these quermassintegrals coincide with the Lipschitz–Killing invariants in Weyl's celebrated tube formula, namely the coefficients of the polynomial $\mu(K_t) = \sum_{i=0}^n \binom{n}{i} W_{n-i}(K) t^i$ for $t \in [0, \epsilon_K]$ and small enough $\epsilon_K > 0$. In particular, when (M, g) is Euclidean and K is convex, these generalized quermassintegrals coincide with their classical counterparts, discovered by Steiner in the 19th century (see e.g. [6] for a very nice account).

As an immediate consequence of Corollary 6.3 we obtain:

Corollary 8.1. (Riemannian Brunn-Minkowski for Geodesic Extensions). *Assume that $(K, g|_K, \mu)$ satisfies the $CD(0,N)$ condition ($1/N \in (-\infty, 1/n]$) and that $II_{\partial K} > 0$ (K is locally strictly-convex). Then the following holds:*

1. *(Generalized Minkowski's second inequality for geodesic extensions)*

$$W_{N-1}(K)^2 \geq W_N(K) W_{N-2}(K) \; , \tag{8.1}$$

or in other words:

$$\delta^1(K)^2 \geq \frac{N}{N-1}\delta^0(K)\delta^2(K) \; .$$

(2) *(Infinitesimal Geodesic Brunn–Minkowski)* $(d/dt)^2 N\mu(K_t)^{1/N}|_{t=0} \leq 0$.

(3) *(Global Geodesic Brunn–Minkowski) The function* $t \mapsto N\mu(K_t)^{1/N}$ *is concave on any interval* $[0,T]$ *so that for all* $t \in [0,T)$, K_t *is* C^2 *smooth, locally strictly-convex, bounded away from* ∂M, *and* $(K_t, g|_{K_t}, \mu|_{K_t})$ *satisfies* $CD(0,N)$.

Proof. The first assertion is precisely the content of Corollary 6.3. The second follows since:

$$(d/dt)^2 N\mu(K_t)^{1/N}|_{t=0} = \mu(K_t)^{1/N-2}\left(\delta_2(K)\delta_0(K) - \frac{N-1}{N}\delta_1(K)^2\right) .$$

The third is an integrated version of the second. □

Remark 8.2. In the non-weighted Riemannian setting, the interpretation of Corollary 6.3 as a Riemannian version of Minkowski's second inequality was already noted by Reilly [60]. We also mention that in Euclidean space, a related Alexandrov–Fenchel inequality was shown to hold by Guan and Li [25] for arbitrary mean-convex star-shaped domains.

8.2 Generalized Minkowski Addition: The Parallel Normal Flow

Let $F_0 : \Sigma^{n-1} \to M^n$ denote a smooth embedding of an oriented submanifold $\Sigma_0 := F_0(\Sigma)$ in (M,g), where Σ is a $n-1$ dimensional compact smooth oriented manifold without boundary. The following geometric evolution equation for $F : \Sigma \times [0,T] \to M$ has been well-studied in the literature:

$$\frac{d}{dt}F(y,t) = \varphi(y,t)\nu_{\Sigma_t}(F(y,t)) \ , \ F(y,0) = F_0 \ , \ y \in \Sigma \ , \ t \in [0,T] \ . \tag{8.2}$$

Here ν_{Σ_t} is the unit-normal (in accordance to the chosen orientation) to $\Sigma_t := F_t(\Sigma)$, $F_t := F(\cdot,t)$, and $\varphi : \Sigma \times [0,T] \to \mathbb{R}_+$ denotes a function depending on the extrinsic geometry of $\Sigma_t \subset M$ at $F(y,t)$. Typical examples for φ include the mean-curvature, the inverse mean-curvature, the Gauss curvature, and other symmetric polynomials in the principle curvatures (see [30] and the references therein).

Motivated by the DeTurck trick in the analysis of Ricci-flow (e.g. [67]), we propose to add another tangential component τ_t to (8.2). Let $\varphi : \Sigma \to \mathbb{R}$ denote a C^2 function which is fixed throughout the flow. Assume that $\mathrm{II}_{\Sigma_t} > 0$ for $t \in [0,T]$ along the following flow:

$$\begin{aligned} \frac{d}{dt}F(y,t) &= \omega_t(F(y,t)) \ , \ F(y,0) = F_0 \ , \ y \in \Sigma \ , \ t \in [0,T] \ , \\ \omega_t &:= \varphi_t \nu_{\Sigma_t} + \tau_t \ \text{ on } \Sigma_t \ , \ \tau_t := \mathrm{II}_{\Sigma_t}^{-1} \nabla_{\Sigma_t}(\varphi_t) \ , \ \varphi_t := \varphi \circ F_t^{-1} \ . \end{aligned} \tag{8.3}$$

For many flows, the tangential component τ_t would be considered an inconsequential diffeomorphism term, which does not alter the set $\Sigma_t = F_t(\Sigma)$, only the individual trajectories $t \mapsto F(y,t)$ for a given $y \in \Sigma$. However, contrary to most flows where $\varphi(y,t)$ depends solely on the geometry of Σ_t at $F_t(y)$, for our flow φ plays a different role, and in particular its value along every trajectory is fixed throughout the evolution. Consequently, this tangential term creates a desirable geometric effect as we shall see below.

Before proceeding, it is useful to note that (8.3) is clearly parametrization invariant: if $\zeta : \Sigma' \to \Sigma$ is a diffeomorphism and F satisfies (8.3) on Σ, then $F'(z,t) := F(\zeta(z),t)$ also satisfies (8.3) with $\varphi'(z) := \varphi(\zeta(z))$. Consequently, we see that (8.3) defines a semi-group of pairs (Σ_t, φ_t), so it is enough to perform calculations at time $t = 0$. In addition, we are allowed to use a convenient parametrization Σ' for our analysis.

8.2.1 Euclidean Setting

We now claim that in Euclidean space, Minkowski summation can indeed be parametrized by the evolution equation (8.3). Given a convex compact set in $\mathbb{R}^n$ containing the origin in its interior ("convex body") with C^2 smooth boundary and outer unit-normal ν_K, by identifying $T_x\mathbb{R}^n$ with $\mathbb{R}^n$, $\nu_K : \partial K \to S^{n-1}$ is the Gauss-map. Note that when K is strictly convex ($\mathrm{II}_K > 0$), the Gauss-map is a diffeomorphism. Finally, the support function h_K is defined by $h_K(x) := \sup_{y\in K} \langle x, y\rangle$, so that $\langle x, \nu_K(x)\rangle = h_K(\nu_K(x))$ and $\langle \nu_K^{-1}(\nu), \nu\rangle = h_K(\nu)$.

Proposition 8.3. *Let K and L denote two strictly-convex bodies in $\mathbb{R}^n$ with C^2 smooth boundaries. Let $F : S^{n-1} \times \mathbb{R}_+ \to \mathbb{R}^n$ be defined by $F(\nu, t) := \nu_{K+tL}^{-1}(\nu)$, so that $\partial(K + tL) = F_t(S^{n-1})$ for all $t \geq 0$. Then F satisfies (8.3) with $\varphi = h_L$ and $F_0 := \nu_K^{-1}$.*

Proof. As the support function is additive with respect to Minkowski addition, then so is the inverse Gauss-map: $\nu_{K+tL}^{-1} = \nu_K^{-1} + t\nu_L^{-1}$. Consequently:

$$\frac{d}{dt} F_t(\nu) = \nu_L^{-1}(\nu) = h_L(\nu)\nu + \left(\nu_L^{-1}(\nu) - \langle \nu_L^{-1}(\nu), \nu\rangle \nu\right) \ .$$

Since $\nu_{F_t(S^{n-1})}(F_t(\nu)) = \nu$, it remains to show (in fact, just for $t = 0$) with the usual identification between $T_x\mathbb{R}^n$ and $\mathbb{R}^n$ that:

$$\nu_{K+tL}(x) = \nu \quad \Rightarrow \quad \mathrm{II}_{\partial(K+tL)}^{-1} \nabla_{\partial(K+tL)} (h_L \circ \nu_{K+tL})(x) = \nu_L^{-1}(\nu) - \langle \nu_L^{-1}(\nu), \nu\rangle \nu \ .$$

Indeed by the chain-rule:

$$\nabla_{\partial K}(h_L(\nu_K(x))) = \nabla_{S^{n-1}} h_L(\nu) \nabla_{\partial K} \nu_K(x) = \mathrm{II}_{\partial K}(x) \nabla_{S^{n-1}} h_L(\nu) \ ,$$

so our task reduces to showing that:

$$\nabla_{S^{n-1}} h_L(\nu) = \nu_L^{-1}(\nu) - \left\langle \nu_L^{-1}(\nu), \nu \right\rangle \nu \ , \ \forall \nu \in S^{n-1} \ .$$

This is indeed the case, and moreover:

$$\nabla_{\mathbb{R}^n} h_L(\nu) = \nu_L^{-1}(\nu) \ , \ \forall \nu \in S^{n-1} \ .$$

The reason is that $\nu_L(x) = \frac{\nabla_{\mathbb{R}^n} \|x\|_L}{\left|\nabla_{\mathbb{R}^n} \|x\|_L\right|}$, and since $\nabla_{\mathbb{R}^n} h_L$ is 0-homogeneous, we obtain:

$$\nabla_{\mathbb{R}^n} h_L \circ \nu_L(x) = \nabla_{\mathbb{R}^n} h_L \circ \nabla_{\mathbb{R}^n} \|x\|_L = x \ , \ \forall x \in \partial L \ ,$$

where the last equality follows since h_L and $\|\cdot\|_L$ are dual norms. This concludes the proof. □

8.2.2 Characterization in Riemannian Setting

The latter observation gives a clear geometric interpretation of what the flow (8.3) is doing in the Euclidean setting: normals to the evolving surface remain constant along trajectories. In the more general Riemannian setting, where one cannot identify between M and $T_x M$ and where the Gauss map is not available, we have the following geometric characterization of the flow (8.3) which extends the latter property: normals to the evolving surface remain *parallel* along trajectories. Consequently, we dub (8.3) the "Parallel Normal Flow".

Proposition 8.4. *Consider the following geometric evolution equation along a time-dependent vector-field* ω_t*:*

$$\frac{d}{dt} F_t(y) = \omega_t(F_t(y)) \ , \ y \in \Sigma \ , \ t \in [0,T] \ ,$$

and assume that $F_t : \Sigma \to \Sigma_t$ *is a local diffeomorphism and that* $II_{\Sigma_t} > 0$ *for all* $t \in [0,T]$*. Then the unit-normal field is parallel along the flow:*

$$\frac{d}{dt} \nu_{\Sigma_t}(F_t(y)) = 0 \ , \ \forall y \in \Sigma \ , \ \forall t \in [0,T] \ ,$$

if and only if there exists a family of functions $f_t : \Sigma_t \to \mathbb{R}$*,* $t \in [0,T]$*, so that:*

$$\omega_t = f_t \nu_{\Sigma_t} + II_{\Sigma_t}^{-1} \nabla_{\Sigma_t} f_t \ , \tag{8.4}$$

Furthermore, the entire normal component of ω_t*, denoted* $\omega_t^\nu := \langle \omega_t, \nu_{\Sigma_t} \rangle \nu_{\Sigma_t}$*, is parallel along the flow:*

$$\frac{d}{dt} \omega_t^\nu(F_t(y)) = 0 \ , \ \forall y \in \Sigma \ , \ \forall t \in [0,T] \ ,$$

if and only if there exists a function $\varphi : \Sigma \to \mathbb{R}$ *so that* $f_t = \varphi \circ F_t^{-1}$ *in (8.4).*

Recall that the derivative of a vector-field X along a path $t \mapsto \gamma(t)$ is interpreted by employing the connection $\frac{d}{dt} X(\gamma(t)) = \nabla_{\gamma'(t)} X$.

Proof. First, observe that $\left\langle \frac{d}{dt} \nu_{\Sigma_t}(F_t(y)), \nu_{\Sigma_t} \right\rangle = \frac{1}{2} \frac{d}{dt} \left\langle \nu_{\Sigma_t}, \nu_{\Sigma_t} \right\rangle (F_t(y)) = 0$, so $\frac{d}{dt} \nu_{\Sigma_t}(F_t(y))$ is tangent to Σ_t. Given $y \in \Sigma$ let $e \in T_y\Sigma$ and set $e_t := dF_t(e) \in T_{F_t(y)}\Sigma_t$. Since $\langle \nu_{\Sigma_t}, e_t \rangle = 0$, we have:

$$\left\langle \frac{d}{dt} \nu_{\Sigma_t}(F_t(y)), e_t \right\rangle = - \left\langle \nu_{\Sigma_t}, \frac{d}{dt} dF_t(e) \right\rangle = - \left\langle \nu_{\Sigma_t}, \nabla_{e_t} \frac{d}{dt} F_t(y) \right\rangle = - \left\langle \nu_{\Sigma_t}, \nabla_{e_t} \omega_t \right\rangle \ .$$

Decomposing ω_t into its normal $\omega_t^\nu = f_t \nu_{\Sigma_t}$ and tangential ω_t^τ components, we calculate:

$$- \left\langle \nu_{\Sigma_t}, \nabla_{e_t} \omega_t \right\rangle = -\nabla_{e_t} f_t - f_t \left\langle \nu_{\Sigma_t}, \nabla_{e_t} \nu_{\Sigma_t} \right\rangle - \left\langle \nu_{\Sigma_t}, \nabla_{e_t} \omega_t^\tau \right\rangle \ .$$

Since $\left\langle \nu_{\Sigma_t}, \nabla_{e_t} \nu_{\Sigma_t} \right\rangle = \frac{1}{2} \nabla_{e_t} \left\langle \nu_{\Sigma_t}, \nu_{\Sigma_t} \right\rangle = 0$, $\left\langle \nu_{\Sigma_t}, \nabla_{e_t} \omega_t^\tau \right\rangle = - \left\langle \mathrm{II}_{\Sigma_t} \omega_t^\tau, e_t \right\rangle$ and since e and hence e_t were arbitrary, we conclude that:

$$\frac{d}{dt} \nu_{\Sigma_t}(F_t(y)) = -\nabla_{\Sigma_t} f_t + \mathrm{II}_{\Sigma_t} \omega_t^\tau \ ,$$

and so the first assertion follows. The second assertion follows by calculating:

$$\frac{d}{dt} \omega_t^\nu(F_t(y)) = \frac{d}{dt} (f_t \nu_{\Sigma_t})(F_t(y)) = (\frac{d}{dt} f_t(F_t(y))) \nu_{\Sigma_t}(F_t(y)) + f_t(F_t(y)) \frac{d}{dt} \nu_{\Sigma_t}(F_t(y)) \ .$$

We see that ω_t^ν is parallel along the flow if and only if both normal and tangential components on the right-hand-side above vanish, reducing to the first assertion in conjunction with the requirement that f_t remain constant along the flow, i.e. $f_t = \varphi \circ F_t^{-1}$. □

Consequently, given a locally strictly-convex compact set $\Omega \subset (M, g)$ with C^2 smooth boundary which is bounded away from ∂M, the region bounded by $F_t(\partial\Omega) \subset (M, g)$ with initial conditions $F_0 = Id$ and $\varphi \in C^2(\partial\Omega)$, if it exists, will be referred to as the "Riemannian Minkowski Extension of Ω by $t\varphi$" and denoted by $\Omega + t\varphi$. Note that this makes sense as long as the Parallel Normal Flow is a diffeomorphism which preserves the aforementioned convexity and boundedness away from ∂M up until time t - we will say in that case that the Riemannian Minkowski extension is "well-posed". When $\varphi \equiv 1$ on $\partial\Omega$, we obtain the usual geodesic extension Ω_t. Note that φ need not be positive to make sense of this operation, and that multiplying φ by a positive constant is just a time re-parametrization of the flow. Also note that this operation only depends on the geometry of (M, g), and not on the measure μ, in accordance with the classical Euclidean setting.

8.2.3 Homogeneous Monge-Ampère Equation and Short-Time Existence in Analytic Case

We now briefly explain the relation of our flow to a homogeneous Monge-Ampère equation on the exterior of Ω. Consider the function u whose t-level sets are precisely the hypersurfaces $\Sigma_t \subset M$:

$$u(F_t(y)) = t \ , \ \ y \in \Sigma.$$

Differentiating this formula in t yields $1 = \left\langle \nabla u(F_t(y)), \frac{d}{dt} F_t(y) \right\rangle = \langle \nabla u(F_t(y)), \omega_t(F_t(y)) \rangle$. But since $\nu_{\Sigma_t}(F_t(y)) = \frac{\nabla u(F_t)}{|\nabla u(F_t)|}$, one immediately obtains:

$$|\nabla u(F_t(y))| = \frac{1}{\varphi(y)}. \tag{8.5}$$

Corollary 8.5. *The vector $\nabla u(F_t(y))$ is parallel along the flow.*

Proof. Immediate since $\nabla u(F_t(u)) = |\nabla u(F_t(y))| \, \nu_{\Sigma_t}(F_t(y))$. $|\nabla u(F_t(y))|$ is constant along the flow by (8.5), and $\nu_{\Sigma_t}(F_t(y))$ is parallel along the flow by Proposition 8.4. $\square$

In particular, we deduce that ω_t is a zero eigenvector for $\nabla^2 u$, as $\nabla^2 u \cdot \omega_t = \nabla_{\omega_t} \nabla u = 0$. We therefore see that u solves the following homogeneous Monge-Ampère boundary value problem:

$$\det \nabla^2 u \equiv 0, \ u|_{\Sigma_0} \equiv 0, \ u_\nu|_{\Sigma_0}(x) = \frac{1}{\varphi_0(x)}, \tag{8.6}$$

and that the trajectories of the Normal Parallel Flow are precisely the characteristic curves of this PDE.

The short-time solution for the above boundary value problem (and hence our original flow) can be established by a standard application of the Cauchy–Kowalevskaya theorem, provided that all the data (including the metric) is analytic and $\varphi_0 > 0$, $II_{\Sigma_0} > 0$; further details will be provided elsewhere. We conclude this subsection with a couple of additional observations regarding the classical Euclidean setting.

The relation between the homogeneous Monge-Ampère equation and interpolation or infimum-convolution in the context of Banach Space Theory or Kähler geometry is known (e.g. [64, 18, 62, 2]). However, even in the Euclidean setting, the above explicit description seems to have been previously unnoted in the literature: given two compact convex bodies $K, L \subset \mathbb{R}^n$ containing the origin (say smooth and strictly convex), the entire family $\{K + tL\}_{t \geq 0}$ is obtained as the t-sub-level sets of a (convex) solution u to the homogeneous Monge-Ampère equation (8.6) on $\mathbb{R}^n \setminus K$ with $\Sigma_0 = \partial K$ and $\varphi_0 = h_L \circ \nu_{\partial K}$.

Similarly, if $K \subset L$ are convex as above, it follows that the homogeneous Monge-Ampère equation:

$$\det \nabla^2 u \equiv 0, \; u|_{\partial K} \equiv 0, \; u|_{\partial L} \equiv 1,$$

admits a convex solution $u : L \setminus K \to [0,1]$ whose t-sub-level sets ($t \in [0,1]$) are precisely $(1-t)K + tL$. Indeed, the corresponding Normal Parallel Flow is then $\partial K \ni x \mapsto (1-t)x + t\nu_{\partial L}^{-1} \circ \nu_{\partial K}(x)$ mapping ∂K onto $\partial((1-t)K + tL)$. The analogous statement in the Riemannian setting will be developed elsewhere.

Lastly, we obtain using the Parallel Normal Flow an explicit map $T : K \to L$ so that $(Id + tT)(K) = K + tL$ for all $t \geq 0$; simply set:

$$T(x) = \|x\|_K \, \nu_{\partial L}^{-1} \circ \nu_{\partial K}(x / \|x\|_K),$$

and use homogeneity and the fact that $(Id + tT)(\partial K) = \partial(K + tL)$. A non-explicit map using Optimal-Transport has been previously constructed by Alesker, Dar and V. Milman [1], and it would be interesting to see how to use our explicit map for deducing Alexandrov–Fenchel inequalities, as in [1].

8.3 Riemannian Brunn-Minkowski

We have seen that Riemannian Minkowski extension coincides with Minkowski summation in the Euclidean setting: $K + t \cdot h_L = K + tL$. We do not go here into analyzing the well-posedness of this operation in the general (non-analytic) case, but rather concentrate on using this operation to derive the following Riemannian generalization of the Brunn-Minkowski inequality.

Theorem 8.6 (Riemannian Brunn–Minkowski Inequality)**.** *Let $\Omega \subset (M,g)$ denote a locally strictly-convex compact set with C^2 smooth boundary which is bounded away from ∂M, and let $\varphi \in C^2(\partial\Omega)$. Let Ω_t denote the Riemannian Minkowski extension $\Omega_t := \Omega + t\varphi$, and assume that it is well-posed for all $t \in [0,T]$. Assume that (M,g,μ) satisfies the $CD(0,N)$ condition ($1/N \in (-\infty, 1/n]$). Then the function*

$$t \mapsto N\mu(\Omega_t)^{1/N}$$

is concave on $[0,T]$.

Proof. Set $\Sigma := \partial\Omega$, $F_0 = Id$, and recall that our evolution equation is:

$$\frac{d}{dt}F_t(y) = \omega_t(F_t(y)) := \varphi(y)\nu_{\Sigma_t}(F_t(y)) + \tau_t(F_t(y)) \; , \; F_0(y) = Id \; , \; y \in \Sigma \; , \; t \in [0,T] \; . \tag{8.7}$$

As previously explained, it is enough to perform all the analysis at time $t = 0$.

Clearly, the first variation of $\mu(\Omega_t)$ only depends on the normal velocity φ, and so we have:

$$\frac{d}{dt}\mu(\Omega_t)|_{t=0} = \int_\Sigma \varphi \exp(-V) d\mathrm{Vol}_\Sigma \ .$$

By the semi-group property, it follows that:

$$\frac{d}{dt}\mu(\Omega_t) = \int_{\Sigma_t} \varphi \circ F_t^{-1} \exp(-V) d\mathrm{Vol}_{\Sigma_t} \ .$$

Since F_t is a diffeomorphism for small $t \geq 0$, we obtain by the change-of-variables formula:

$$\frac{d}{dt}\mu(\Omega_t) = \int_\Sigma \varphi \exp(-V \circ F_t) \ \mathrm{Jac} F_t \ d\mathrm{Vol}_\Sigma \ , \tag{8.8}$$

where $\mathrm{Jac}F_t(y)$ denotes the Jacobian of $(\Sigma, g|_\Sigma) \ni y \mapsto F_t(y) \in (\Sigma_t, g|_{\Sigma_t})$, i.e. the determinant of $d_y F_t : (T_y\Sigma, g_y) \to (T_{F_t(y)}\Sigma_t, g_{F_t(y)})$.

As is well known, $\frac{d}{dt}\mathrm{Jac}F_t = \mathrm{div}_{\Sigma_t} \frac{d}{dt} F_t$; we briefly sketch the argument. It is enough to show this for $t = 0$ and for a $y \in \Sigma$ so that $\left\langle \frac{d}{dt} F_t(y), \nu_\Sigma(y) \right\rangle \neq 0$. Fix an orthonormal frame $e_1, \ldots, e_n$ in TM so that e_n coincides with ν_{Σ_t} in a neighborhood of $(y, 0)$ in $M \times \mathbb{R}$, and hence $e_1, \ldots, e_{n-1}$ is a basis for $T_{F(y,t)}\Sigma$. Since $dF_0 = Id$, it follows that:

$$\frac{d}{dt}\mathrm{Jac}F_t(y) = tr\left(\frac{d}{dt} d_y F_t\right) = \sum_{i=1}^{n-1} \frac{d}{dt} \left\langle d_y F_t(e_i(y)), e_i(F_t(y)) \right\rangle \ .$$

Now as $F_0 = Id$ and $\frac{d}{dt} F_t|_{t=0} = \omega_0$, we have at $(y, 0)$ (denoting $\omega = \omega_0$):

$$\frac{d}{dt} \left\langle d_y F_t(e_i), e_i(F_t(y)) \right\rangle |_{t=0} = \left\langle \frac{d}{dt} d_y F_t(e_i)|_{t=0}, e_i \right\rangle + \left\langle e_i, \nabla_{\frac{d}{dt} F_t|_{t=0}} e_i \right\rangle = \left\langle \nabla_{e_i} \omega, e_i \right\rangle + \left\langle e_i, \nabla_\omega e_i \right\rangle \ .$$

But $\langle e_i, \nabla_\omega e_i \rangle = \frac{1}{2} \nabla_\omega \langle e_i, e_i \rangle = 0$, and so we confirm that $\frac{d}{dt}\mathrm{Jac}F_t = \mathrm{div}_{\Sigma_t} \omega_t$.

Now, taking the derivative of (8.8) in t, we obtain:

$$\frac{d^2}{(dt)^2}\mu(\Omega_t)|_{t=0} = \int_\Sigma \varphi(\mathrm{div}_\Sigma \omega - \langle \nabla V, \omega \rangle) \exp(-V) d\mathrm{Vol}_\Sigma \ .$$

Recall that $\omega = \varphi \nu_\Sigma + \tau$ (denoting $\tau = \tau_0$), so:

$$\mathrm{div}_\Sigma \omega - \langle \nabla V, \omega \rangle = \varphi(H_{\Sigma,g} - \langle \nabla V, \nu_\Sigma \rangle) + \mathrm{div}_\Sigma \tau - \langle \nabla_\Sigma V, \tau \rangle = \varphi H_{\Sigma,\mu} + \mathrm{div}_{\Sigma,\mu} \tau \ .$$

Plugging this above and integrating by parts, we obtain:

$$\frac{d^2}{(dt)^2}\mu(\Omega_t)|_{t=0} = \int_\Sigma H_{\Sigma,\mu} \varphi^2 d\mu - \int_\Sigma \langle \nabla_\Sigma \varphi, \tau \rangle \, d\mu.$$

Recalling that $\tau = \mathrm{II}_\Sigma^{-1} \nabla_\Sigma \varphi$ and applying Theorem 5.1, we deduce that:

$$\frac{d^2}{(dt)^2}\mu(\Omega_t)|_{t=0} \leq \frac{N-1}{N}\frac{(\int_\Sigma \varphi d\mu)^2}{\mu(\Omega)} = \frac{N-1}{N}\frac{\left(\frac{d}{dt}\mu(\Omega_t)|_{t=0}\right)^2}{\mu(\Omega)} ,$$

which is precisely the content of the assertion. □

Remark 8.7. Other more standard generalizations of the Brunn–Minkowski inequality in the weighted Riemannian setting and in the even more general metric-measure space setting, for spaces satisfying the $CD(\rho, N)$ condition, have been obtained by Cordero-Erausquin–McCann–Schmuckenshläger [19, 20], Sturm [65], Lott–Villani [41] and Ohta [54], using the theory of optimal-transport. In those versions, Minkowski interpolation $(1-t)K + tL$ is replaced by geodesic interpolation of two domains, an operation whose existence does not require any a-priori justification, and which is not confined to convex domains. However, our version has the advantage of extending Minkowski summation $K + tL$ as opposed to interpolation, so we just need a single domain Ω_0 and an initial condition φ_0 on the normal derivative to $\partial\Omega_0$; this may consequently be better suited for compensating the lack of homogeneity in the Riemannian setting and obtaining isoperimetric inequalities. There seem to be some interesting connections between the Parallel Normal Flow and an appropriate optimal-transport problem and Monge–Ampère equation, but this is a topic for a separate note. In this context, we mention the work by V. I. Bogachev and the first named author [10], who showed a connection between the Gauss curvature flow and an appropriate optimal transport problem.

Remark 8.8. While we do not go into this here, it is clear that in analogy to the Euclidean setting, one may use the Riemannian Minkowski extension operation to define the k-th Riemannian mixed volume of a locally strictly-convex K and $\varphi \in C^2(\partial K)$ by taking the k-th variation of $t \mapsto \mu(K + t\varphi)$ and normalizing appropriately. It is then very plausible to expect that these mixed volumes should satisfy Alexandrov–Fenchel type inequalities, in analogy to the original inequalities in the Euclidean setting.

8.4 Comparison with the Borell–Brascamp–Lieb Theorem

Let μ denote a Borel measure with convex support Ω in Euclidean space $(\mathbb{R}^n, |\cdot|)$. In this Euclidean setting, it was shown by Borell [11] and independently by Brascamp and Lieb [12], that if $(\Omega, |\cdot|, \mu)$ satisfies $CD(0, N)$, $1/N \in [-\infty, 1/n]$, then for all Borel subsets $A, B \subset \mathbb{R}^n$ with $\mu(A), \mu(B) > 0$:

$$\mu((1-t)A + tB) \geq \left((1-t)\mu(A)^{\frac{1}{N}} + t\mu(B)^{\frac{1}{N}}\right)^N \;,\; \forall t \in [0,1] \;. \tag{8.9}$$

Consequently, since $(1-t)K + tK = K$ when K is convex, by using $A = K + t_1 L$ and $B = K + t_2 L$ for two convex subsets $K, L \subset \Omega$, it follows that the function:

$$t \mapsto N\mu(K + tL)^{\frac{1}{N}}$$

is concave on $\mathbb{R}_+$. Clearly, Corollary 8.1 and Theorem 8.6 are generalizations to the Riemannian setting of this fact, and in particular provide an alternative proof in the Euclidean setting. The above reasoning perhaps provides some insight as to the reason behind the restriction to convex domains in the concavity results of this section.

We mention in passing that when the measure μ is homogeneous (in the Euclidean setting), one does not need to restrict to convex domains, simply by rescaling A in (8.9). See [48] for isoperimetric applications.

8.5 The Weingarten Curvature Wave Equation

To conclude this section, we observe that there is another natural evolution equation which yields the concavity of $N\mu(\Omega_t)^{1/N}$. Assume that φ in (8.2) evolves according to the following heat-equation on the evolving weighted-manifold $(\Sigma_t, \mathrm{II}_{\Sigma_t}, \mu_{\Sigma_t})$ equipped the Weingarten metric II_{Σ_t} and the measure $\mu_{\Sigma_t} := \exp(-V) dVol_{g_{\Sigma_t}}$, $g_{\Sigma_t} := g|_{\Sigma_t}$:

$$\frac{d}{dt} \log \varphi(y,t) = L_{(\Sigma_t, \mathrm{II}_{\Sigma_t}, \mu_{\Sigma_t})}(\varphi_t)(F_t(y)) \; , \; \varphi_t := \varphi(F_t^{-1}(\cdot), t) \; , \; \varphi(\cdot, 0) = \varphi_0 \; . \quad (8.10)$$

Here $L = L_{(\Sigma_t, \mathrm{II}_{\Sigma_t}, \mu_{\Sigma_t})}$ denotes the weighted-Laplacian operator associated to this weighted-manifold, namely:

$$L(\psi) = \mathrm{div}_{\mathrm{II}_{\Sigma_t}, \mu}(\nabla_{\mathrm{II}_{\Sigma_t}} \psi) = \mathrm{div}_{g_{\Sigma_t}, \mu}(\mathrm{II}_{\Sigma_t}^{-1} \nabla_{g_{\Sigma_t}} \psi) \; . \quad (8.11)$$

The last transition in (8.11) is justified since for any test function f:

$$\int_{\Sigma_t} f \cdot \mathrm{div}_{\mathrm{II}_{\Sigma_t}, \mu}(\nabla_{\mathrm{II}_{\Sigma_t}} \psi) d\mu = - \int_{\Sigma_t} \mathrm{II}_{\Sigma_t}(\nabla_{\mathrm{II}_{\Sigma_t}} f, \nabla_{\mathrm{II}_{\Sigma_t}} \psi) d\mu$$
$$= - \int_{\Sigma_t} g_{\Sigma_t}(\nabla_{g_{\Sigma_t}} f, \mathrm{II}_{\Sigma_t}^{-1} \nabla_{g_{\Sigma_t}} \psi) d\mu = \int_{\Sigma_t} f \cdot \mathrm{div}_{g_{\Sigma_t}, \mu}(\mathrm{II}_{\Sigma_t}^{-1} \nabla_{g_{\Sigma_t}} \psi) d\mu \; .$$

Note that (8.10) is precisely the (logarithmic) gradient flow in $L^2(\Sigma_t, \mu_{\Sigma_t})$ for the Dirichlet energy functional on $(\Sigma_t, \mathrm{II}_{\Sigma_t}, \mu_{\Sigma_t})$:

$$\varphi \mapsto E(t, \varphi) := \frac{1}{2} \int_{\Sigma_t} \mathrm{II}_{\Sigma_t}(\nabla_{\mathrm{II}_{\Sigma_t}}(\varphi_t), \nabla_{\mathrm{II}_{\Sigma_t}}(\varphi_t)) d\mu_{\Sigma_t} \; .$$

Coupling (8.2) and (8.10), it seems that an appropriate name for the resulting flow would be the "Weingarten Curvature Wave Equation", since the second derivative in time of F_t in the normal direction to the evolving surface Σ_t is equal to the weighted Laplacian on $(\Sigma_t, \mathrm{II}_{\Sigma_t}, \mu_{\Sigma_t})$. We do not go at all into justifications of existence of such a flow, but rather observe the following:

Theorem 8.9 (Weingarten Curvature Wave Equation is N-concave)**.** *Assume that there exists a smooth solution (F, φ) to the system of coupled equations (8.2) and (8.10) on $\Sigma \times [0,T]$, so that $F_t : \Sigma \to \Sigma_t \subset (M,g)$ is a diffeomorphism for all $t \in [0,T]$. Assume that (M,g,μ) satisfies the $CD(0,N)$ condition ($1/N \in (-\infty, 1/n]$), that Σ_t are strictly locally-convex ($II_{\Sigma_t} > 0$) and bounded away from ∂M for all $t \in [0,T]$. Assume that Σ_t are the boundaries of compact domains Ω_t having ν_t as their exterior unit-normal field. Then the function*

$$t \mapsto N\mu(\Omega_t)^{1/N}$$

is concave on $[0,T]$.

Proof. Denote $\Phi_t := \frac{d}{dt}\varphi(F_t^{-1}(\cdot), t)$. It is easy to verify as in the proof of Theorem 8.6 that:

$$\frac{d}{dt}\mu(\Omega_t) = \int_{\partial\Omega_t} \varphi_t d\mu \ , \ \frac{d}{dt}\mu(\partial\Omega_t) = \int_{\partial\Omega_t} H_\mu \varphi_t d\mu \ ,$$

and that:

$$\frac{d^2}{(dt)^2}\mu(\Omega_t) = \int_{\partial\Omega_t} (\Phi_t + H_\mu \varphi_t^2) d\mu \ .$$

Plugging the evolution equation (8.10) above and integrating by parts, we obtain:

$$\begin{aligned} \frac{d^2}{(dt)^2}\mu(\Omega_t) &= \int_{\partial\Omega_t} (\varphi_t \mathrm{div}_{g_{\Sigma_t},\mu} \left(\mathrm{II}_{\Sigma_t}^{-1} \nabla_{g_{\Sigma_t}} \varphi_t\right) + H_\mu \varphi_t^2) d\mu \\ &= \int_{\partial\Omega_t} (H_\mu \varphi_t^2 - \left\langle \mathrm{II}_{\Sigma_t}^{-1} \nabla_{g_{\Sigma_t}} \varphi_t, \nabla_{g_{\Sigma_t}} \varphi_t \right\rangle) d\mu \ . \end{aligned}$$

Applying Theorem 5.1, we deduce that:

$$\frac{d^2}{(dt)^2}\mu(\Omega_t) \le \frac{N-1}{N} \frac{(\int_{\partial\Omega_t} \varphi_t d\mu)^2}{\mu(\Omega_t)} = \frac{N-1}{N} \frac{\left(\frac{d}{dt}\mu(\Omega_t)\right)^2}{\mu(\Omega_t)} \ ,$$

which is precisely the content of the assertion. □

9 Isoperimetric Applications

We have seen in the previous section that under the $CD(0,N)$ condition and for various geometric evolution equations, including geodesic extension, the function $t \mapsto$

$N\mu(\Omega_t)^{1/N}$ is concave as long as Ω_t remain strictly locally-convex, C^2 smooth, and bounded away from ∂M. Consequently, the following derivative exists in the wide-sense:

$$\mu^+(\Omega) := \frac{d}{dt}\mu(\Omega_t)|_{t=0} = \lim_{t\to 0} \frac{\mu(\Omega_t) - \mu(\Omega)}{t} \; .$$

$\mu^+(\Omega)$ is the induced "boundary measure" of Ω with respect to μ and the underlying evolution $t \mapsto \Omega_t$. It is well-known and easy to verify (as in the proof of Theorem 8.6) that in the case of geodesic extension, $\mu^+(\Omega)$ coincides with $\mu(\partial\Omega) = \int_{\partial\Omega} d\mu$. We now mention several useful isoperimetric consequences of the latter concavity. For simplicity, we illustrate this in the Euclidean setting, but note that all of the results remain valid in the Riemannian setting as long as the corresponding generalizations described in the previous section are well-posed.

Denote by $\mu^+_L(K)$ the boundary measure of K with respect to μ and the Minkowski extension $t \mapsto K+tL$, where L is a compact convex set having the origin in its interior.

Proposition 9.1. *Let Euclidean space $(\mathbb{R}^n, |\cdot|)$ be endowed with a measure μ with convex support Ω, so that $(\Omega, |\cdot|, \mu)$ satisfies the $CD(0,N)$ condition ($1/N \in (-\infty, 1/n]$). Let $K \subset \Omega$ and $L \subset \mathbb{R}^n$ denote two strictly convex compact sets with non-empty interior and C^2 boundary. Then:*

1. *The function $t \mapsto N\mu(K+tL)^{1/N}$ is concave on $\mathbb{R}_+$.*
2. *The following isoperimetric inequality holds:*

$$\mu^+_L(K) \geq \mu(K)^{\frac{N-1}{N}} \sup_{t>0} N \frac{\mu(K+tL)^{1/N} - \mu(K)^{1/N}}{t} \; .$$

In particular, if the L-diameter of Ω is bounded above by $D < \infty$ ($\Omega - \Omega \subset DL$), we have:

$$\mu^+_L(K) \geq \frac{N}{D}\mu(K)^{\frac{N-1}{N}} \left(\mu(\Omega)^{1/N} - \mu(K)^{1/N}\right) \; .$$

Alternatively, if $\mu(\Omega) = \infty$ and $N \in [n, \infty]$, we have:

$$\mu^+_L(K) \geq \mu(K)^{\frac{N-1}{N}} \limsup_{t\to\infty} \frac{N\mu(tL)^{1/N}}{t} \; . \tag{9.1}$$

3. *Define the following "convex isoperimetric profile":*

$$\mathcal{I}^c_L(v) := \inf \left\{ \mu^+_L(K) \; ; \; \mu(K) = v \; , \; K \subset \Omega \text{ has } C^2 \text{ smooth boundary and } II_{\partial K} > 0 \right\} \; .$$

Then the function $v \mapsto (\mathcal{I}^c_L(v))^{\frac{N}{N-1}}/v$ is non-increasing on its domain.

Remark 9.2. Given a weighted-manifold (M, g, μ), recall that the usual isoperimetric profile is defined as:

$$\mathcal{I}(v) := \inf \left\{ \mu(\partial A) \; ; \; \mu(A) = v \; , \; A \subset M \text{ has } C^2 \text{ smooth boundary} \right\} \; .$$

When (M, g, μ) satisfies the $CD(0, N)$ condition with $N \in [n, \infty]$ and $\mathrm{II}_{\partial M} \geq 0$ (M is locally convex), it is known that $v \mapsto \mathcal{I}(v)^{\frac{N}{N-1}}$ is in fact concave on its domain, implying that $v \mapsto \mathcal{I}(v)^{\frac{N}{N-1}}/v$ is non-increasing (see [44, 45, 46] and the references therein). The proof of this involves crucial use of regularity results from Geometric Measure Theory, and a major challenge is to give a softer proof. In particular, even in the Euclidean setting, an extension of these results to a non-Euclidean boundary measure $\mu^+_L(A)$ is not known and seems technically challenging. The last assertion provides a soft proof for the class of *convex* isoperimetric minimizers, which in fact remains valid for $N < 0$.

Remark 9.3. As explained in Subsection 8.4, it is possible to prove the above assertions using the Borell–Brascamp–Lieb theorem. Another possibility is to invoke the localization method (see [31, 9]). However, these two approaches would be confined to the Euclidean setting, whereas the proof we give below is not.

Proof of Proposition 9.1.

(1) The first assertion is almost an immediate consequence of the concavity calculation performed in the previous section for the classical Minkowski extension operation $t \mapsto K + tL$. However, in that section we assumed that $K + tL$ is bounded away from the boundary $\partial\Omega$, and we now explain how to remove this restriction. Note that if $y \in \partial K$, then $F_t(y) = y + t\nu_L^{-1}(\nu_K(y))$ is a straight line, as verified in Proposition 8.3. By the convexity of Ω, this means that this line can at most exit Ω once, never to return. It is easy to verify that this incurs a non-positive contribution to the calculation of the second variation of $t \mapsto \mu(K + tL)$ in the proof of Theorem 8.6; the rest of the proof remains the same (with the first variation interpreted as the left-derivative).

    More generally, we note here that the concavity statement remains valid if instead of using φ which remains constant on the trajectories of the flow, it is allowed to decrease along each trajectory.

(2) By the concavity from the first assertion, it follows that for every $0 < s \leq t$:

$$N \frac{\mu(K + sL)^{1/N} - \mu(K)^{1/N}}{s} \geq N \frac{\mu(K + tL)^{1/N} - \mu(K)^{1/N}}{t} \; .$$

    Taking the limit as $s \to 0$, the second assertion follows.

(3) Given $K \subset \Omega$ with C^2 smooth boundary and $\mathrm{II}_{\partial K} > 0$, denote $V(t) := \mu(K+tL)$ and set $\mathcal{I}_K := V' \circ V^{-1}$, expressing the boundary measure of $K+tL$ as a function of its measure. Note that $\mathcal{I}_K^{\frac{N}{N-1}}(v)/v$ is non-increasing on its domain. Indeed, assuming that V is twice-differentiable, we calculate:

$$\frac{d}{dv} \frac{\mathcal{I}_K^{\frac{N}{N-1}}(v)}{v} = \left(\left(\frac{N}{N-1} \frac{VV''}{V'} - V' \right) \frac{(V')^{\frac{1}{N-1}}}{V^2} \right) \circ V^{-1}(v) \leq 0 \; ,$$

and the general case follows by approximation. But since $I_L^c := \inf_K \mathcal{I}_K$ where the infimum is over K as above, the third assertion readily follows.

□

Remark 9.4. When in addition $N\mu^{1/N}$ is homogeneous, i.e. $N\mu(tL)^{1/N} = tN\mu(L)^{1/N}$ for all $t > 0$, it follows by (9.1) that for convex K and $N \in [n, \infty]$:

$$\mu_L^+(K) \geq \mu(K)^{\frac{N-1}{N}} N\mu(L)^{1/N} \; .$$

In particular, among all convex sets, homothetic copies of L are isoperimetric minimizers. As already eluded to in Subsection 8.4, this is actually known to hold for arbitrary Borel sets K (see [14, 48]).

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