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Mini-Workshop: New Developments in Newton-Okounkov Bodies

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Abstract. The theory of Newton-Okounkov bodies, also called Okounkov bodies, is a new connection between algebraic geometry and convex geometry. It generalizes the well-known and extremely rich correspondence between geometry of toric varieties and combinatorics of convex integral polytopes. Okounkov bodies were first introduced by Andrei Okounkov, in a construction motivated by a question of Khovanskii concerning convex bodies govering the multiplicities of representations. Recently, Kaveh-Khovanskii and Lazarsfeld-Mustata have generalized and systematically developed Okounkov's construction, showing the existence of convex bodies which capture much of the asymptotic information about the geometry of (X, D) where X is an algebraic variety and D is a big divisor. The study of Okounkov bodies is a new research area with many open questions. The goal of this mini-workshop was to bring together a core group of algebraic/symplectic geometers currently working on this topic to establish the groundwork for future development of this area.

Mathematics Subject Classification (2000): 14xx, 05xx.

Introduction by the Organisers

The mini-workshop *New developments in Newton-Okounkov bodies*, organised by Megumi Harada (McMaster University), Kiumars Kaveh (University of Pittsburgh), and Askold Khovanskii (University of Toronto), was held August 21 to August 27, 2011. The goal of the meeting was to explore the new and rapidly developing research area centred on Newton-Okounkov bodies (also called Okounkov bodies[1]). These are convex bodies associated to algebraic varieties in a very general setting, and may be viewed as a vast generalization of the theory of toric varieties.

The meeting was attended by 16 participants, with broad geographic representation. In addition to the official participants, one Oberwolfach Leibniz Postdoctoral Fellow actively took part in the research activities. There were 13 research talks total and 4 informal discussion sessions which focused on open problems and future directions for research.

In the remaining part of this introduction we attempt to briefly describe some of the motivation behind the subject of Okounkov bodies. It should be emphasized that our mini-workshop provided ample evidence that the theory of Okounkov bodies touches upon many other subjects and that there is a wealth of possible areas of application; as such, we make no claim that our brief overview below is is in any way complete.

A central theme in algebraic geometry is to associate to a variety a combinatorial object – in particular a convex polytope – in such a way that questions about the original variety (such as intersection theory) be answered from the geometry of the associated polytope. A setting in which this geometry-combinatorics dictionary works out perfectly is the extremely useful and popular theory of toric varieties. The theory of Okounkov bodies is a new general frame work to associate convex bodies to algebraic varieties; as such it is a vast generalization of the theory of toric varieties, as we now explain.

Recall that the celebrated Bernstein-Kushnirenko theorem from Newton polyhedra theory relates the number of solutions of a system of polynomial equations with the volume of their corresponding Newton polytopes. Indeed, this theorem motivated the development of the theory of toric varieties. In the more recent setting of symplectic manifolds and Hamiltonian actions, the Atiyah-Guillemin-Sternberg and Kirwan convexity theorems link equivariant symplectic and algebraic geometry to the combinatorics of moment map polytopes. In the case of a toric variety X, the moment map polytope Δ coincides with its Newton polytope and therefore fully encodes the geometry of X, but this fails in the general case. In ground-breaking work, Okounkov constructs, for an (irreducible) projective variety $X \subseteq \mathbb{P}(V)$ equipped with an action of a reductive algebraic group G, a convex body $\tilde{\Delta}$ and a natural projection from $\tilde{\Delta}$ to the moment map polytope Δ of X. The volumes of the fibers of this projection encode the so-called Duistermaat-Heckman measure, and in particular, one recovers the degree of X (i.e. the symplectic volume) from $\tilde{\Delta}$. The recent work of Kaveh-Khovanskii and Lazarsfeld-Mustata, which generalizes and systematically develops Okounkov's ideas, yield constructions of such $\tilde{\Delta}(X, D)$ (associated to X and a choice of (big) divisor D) – the *Okounkov body* – even without presence of any group action. Crucially, in their

[1]One of the suggestions made at our workshop was to use the term "NObodies", for short. Some of the workshop participants have embraced this new terminology, as can be seen from the abstracts.

construction the polytope $\tilde{\Delta}(X, D)$ has (real) dimension precisely the (complex) dimension of X, just as in the case of toric varieties. In this sense the polytope has the maximal possible dimension. As a first application, Kaveh and Khovanskii use their construction to prove a far-reaching generalization of the Bernstein-Kushnirenko theorem to arbitrary varieties which relates the self-intersection number of a divisor with the volume of the corresponding Newton-Okounkov body.

Thus, by generalizing the case of toric varieties as well as many other cases of varieties with a group action, the work of Okounkov, Lazarsfeld-Mustata, and Kaveh-Khovanskii show that there *are* combinatorial objects of 'maximal' dimension associated to X. The fundamental question is:

What geometric data of (X, D) do the combinatorics of these Newton-Okounkov bodies encode, and how?

We now mention briefly a small sample of the exciting directions for future research which were discussed at the workshop. First, we have already mentioned that, generalizing the Bernstein-Kushnirenko theorem in toric geometry, the volumes of the Okounkov bodies $\tilde{\Delta}$ give the intersection numbers of divisors on X. Motivated by this, we may ask another important and general question: what topological or geometric data can we extract from other similar invariants of the Newton-Okounkov bodies, e.g., the volume of its boundary $\partial\tilde{\Delta}$? Are they related to the other coefficients of the Hilbert function of X, and if so, how? Secondly, we expect rich applications of Okounkov bodies to, and interactions with, geometric representation theory and Schubert calculus. Kiritchenko's talk on convex chains for Schubert varieties and Kaveh's talk on crystal bases and Okounkov bodies already indicate some of the possibilities. Third, we must better understand conditions under which the Okounkov body is actually a convex polytope (and hence amenable to combinatorial methods). Dave Anderson took important first steps in this direction, but more is needed. Anderson's talk at our workshop addressed precisely this question in the context of Bott-Samelson varieties. Fourth, the toric degeneration construction of Dave Anderson and the resulting gradient-Hamiltonian-flow construction explained in Kaveh's talk on 'Integrable systems via Okounkov bodies' suggests that there exists an integrable system on a variety X in rather broad generality.

The above (very incomplete) sampling of open research problems discussed at our workshop already ilustrates that the theory of Okounkov bodies lies in the exciting intersection of equivariant algebraic geometry, convex geometry, representation theory, symplectic geometry, and commutative algebra. Lozovanu's talk also suggests connections with number theory, while Huh's talk indicates that tropical geometry and the theory of matroids should also be relevant. We expect this theory not only to significantly contribute to each of these areas, but also to establish previously unknown connections between them. The theory is evidently quite powerful in that it unifies seemingly unrelated constructions in different research areas, such as the Newton polytope of a toric variety, the moment polytope of a Hamiltonian action on a symplectic manifold, and the Gelfand-Cetlin polytopes (or more generally the Littelmann-Berenstein-Zelevinsky string polytopes of representation theory). Furthermore, the Okounkov-body theory allows us also to employ (or use as guiding principles) the well-known and powerful methods of toric geometry to a very large class of varieties.

As can be seen from the discussion above, the theory of Okounkov bodies is still in its infancy and the subject is wide open. We hope that this quick overview has interested the reader in the subject. The abstracts which follow contain a remarkable breadth of topics which further develop the themes we only sketched above (or introduce new themes altogether). Our Oberwolfach Mini-Workshop brought together a core group of algebraic/symplectic geometers working on this topic in order to lay the groundwork for the future development of this area. By all accounts, it was a remarkable and enjoyable success. We hope and expect that this will be the beginning of a long and illustrious history of such gatherings.

Mini-Workshop: New Developments in Newton-Okounkov Bodies

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Abstracts

Interplay between Algebraic and Convex Geometries

Askold Khovanskii
(joint work with Kiumars Kaveh)

It has been a standard view that the connection between convex geometry and algebraic geometry belongs only to the framework of toric varieties. But recently I and my former student K. Kaveh found an unexpected general theory, which we call the theory of Newton-Okounkov bodies, that connects general algebraic varieties and convex geometry (see [1, 2, 3, 4, 5] and also [6]). We obtain several results: we show that for a large class of graded algebras, the Hilbert functions have polynomial growth and their growth coefficients satisfy a Brunn-Minkowski type inequality. We prove analogues of the Fujita approximation theorem for semigroups of integral points, graded algebras and arbitrary linear systems respectively. Applications include a far-reaching generalization of the Kushnirenko theorem, a new version of the Hodge inequality, and an elementary proof of the Alexandrov-Fenchel inequality in convex geometry and its analogue in algebraic geometry. There are also local versions of these algebraic inequalities dealing with the Samuel multiplicities of primary ideals. These local algebraic inequalities suggest a new geometric Alexandrov-Fenchel type inequality for the so-called mixed co-volumes of convex bodies inscribed in a fixed convex cone.

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Okounkov bodies on projectivizations of rank two toric vector bundles

José Luis González

In his work on log-concavity of multiplicities, e.g. [8], [9], A. Okounkov introduced a procedure to associate convex bodies to linear systems on projective varieties. This construction was systematically studied by R. Lazarsfeld and M. Mustata in the case of big line bundles in [5], and by K. Kaveh and A. Khovanskii in [6] where they use a similar procedure to associate convex bodies to finite dimensional subspaces of the function field $\mathrm{K}(X)$ of a variety X.

The construction of these *Okounkov bodies* depends on a fixed flag of subvarieties and produces a convex compact set for each Cartier divisor on a projective variety. The Okounkov body of a divisor encodes asymptotic invariants of the divisor's linear system, and it is determined solely by the divisor's numerical equivalence class. Moreover, these bodies vary as fibers of a linear map defined on a closed convex cone as one moves in the space of numerical equivalence classes of divisors on the variety. As a consequence, one can expect to obtain results about line bundles by applying methods from convex geometry to the study of these Okounkov bodies.

Let us consider an n-dimensional projective variety X over an algebraically closed field, endowed with a flag $X_\bullet \colon X = X_n \supseteq \cdots \supseteq X_0 = \{pt\}$, where X_i is an i-dimensional subvariety that is nonsingular at the point X_0. In [5], Lazarsfeld and Mustata established the following:

(a) For each big rational numerical divisor class ξ on X, Okounkov's construction yields a convex compact set $\Delta(\xi)$ in $\mathbf{R}^n$, now called the *Okounkov body* of ξ, whose Euclidean volume satisfies

$$\mathrm{vol}_{\mathbf{R}^n}\big(\Delta(\xi)\big) \;=\; \frac{1}{n!}\cdot \mathrm{vol}_X(\xi).$$

The quantity $\mathrm{vol}_X(\xi)$ on the right is the *volume* of the rational class ξ, which is defined by extending the definition of the volume of an integral Cartier divisor D on X, namely,

$$\mathrm{vol}_X(D) \;=_{\mathrm{def}}\; \lim_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

(b) Moreover, there exists a closed convex cone $\Delta(X) \subseteq \mathbf{R}^n \times N^1(X)_{\mathbf{R}}$ characterized by the property that in the diagram

$$\begin{array}{ccc} \Delta(X) & \hookrightarrow & \mathbf{R}^n \times N^1(X)_{\mathbf{R}} \\ & \searrow \quad \swarrow & \\ & N^1(X)_{\mathbf{R}}, & \end{array}$$

the fiber $\Delta(X)_\xi \subseteq \mathbf{R}^n \times \{\xi\} = \mathbf{R}^n$ of $\Delta(X)$ over any big class $\xi \in N^1(X)_{\mathbf{Q}}$ is $\Delta(\xi)$. $\Delta(X)$ is called the *global Okounkov body* of X.

In this talk, we are interested in describing the Okounkov bodies of the divisors on the projectivization $\mathbf{P}(\mathcal{E})$ of a toric vector bundle $\mathcal{E}$ over a smooth projective toric variety X. Such vector bundles were described by A. Klyachko in [7] in terms of certain filtrations of a suitable vector space, and they have been the focus of some recent activity, e.g. [4], [10], [11]. As we will see, these filtrations can be used to compute the sections of all line bundles on $\mathbf{P}(\mathcal{E})$. For our main result, we restrict to the case of rank two toric vector bundles, where the Klyachko filtrations are considerably simpler. Using the data from the filtrations, we construct a flag of torus invariant subvarieties $Y_\bullet$ on $\mathbf{P}(\mathcal{E})$ and produce finitely many linear inequalities defining the global Okounkov body of $\mathbf{P}(\mathcal{E})$ with respect to this flag. In particular, we see that this is a rational polyhedral cone.

In the construction of Okounkov bodies on a variety one constructs a valuation-like function $\nu_{Y_\bullet}$ and then one obtains an associated semigroup $\mathrm{S}_{Y_\bullet}$ encoding all images of $\nu_{Y_\bullet}$. We use our description of the global Okounkov body of a projectivized rank two toric vector bundle to prove the finite generation of the semigroup $\mathrm{S}_{Y_\bullet}$ in this setting. As an application one obtains a proof of the finite generation of the total coordinate rings or Cox rings of these varieties (see [3], [2] for alternative arguments for the finite generation of these Cox rings). A reference for the results presented in this talk is the article [1].

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Crystal bases and Newton-Okounkov bodies

Kiumars Kaveh

Let G be a connected reductive algebraic group over $\mathbb{C}$. The purpose of the talk is to show that some important convex polytopes arising in the representation theory of G, as well as in the theory of spherical G-varieties, are instances of Newton-Okounkov bodies for flag and spherical varieties respectively.

The main result concerns a geometric description of the so-called string parametrization of a crystal basis for an irreducible representation of G. More precisely, we show that the string parametrization of a crystal basis coincides with a valuation on the field of rational functions on the flag variety G/B, constructed out of a natural coordinate system on a Bott-Samelson variety. We regard the elements of the irreducible representation as polynomials on the (opposite) open cell in G/B and hence rational functions on G/B. This realization of the string parametrization shows that the so-called string polytopes of Littelmann and Berenstein-Zelevinsky ([Litt98, B-Z01]) associated to irreducible representations of G, can be realized as Newton-Okounkov bodies for the flag variety of G. This provides a new point of view on crystal bases and we expect it to make several properties of the crystal bases more transparent. As an example, one can readily deduce a multiplicativity property of the dual canonical basis for the covariant algebra $\mathbb{C}[G]^U$ due to P. Caldero ([Cal02]).

The motivation for this result goes back to a result of A. Okounkov who showed that when $G = \mathrm{Sp}(2n, \mathbb{C})$, the set of integral points in the Gelfand-Cetlin polytope of an irreducible representation of G can be identified with the collection of initial terms of elements of this representation regarded as polynomials on the open cell in the flag variety ([Ok98]).

Let V_λ be a finite dimensional irreducible representation of G with highest weight λ. There are remarkable bases for V_λ, consisting of weight vectors, called *crystal bases* which combinatorially encode the action of $\mathrm{Lie}(G)$ ([Kash90]). Crystal bases play a fundamental role in the representation theory of G. There is a nice parametrization of the elements of a crystal basis, called the *string parametrization*, by the set of integral points in a certain polytope in $\mathbb{R}^N$, where $N = \dim(G/B)$ is the number of positive roots ([Litt98], [B-Z01]). This parametrization depends on a choice of a reduced decomposition for the longest element w_0 in the Weyl group, i.e., an N-tuple of simple roots $\underline{w}_0 = (\alpha_{i_1}, \dots, \alpha_{i_N})$ with

$$w_0 = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_N}}.$$

The polytope associated to V_λ and a reduced decomposition $\underline{w}_0$ is called a *string polytope*. We denote it by $\Delta_{\underline{w}_0}(\lambda)$. The number of integral points in the polytope $\Delta_{\underline{w}_0}(\lambda)$ is equal to $\dim(V_\lambda)$. The string polytopes are generalizations of the well-known Gelfand-Cetlin polytopes of representations of $\mathrm{GL}(n, \mathbb{C})$ ([G-C50]).

Let $X = G/B$ be the flag variety of G, and let $X_w \subset X$ denote the Schubert variety corresponding to a Weyl group element w. Fix a reduced decomposition $\underline{w}_0 = (\alpha_{i_1}, \dots, \alpha_{i_N})$. It gives rise to a sequence of Schubert varieties in G/B:

$$\{o\} = X_{w_N} \subset \cdots \subset X_{w_0} = X,$$

where $w_k = s_{\alpha_{i_{k+1}}} \cdots s_{\alpha_{i_N}}$, and $o = eB$ is the unique B-fixed point in X. The reduced decomposition $\underline{w}_0$ also gives rise to a sequence of Bott-Samelson varieties

$$\{o\} = \tilde{X}_N \subset \cdots \subset \tilde{X}_0 = \tilde{X}_{\underline{w}_0},$$

together with a birational isomorphism $\pi : \tilde{X}_{\underline{w}_0} \to X$, such that for each k, $\tilde{X}_k$ is smooth and $\pi_{|\tilde{X}_k} : \tilde{X}_k \to X_{w_k}$ is also a birational isomorphism. One constructs a natural coordinate system $u_1, \ldots, u_N$, in a Zariski open subset $\tilde{\mathcal{U}}$ of $X_{\underline{w}_0}$ lying above the opposite open Schubert cell of X, such that in the open set $\tilde{\mathcal{U}}$ each $\tilde{X}_k$ is given by the equations $u_1 = \cdots = u_k = 0$. The coordinate system $u_1, \ldots, u_N$ then defines a *highest* term valuation $v_{\underline{w}_0}$ on the algebra of polynomials on $\tilde{\mathcal{U}}$, and hence a valuation on the field of rational functions $\mathbb{C}(X)$. As usual this valuation extends to define a valuation on the ring of sections of any line bundle on X.

Let L_λ denote the G-linearized line bundle on X associated to a dominant weight λ. The space of sections $H^0(X, L_\lambda)$ is isomorphic to the dual representation V_λ^*. The main result is the following:

Theorem 1. *The string parametrization for a dual crystal basis in V_λ^* coincides with the valuation $v_{\underline{w}_0}$ on $H^0(X, L_\lambda)$. It follows that the string polytope $\Delta_{\underline{w}_0}(\lambda)$ coincides with the Newton-Okounkov body of the algebra of sections $R(L_\lambda) = \bigoplus_k H^0(X, L_\lambda^{\otimes k})$ and the valuation $v_{\underline{w}_0}$.*

A variety X is *spherical* if a (and hence any) Borel subgroup has a dense orbit. When X is spherical the space of sections of any G-linearized line bundle on X is a multiplicity-free G-module. Flag varieties are spherical.

Generalizing the notion of the Newton polytope of a toric variety, to a G-linearlized line bundle L on a spherical variety X one associates a polytope $\Delta_{\underline{w}_0}(X, L)$ (see [Ok97] and [A-B04]). The construction depends on a reduced decomposition $\underline{w}_0$. It combines the so-called *moment polytope* of (X, L) and the string polytopes $\Delta_{\underline{w}_0}(\lambda)$. From the above result one shows:

Corollary 1. *There is a natural geometric valuation $\tilde{v}_{\underline{w}_0}$ on $\mathbb{C}(X)$ such that the polytope $\Delta_{\underline{w}_0}(X, L)$ can be realized as the Newton-Okounkov body of the ring of sections $R(L) = \bigoplus_k H^0(X, L^{\otimes k})$ and the valuation $\tilde{v}_{\underline{w}_0}$.*

Fix a total order $<$ on $\mathbb{Z}^n$ respecting addition. Let A be a subalgebra of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. A subset $f_1, \ldots, f_r \in A$ is called a *SAGBI basis* for A (Subalgebra Analogue of Gröbner Basis for Ideals) if the set of highest terms of the f_i (with respect to $<$) generates the semigroup of highest terms in A (in particular this semigroup is finitely generated). Given a SAGBI basis for A one can represent each $f \in A$ as a polynomial in the f_i via a simple classical algorithm (known as the *subduction algorithm*). There are not many examples of subalgebras known to have a SAGBI basis. It is an important unsolved problem to determine which subalgebras have a SAGBI basis.

We generalize the notion of SAGBI basis to the context of valuations on graded algebras. Using the previous result we show that:

Corollary 2. *With respect to the valuation $\tilde{v}_{\underline{w}_0}$, the ring of sections $R(L)$ of any G-linearized very ample line bundle L on a projective spherical variety X has a SAGBI basis. It follows that (X, L) has a flat degeneration to the toric variety associated to the polytope $\Delta_{\underline{w}_0}(X, L)$.*

This recovers toric degeneration results in [A-B04], [Cal02] and [Kav05].

It is expected that the Gelfand-Cetlin and more generally the string polytopes carry a lot of information about the geometry of the flag variety (and more generally spherical varieties). In fact, there is a general philosophy that these polytopes play a role for the flag variety similar to the role of Newton polytopes for toric varieties. The results presented here provide strong evidence in this direction. More evidence for this similarity is obtained in the recent works of V. Kiritchenko who made an interesting connection between the combinatorics of the faces of the Gelfand-Cetlin polytope and the Schubert calculus (in type A) [Kir09]. This talk is also closely related to the work of [And10].

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Newton bodies consisting of orbits of the coadjoint action

Boris Kazarnovskii

1. **Formulations**. Let G be a complex reductive Lie group, $f_1, \cdots, f_k$ be a system of matrix functions of finite dimensional holomorphic representations $\pi_1, \cdots, \pi_k$ of G, respectively. (A matrix function is a linear combination of matrix elements of representation). Define

$$X_k = \{g \in G\colon f_1(g) = \cdots = f_k(g) = 0\}. \tag{1}$$

Theorem 1. *For $k = \dim G$ and general system (1), the number of points of X_k is equal to $k!V_k(\Delta_{\pi_1}, \cdots, \Delta_{\pi_k})$, where Δ_π – defined below k-dimensional convex body depending on representation π, and V_k – mixed volume of convex bodies.*

Remark 1. Other formulas for the number of points of X_k are in [2, 3, 4].

The body Δ_π depends on the choice of the maximal compact subgroup $K \subset G$. Let $\mathfrak{g}$ be a Lie algebra of G, $\mathrm{Im}\mathfrak{g} \subset \mathfrak{g}$ be a Lie algebra of K and $\mathrm{Re}\mathfrak{g} = \sqrt{-1}\mathrm{Im}\mathfrak{g}$. In the dual space $\mathfrak{g}^*$, we consider the real subspaces $\mathrm{Re}\mathfrak{g}^*$ and $\mathrm{Im}\mathfrak{g}^*$ orthogonal to $\mathrm{Im}\mathfrak{g}$ and $\mathrm{Re}\mathfrak{g}$, respectively, with respect to the real part of the complex pairing. These subspaces are invariant under the coadjoint action of K.

The body Δ_π lies in the space $\mathrm{Re}\mathfrak{g}^*$ and consists of orbits of coadjoint action of group K. Let τ be a maximal torus of group K (Δ_π doesn't depend on the choice of τ). Define β^* as the subspace of fixed points of the coadjoint action of τ on the space $\mathrm{Re}\mathfrak{g}^*$. The space β^* is dual to the Lie algebra of τ. We place the weights of the representation $d\pi$ of Lie algebra $\mathfrak{g}$ in the space β^*. We define Δ_π as the union of coadjoint K-orbits in $\mathrm{Re}\mathfrak{g}^*$ intersecting the weight polytope (the convex hull of the weights) .

If $G = GL(m, \mathbb{C})$, then Δ_π consists of Hermitian matrices with the set of eigenvalues lying in the weight polytope.

Proposition 1. For any π the body Δ_π is convex.

Definition 1. (1) The increment of a finite-dimensional representation π of a Lie group G is the function on $\mathfrak{g}$ equal to the maximum real part of eigenvalues of the representation operator $d\pi$.

(2) The reductive increment h_π of holomorphic representation π is the function on $\mathfrak{g}$ that coincides with the increment on the subspace $\mathrm{Re}\mathfrak{g}$ and is constant along the subspace $\mathrm{Im}\mathfrak{g}$.

Corollary 1. *The reductive increment h_π is the support function of Δ_π.*

Remark 2. If $G = (\mathbb{C} \setminus 0)^N$ then the functions increment, reductive increment and support function of the weight polytope are the same.

There is an analogue of Theorem 1 for the case $k \le \dim G$.

Theorem 2. [1] *The asymptotic density (defined below) of varieties of type* (1) *exists and is equal to $dd^c h_{\pi_1} \wedge \cdots \wedge dd^c h_{\pi_k}$.*

Remark 3. If $G = (\mathbb{C} \setminus 0)^N$ then the current $dd^c h_1 \wedge \cdots \wedge \cdots \wedge dd^c h_k$ is, in fact, some recoding of the so-called tropicalization of variety X_k [6]. So Theorem 2 can be placed in the context of a more general question: "Is there a way of survival of tropical geometry after passing from torus to an arbitrary complex reductive Lie group ?".

2. **Comments on Theorem 2**.

2.1. **Monge-Ampere operator**. The mixed Monge-Ampere operator of degree k is called a map $(h_1, \cdots, h_k) \mapsto dd^c h_1 \wedge \cdots \wedge dd^c h_k$ (for a function g on a complex manifold $d^c g(x_t) = dg(\sqrt{-1}x_t)$ for any tangent vector x_t). Values of the operator are considered as currents (functionals on the space of smooth compactly supported differential forms) of degree $2k$. If h_i are continuous plurisubharmonic functions on the complex manifold (eg, *convex functions on* $\mathfrak{g}$), then the current $dd^c h_1 \wedge \cdots \wedge dd^c h_k$ is well defined. This means that approximating h_i with smooth plurisubharmonic functions one gets the sequence of values of Monge-Ampere operator weakly converging to the current independent of the choice of approximation. This functional can be continued to a functional on the space of continuous compactly supported forms (i.e. is the current of measure type).

Proposition 2. [6] Let h_i be a support function of a convex body $A_i \subset \mathrm{Re}\mathbb{C}^{\mathrm{n}*}$. Then the current $dd^c h_1 \wedge \cdots \wedge dd^c h_n$ is the Euclidean measure on subspace $\mathrm{Im}\mathbb{C}^{\mathrm{n}}$ with multiplicity equal to $n!$ times the mixed volume of $A_1, \cdots, A_n$.

2.2. **Asymptotic density**. Now we consider the *averaged root distribution* $\bar{X}_k$. This is the averaging of root varieties (1) over all systems of matrix functions of our representations. Formula for the averaged root distribution current belongs to the family of "Crofton formulas" [5]

$$\bar{X}_k = dd^c \log \mathrm{Tr}\,(\pi_1(g)\pi_1(g)^*) \wedge \cdots \wedge dd^c \log \mathrm{Tr}\,(\pi_k(g)\pi_k(g)^*).$$

Any $g \in G$ is uniquely represented as $\{g = \exp(y)\kappa\colon\ y \in \mathrm{Re}\mathfrak{g},\ \kappa \in \mathrm{K}\}$ (this decomposition is called the polar decomposition or the Cartan decomposition). Let the map $\exp_r\colon \mathfrak{g} \to G$ is defined as

$$\exp_r\colon \xi \to \exp(\mathrm{Re}\xi)\ \exp(\mathrm{Im}\xi),$$

and $\log_r(\bar{X}_k) = \exp_r^* \bar{X}_k$.

Consider the scaling map $r_t\colon z \mapsto \mathrm{Rez/t} + \mathrm{Imz}$ of the space $\mathbb{C}^N$. Let Z be a current on the space $\mathbb{C}^N$. If $(r_t)_* Z = \sigma_r + o(1)$ as $t \to +\infty$, then say that the current σ_r is the *reductive density* of Z.

Definition 2. The asymptotic density of varieties of type (1) is the reductive density of the current $\log_r(\bar{X}_k)$.

3. **Sketch of proof**. The map $\exp_r\colon \mathfrak{g} \to G$ is not holomorphic, but nevertheless (it requires justification) it is true that

$$\log_r(\bar{X}_k) = dd^c \log \mathrm{Tr}\ \exp(2d\pi_1(\mathrm{Re}\xi)) \wedge \cdots \wedge \mathrm{dd^c} \log \mathrm{Tr}\ \exp(2\mathrm{d}\pi_\mathrm{k}(\mathrm{Re}\xi)).$$

So the asymptotic density equals to

$$\lim_{t\to\infty} r_t^*(\ dd^c \log \mathrm{Tr}\ \exp(2d\pi_1(\mathrm{Re}\xi))\ ) \wedge \cdots \wedge \mathrm{r_t^*}(\ \mathrm{dd^c} \log \mathrm{Tr}\ \exp(2\mathrm{d}\pi_\mathrm{k}(\mathrm{Re}\xi)\ )$$

Proposition 3. Let f be a function on $\mathbb{C}^n$ such that $f(x + \sqrt{-1}y) = f(x)$ for any $x, y \in \mathrm{Re}\mathbb{C}^{\mathrm{n}}$. Then

$$(r_t)^* d^c f = \frac{1}{t} d^c (r_t)^* f$$

.

So (of proposition 3) the asymptotic density equals to

$$\lim_{t\to\infty} dd^c \left(\frac{1}{t} \log \mathrm{Tr}\ \exp(2d\pi_1(t\mathrm{Re}\xi))\right) \wedge \cdots \wedge dd^c \left(\frac{1}{t} \log \mathrm{Tr}\ \exp(2d\pi_k(t\mathrm{Re}\xi))\right).$$

Now the proof ends with a final reference to the following well known statement (to justify the correctness of the reference we must show that the function

$\log \operatorname{Tr}(\exp(tA)\ \exp(tA^*))$ is plurisubharmonic)

$$\lim_{t\to+\infty} \frac{1}{t} \log \operatorname{Tr}(\exp(tA)\ \exp(tA^*)) = h(A),$$

where A is a square matrix, $h(A)$ is the increment of A, and the convergence is locally uniform on the set of $n \times n$ matrices.

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Convex chains for Schubert varieties

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(joint work with Evgeny Smirnov and Vladlen Timorin)

In [4], we constructed generalized Newton polytopes for Schubert subvarieties in the variety of complete flags in $\mathbb{C}^n$. Every such "polytope" is a union of faces of a Gelfand–Zetlin polytope (the latter is a well-known Newton–Okounkov body for the flag variety). These unions of faces are responsible for Demazure characters of Schubert varieties and were originally used for Schubert calculus.

The methods of [4] lead to an extension of Demazure (or divided difference) operators from representation theory and topology to the setting of convex geometry. Below I define divided difference operators acting on convex polytopes and outline some applications such as a simple inductive construction of Gelfand-Zetlin polytopes and their generalizations.

The definition is based on the following observation. Let $\Pi(\mu,\nu)$ where μ, $\nu \in \mathbb{Z}^m$ denote the integer *coordinate parallelepiped* $\{(x_1,\ldots,x_m) | \mu_i \le x_i \le \nu_i\} \subset \mathbb{R}^m$, and let $\sigma(x)$ for $x \in \mathbb{R}^m$ denote the sum of coordinates $\sum_{i=1}^m x_i$. Given a parallelepiped $\Gamma = \Pi(\mu,\nu) \subset \mathbb{R}^m$ of dimension $m-1$ (assume that $\mu_m = \nu_m$) and an integer C, there is a unique parallelepiped $\Pi = \Pi(\mu,\nu') \subset \mathbb{R}^m$ such that $\Gamma = \Pi \cap \{x_m = \mu_m\}$ (that is, $\nu'_i = \nu_i$ for $i < m$) and

$$\sum_{x\in\Pi\cap\mathbb{Z}^d} t^{\sigma(x)} = D_C(\sum_{x\in\Gamma\cap\mathbb{Z}^d} t^{\sigma(x)}), \qquad (*)$$

where D_C is a Demazure-type operator on the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials in t:

$$D_C(f) := \frac{f - tf^*}{1-t}, \quad f^* := t^C f(t^{-1}).$$

Indeed, an easy calculation (using the formula for the sum of a geometric progression) shows that $\sum_{i=1}^{m}(\mu_i + \nu_i') = C$ which yields the value of ν_m'. Note that Γ is a facet of Π unless $\Pi = \Gamma$.

We now use this observation in a more general context. A *root space* of rank n is a coordinate space $\mathbb{R}^d$ together with a direct sum decomposition

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_n}$$

and a collection of linear functions $l_1, \ldots, l_n \in (\mathbb{R}^d)^*$ such that l_i vanishes on $\mathbb{R}^{d_i}$. We always assume that the summands are coordinate subspaces so that $\mathbb{R}^{d_1}$ is spanned by the first d_1 basis vectors etc.

Let $P \subset \mathbb{R}^d$ be a convex polytope in the root space. It is called a *parapolytope* if for all $i = 1,\ldots, n$, the intersection of P with any parallel translate of $\mathbb{R}^{d_i}$ is a coordinate parallelepiped. For instance, if $d = n$, that is, $d_1 = \ldots = d_n = 1$, then every polytope is a parapolytope. For each $i = 1,\ldots, n$, we now define a *divided difference operator* A_i on parapolytopes. In general, the operator A_i takes values in *convex chains* in $\mathbb{R}^d$ (see [3] for a definition) but in many cases of interest (see examples below) these convex chains will just be single convex parapolytopes.

First, consider the case where $P \subset (c + \mathbb{R}^{d_i})$ for some $c \in \mathbb{R}^d$, i.e. $P = P(\mu, \nu)$ is a coordinate parallelepiped. Here $\mu = (\mu_1, \ldots, \mu_d)$, $\nu = (\nu_1, \ldots, \nu_d)$. Put $N_i := d_1 + \ldots + d_i$ and $N_0 = 0$. Assume that $\dim(P) < d_i$. Choose the smallest $j \in [N_{i-1} + 1, N_i]$ such that $\mu_j = \nu_j$. Define $A_i(P)$ to be the coordinate parallelepiped $\Pi(\mu, \nu')$, where $\nu_k' = \nu_k$ for all $k \neq j$ and ν_j' is chosen so that

$$\sum_{k=N_{i-1}+1}^{N_i} (\mu_k + \nu_k') = l_i(c), \tag{$**$}$$

that is, an analog of formula $(*)$ holds for $\Gamma = P$, $\Pi = A_i(P)$ and $C = l_i(c)$. The definition yields a non-virtual coordinate parallelepiped if $l_i(c)$ is sufficiently large and can be extended to other values of $l_i(c)$ by linearity.

For an arbitrary parapolytope $P \subset \mathbb{R}^d$ define $A_i(P)$ as the union of $A_i(P \cap (c + \mathbb{R}^{d_i}))$ over all $c \in \mathbb{R}^d$:

$$A_i(P) = \bigcup_{c \in \mathbb{R}^d} \{A_i(P \cap (c + \mathbb{R}^{d_i}))\}$$

(assuming that $\dim(P \cap (c + \mathbb{R}^{d_i})) < d_i$ for all $c \in \mathbb{R}^d$). In other words, we first slice P by subspaces parallel to $\mathbb{R}^{d_i}$ and then replace each slice with another parallelepiped according to $(**)$. Note that P is a facet of $A_i(P)$ unless $A_i(P) = P$. It is easy to check that $A_i^2 = A_i$ (the same identity as for the classical Demazure operators).

Examples: (1) The simplest meaningful example is $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R} = \{(x, y)\}$ with the functions $l_1 = y$ and $l_2 = x$. If $P = (a, b)$ is a point, then $A_1(P)$ and

$A_2(P)$ are segments:

$$A_1(P) = [(a, b), (b - a, b)], \quad A_2(P) = [(a, b), (a, a - b)],$$

assuming that $\frac{1}{2}b \geq a \geq 2b$. If $b < 2a$, then $A_1(P)$ is a virtual segment. If $2b > a$, then $A_2(P)$ is virtual.

If $P = [(a, b), (a', b)]$ is a horizontal segment, then $A_2(P)$ is the trapezoid (or a skew trapezoid) with the vertices (a, b), (a', b), $(a, a - b)$, $(a', a' - b)$.

(2) A more interesting example is $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R} = \{(x, y, z)\}$ with the functions $l_1 = z$ and $l_2 = x+y$. If $P = [(a, b, c), (a', b, c)]$ is a segment in $\mathbb{R}^2$, then $A_1(P)$ is the rectangle with the vertices (a, b, c), (a', b, c), $(a, c-a-a'-b, c)$, $(a', c-a-a'-b, c)$. Using this calculation and those in (1), it is easy to show that if $P = (b, c, c)$ is a point and $-b - c > b > c$, then $A_1A_2A_1(P)$ is the 3-dimensional Gelfand–Zetlin polytope given by the inequalities $a \geq x \geq b$, $b \geq y \geq c$ and $x \geq z \geq y$, where $a + b + c = 0$.

(3) Generalizing the last example we now construct Gelfand–Zetlin polytopes for arbitrary n via divided difference operators. For $n \in \mathbb{N}$, put $d = \frac{n(n-1)}{2}$. Consider the root space $\mathbb{R}^d = \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-2} \oplus \ldots \oplus \mathbb{R}^1$ of rank $(n - 1)$ with the functions l_i given by the formula: $l_i(x) = \sigma_{i-1}(x) + \sigma_{i+1}(x)$. Here $\sigma_i(x)$ denotes the sum of those coordinates of $x \in \mathbb{R}^d$ that correspond to the subspace $\mathbb{R}^{d_i}$ (put $\sigma_0 = \sigma_n = 0$).

For every strictly dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ (that is, $\lambda_1 > \ldots > \lambda_n$) of GL_n such that $\lambda_1 + \ldots + \lambda_n = 0$, the Gelfand–Zetlin polytope Q_λ coincides with the polytope

$$[(A_1 \ldots A_{n-1})(A_1 \ldots A_{n-2}) \ldots (A_1)]\,(p),$$

where $p \in \mathbb{R}^d$ is the point $(\lambda_2, \ldots, \lambda_n; \lambda_3, \ldots, \lambda_n; \ldots; \lambda_n)$.

Similarly, divided difference operators for suitable root spaces allow one to construct the classical Gelfand–Zetlin polytopes for symplectic and orthogonal groups. They also yield an elementary description of more general *string polytopes* defined in [5] and might help to extend the results of [4] to arbitrary semisimple groups.

As outlined below, these convex geometric operators are well suited for inductive constructions of Newton–Okounkov polytopes for line bundles on Bott towers and on Bott-Samelson varieties (for natural choice of a geometric valuation). The former polytopes were described in [2] and the latter are currently being computed by Dave Anderson.

Bott towers. Consider a root space with $d = n$, that is, $d_1 = \ldots = d_n = 1$. We have the decomposition

$$\mathbb{R}^n = \underbrace{\mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{n}; \quad y = (y_1, \ldots, y_n)$$

into coordinate lines. Assume that the linear function l_i for $i < n$ does not depend on $y_1, \ldots, y_i$, and $l_n = y_1$. I can show that the polytope $P := A_1 \ldots A_n(p)$ (for a point $p \in \mathbb{R}^n$) coincides with the Newton–Okounkov body for a *Bott tower* (that depends on $l_1, \ldots, l_n$) together with a line bundle (that depends on p). For $n = 2$, a Bott tower is a Hirzebruch surface and P is a trapezoid (or a skew trapezoid) constructed similarly to the one in example (1). In general, a Bott tower is a toric variety obtained by successive projectivizations of rank two split vector bundles, and P is a multidimensional version of a trapezoid.

Bott–Samelson resolutions. Let $X = R(i_1, \ldots, i_l)$ be the *Bott-Samelson variety* corresponding to any sequence $(\alpha_{i_1}, \ldots, \alpha_{i_l})$ of roots of the group GL_n. It can be obtained by successive projectivizations of rank two (usually non-split) vector bundles. Consider the root space $\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2} \oplus \ldots \oplus \mathbb{R}^{d_{n-1}}$ with the functions l_i given by the formula $l_i(x) = \sigma_{i-1}(x) + \sigma_{i+1}(x)$, where d_i is the number of times the root α_i occurs in the sequence $(\alpha_{i_1}, \ldots, \alpha_{i_l})$. Denote by T_v the parallel translation in the root space by a vector $v \in \mathbb{R}^d$. Consider the polytope

$$P = \left[A_{i_1} T_{v_1} A_{i_2} \ldots T_{v_{l-1}} A_{i_l}\right](p).$$

In his talk, Dave Anderson described an algorithm for computing the Newton–Okounkov body of a line bundle on X with respect to the valuation given by the flag of subvarieties $\{\ldots \supset R(i_{l-1}, i_l) \supset R(i_l)\}$. Based on his computations for $l = 3$ [1], I conjecture that this Newton–Okounkov body coincides with P for suitable choice of a point $p \in \mathbb{R}^d$ and vectors $v_j \in \mathbb{R}^{d_{i_j}}$ for $j = 1, \ldots, l-1$.

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Combinatorial 2-truncated cubes and applications

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(joint work with Vadim Volodin)

Our interest in the family of 2-truncated cubes arises from [Bu2] where it was shown that associahedra (Stasheff polytopes) are 2-truncated cubes, i.e. can be obtained from a cube by truncations only of faces of codimension 2 (2-truncations).

The F-polynomials (see [Bu1]) of n-polytopes P is defined by

$$F(P)(\alpha, t) = \alpha^n + f_{n-1}\alpha^{n-1}t + \cdots + f_1\alpha t^{n-1} + f_0 t^n,$$

where f_i are the components of f-vector of P. The H-polynomial is defined by

$$H(P)(\alpha, t) = F(P)(\alpha - t, t) = h_0\alpha^n + h_1\alpha^{n-1}t + \cdots + h_{n-1}\alpha t^{n-1} + h_n t^n$$

The numbers h_i are the components of h-vector of P. The classical Dehn-Sommerville equations imply that $H(P)$ is symmetric for any simple polytope (see [Bu1]).

Therefore, it can be represented as a polynomial of $a = \alpha + t$ and $b = \alpha t$:

$$H(P) = \sum_{i=0}^{[\frac{n}{2}]} \gamma_i (\alpha t)^i (\alpha + t)^{n-2i}.$$

By definition, γ-vector is $(\gamma_0, \gamma_1, \dots, \gamma_{[\frac{n}{2}]})$ and γ-polynomial is

$$\gamma(P)(\tau) = \gamma_0 + \gamma_1 \tau + \cdots + \gamma_{[\frac{n}{2}]} \tau^{[\frac{n}{2}]}$$

.

One of the well-known problems is to find a geometric interpretation of the γ-vector. The famous Charney-Davis conjecture for polytopes is equivalent to the statement that $\gamma_{[\frac{n}{2}]}$ is nonnegative for flag simple polytopes. It has a natural generalization which in the case of polytopes is formulated as the following.

Conjecture 1 (Gal, 2005)**.** *Any flag simple n-polytope P satisfies $\gamma_i(P^n) \geq 0, i = 0, \dots, [\frac{n}{2}]$.*

Gal's conjecture is proved in particular cases. One can check that any 2-truncated cube is a flag simple polytope.

Theorem 1. *Gal's conjecture holds for any 2-truncated cube.*

The proof is based on the formula $\gamma(Q) = \gamma(P) + \tau\gamma(G)\gamma(\Delta^{n-k-2})$, where Q is obtained from P by truncation of the face G of dimension k. When $k = n-2$ the formula becomes $\gamma(Q) = \gamma(P) + \tau\gamma(G)$ and we can apply induction.

It was shown in [Bu1] that the g-vector is obtained from the γ-vector by multiplication on the matrix with nonnegative coefficients. They yield the following implications for simple n-polytopes P_1 and P_2.

$$\gamma_i(P_1) \leq \gamma_i(P_2) \Rightarrow g_i(P_1) \leq g_i(P_2) \Rightarrow h_i(P_1) \leq h_i(P_2) \Rightarrow f_i(P_1) \leq f_i(P_2).$$

So, if we have upper and lower bounds for the components of γ-vector on some class of simple polytopes, then it implies upper and lower bounds for components of g-,h- and f-vectors on this class of simple polytopes.

Every Delzant polytope, particularly every 2-truncated cube P^n is the image of moment map for some smooth toric variety M_P^{2n}. Odd Betti numbers b_{2i-1} for Hamiltonian toric manifold M_P^{2n} are zero and even Betti numbers b_{2i} are equal to components $h_i(P)$ of the h-vector of P. Then we have lower bounds for Betti numbers of smooth toric manifolds corresponding 2-truncated cubes.

Corollary 1. *If the image of the moment map for smooth toric variety M_P^{2n} is a 2-truncated cube P^n, then*

$$b_{2i}(M_P^{2n}) \geq h_i(I^n) = \binom{n}{i}.$$

Let $\sigma(M^{2n})$ be the classical signature of M^{2n}. If n is odd $\sigma(M_P^{2n})$ is zero, but if $n = 2q$ is even, then

$$\sigma(M_P^{4q}) = \sum_{k=0}^{2q} (-1)^k h_k(P^{2q}) = (-1)^q \gamma_q(P^{2q}). \tag{1}$$

Truncation of a face corresponds to a blow-up for the associated smooth toric variety. So, when the image of the moment map P^n is a 2-truncated cube, then the smooth toric variety M_P^{2n} is obtained from a product of complex projective lines by a sequence of blow-ups. Moreover if the dimension of the 2-truncated cube is even, the following result holds.

Corollary 2. *If the image of the moment map for smooth toric variety M^{4q} is a 2-truncated cube P^{2q}, then*

$$(-1)^q\sigma(M^{4q}) \geq 0.$$

One of our main results is that many famous classes of simple polytopes are 2-truncated cubes.

An important class of simple polytopes is nestohedra. These polytopes arose in the work of C. De Concini and C. Procesi. Nestohedra were constructed as Minkowski sums of certain sets of simplexes corresponding to some building set. Attention was drawn to nestohedra due to the work of A. Postnikov, V. Reiner, L. Williams where they obtained important results about their combinatorics and in particular proved Gal's conjecture for chordal building sets.

Theorem 2. *A flag nestohedron is a 2-truncated cube if and only if it is a flag polytope.*

Corollary 3. *Gal's conjecture holds for every flag nestohedron.*

A wide class of flag nestohedra are graph-associahedra was introduced by M. Carr and S. Devadoss. Among them are the Stasheff polytopes (associahedra), Bott-Taubes polytopes (cyclohedra) and permutohedra. Graph-associahedra can be described as nestohedra where the building set is constructed by natural way from a graph. We prove the following bounds for the graph-associahedra.

Theorem 3. *There are following unimprovable bounds:*

1) $\gamma_i(As^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(Pe^n)$ *for any connected graph* Γ_{n+1} *on* $[n+1]$*;*
2) $\gamma_i(Cy^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(Pe^n)$ *for any Hamiltonian graph* Γ_{n+1} *on* $[n+1]$*;*
3) $\gamma_i(As^n) \leq \gamma_i(P_{\Gamma_{n+1}}) \leq \gamma_i(St^n)$ *for any tree* Γ_{n+1} *on* $[n+1]$.

The similar bounds hold for f-, g- and h-vectors.

The theorem and (1) implies bounds for signature and Betti numbers for corresponding smooth toric varieties.

We describe geometric operations that transform an n-dimensional graph-associahedron to an $(n+1)$-dimensional one. It allows us to consider a series of graph-associahedra and to describe their combinatorics in terms of differential and functional equations on generating function of face polynomials. Similar equations were obtained using the ring of simple polytopes (see [Bu1]). For example, the functional equation on the generating series of the H-vectors of Stasheff polytopes is

$$H_{As}(x) = (1+\alpha x H_{As}(x))(1+txH_{As}(x)), \text{ where } H_{As}(x) = \sum_{n=0}^{\infty} H(As^n)x^n.$$

S. Fomin and A. Zelevinsky introduced a new class of polytopes corresponding to cluster algebras related to Dynkin diagrams. It was shown by M. Goresky that the polytopes corresponding to diagrams of the D-series are not nestohedra but each of them is a 2-truncated cube.

In the work of S. L. Devadoss, T. Heath, W. Vipismakul it was shown that some moduli spaces of marked bordered surfaces has a polytopal stratification and there they introduced a class of simple polytopes called graph cubeahedra generalizing the polytopes associated with moduli spaces. This class contains as well-known series (for example, associahedra) as a new sequence of polytopes called halohedra. We introduce a class of simple n-polytopes $NP(P, B)$ (called nested polytopes). Each of them is defined by a pair (P, B), where P is a simple n-polytope with fixed order of facets and B is a building set on $[n]$.

Theorem 4. *(1) The nested polytope $NP(P, B)$ is flag if both polytope P and nestohedron P_B are flag.*
(2) The nested polytope $NP(P, B)$ is a 2-truncated cube if P is a 2-truncated cube and P_B is a flag polytope.
(3) The nested polytope $NP(I^n, B(\Gamma_n))$ is equivalent to graph cubeahedron corresponding Γ_n. Here Γ_n is a simple graph on n nodes and $B(\Gamma_n)$ is a graphical building set associated with Γ_n.

Corollary 4. *(1) Every graph cubeahedron is a 2-truncated cube.*
(2) Gal's conjecture holds for every graph cubeahedron.

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Volumes of NObodies

Victor Lozovanu

(joint work with Alex Küronya and Catriona Maclean)

An important reason for organizing this mini-workshop, as we have seen, is to see applications of the theory of Newton-Okounkov bodies ("NObodies") to different areas of mathematics. The purpose of this talk is to look at NObodies and volumes of divisors from an arithmetic point of view.

In the last thirty years or so, it became clear that it is helpful and useful to tackle many problems appearing in algebraic geometry from an asymptotic perspective. One of these is the Riemann-Roch problem:

Riemann-Roch Problem 0.1. Compute the dimensions:

$$h^0(X, kD) \ := \ \dim_{\mathbb{C}}(H^0(X, \mathcal{O}_X(kD)))$$

as a function of k, where D is a Cartier divisor on a complex projective variety X of dimension n.

The history of this problem is very rich and dates back to Zariski. On surfaces, based on Zariski's work, Cutkosky and Srinivas in [3] show that the function $h^0(X, kD)$ grows as a sum of a quadratic polynomial and a periodical function for large k. In higher dimensions, the problem is more complex and Cutkosky was first to realize that it is helpful to look at the problem from an asymptotic perspective. This was materialized first in the work of Ein and Lazarsfeld, who developed the theory of volume. Later, based on work of Okounkov [11], [12], Lazarsfeld and Mustata in [10] and independently Kaveh and Khovanskii in [6] introduced the notion of Newton-Okounkov bodies.

If D is a Cartier divisor on some irreducible projective variety X of dimension n, then *the volume of* D is defined to be

$$\mathrm{vol}_X(D) \ = \ \limsup_{k \to \infty} \frac{\dim_{\mathbb{C}}(H^0(X, \mathcal{O}_X(kD)))}{k^n/n!}.$$

The volume is one of the first asymptotic invariants of divisors that has been studied. It first appeared in some form in [2] (and is elegantly explained in [9]), where Cutkosky used the irrationality of a volume of a divisor to show the non-existence of birational Zariski decompositions with rational coefficients. For a complete account, the reader is invited to look at [9] and the recent paper [10].

Since it is invariant with respect to numerical equivalence of divisors, the volume can be considered as a function on the Néron–Severi group. This function turns out to be homogeneous, log-concave, and extends continuously to divisors classes with real coefficients. It can be explcitly determined on toric varieties [5], on surfaces [1], and on abelian varieties and homogeneous spaces for example. In every case, the volume reveals a fair amount of the underlying geometry.

Our main focus here is the multiplicative submonoid of positive real numbers consisting of volumes of integral divisors, which we denote by $\mathcal{V}$. First, we know that the volume of a divisor with finitely generated section ring is rational. Looking at the case of surfaces, an immediate consequence of Zariski decomposition gives that every divisor there — even the ones with non-finitely generated section ring — has rational volume. Conversely, based on Cutkosky's construction [9, Example 2.3.6], there are examples of integral divisors such that any positive rational number can be displayed as the volume of one of those divisors.

In higher dimensions, we mentioned that the volume of an integral divisor need not be rational. Although the example Cutkosky obtains is algebraic, the question remains whether any non-negative real number is contained in the monoid $\mathcal{V}$. This issue was addressed in [8] from two somewhat complementary directions.

Theorem. *The set of volumes satisfies the following two properties:*

(1) $\mathcal{V}$ is countable;
(2) $\mathcal{V}$ contains transcendental elements.

Let us briefly give an idea why the above results appearing originally in [8] hold. The transcendency of volumes of integral divisors is an application of Cutkosky's principle to look for examples among projectivized vector bundles over varieties whose behaviour we understand quite well. In our case we consider $\mathcal{O}(1)$ of the projectivization of a rank three vector bundle on the self-product of a general elliptic curve. We exploit the non-linear shape of the nef cone on the abelian surface to show the transcendency of $\mathrm{vol}(\mathcal{O}(1))$. This example also shows that divisors with transcendental volume show up quite naturally and often in a non-finitely generated setting.

As far as the cardinality of $\mathcal{V}$ is concerned, it is a direct consequence of a much stronger countability result: building on the existence of multigraded Hilbert schemes as proved in [4], we establish the fact that there exist altogether countably many volume functions and ample/nef/big/pseudo-effective cones for all irreducible varieties in all dimensions.

Getting back to the issue of transcendental volumes, it is an interesting fact that the irregular values obtained so far by Cutkosky's construction have all been produced by evaluating integrals of polynomials over algebraic domains. In fact, all volumes computed to date can be put in such a form quite easily. Such numbers are called periods, and are studied extensively in various branches of mathematics, including number theory, modular forms, and partial differential equations. An enjoyable account of periods can be found in [7].

To some degree the phenomenon that all known volumes are periods is explained and accounted for by the existence of Newton-Okounkov bodies. Expanding earlier ideas of Okounkov [11, 12], Lazarsfeld and Mustata and independently Kaveh and Khovanskii associate a convex body to any divisor with asymptotically sufficiently many sections. The actual NObody depends on the choice of an appropiate complete flag of subvarieties; however, it is not difficult to see that the volume of a divisor D on an n-dimensional irreducible projective variety X is proportional to the n-dimensional Lebesgue measure of the corresponding NObody. Consequently, whenever the NObody of a divisor with respect to a judiciously chosen flag is an algebraic domain, the volume of D will be a period, which indeed happens in all known cases.

This gives rise to the following two questions: (1). Is it true that the volume of a line bundle on a smooth projective variety is always a period?, (2) For any integral divisor can one find a flag with respect to which the NObody is given by inequalities of polynomial with rational coefficients?

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Multigraded Fujita Approximation

Shin-Yao Jow

Let X be an irreducible variety of dimension d over an algebraically closed field $\mathbf{K}$, and let D be a (Cartier) divisor on X. When X is projective, the following limit, which measures how fast the dimension of the section space $H^0\big(X, \mathcal{O}_X(mD)\big)$ grows, is called the *volume* of D:

$$\operatorname{vol}(D) = \operatorname{vol}_X(D) = \lim_{m\to\infty} \frac{h^0\big(X, \mathcal{O}_X(mD)\big)}{m^d/d!}.$$

One says that D is *big* if $\operatorname{vol}(D) > 0$. It turns out that the volume is an interesting numerical invariant of a big divisor ([Laz04, Sec 2.2.C]), and it plays a key role in several recent works in birational geometry ([BDPP04], [Tsu00], [HM06], [Tak06]).

When D is ample, one can show that $\operatorname{vol}(D) = D^d$, the self-intersection number of D. This is no longer true for a general big divisor D, since D^d may even be negative. However, it was shown by Fujita [Fuj94] that the volume of a big divisor can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of X. This theorem, known as *Fujita approximation*, has several implications on the properties of volumes, and is also a crucial ingredient in [BDPP04] (see [Laz04, Sec 11.4] for more details).

In their recent paper [LM08], Lazarsfeld and Mustata obtained, among other things, a generalization of Fujita approximation to *graded linear series*. Recall that a graded linear series $W_\bullet = \{W_k\}$ on a (not necessarily projective) variety X associated to a divisor D consists of finite dimensional vector subspaces

$$W_k \subseteq H^0\big(X, \mathcal{O}_X(kD)\big)$$

for each $k \geq 0$, with $W_0 = \mathbf{K}$, such that

$$W_k \cdot W_\ell \subseteq W_{k+\ell}$$

for all $k, \ell \geq 0$. Here the product on the left denotes the image of $W_k \otimes W_\ell$ under the multiplication map $H^0\big(X, \mathcal{O}_X(kD)\big) \otimes H^0\big(X, \mathcal{O}_X(\ell D)\big) \longrightarrow H^0\big(X, \mathcal{O}_X((k+\ell)D)\big)$. In order to state the Fujita approximation for $W_\bullet$, they defined, for each fixed positive integer p, a graded linear series $W_\bullet^{(p)}$ which is the graded linear subseries of $W_\bullet$ generated by W_p:

$$W_m^{(p)} = \begin{cases} 0, & \text{if } p \nmid m; \\ \operatorname{Im}\big(S^k W_p \longrightarrow W_{kp}\big), & \text{if } m = kp. \end{cases}$$

Then under mild hypotheses, they showed that the volume of $W_\bullet^{(p)}$ approaches the volume of $W_\bullet$ as $p \to \infty$. See [LM08, Theorem 3.5] for the precise statement, as well as [LM08, Remark 3.4] for how this is equivalent to the original statement of Fujita when X is projective and $W_\bullet$ is the complete graded linear series associated to a big divisor D (i.e. $W_k = H^0\big(X, \mathcal{O}_X(kD)\big)$ for all $k \geq 0$).

Our goal is to generalize the Fujita approximation theorem to *multigraded linear series*. We will adopt the following notations from [LM08, Sec 4.3]: let $D_1, \ldots, D_r$ be divisors on X. For $\vec{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$ we write $\vec{m}D = \sum m_i D_i$, and we put $|\vec{m}| = \sum |m_i|$.

Definition 1. A *multigraded linear series* $W_{\vec{\bullet}}$ on X associated to the D_i's consists of finite-dimensional vector subspaces

$$W_{\vec{k}} \subseteq H^0\big(X, \mathcal{O}_X(\vec{k}D)\big)$$

for each $\vec{k} \in \mathbb{N}^r$, with $W_{\vec{0}} = \mathbf{K}$, such that

$$W_{\vec{k}} \cdot W_{\vec{m}} \subseteq W_{\vec{k}+\vec{m}},$$

where the multiplication on the left denotes the image of $W_{\vec{k}} \otimes W_{\vec{m}}$ under the natural map

$$H^0\big(X, \mathcal{O}_X(\vec{k}D)\big) \otimes H^0\big(X, \mathcal{O}_X(\vec{m}D)\big) \longrightarrow H^0\big(X, \mathcal{O}_X((\vec{k}+\vec{m})D)\big).$$

Given $\vec{a} \in \mathbb{N}^r$, denote by $W_{\vec{a},\bullet}$ the singly graded linear series associated to the divisor $\vec{a}D$ given by the subspaces $W_{k\vec{a}} \subseteq H^0\big(X, \mathcal{O}_X(k\vec{a}D)\big)$. Then put

$$\operatorname{vol}_{W_{\vec{\bullet}}}(\vec{a}) = \operatorname{vol}(W_{\vec{a},\bullet})$$

(assuming that this quantity is finite). It will also be convenient for us to consider $W_{\vec{a},\bullet}$ when $\vec{a} \in \mathbb{Q}^r_{\geq 0}$, given by

$$W_{\vec{a},k} = \begin{cases} W_{k\vec{a}}, & \text{if } k\vec{a} \in \mathbb{N}^r; \\ 0, & \text{otherwise.} \end{cases}$$

Our multigraded Fujita approximation, similar to the singly-graded version, is going to state that (under suitable conditions) the volume of $W_{\vec{\bullet}}$ can be approximated by the volume of the following finitely generated multigraded linear subseries of $W_{\vec{\bullet}}$:

Definition 2. Given a multigraded linear series $W_{\vec{\bullet}}$ and a positive integer p, define $W^{(p)}_{\vec{\bullet}}$ to be the multigraded linear subseries of $W_{\vec{\bullet}}$ generated by all $W_{\vec{m}_i}$ with $|\vec{m}_i| = p$, or concretely

$$W^{(p)}_{\vec{m}} = \begin{cases} 0, & \text{if } p \nmid |\vec{m}|; \\ \displaystyle\sum_{\substack{|\vec{m}_i| = p \\ \vec{m}_1 + \cdots + \vec{m}_k = \vec{m}}} W_{\vec{m}_1} \cdots W_{\vec{m}_k}, & \text{if } |\vec{m}| = kp. \end{cases}$$

We now state our multigraded Fujita approximation when $W_{\vec{\bullet}}$ is a complete multigraded linear series, since this is the case of most interest and allows for a more streamlined statement.

Main Theorem ([Jow11]). *Let X be an irreducible projective variety of dimension d, and let $D_1, \ldots, D_r$ be big divisors on X. Let $W_{\vec{\bullet}}$ be the complete multigraded linear series associated to the D_i's, namely*

$$W_{\vec{m}} = H^0\big(X, \mathcal{O}_X(\vec{m}D)\big)$$

for each $\vec{m} \in \mathbb{N}^r$. Then given any $\varepsilon > 0$, there exists an integer $p_0 = p_0(\varepsilon)$ having the property that if $p \geq p_0$, then

$$\left| 1 - \frac{\mathrm{vol}_{W^{(p)}_{\vec{\bullet}}}(\vec{a})}{\mathrm{vol}_{W_{\vec{\bullet}}}(\vec{a})} \right| < \varepsilon$$

for all $\vec{a} \in \mathbb{N}^r$.

The main tool in our proof is the theory of *Newton-Okounkov bodies* developed systematically in [LM08]. Given a graded linear series $W_\bullet$ on a d-dimensional variety X, its Newton-Okounkov body $\Delta(W_\bullet)$ is a convex body in $\mathbb{R}^d$ that encodes many asymptotic invariants of $W_\bullet$, the most prominent one being the volume of $W_\bullet$, which is precisely $d!$ times the Euclidean volume of $\Delta(W_\bullet)$. The idea first appeared in Okounkov's papers [Oko96] and [Oko03] in the case of complete linear series of ample line bundles on a projective variety. Later it was further developed and applied to much more general graded linear series by Lazarsfeld-Mustata [LM08], and also independently by Kaveh-Khovanskii [KK08, KK09].

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Do we have Okounkov bodies in tropical geometry?

June Huh

(joint work with Eric Katz)

We discuss several log-concavity conjectures, in particular the log-concavity conjecture of Rota, Heron and Welsh on the coefficients of the characteristic polynomial of a matroid M. Our approach to the conjecture is based on the observation that log-concave sequences correspond to homology classes of $\mathbb{P}^n \times \mathbb{P}^m$ which are representable by an irreducible subvariety [2, Theorem 20]. More precisely, if we write a homology class $\xi \in A_k(\mathbb{P}^n \times \mathbb{P}^m)$ as an integral linear combination

$$\xi = \sum_i e_i \big[\mathbb{P}^{k-i} \times \mathbb{P}^i\big],$$

then the following holds:

1. If ξ is an integer multiple of either
$$\big[\mathbb{P}^n \times \mathbb{P}^m\big], \big[\mathbb{P}^n \times \mathbb{P}^0\big], \big[\mathbb{P}^0 \times \mathbb{P}^m\big], \big[\mathbb{P}^0 \times \mathbb{P}^0\big],$$
then ξ is representable by a subvariety iff the integer is 1.
2. If otherwise, some positive integer multiple of ξ is representable by a subvariety iff e_i form a nonzero log-concave sequence of nonnegative integers with no internal zeros.

Therefore, to show that the coefficients of the reduced characteristic polynomial

$$\overline{\chi}_M(q) := \chi_M(q)/(q-1) = \sum_{i=0}^{r}(-1)^i \mu^i q^{r-i}.$$

form a log-concave sequence, it is natural to look for a subvariety of $\mathbb{P}^n \times \mathbb{P}^m$ which has the homology class corresponding to the sequence μ^i. If the matroid M is realizable over a field, then there is such a subvariety defined over the same field [3, Theorem 1.1]; writing $\mathcal{A}$ for an arrangement of $n+1$ hyperplanes on an r-dimensional projective subspace $V \subset \mathbb{P}^n$ realizing M, the closure $\widetilde{V} \subset \mathbb{P}^n \times \mathbb{P}^n$ of the graph of the Cremona transformation

$$\mathrm{Crem} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \qquad (z_0 : \cdots : z_n) \mapsto (z_0^{-1} : \cdots : z_n^{-1})$$

restricted to $V \setminus \mathcal{A}$ satisfies

$$[\widetilde{V}] = \sum_{i=0}^{r} \mu^i [\mathbb{P}^{r-i} \times \mathbb{P}^i] \in A_r(\mathbb{P}^n \times \mathbb{P}^n).$$

To justify the above equality between homology classes, we prove its tropical analogue which in fact applies to all matroids. More precisely, we show

$$\mu^k = \deg\big(\alpha^{r-k} \cdot (\mathrm{Crem}^* \alpha)^k \cdot \Delta_M\big)$$

where α is the piecewise linear function $\min(0, x_1, \ldots, x_n)$ on $\mathbb{R}^n$ and Δ_M is the Bergman fan of M studied by Ardila-Klivans [1]. Since the log-concavity of the intersection numbers is a formal consequence of the existence of Okounkov bodies of line bundles in the classical case [4, 5], one might ask: can we construct Okounkov bodies in tropical setting? The question is closely related to the problem of defining irreducible tropical varieties. Several necessary conditions were discussed during the workshop.

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Integrable systems via Okounkov bodies

Kiumars Kaveh
(joint work with Megumi Harada)

A (completely) integrable system on a symplectic manifold is a Hamiltonian system which admits a maximal number of first integrals (also called 'conservation laws'). A first integral is a function which is constant along the Hamiltonian flow; when there are a maximal number of such, then one can describe the integral curves of the Hamiltonian vector field implicitly by setting the first integrals equal to constants. In this sense an integrable system is very well-behaved. For a modern overview of this vast subject, see [2] and its extensive bibliography. The theory of integrable systems in symplectic geometry is rather dominated by specific examples (e.g. 'spinning top', 'Calogero-Moser system', 'Toda lattice'). The main contribution of this work (joint work-in-progress with M. Harada), summarized in Theorem 1 below, is a construction of an integrable system on (an open dense subset of) a variety X under only very mild hypotheses. (Details still remain to be checked as of this writing.) Our result therefore substantially contributes to the set of known examples, with a corresponding expansion of the possible applications of integrable systems theory to other research areas.

We begin with a definition. For details see e.g. [3]. Let (X, ω) be a symplectic manifold of real dimension $2n$. Let $\{f_1, f_2, \ldots, f_n\}$ be functions on X.

Definition 1. The functions $\{f_1, \ldots, f_n\}$ form an **integrable system** on X if they pairwise *Poisson-commute*, i.e. $\{f_i, f_j\} = 0$ for all i, j, and if they are *functionally independent*, i.e. their derivatives $df_1, \ldots, df_n$ are linearly independent almost everywhere on X.

We recall two examples which may be familiar to researchers in algebraic geometry.

Example. A (smooth projective) toric variety X is a symplectic manifold, equipped with the pullback of the standard Fubini-Study form on projective space. The (compact) torus action on X is in fact Hamiltonian in the sense of symplectic geometry and its moment map image is precisely the polytope corresponding to X. The torus has real dimension $n = \frac{1}{2}\dim_{\mathbb{R}}(X)$, and the n components of its moment map form an integrable system on X.

Example. Let $X = GL(n, \mathbb{C})/B$ be the flag variety of nested subspaces in $\mathbb{C}^n$. For λ a regular highest weight, consider the usual Plücker embedding $X \hookrightarrow \mathbb{P}(V_\lambda)$ where V_λ denotes the irreducible representation of $GL(n, \mathbb{C})$ with highest weight λ. Equip X with the Kostant-Kirillov-Souriau symplectic form coming from its identification with the coadjoint orbit $\mathcal{O}_\lambda$ of $U(n, \mathbb{C})$ which meets the positive Weyl chamber at precisely λ. Then Guillemin-Sternberg build an integrable system on X by viewing the coadjoint orbit $\mathcal{O}_\lambda$ as a subset of hermitian $n \times n$ matrices and taking eigenvalues (listed in increasing order) of the upper-left $k \times k$ submatrices for all $1 \leq k \leq n - 1$. This is the Guillemin-Sternberg/Gel'fand-Cetlin integrable system on the flag variety. (See [4] for details.)

More generally, suppose now X is a projective variety and $\mathcal{L}$ a very ample line bundle on X. Let $n = \dim_{\mathbb{C}}(X)$. Pick ν a valuation (corresponding to some choice of flag of subvarieties) and let $\Delta(X, \mathcal{L}, \nu)$ denote the corresponding Okounkov body. Denote by $S := S(X, \mathcal{L}, \nu)$ the value semigroup in $\mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$.

Theorem 1. *In the setting above, suppose that S is a finitely generated semigroup. (Recall that this implies that $\Delta(X, \mathcal{L}, \nu)$ is a rational polytope.) Then there exist $f_1, \ldots, f_n$ functions on X such that*

- *the f_i are continuous on X and differentiable on an open dense subset U of X,*
- *the f_i pairwise Poisson-commute on U,*
- *the image of X under $\mu := (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is precisely the Okounkov body $\Delta(X, \mathcal{L}, \nu)$.*

Remark. Among other things, our theorem addresses a question posed to us by Julius Ross and by Steve Zelditch: does there exist, in general, a 'reasonable' map from a variety X to its Okounkov body? At least under the technical assumption that the value semigroup S is finitely generated, our theorem suggests that the answer is yes.

We now we briefly sketch the idea of our proof. The essential ingredient is the toric degeneration from X to the toric variety X_0 corresponding to $\Delta(X, \mathcal{L}, \nu)$, constructed by Dave Anderson [1]. Let $f : \mathcal{X} \to \mathbb{C}$ denote the flat family with special fiber $f^{-1}(0) \cong X_0$ and $f^{-1}(t) = X_t \cong X$ for $t \neq 0$. Since toric varieties are integrable systems (see example above), the idea is to "pull back" the integrable system on X_0 to one on X. To accomplish this we use the so-called 'gradient Hamiltonian vector field' (first defined by Ruan and also used by Nishinou-Nohara-Ueno, cf. [6, 5]) on $\mathcal{X}$, where we think of $\mathcal{X}$ as a symplectic space by embedding it into an appropriate product of projective spaces. The main technicality which must be overcome to make this sketch rigorous is to appropriately deal with the singular points of $\mathcal{X}$ such that the f_i are continuous on all of $\mathcal{X}$ (not just at smooth points). It turns out that, in order to deal with this issue, we need a subtle generalization of the famous Lojasiewicz inequality.

Note added in proof. As we prepared this abstract and our manuscript, it came to our attention that Allen Knutson had more or less precisely predicted our result in a MathOverflow discussion thread. (His post is dated January 2010, but he apparently had the ideas long before.)

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Polyhedral effective cones and polyhedral Okounkov bodies

Dave Anderson

Let X be a normal projective variety of dimension d, let $\mathcal{L}$ be a big line bundle on X, and let $Y_\bullet$ be an admissible flag of subvarieties. A key problem is to find methods for computing the Okounkov body $\Delta = \Delta_{Y_\bullet}(\mathcal{L})$. Still more refined information is contained in the semigroup $\Gamma = \Gamma_{Y_\bullet}(\mathcal{L})$ used to define the Okounkov body. When Γ is finitely generated, it follows immediately that Δ is a rational polytope, and by the results of [1] there is a flat degeneration to a corresponding toric variety.

With this as motivation, we seek some geometric criteria for finite generation of the semigroup Γ. A first approach comes from [1], using the notion of a *maximal divisor*:

Definition. Let $Y \subset X$ be a subvariety of codimension 1, and let

$$D = aY + b_1B_1 \cdots + b_kB_k + e_1E_1 + \cdots + e_\ell E_\ell$$

be an effective divisor in $H^0(X, \mathcal{L})$, written in terms of its irreducible components. We say D is *maximal with respect to* Y if each b_iB_i is contained in the base scheme of $\mathcal{L}$, and all E_i's lie on a facet of the pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ not containing Y.

The significance of this notion is that orders of vanishing along Y are bounded — not only for sections of $\mathcal{L}$, but also for sections of all tensor powers $\mathcal{L}^{\otimes m}$. It follows that the projection of Γ onto the first two coordinates is a finitely generated semigroup, and one obtains an inductive criterion for finite generation of Γ this way [1, Corollary 4.11].

The inductive condition described above is somewhat complicated, and it would be desirable to have a criterion that is easier to check. We propose a second method, and illustrate it with the example of a three-dimensional Bott-Samelson variety.

Let $X = X(\alpha, \beta, \alpha)$ be the Bott-Samelson variety

$$(P_\alpha \times P_\beta \times P_\alpha)/B^3,$$

where P_α, P_β are the standard parabolic subgroups in GL_3 containing the Borel group B of upper triangular matrices, and B^3 acts (on the right) by $(p_1, p_2, p_3) \cdot (b_1, b_2, b_3) = (p_1b_1, b_1^{-1}p_2b_2, b_2^{-1}p_3b_3)$. The Picard group of X is generated by the classes of the three divisors

$$X_i = \{[p_1, p_2, p_3] \,|\, p_i = e\}$$

(where $e \in GL_3$ is the identity element). Thus every line bundle on X can be written as $\mathcal{L} = \mathcal{O}(n_1X_1 + n_2X_2 + n_3X_3)$.

There is a natural choice of flag $Y_\bullet$ on X:

$$X \supseteq \{[\dot{s}_\alpha, *, *]\} \supset \{[\dot{s}_\alpha, \dot{s}_\beta, *]\} \supset \{[\dot{s}_\alpha, \dot{s}_\beta, \dot{s}_\alpha]\},$$

where $\dot{s}_{\in}P$ is a representative of a simple reflection. We show that the *global Okounkov cone* of X with respect to this flag is defined by the inequalities

$$\begin{aligned} n_1, n_2, n_3 &\geq 0, \\ t_1, t_2, t_3 &\geq 0, \\ n_3 - t_3 &\geq 0, \\ n_2 - t_2 - t_3 &\geq 0, \\ 2n_2 - n_3 - t_2 - t_3 &\geq 0, \\ n_1 - t_1 &\geq 0, \\ n_1 + n_2 - t_1 - t_2 - t_3 &\geq 0, \\ n_1 + n_3 - t_1 - t_2 - t_3 &\geq 0 \end{aligned}$$

inside $N^1(X)_\mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$. The inequalities are obtained by examining the pseudoeffective cones of each divisor X_i; these cones are known to be polyhedral, and this plays a role in the proof. We suggest that these methods should generalize beyond Bott-Samelson varieties to compute Okounkov bodies for varieties with recursive structure, and where one has a good understanding of pseudoeffective cones of divisors.

Setting $(n_1, n_2, n_3) = (1, 2, 1)$ in the above example, one recovers the Okounkov body of the projectivized tangent bundle of $\mathbb{P}^2$, an example computed by J. L. Gonzalez.

Using a different flag and different methods, K. Kaveh identifies the Okounkov bodies of line bundles on certain Bott-Samelson varieties with the *string polytopes* coming from the theory of crystal bases [2]. A promising future direction is to use the approach outlined here to arrive at a new geometric description of crystal bases.

References

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Okounkov bodies of complexity-one *T*-varieties

Lars Petersen

In the first half of this talk I gave a brief introduction to the language of polyhedral divisors, divisorial fans and marked fansy divisors. Generalizing the well known correspondence between polyhedral fans and normal toric varieties, these notions not only provide a handy description of T-varieties, i.e. normal varieties with an effective torus action, but also facilitate, for example, the investigation of orbit structures and invariant Cartier divisors together with their global sections. For more details, we refer the reader to [1].

In the second half of this talk, I presented some recent results on the computation of Okounkov bodies of projective complexity-one T-varieties formulated in terms of the notions mentioned above. Namely, fixing a T-invariant flag and a T-invariant big divisor D on the variety X, the associated Okounkov body is a rational polytope which can be computed explicitly in terms of the essentially combinatorial data describing D. Moreover, assuming X to be rational, it can be shown that the global Okounkov body is a rational polyhedral cone (see [3]) which generalizes a result obtained by González for projectivized toric vector bundles of rank two over smooth projective toric varieties, cf. [4].

Let T be a d-dimensional algebraic torus and denote by M and N its mutually dual lattices of characters and one-parameter subgroups. A *marked fansy divisor* over the smooth projective curve C consists of a triple $(\mathcal{S}, \Sigma, \mathcal{M})$ where

$$\mathcal{S} = \sum_{P \in C} \mathcal{S}_P \otimes P$$

is a formal sum in which the $\mathcal{S}_P$ are polyhedral subdivisions of $N_{\mathbb{Q}}$ all with the same complete polyhedral *(tail)*fan Σ. In addition, $\mathcal{M} \subset \Sigma$ denotes the set of *marked* cones. For the remaining *coherence conditions* regarding $\mathcal{M}$ and those imposed upon the specific arrangement of the subdivisions the reader is referred to [2]. Now, given a marked fansy divisor $\mathcal{S}$ over C one can associate to it a complete complexity-one T-variety $\mathrm{TV}(\mathcal{S}, \Sigma, \mathcal{M})$. Conversely, every complete T-variety X of complexity one can be described in these terms.

Moreover, every marked fansy divisor $(\mathcal{S}, \Sigma, \mathcal{M})$ over C comes with a natural partner $\mathrm{TV}(\mathcal{S}, \Sigma, \emptyset)$ together with a proper T-equivariant morphism

$$r : \mathrm{TV}(\mathcal{S}, \Sigma, \emptyset) \to \mathrm{TV}(\mathcal{S}, \Sigma, \mathcal{M})$$

and a quotient map $\pi : \mathrm{TV}(\mathcal{S}, \Sigma, \emptyset) \to C$ with general fiber equal to the toric variety $\mathrm{TV}(\Sigma)$. In this setting, $\mathcal{M}$ encodes the set of T-orbits in $\mathrm{TV}(\mathcal{S}, \Sigma, \emptyset)$ which are identified via the map r. Together with the quotient map π these data completely determine the orbit structure of $\mathrm{TV}(\mathcal{S}, \Sigma, \mathcal{M})$. In particular, one can immediately read off the set of invariant prime divisors.

On the other hand, invariant Cartier divisors on $\mathrm{TV}(\mathcal{S}, \Sigma, \mathcal{M})$ correspond to *divisorial support functions* $h = (h_P)_{P \in C}$ on $(\mathcal{S}, \Sigma, \mathcal{M})$. This is a collection of continuous piecewise affine linear functions $h_P : |\mathcal{S}_P| \to \mathbb{Q}$ with the following properties:

- h_P has integral slope and integral translation on every polyhedron in $\mathcal{S}_P$.
- the linear part $\underline{h}(v) := \lim_{k \to \infty} h_P(kv)/k$ for $v \in N_{\mathbb{Q}}$ is independent of P, i.e. $\underline{h} : \Sigma \to \mathbb{Q}$ defines a piecewise linear function on Σ.
- $h_P \neq \underline{h}$ for only finitely many $P \in C$.
- $(h_P|_{\mathcal{S}_P(\sigma)})_P$ is principal for every maximal cone $\sigma \in \mathcal{M} \cap \Sigma(d)$, see [1, Section 7] for more details.

We denote the associated T-invariant Cartier divisor by D_h. An explicit description of its global sections is then given as follows (see also [1, Section 9]):

$$\Gamma(\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M}),D_h)=\bigoplus_{u\in M}\Gamma(\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M}),D_h)_u=\bigoplus_{u\in\Box_h\cap M}\Gamma(C,h^*(u)),$$

where $\Box_h:=\{u\in M_{\mathbb{Q}}\mid\langle u,v\rangle\geq\underline{h}(v)\ \forall v\in N_{\mathbb{Q}}\}$ and

$$h^*(u):=\sum_{P\in C}h^*_P(u)P:=\sum_{P\in C}\min_{v\in\mathcal{S}_P(0)}(u-h_p)P,$$

i.e. we can consider h^* as a map from $\Box_h\to\mathrm{CaDiv}_{\mathbb{Q}}\,C$.

For the construction of a T-invariant flag $Y_\bullet:=Y_0\supset Y_1\supset\cdots\supset Y_{d+1}$ on $\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M})$, it is sufficient to perform a detailed analysis of its orbit structure. Fixing a smooth T-invariant fixed point Y_{d+1} in $\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M})$, one can then construct essentially two different types (*general* and *toric*), depending upon the local structure of $\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M})$ around Y_{d+1}.

Theorem. Let $\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M})$ be a projective T-variety of complexity one together with a fixed general flag $Y_\bullet$ and a big T-invariant Cartier divisor D_h. Then we have that

$$\Delta_{Y_\bullet}(D_h)=\left\{(x,w)\in\mathbb{R}\times\mathbb{R}^d\;\middle|\;w\in\Delta_{Y_{\geq1}}(D_{\underline{h}}),\ 0\leq x\leq\deg h^*(w)\right\}$$

up to a shift of $\Delta_{Y_{\geq1}}(D_{\underline{h}})$ where the latter denotes the Okounkov body of the induced Cartier divisor $D_{\underline{h}}$ on the general fiber $\mathrm{TV}(\Sigma)$ with respect to an induced flag $Y_{\geq1}$. In particular, $\Delta_{Y_\bullet}(D_h)$ is a rational polytope.

Results of similar type also hold for a fixed toric flag and can be derived almost entirely by only using properties of the function h^*.

Finally we consider the global Okounkov body of a rational projective complexity-one T-variety $\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M})$. A key ingredient used in its investigation is an explicit representation of the divisor class group of $\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M})$ as the cokernel of a linear map of lattices associated to $(\mathcal{S},\Sigma,\mathcal{M})$, cf. [1, Section 7]. This representation also provides a description of the pseudoeffective cone of $\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M})$ which is rational polyhedral. The following result can then be proved by closely analyzing the properties of the function $h\mapsto h^*$ and the fact that the global Okounkov body can be represented as the convex hull of the graph of the latter function over a rational polyhedral cone of one dimension less.

Theorem. Let $\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M})$ be a rational projective T-variety of complexity one together with a fixed general or toric T-invariant flag $Y_\bullet$. Then the global Okounkov body $\Delta_{Y_\bullet}(\mathrm{TV}(\mathcal{S},\Sigma,\mathcal{M}))$ is a rational polyhedral cone.

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