

Homogeneous components in the moduli space of sheaves and Virasoro characters

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ABSTRACT

The moduli space $\mathcal{M}(r, n)$ of framed torsion free sheaves on the projective plane with rank r and second Chern class equal to n has the natural action of the $(r + 2)$ -dimensional torus. In this paper, we look at the fixed point set of different one-dimensional subtori in this torus. We prove that in the homogeneous case the generating series of the numbers of the irreducible components has a beautiful decomposition into an infinite product. In the case of odd r , these infinite products coincide with certain Virasoro characters. We also propose a conjecture in a general quasihomogeneous case.

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1. Introduction

Let $\mathcal{M}(r, n)$ be the moduli space of framed torsion free sheaves on \mathbb{P}^2 with rank r and second Chern class c_2 equal to n . It is a smooth irreducible quasi-projective variety of dimension $2rn$. In the case $r = 1$, it is isomorphic to the Hilbert scheme of n points on the plane. The moduli space $\mathcal{M}(r, n)$ has a simple quiver description and we recall it in Section 2.1. In principle, one can use this description as a definition of $\mathcal{M}(r, n)$. We refer the reader to [1,2] for a more detailed discussion of the moduli space $\mathcal{M}(r, n)$.

There is a natural action of the $(r + 2)$ -dimensional torus $T = (\mathbb{C}^*)^{r+2}$ on $\mathcal{M}(r, n)$. It is induced by the $(\mathbb{C}^*)^2$ -action on \mathbb{P}^2 and by the action of $(\mathbb{C}^*)^r$ on the framing. Consider a vector

$$\vec{w} = (w_1, w_2, \dots, w_r) \in \mathbb{Z}^r$$

and integers $\alpha, \beta \geq 1$, such that $\gcd(\alpha, \beta) = 1$. Let $T_{\alpha, \beta}^{\vec{w}}$ be the one-dimensional subtorus of T defined by

$$T_{\alpha, \beta}^{\vec{w}} = \{(t^\alpha, t^\beta, t^{w_1}, t^{w_2}, \dots, t^{w_r}) \in T \mid t \in \mathbb{C}^*\}.$$

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For $0 \leq m \leq r$, let $\vec{w}(m)$ be the vector $(\underbrace{1, \dots, 1}_m, 0, \dots, 0) \in \mathbb{Z}^r$. We denote by $h_0(X)$ the number of connected components of a manifold X . We will use the classical q -series notations:

$$(a)_n = (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}),$$

$$(a_1, a_2, \dots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty.$$

Now we can state our main result.

Theorem 1.1. For any $0 \leq m \leq r$ we have

$$\sum_{n \geq 0} h_0 \left(\mathcal{M}(r, n)^{T_{1,1}^{\vec{w}(m)}} \right) q^n = \frac{(-q)_\infty}{(q)_\infty} (q^{m+1}, q^{r-m+1}, q^{r+2}, q^{r+2})_\infty. \tag{1}$$

In the case of odd r the right-hand side of (1) up to the factor $(-q)_\infty$ coincides with a certain Virasoro character. We discuss it in Section 1.2. In Section 1.4 we formulate a conjecture in the case of arbitrary α, β . We also give a conjectural formula for the two-variable generating function of the Betti numbers of $\mathcal{M}(2, n)^{T_{1,1}^{\vec{w}(m)}}$ for $m = 0, 1$.

A connection between the moduli space $\mathcal{M}(r, n)$ and the Virasoro characters (or more generally W_n -characters) was also found in [3]. It appears in a different context and we do not know how to relate it to our work. However in Section 1.3 we review briefly the paper [3], because we use the characters defined there in our Conjecture 1.2.

Our proof of Theorem 1.1 is combinatorial but we can propose another way to prove it using the representation theory of the toroidal Yangian. These ideas are under development and we briefly discuss them in Section 1.6. We are going to write the details in the forthcoming paper.

This work is a continuation of [4,5]. In [4] the first author studied cohomology groups of $\mathcal{M}(1, n)^{T_{\alpha,\beta}}$. In [5] the first author computed Betti numbers of $\mathcal{M}(r, n)^{(\mathbb{C}^*)^2}$ and showed that they coincide with certain coefficients in a generalization of the MacMahon’s formula.

1.1. Moduli space of sheaves on \mathbb{P}^2

The moduli space $\mathcal{M}(r, n)$ is defined by

$$\mathcal{M}(r, n) = \left\{ (E, \Phi) \left| \begin{array}{l} E : \text{a torsion free sheaf on } \mathbb{P}^2 \\ \text{rank}(E) = r, c_2(E) = n \\ \Phi : E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r} : \text{framing at infinity} \end{array} \right. \right\} / \text{isomorphism},$$

where $l_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2\} \subset \mathbb{P}^2$ is the line at infinity.

Let T be the maximal torus of $GL_r(\mathbb{C})$ consisting of diagonal matrices and let $T = (\mathbb{C}^*)^2 \times \tilde{T}$. The action of T on $\mathcal{M}(r, n)$ is defined as follows. For $(t_1, t_2) \in (\mathbb{C}^*)^2$ let F_{t_1, t_2} be the automorphism of \mathbb{P}^2 defined by

$$F_{t_1, t_2}([z_0 : z_1 : z_2]) = [z_0 : t_1 z_1 : t_2 z_2].$$

For $\text{diag}(e_1, \dots, e_r) \in \tilde{T}$ let G_{e_1, \dots, e_r} denote the isomorphism of $\mathcal{O}_{l_\infty}^{\oplus r}$ given by

$$\mathcal{O}_{l_\infty}^{\oplus r} \ni (s_1, \dots, s_r) \mapsto (e_1 s_1, \dots, e_r s_r).$$

Then for $(E, \Phi) \in \mathcal{M}(r, n)$ we define

$$(t_1, t_2, e_1, \dots, e_r) \cdot (E, \Phi) = ((F_{t_1, t_2}^{-1})^* E, \Phi'),$$

where Φ' is the composition of the homomorphisms

$$(F_{t_1, t_2}^{-1})^* E|_{l_\infty} \xrightarrow{(F_{t_1, t_2}^{-1})^* \Phi} (F_{t_1, t_2}^{-1})^* \mathcal{O}_{l_\infty}^{\oplus r} = \mathcal{O}_{l_\infty}^{\oplus r} \xrightarrow{G_{e_1, \dots, e_r}} \mathcal{O}_{l_\infty}^{\oplus r}.$$

1.2. Virasoro characters

We recall several results from the representation theory of the Virasoro algebra. There are modules $M(p, p')_2$ that are called the Virasoro minimal models and labelled by coprime integers p and p' for which $1 < p < p'$. They contain irreducible modules labelled by r and s with $1 \leq r < p$ and $1 \leq s < p'$. In [6,7], the characters of these modules were computed to be $\widehat{\chi}_{r,s}^{p,p'} = q^{\Delta_{r,s}^{p,p'}} \overline{\chi}_{r,s}^{p,p'}$, where $\overline{\chi}_{r,s}^{p,p'}$ is called the normalized character and is given by:

$$\overline{\chi}_{r,s}^{p,p'} = \frac{1}{(q)_\infty} \sum_{\lambda=-\infty}^{\infty} \left(q^{\lambda^2 p p' + \lambda(p'r - ps)} - q^{(\lambda p + r)(\lambda p' + s)} \right),$$

and the number $\Delta_{r,s}^{p,p'}$ is called the conformal dimension and is given by:

$$\Delta_{r,s}^{p,p'} = \frac{(p'r - ps)^2 - (p' - p)^2}{4pp'}.$$

Now let us return to **Theorem 1.1**. The right-hand side of (1) is known to be equal to $(-q)_\infty \bar{\chi}_{1,m+1}^{2,r+2}$ when r is odd (see e.g. [8]). Thus we have the following equation

$$\sum_{n \geq 0} h_0 \left(\mathcal{M}(2k + 1, n)^{T_{1,1}^{\vec{w}(m)}} \right) q^n = (-q)_\infty \bar{\chi}_{1,m+1}^{2,2k+3}.$$

1.3. Quantum continuous \mathfrak{gl}_∞

In [3] the authors study representations of the associative algebra which they denote by \mathcal{E} and call quantum continuous \mathfrak{gl}_∞ . This algebra depends on parameters q_1 and q_2 . They construct representations $\mathcal{F}_1(u_1) \otimes \dots \otimes \mathcal{F}_s(u_s)$ that depend on parameters $q_1, q_2, u_1, \dots, u_s$. In [3] these representations are constructed purely algebraically but they have the following geometrical meaning. The space $\mathcal{F}_1(u_1) \otimes \dots \otimes \mathcal{F}_s(u_s)$ can be identified with the equivariant K -theory of $\coprod_{n \geq 0} \mathcal{M}(s, n)$ and the algebra \mathcal{E} acts there through a slight generalization of the correspondences from [9].

The authors of [3] impose the following conditions on the parameters $q_1, q_2, u_1, \dots, u_s$:

$$u_i = u_{i+1} q_1^{a_i+1} q_2^{b_i+1}, \quad i = 1, \dots, s-1, \quad q_1^p q_2^{p'} = 1,$$

where $\vec{a} = (a_1, \dots, a_{s-1}) \in \mathbb{Z}_{\geq 0}^{s-1}$ and $\vec{b} = (b_1, \dots, b_{s-1}) \in \mathbb{Z}_{\geq 0}^{s-1}$ are arbitrary vectors and $p, p' \in \mathbb{Z}_{\geq 0}$ are integers such that $p \neq p'$ and

$$p - 1 - \sum_{i=1}^{s-1} (a_i + 1) \geq 0, \quad p' - 1 - \sum_{i=1}^{s-1} (b_i + 1) \geq 0.$$

They construct a new \mathcal{E} -module as a subquotient of $\mathcal{F}_1(u_1) \otimes \dots \otimes \mathcal{F}_s(u_s)$. This module is denoted by $\mathcal{M}_{a,b}^{p,p'}$ and its character is denoted by $\chi_{a,b}^{p,p'}$. These characters are connected with the Virasoro characters in the following way. In [3] it is proved that if $p' > p > 1$, $\text{gcd}(p', p) = 1$ and $s = 2$, then

$$\chi_{a_1, b_1}^{p,p'} = \frac{1}{(q)_\infty} \bar{\chi}_{a_1+1, b_1+1}^{p,p'}.$$

Let us make a remark about the symmetries of the character $\chi_{a,b}^{p,p'}$. For a vector $\vec{c} \in \mathbb{Z}^{s-1}$ and an integer m we define the vectors $\tau(\vec{c}, m) = (\tau(\vec{c}, m)_1, \dots, \tau(\vec{c}, m)_{s-1})$ and $\sigma(\vec{c}, m) = (\sigma(\vec{c}, m)_1, \dots, \sigma(\vec{c}, m)_{s-1})$ as follows:

$$\tau(\vec{c}, m)_i = c_{i+1}, \quad \sigma(\vec{c}, m)_i = c_{s+1-i},$$

where $c_s = m - s - \sum_{i=1}^{s-1} c_i$. Then we have (see [3])

$$\chi_{a,b}^{p,p'} = \chi_{\tau(\vec{a}, p), \tau(\vec{b}, p')}^{p,p'} = \chi_{\sigma(\vec{a}, p), \sigma(\vec{b}, p')}^{p,p'}. \tag{2}$$

1.4. Conjecture 1: arbitrary α, β

Consider a vector $\vec{w} \in \mathbb{Z}^r$ and numbers $\alpha, \beta \geq 1$ such that $0 \leq w_i < \alpha + \beta$ and $\text{gcd}(\alpha, \beta) = 1$. Let a_i be the number of j such that $w_j = i$, i.e. $a_i = \#\{j | w_j = i\}$. The numbers α and $\alpha + \beta$ are coprime, therefore there exists the unique number α' such that $0 \leq \alpha' < \alpha + \beta$ and $\alpha' \alpha = 1 \pmod{\alpha + \beta}$. We define the vector $\vec{a}' = (a'_0, a'_1, \dots, a'_{\alpha+\beta-1})$ as follows

$$a'_i = a_{\alpha' i \pmod{\alpha + \beta}}.$$

We define the vector $\vec{a}'' \in \mathbb{Z}^{\alpha+\beta-1}$ as the vector \vec{a}' without the last coordinate. Let $\vec{0} = (0, 0, \dots, 0) \in \mathbb{Z}^{\alpha+\beta-1}$.

Conjecture 1.2.

$$\sum_{n \geq 0} h_0 \left(\mathcal{M}(r, n)^{T_{\vec{0}, \vec{a}''}^{\vec{w}}} \right) q^n = (q^{\alpha+\beta}; q^{\alpha+\beta})_\infty \chi_{\vec{0}, \vec{a}'}^{\alpha+\beta, \alpha+\beta+r}.$$

Remark 1.3. We used the multiplication by $\alpha^{-1} \pmod{\alpha + \beta}$ in the definition of \vec{a}' . If one uses the multiplication by $\beta^{-1} \pmod{\alpha + \beta}$, then the character χ will be the same. It follows from (2) and the fact that $\beta^{-1} = -\alpha^{-1} \pmod{\alpha + \beta}$.

1.5. Conjecture 2: Betti numbers

We denote by $P_q(X)$ the Poincare polynomial $\sum_{i \geq 0} \dim H_i(X) q^{\frac{i}{2}}$ of a manifold X .

In [4] we proposed the following conjecture

$$\sum_{n \geq 0} P_q(\mathcal{M}(1, n)^{T_{\alpha, \beta}}) t^n = \prod_{\substack{i \geq 1 \\ (\alpha + \beta) | i}} \frac{1}{1 - t^i} \prod_{i \geq 1} \frac{1}{1 - qt^{(\alpha + \beta)i}}. \tag{3}$$

We conjecture an analogue of (3) for the case $r = 2$ and $\alpha = \beta = 1$.

Conjecture 1.4.

$$\sum_{n \geq 0} P_q(\mathcal{M}(2, n)^{T_{1,1}^{(0,0)}}) t^n = \prod_{4|i} \frac{1}{(1 - t^i)(1 - qt^i)} \prod_{i \geq 1} \frac{1}{(1 - qt^{4i})(1 - q^2t^{4i})},$$

$$\sum_{n \geq 0} P_q(\mathcal{M}(2, n)^{T_{1,1}^{(0,1)}}) t^n = \prod_{n \geq 1} \frac{(1 - t^{4n-2})}{(1 - t^{2n-1})^2(1 - qt^{4n-2})^2(1 - q^2t^{4n-2})(1 - qt^{4n})^2}.$$

1.6. The toroidal Yangian

For simplicity in this section we consider the case $\vec{w} = \vec{0}$.

At the moment we cannot relate the varieties $\mathcal{M}(r, n)^{T_{\alpha, \beta}}$ to \widehat{gl}_1 -toroidal algebra. However we can see a relation with the toroidal algebra $\widehat{gl}_{\alpha + \beta}$. Let $\Gamma_{\alpha + \beta}$ be the subgroup of $\mathbb{C}^* \times \mathbb{C}^*$ defined by

$$\Gamma_{\alpha + \beta} = \left\{ (\zeta^s, \zeta^{-s}) \mid s = 0, 1, \dots, \alpha + \beta - 1, \zeta = e^{\frac{2\pi i}{\alpha + \beta}} \right\}.$$

In [10] it is proved that the toroidal Yangian acts on the equivariant homology groups $\bigoplus_n H_*^{\mathbb{C}^* \times \mathbb{C}^*}(\mathcal{M}(r, n)^{\Gamma_{\alpha + \beta}})$. We want to consider the localized homology groups $\bigoplus_n H_*^{T_{\alpha, \beta}}(\mathcal{M}(r, n)^{\Gamma_{\alpha, \beta}})$ and a filtration in them given by a dimension of a support. This filtration is increasing and the lowest level of it has a basis enumerated by the irreducible components of $\mathcal{M}(r, n)^{T_{\alpha, \beta}}$. The toroidal Yangian also has a filtration such that the lowest level is isomorphic to $\widehat{sl}_{\alpha + \beta}$. The filtration on the Yangian induces a filtration on the representation and we suppose that it is exactly the filtration given by a dimension of a support. Then the lowest level of this filtration is the irreducible integrable representation of $\widehat{sl}_{\alpha + \beta}$ of level r . Suppose $\alpha = \beta = 1$. It is well-known that the characters of the integrable \widehat{sl}_2 -modules of level r in the principal grading coincide (up to the factor $(-q)_\infty$) with the characters of the irreducible representations of the Virasoro algebra that come from $(2, r + 2)$ -models. Thus we get (1). Conjecture 1.2 corresponds to the case of general α, β . There is also a possible way to apply the representation theory of the toroidal Yangian to the proof of the other conjectures in this paper. We hope to develop these ideas in a forthcoming paper.

1.7. Organization of the paper

In Section 2 we recall the quiver description of the moduli space $\mathcal{M}(r, n)$ and find a sufficient condition for the varieties $\mathcal{M}(r, n)^{T_{\alpha, \beta}}$ to be compact. Compactness of the varieties $\mathcal{M}(r, n)^{T_{1,1}^{(m)}}$ is important in the proof of Theorem 1.1. In Section 3 we construct a cellular decomposition of $\mathcal{M}(r, n)^{T_{1,1}^{(m)}}$ and obtain a combinatorial formula for the number of the irreducible components. In Section 4 we analyse this combinatorial formula and give a proof of Theorem 1.1.

2. Moduli space of sheaves on \mathbb{P}^2

Here we recall the quiver description of the moduli space $\mathcal{M}(r, n)$ and find a sufficient condition for the varieties $\mathcal{M}(r, n)^{T_{\alpha, \beta}}$ to be compact.

2.1. Quiver description of $\mathcal{M}(r, n)$

The variety $\mathcal{M}(r, n)$ has the following quiver description (see e.g. [1]).

$$\mathcal{M}(r, n) \cong \left\{ (B_1, B_2, i, j) \mid \begin{array}{l} (1) [B_1, B_2] + ij = 0 \\ (2) \text{ (stability) There is no subspace } \\ S \subsetneq \mathbb{C}^n \text{ such that } B_\alpha(S) \subset S \ (\alpha = 1, 2) \\ \text{and } \text{Im}(i) \subset S \end{array} \right\} / GL_n(\mathbb{C}),$$

where $B_1, B_2 \in \text{End}(\mathbb{C}^n)$, $i \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ and $j \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ with the action of $GL_n(\mathbb{C})$ given by

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$$

for $g \in GL_n(\mathbb{C})$.

In terms of Section 2.1 the T -action on $\mathcal{M}(r, n)$ is given by (see e.g. [2])

$$(t_1, t_2, e_1, e_2, \dots, e_r) \cdot [(B_1, B_2, i, j)] = [(t_1 B_1, t_2 B_2, i e^{-1}, t_1 t_2 e_j)].$$

2.2. Compactness of $\mathcal{M}(r, n)^{T_{\alpha, \beta}^{\bar{w}}}$

Proposition 2.1. *Suppose that $\max_{1 \leq i \leq r} w_i - \min_{1 \leq i \leq r} w_i < \alpha + \beta$, then for any n the variety $\mathcal{M}(r, n)^{T_{\alpha, \beta}^{\bar{w}}}$ is compact.*

Proof. By definition, a point $[(B_1, B_2, i, j)] \in \mathcal{M}(r, n)$ is fixed under the action of $T_{\alpha, \beta}^{\bar{w}}$ if and only if there exists a homomorphism $\lambda : \mathbb{C}^* \rightarrow GL_n(\mathbb{C})$ satisfying the following conditions:

$$\begin{aligned} t^\alpha B_1 &= \lambda(t)^{-1} B_1 \lambda(t), \\ t^\beta B_2 &= \lambda(t)^{-1} B_2 \lambda(t), \\ i \circ \text{diag}(t^{w_1}, t^{w_2}, \dots, t^{w_r})^{-1} &= \lambda(t)^{-1} i, \\ t^{\alpha+\beta} \text{diag}(t^{w_1}, t^{w_2}, \dots, t^{w_r}) \circ j &= j \lambda(t). \end{aligned} \tag{4}$$

Suppose that $[(B_1, B_2, i, j)]$ is a fixed point. Then we have the weight decomposition of \mathbb{C}^n with respect to $\lambda(t)$, i.e. $\mathbb{C}^n = \bigoplus_{k \in \mathbb{Z}} V_k$, where $V_k = \{v \in \mathbb{C}^n \mid \lambda(t) \cdot v = t^k v\}$. We also have the weight decomposition of \mathbb{C}^r , i.e. $\mathbb{C}^r = \bigoplus_{k \in \mathbb{Z}} W_k$, where $W_k = \{v \in \mathbb{C}^r \mid \text{diag}(t^{w_1}, \dots, t^{w_r}) \cdot v = t^k v\}$. From the conditions (4) it follows that the only components of B_1, B_2, i and j that might survive are

$$B_1 : V_k \rightarrow V_{k-\alpha}, \tag{5}$$

$$B_2 : V_k \rightarrow V_{k-\beta}, \tag{6}$$

$$i : W_k \rightarrow V_k,$$

$$j : V_k \rightarrow W_{k-\alpha-\beta}. \tag{7}$$

From the stability condition it follows that

$$V_k = 0, \quad \text{if } k > \max_{1 \leq i \leq r} w_i.$$

Then from the condition $\max_{1 \leq i \leq r} w_i - \min_{1 \leq i \leq r} w_i < \alpha + \beta$ and (7) it follows that $j = 0$.

Consider the variety $\mathcal{M}_0(r, n)$ from [2]. It is defined as an affine algebro-geometric quotient

$$\mathcal{M}_0(r, n) = \{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = 0\} // GL_n(\mathbb{C}).$$

It can be viewed as the set of closed orbits in $\{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = 0\}$. There is a morphism $\pi : \mathcal{M}(r, n) \rightarrow \mathcal{M}_0(r, n)$. It maps a point $[(B_1, B_2, i, j)] \in \mathcal{M}(r, n)$ to the unique closed orbit that is contained in the closure of the orbit of (B_1, B_2, i, j) in $\{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = 0\}$. The variety $\mathcal{M}_0(r, n)$ is affine and the morphism π is projective (see e.g. [2]).

By [11] the coordinate ring of $\mathcal{M}_0(r, n)$ is generated by the following two types of functions:

- (a) $\text{tr}(B_{a_N} B_{a_{N-1}} \cdots B_{a_1} : \mathbb{C}^n \rightarrow \mathbb{C}^n)$, where $a_i = 1$ or 2 .
- (b) $\chi(j B_{a_N} B_{a_{N-1}} \cdots B_{a_1} i)$, where $a_i = 1$ or 2 , and χ is a linear form on $\text{End}(\mathbb{C}^r)$.

From (5) and (6) it follows that the equation

$$\pi^* f|_{\mathcal{M}(r, n)^{T_{\alpha, \beta}^{\bar{w}}}} = 0 \tag{8}$$

holds for any function f of type (a). We observed that for any point $[(B_1, B_2, i, j)] \in \mathcal{M}(r, n)^{T_{\alpha, \beta}^{\bar{w}}}$ we have $j = 0$. Hence, (8) holds for any function f of type (b).

We see that the image of $\mathcal{M}(r, n)^{T_{\alpha, \beta}^{\bar{w}}}$ under the map π is a point. Therefore the variety $\mathcal{M}(r, n)^{T_{\alpha, \beta}^{\bar{w}}}$ is compact. \square

3. Cellular decomposition of $\mathcal{M}(r, n)^{T_{1,1}^{\bar{w}(m)}}$

In this section we construct a cellular decomposition of $\mathcal{M}(r, n)^{T_{1,1}^{\bar{w}(m)}}$ and obtain a combinatorial formula for the number of the irreducible components.

For a partition $\lambda = \lambda_1, \lambda_2, \dots, \lambda_k, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$ let $|\lambda| = \sum_{i=1}^k \lambda_i$ and $l(\lambda) = k$. We denote by \mathcal{P} the set of all partitions and by \mathcal{DP} the set of partitions with distinct parts.

Let $S(r, m)$ be the set of r -tuples $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ of partitions $\lambda^{(i)} \in \mathcal{DP}$ such that $\lambda_1^{(i)} \leq l(\lambda^{(i+1)}) + \delta_{i,m}$, for $1 \leq i \leq r - 1$. Let

$$S(r, m)_n = \left\{ (\lambda^{(1)}, \dots, \lambda^{(r)}) \in S(r, m) \mid \sum_{i=1}^r |\lambda^{(i)}| = n \right\}.$$

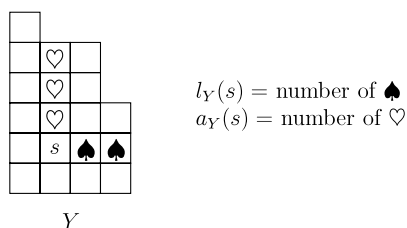


Fig. 1.

Proposition 3.1.

$$h_0 \left(\mathcal{M}(r, n)^{T_{1,1}^{\vec{w}(m)}} \right) = \#S(r, m)_n.$$

Proof. The set of fixed points of the T -action on $\mathcal{M}(r, n)$ is finite and is parametrized by the set of r -tuples $D = (D_1, D_2, \dots, D_r)$ of Young diagrams D_i , such that $\sum_{i=1}^r |D_i| = n$ (see e.g. [2]).

For a Young diagram Y let

$$r_l(Y) = |\{(i, j) \in D | j = l\}|,$$

$$c_l(Y) = |\{(i, j) \in D | i = l\}|.$$

For a point $s = (i, j) \in \mathbb{Z}_{\geq 0}^2$ let

$$l_Y(s) = r_j(Y) - i - 1,$$

$$a_Y(s) = c_i(Y) - j - 1,$$

see Fig. 1. Note that $l_Y(s)$ and $a_Y(s)$ are negative if $s \notin Y$.

Let p be the fixed point of the T -action corresponding to an r -tuple D . Let $R(T) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, e_1^{\pm 1}, e_2^{\pm 1}, \dots, e_r^{\pm 1}]$ be the representation ring of T . Then the weight decomposition of the tangent space $T_p(\mathcal{M}(r, n))$ of the variety $\mathcal{M}(r, n)$ at the point p is given by (see e.g. [2])

$$T_p(\mathcal{M}(r, n)) = \sum_{i,j=1}^r e_j e_i^{-1} \left(\sum_{s \in D_i} t_1^{-l_{D_j}(s)} t_2^{a_{D_i}(s)+1} + \sum_{s \in D_j} t_1^{l_{D_i}(s)+1} t_2^{-a_{D_j}(s)} \right). \tag{9}$$

Consider an integer γ and an integer vector $\vec{v} = (v_1, \dots, v_r)$ such that

$$v_1 \gg v_2 \gg \dots \gg v_r \gg \gamma \gg 1. \tag{10}$$

It is easy to see that $\mathcal{M}(r, n)^T = \mathcal{M}(r, n)^{T_{1,\gamma}^{\vec{v}}}$. For a fixed point $p \in \mathcal{M}(r, n)^T$ let $C_p = \{z \in \mathcal{M}(r, n)^{T_{1,1}^{\vec{w}(m)}} \mid \lim_{t \rightarrow 0, t \in T_{1,\gamma}^{\vec{v}}} tz = p\}$.

By Proposition 2.1 the variety $\mathcal{M}(r, n)^{T_{1,1}^{\vec{w}(m)}}$ is compact, hence it has a cellular decomposition with the cells C_p (see [12,13]). From (10) and (9) it follows that the complex dimension of the cell C_p is equal to

$$\sum_{i=1}^r |\{s \in D_i | a_{D_i}(s) + 1 = l_{D_i}(s)\}| + \sum_{r \geq i > j \geq 1} |\{s \in D_i | w_j - w_i - l_{D_j}(s) + a_{D_i}(s) + 1 = 0\}|$$

$$+ \sum_{r \geq i > j \geq 1} |\{s \in D_j | w_j - w_i + l_{D_i}(s) + 1 - a_{D_j}(s) = 0\}|,$$

where $(w_1, \dots, w_r) = \vec{w}(m)$. Therefore, the dimension of the cell C_p is equal to 0 if and only if the following three conditions hold

$$\{s \in D_i | a_{D_i}(s) + 1 = l_{D_i}(s)\} = \emptyset, \quad \forall 1 \leq i \leq r, \tag{11}$$

$$\{s \in D_i | w_j - w_i - l_{D_j}(s) + a_{D_i}(s) + 1 = 0\} = \emptyset, \quad \forall r \geq i > j \geq 1, \tag{12}$$

$$\{s \in D_j | w_j - w_i + l_{D_i}(s) + 1 - a_{D_j}(s) = 0\} = \emptyset, \quad \forall r \geq i > j \geq 1. \tag{13}$$

It is sufficient to prove that these equations are equivalent to the following system

$$D_i \in \mathcal{DP}, \tag{14}$$

$$c_0(D_i) \leq r_0(D_{i+1}) + \delta_{i,m}, \tag{15}$$

where $D_i \in \mathcal{DP}$ means that nonzero lengths of columns of a Young diagram D_i are distinct.

Suppose that Eqs. (11)–(13) hold. Condition (14) easily follows from (11). Suppose that $c_0(D_i) > r_0(D_{i+1}) + \delta_{i,m}$. For a point $s = (0, c_0(D_i) - 1)$ we have

$$\delta_{i,m} + l_{D_{i+1}}(s) + 1 - a_{D_i}(s) = \delta_{i,m} + l_{D_{i+1}}(s) + 1 \geq 0. \tag{16}$$

For a point $s = (0, 0)$ we have

$$\delta_{i,m} + l_{D_{i+1}}(s) + 1 - a_{D_i}(s) = \delta_{i,m} + r_0(D_{i+1}) - c_0(D_i) + 1 \leq 0. \tag{17}$$

Note that for two points $s_1 = (0, y)$ and $s_2 = (0, y + 1)$, where $0 \leq y < c_0(D_i) - 1$, we have

$$(l_{D_{i+1}}(s_2) + 1 - a_{D_i}(s_2)) - (l_{D_{i+1}}(s_1) + 1 - a_{D_i}(s_1)) = l_{D_{i+1}}(s_2) - l_{D_{i+1}}(s_1) + 1 \leq 1. \tag{18}$$

From (16)–(18) it follows that there exists a number $0 \leq y \leq c_0(D_i) - 1$ such that for a point $s = (0, y)$ we have

$$\delta_{i,m} + l_{D_{i+1}}(s) + 1 - a_{D_i}(s) = 0.$$

This contradicts (13). Thus, we have proved (15).

Suppose that Eqs. (14), (15) hold. It is easy to see that (11) follows from (14). Let us prove (13). Consider a point $s = (x, y) \in D_j$ and let $r \geq i > j \geq 1$. Let $s_1 = (x, 0)$ and $s_2 = (0, 0)$, we have

$$\begin{aligned} w_j - w_i + l_{D_i}(s) + 1 - a_{D_j}(s) &\stackrel{\text{by (14)}}{\geq} w_j - w_i + l_{D_i}(s_1) + 1 - a_{D_j}(s_1) \\ &\stackrel{\text{by (14)}}{\geq} w_j - w_i + l_{D_i}(s_2) + 1 - a_{D_j}(s_2) \\ &= w_j - w_i + r_0(D_i) - c_0(D_j) + 1 \stackrel{\text{by (15)}}{>} 0. \end{aligned}$$

Let us prove (12). Suppose $s \in D_i$ and $r \geq i > j \geq 1$. From (14) and (15) it follows that

$$l_{D_j}(s) \leq l_{D_i}(s) + w_j - w_i.$$

Thus

$$w_j - w_i - l_{D_j}(s) + a_{D_i}(s) + 1 \geq -l_{D_i}(s) + a_{D_i}(s) + 1 \stackrel{\text{by (14)}}{\geq} 1.$$

This completes the proof of the proposition. \square

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. By Proposition 3.1 we have

$$\sum_{n \geq 0} h_0 \left(\mathcal{M}(r, n)_{1,1}^{\tilde{w}(m)} \right) q^n = \sum_{(\lambda^{(1)}, \dots, \lambda^{(r)}) \in S(r,m)} q^{\sum_{i=1}^r |\lambda^{(i)}|}.$$

In Section 4.1 we obtain fermionic expressions for the right-hand side of this equation. The main idea is to transform them to a known fermionic formula for the second infinite product on the right-hand side of (1). In Section 4.2 we use the Gordon’s generalization of the Rogers–Ramanujan identities to finish the proof of the theorem in the case when r is odd. The case of even r is covered by an identity from [14], we do it in Section 4.3.

Clearly, $S(r, r) = S(r, 0)$. Therefore we have $h_0 \left(\mathcal{M}(r, n)_{1,1}^{\tilde{w}(r)} \right) = h_0 \left(\mathcal{M}(r, n)_{1,1}^{\tilde{w}(0)} \right)$. It is also obvious that in the case $m = r$ the right-hand side of (1) is the same as in the case $m = 0$. Thus it is enough to prove the theorem in the case $0 \leq m \leq r - 1$.

4.1. Fermionic expressions for the generating series

Let $\lambda = \lambda_1, \lambda_2, \dots, \lambda_s$ be a partition. We will use the standard notation

$$(q)_\lambda = (q)_{\lambda_1 - \lambda_2} \cdots (q)_{\lambda_{s-1} - \lambda_s} (q)_{\lambda_s}.$$

Proposition 4.1. *Let $0 \leq m \leq r - 1$. Then we have*

$$\sum_{(\lambda^{(1)}, \dots, \lambda^{(r)}) \in S(r,m)} q^{\sum_{i=1}^r |\lambda^{(i)}|} = \sum_{\rho_1 \geq \dots \geq \rho_r} \frac{q^{\sum_{i=1}^r \frac{\rho_i^2 + \rho_i}{2}}}{(q)_\rho} \left(1 + \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\rho_{r-m+j+1})} \right).$$

Proof. The q -binomial coefficients are defined by

$$\begin{bmatrix} M \\ N \end{bmatrix}_q = \begin{cases} \frac{(q)_M}{(q)_N (q)_{M-N}}, & \text{if } M \geq N \geq 0, \\ 0, & \text{in other cases.} \end{cases}$$

We have (see e.g. [15])

$$\begin{bmatrix} M \\ N \end{bmatrix}_q = \begin{bmatrix} M-1 \\ N \end{bmatrix}_q + q^{M-N} \begin{bmatrix} M-1 \\ N-1 \end{bmatrix}_q, \tag{19}$$

$$\sum_{\substack{\lambda \in \mathcal{P} \\ \lambda_1 \leq M, l(\lambda) \leq N}} q^{|\lambda|} = \begin{bmatrix} M+N \\ N \end{bmatrix}_q. \tag{20}$$

From (20) it follows that

$$\sum_{\substack{\lambda \in \mathcal{D}\mathcal{P} \\ l(\lambda) = N, \lambda_1 \leq M}} q^{|\lambda|} = q^{\frac{N^2+N}{2}} \begin{bmatrix} M \\ N \end{bmatrix}_q.$$

Therefore, we have

$$\sum_{(\lambda^{(1)}, \dots, \lambda^{(r)}) \in S(r, m)} q^{\sum_{i=1}^r |\lambda^{(i)}|} = \sum_{\rho_1, \dots, \rho_r} q^{\sum_{i=1}^r \frac{\rho_i^2 + \rho_i}{2}} \prod_{i=0}^{r-1} \begin{bmatrix} \rho_i + \delta_{i, r-m} \\ \rho_{i+1} \end{bmatrix},$$

where we define ρ_0 to be equal to ∞ . Using (19), we get

$$\begin{aligned} & \sum_{\rho_1, \dots, \rho_r} q^{\sum_{i=1}^r \frac{\rho_i^2 + \rho_i}{2}} \prod_{i=0}^{r-1} \begin{bmatrix} \rho_i + \delta_{i, r-m} \\ \rho_{i+1} \end{bmatrix} \\ &= \sum_{\rho_1, \dots, \rho_r} q^{\sum_{i=1}^r \frac{\rho_i^2 + \rho_i}{2}} \prod_{i=0}^{r-1} \begin{bmatrix} \rho_i \\ \rho_{i+1} \end{bmatrix} + q^{\rho_r - m - \rho_{r-m+1} + 1} \sum_{\rho_1, \dots, \rho_r} q^{\sum_{i=1}^r \frac{\rho_i^2 + \rho_i}{2}} \prod_{i=0}^{r-1} \begin{bmatrix} \rho_i \\ \rho_{i+1} - \delta_{i, r-m} \end{bmatrix} \\ &= \sum_{\rho_1, \dots, \rho_r} q^{\sum_{i=1}^r \frac{\rho_i^2 + \rho_i}{2}} \prod_{i=0}^{r-1} \begin{bmatrix} \rho_i \\ \rho_{i+1} \end{bmatrix} + q^{\rho_r - m + 1} \sum_{\rho_1, \dots, \rho_r} q^{\sum_{i=1}^r \frac{\rho_i^2 + \rho_i}{2}} \prod_{i=0}^{r-1} \begin{bmatrix} \rho_i + \delta_{i, r-m+1} \\ \rho_{i+1} \end{bmatrix} \\ &= \dots = \sum_{\rho_1 \geq \dots \geq \rho_r} \frac{q^{\sum_{i=1}^r \frac{\rho_i^2 + \rho_i}{2}}}{(q)_\rho} \left(1 + \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\rho_{r-m+j+1})} \right). \end{aligned}$$

The proposition is proved. \square

Proposition 4.2. Let $0 \leq m \leq r - 1$. Then we have

$$\sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_\lambda} \left(1 + \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\lambda_{r-m+j+1})} \right) = \sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_\lambda} \left(1 + \sum_{i=0}^{m'-1} q^{\sum_{j=0}^i (\lambda_{r-1-2j+1})} \right),$$

where $m' = \min(m, r - m)$.

Before proving this proposition we introduce the following notation. Suppose $P(x_1, \dots, x_r, q)$ and $Q(x_1, \dots, x_r, q)$ are polynomials in x_1, \dots, x_r and q . We will write $P \approx Q$ if

$$\sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_\lambda} (P(q^{\lambda_1}, \dots, q^{\lambda_r}, q) - Q(q^{\lambda_1}, \dots, q^{\lambda_r}, q)) = 0.$$

Proposition 4.2 says that

$$\sum_{i=0}^{m-1} q^i \prod_{j=0}^i x_{r-m+j} \approx \sum_{i=0}^{\min(m, r-m)-1} q^i \prod_{j=0}^i x_{r-1-2j}.$$

We will prove a more general statement.

Proposition 4.3. Suppose $0 \leq s \leq l - 1$ and $l \leq r$. Then

$$\left(\sum_{i=0}^{s-1} q^i \prod_{j=0}^i x_{l-s+j} \right) P(x_{\geq l}, q) \approx \left(\sum_{i=0}^{\min(s, l-s)-1} q^i \prod_{j=0}^i x_{l-1-2j} \right) P(x_{\geq l}, q), \tag{21}$$

where $P(x_{\geq l}, q)$ is any polynomial that does not depend on x_1, \dots, x_{l-1} .

Proof. We adopt the following conventions, $x_{<1} = 0$ and $x_{>r} = 1$.

Lemma 4.4. For $1 \leq s \leq r$ we have

$$x_s(1 + qx_{s+1})P(x_{\neq s}, q) \approx x_{s+1}(1 + qx_{s-1})P(x_{\neq s}, q),$$

where $P(x_{\neq s}, q)$ is a polynomial that does not depend on x_s .

Proof. We have

$$\begin{aligned} & \sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_\lambda} (q^{\lambda_s} - q^{\lambda_{s+1}} + q^{\lambda_s + \lambda_{s+1} + 1} - q^{\lambda_{s-1} + \lambda_{s+1} + 1}) P(q^{\lambda_{\neq s}}, q) \\ &= - \sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_\lambda} q^{\lambda_{s+1}} (1 - q^{\lambda_s - \lambda_{s+1}}) P(q^{\lambda_{\neq s}}, q) \end{aligned} \tag{22}$$

$$+ \sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_\lambda} q^{\lambda_s + \lambda_{s+1} + 1} (1 - q^{\lambda_{s-1} - \lambda_s}) P(q^{\lambda_{\neq s}}, q). \tag{23}$$

In (22) we make the shift $\lambda_i \mapsto \lambda_i + 1$ for $i = 1, \dots, s$ and in the sum (23) we make the shift $\lambda_i \mapsto \lambda_i + 1$ for $i = 1, \dots, s - 1$. We get

$$\begin{aligned} & - \sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_\lambda} q^{\lambda_{s+1}} q^{\sum_{i=1}^s (\lambda_i + 1)} P(q^{\lambda_1 + 1}, \dots, q^{\lambda_{s-1} + 1}, q^{\lambda_{\geq s+1}}, q) \\ & + \sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_\lambda} q^{\lambda_s + \lambda_{s+1} + 1} q^{\sum_{i=1}^{s-1} (\lambda_i + 1)} P(q^{\lambda_1 + 1}, \dots, q^{\lambda_{s-1} + 1}, q^{\lambda_{\geq s+1}}, q) = 0. \end{aligned}$$

The lemma is proved. \square

Lemma 4.5. Suppose that $l \leq r$ and $0 \leq s \leq \frac{l-2}{2}$, then

$$(1 + qx_i) \prod_{i=0}^s x_{l-1-2i} P(x_{\geq l}, q) \approx (1 + qx_{l-2s-2}) \prod_{i=0}^s x_{l-2i} P(x_{\geq l}, q).$$

Proof. By Lemma 4.4

$$\begin{aligned} (1 + qx_i)x_{l-1}x_{l-3} \cdots x_{l-1-2s} P(x_{\geq l}, q) & \approx x_l(1 + qx_{l-2})x_{l-3}x_{l-5} \cdots x_{l-1-2s} P(x_{\geq l}, q) \\ & \approx x_l x_{l-2} (1 + qx_{l-4})x_{l-5} \cdots x_{l-1-2s} P(x_{\geq l}, q) \\ & \approx \cdots \approx x_l x_{l-2} \cdots x_{l-2s} (1 + qx_{l-2-2s}) P(x_{\geq l}, q). \end{aligned}$$

The lemma is proved. \square

We will prove (21) by induction on s . The case $s = 1$ is trivial and the case $s = 2$ follows from Lemma 4.4. Suppose $s \geq 3$. We have

$$\left(\sum_{i=0}^{s-1} q^i \prod_{j=0}^i x_{l-s+j} \right) P(x_{\geq l}, q) = \left((1 + qx_{l-1}) \sum_{i=0}^{s-2} q^i \prod_{j=0}^i x_{l-s+j} - qx_{l-1} \sum_{i=0}^{s-3} q^i \prod_{j=0}^i x_{l-s+j} \right) P(x_{\geq l}, q).$$

Suppose that $2s \leq l + 1$, then

$$\begin{aligned} & \left((1 + qx_{l-1}) \sum_{i=0}^{s-2} q^i \prod_{j=0}^i x_{l-s+j} - qx_{l-1} \sum_{i=0}^{s-3} q^i \prod_{j=0}^i x_{l-s+j} \right) P(x_{\geq l}, q) \\ & \stackrel{\text{by the induction assumption}}{\approx} \left((1 + qx_{l-1}) \sum_{i=0}^{s-2} q^i \prod_{j=0}^i x_{l-2-2j} - \sum_{i=1}^{s-2} q^i \prod_{j=0}^i x_{l-1-2j} \right) P(x_{\geq l}, q) \\ & \stackrel{\text{by Lemma 4.5}}{\approx} \left(\sum_{i=0}^{s-2} (1 + qx_{l-3-2i}) q^i \prod_{j=0}^i x_{l-1-2j} - \sum_{i=1}^{s-2} q^i \prod_{j=0}^i x_{l-1-2j} \right) P(x_{\geq l}, q) \\ & = \left(\sum_{i=0}^{s-1} q^i \prod_{j=0}^i x_{l-1-2j} \right) P(x_{\geq l}, q). \end{aligned}$$

We see that we have done the induction step in the case $2s \leq l$. If $2s = l + 1$, then it remains to note that

$$\left(\sum_{i=0}^{s-1} q^i \prod_{j=0}^i x_{2s-2-2j} \right) P(x_{\geq 2s-1}, q) = \left(\sum_{i=0}^{s-2} q^i \prod_{j=0}^i x_{2s-2-2j} \right) P(x_{\geq 2s-1}, q).$$

Suppose that $2s \geq l + 2$, then

$$\begin{aligned} & \left((1 + qx_{l-1}) \sum_{i=0}^{s-2} q^i \prod_{j=0}^i x_{l-s+j} - qx_{l-1} \sum_{i=0}^{s-3} q^i \prod_{j=0}^i x_{l-s+j} \right) P(x_{\geq l}, q) \\ & \stackrel{\text{by the induction assumption}}{\approx} \left((1 + qx_{l-1}) \sum_{i=0}^{l-s-1} q^i \prod_{j=0}^i x_{l-2-2j} - \sum_{i=1}^{l-s} q^i \prod_{j=0}^i x_{l-1-2j} \right) P(x_{\geq l}, q) \\ & \stackrel{\text{by Lemma 4.5}}{\approx} \left(\sum_{i=0}^{l-s-1} (1 + qx_{l-3-2i}) q^i \prod_{j=0}^i x_{l-1-2j} - \sum_{i=1}^{l-s} q^i \prod_{j=0}^i x_{l-1-2j} \right) P(x_{\geq l}, q) \\ & = \left(\sum_{i=0}^{l-s-1} q^i \prod_{j=0}^i x_{l-1-2j} \right) P(x_{\geq l}, q). \end{aligned}$$

The proposition is proved. \square

From Proposition 4.2 it follows that

$$h_0 \left(\mathcal{M}(r, n)^{T_{1,1}^{\bar{w}(m)}} \right) = h_0 \left(\mathcal{M}(r, n)^{T_{1,1}^{\bar{w}(r-m)}} \right).$$

We can also see that the substitution $m \mapsto r - m$ does not change the right-hand side of (1). So in the rest of the proof of the theorem we assume that $m \leq \frac{r}{2}$.

4.2. The case $r = 2k + 1$

We have $0 \leq m \leq k$.

Proposition 4.6.

$$\sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_\lambda} \left(1 + \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\lambda_{r-1-2j+1})} \right) \tag{24}$$

$$= (-q)_\infty \sum_{\lambda_1 \geq \dots \geq \lambda_k} \frac{q^{\sum_{i=1}^k (\lambda_i^2 + \lambda_i)}}{(q)_\lambda} \left(1 + \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\lambda_{k-j+1})} \right). \tag{25}$$

Proof. We have the following equation

$$\sum_{i=\beta}^{\alpha} \frac{q^{\frac{i(i+1)}{2}} q^{\frac{\beta(\beta+1)}{2}}}{(q)_{\alpha-i}(q)_{i-\beta}} = q^{\beta(\beta+1)} \frac{(-q^{\beta+1})_{\alpha-\beta}}{(q)_{\alpha-\beta}}. \tag{26}$$

It can be easily derived from the q -binomial formula (see e.g. [15]).

We can fix $\lambda_2, \lambda_4, \dots, \lambda_{2k}$ in (24) and sum over $\lambda_1, \lambda_3, \dots, \lambda_{2k+1}$ using (26). Then we get exactly the sum (25). \square

Proposition 4.7.

$$\sum_{\lambda_1 \geq \dots \geq \lambda_k} \frac{q^{\sum_{i=1}^k (\lambda_i^2 + \lambda_i)}}{(q)_{\lambda}} \left(1 + \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\lambda_{k-j} + 1)} \right) = \frac{(q^{m+1}, q^{2k-m+2}, q^{2k+3}; q^{2k+3})_{\infty}}{(q)_{\infty}}.$$

Proof. Consider the functions $J_{k,i}(a, x, q)$ from Ch. 7 of the book [15]. We only need the following two properties of $J_{k,i}$ (see [15, Ch. 7]).

$$J_{k,i}(0, x, q) = \sum_{\lambda_1 \geq \dots \geq \lambda_{k-1}} x^{|\lambda|} \frac{q^{\lambda_1^2 + \dots + \lambda_{k-1}^2 + \lambda_i + \dots + \lambda_{k-1}}}{(q)_{\lambda}},$$

$$J_{k,i}(0, x, q) - J_{k,i-1}(0, x, q) = (xq)^{i-1} J_{k,k-i+1}(0, xq, q). \tag{27}$$

We have

$$\begin{aligned} J_{k+1,m+1}(0, 1, q) &= J_{k+1,m}(0, 1, q) + q^m J_{k+1,k-m+1}(0, q, q) \\ &= J_{k+1,m-1}(0, 1, q) + q^{m-1} J_{k+1,k-m+2}(0, q, q) + q^m J_{k+1,k-m+1}(0, q, q) \\ &= \dots = J_{k+1,1}(0, 1, q) + \sum_{i=0}^{m-1} q^{i+1} J_{k+1,k-i}(0, q, q) \\ &= \sum_{\lambda_1 \geq \dots \geq \lambda_k} \frac{q^{\sum_{i=1}^k (\lambda_i^2 + \lambda_i)}}{(q)_{\lambda}} \left(1 + \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\lambda_{k-j} + 1)} \right). \end{aligned}$$

On the other hand we have (see [15, Ch. 7])

$$J_{k+1,m+1}(0, 1, q) = \frac{(q^{m+1}, q^{2k-m+2}, q^{2k+3}; q^{2k+3})_{\infty}}{(q)_{\infty}}.$$

This completes the proof of the proposition. \square

Propositions 4.6 and 4.7 conclude the proof of the theorem in the case when r is odd.

4.3. The case $r = 2k$

We have $0 \leq m \leq k$.

Proposition 4.8.

$$\sum_{\lambda_1 \geq \dots \geq \lambda_r} \frac{q^{\sum_{i=1}^r \frac{\lambda_i^2 + \lambda_i}{2}}}{(q)_{\lambda}} \left(1 + \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\lambda_{r-1-2j} + 1)} \right) = \sum_{\lambda_1 \geq \dots \geq \lambda_k} \frac{(-q)_{\lambda_1} q^{\frac{\lambda_1^2 + \lambda_1}{2} + \sum_{i=2}^k (\lambda_i^2 + \lambda_i)}}{(q)_{\lambda}} \left(1 + \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\lambda_{k-j} + 1)} \right).$$

Proof. Similar to the proof of Proposition 4.6. \square

Proposition 4.9. We have

$$\sum_{\lambda_1 \geq \dots \geq \lambda_k} \frac{(-q)_{\lambda_1} q^{\frac{\lambda_1^2 + \lambda_1}{2} + \sum_{i=2}^k (\lambda_i^2 + \lambda_i)}}{(q)_{\lambda}} \left(2 \sum_{i=0}^{m-1} q^{\sum_{j=0}^i (\lambda_{k-j} + 1)} + q^{\sum_{j=0}^{m-1} (\lambda_{k-j} + 1)} \right) = \sum_{\lambda_1 \geq \dots \geq \lambda_k} \frac{(-q)_{\lambda_1} q^{\frac{\lambda_1^2 - \lambda_1}{2} + \sum_{i=2}^k \lambda_i^2 + \sum_{i=m+1}^k \lambda_i}}{(q)_{\lambda}}.$$

Proof. Suppose $P(x_1, \dots, x_k, q)$ and $Q(x_1, \dots, x_k, q)$ are polynomials in variables x_1, \dots, x_k and q . We will write $P \approx_2 Q$ if

$$\sum_{\lambda_1 \geq \dots \geq \lambda_k} \frac{(-q)^{\lambda_1} q^{\frac{\lambda_1^2 - \lambda_1}{2} + \sum_{i=2}^k \lambda_i^2}}{(q)_{\lambda}} (P(q^{\lambda_1}, \dots, q^{\lambda_k}, q) - Q(q^{\lambda_1}, \dots, q^{\lambda_k}, q)) = 0.$$

The proposition says that

$$\prod_{i=m+1}^k x_i \approx_2 \left(2 \sum_{i=0}^{m-1} q^i \prod_{j=k-i+1}^k x_j + q^m \prod_{i=k-m+1}^k x_i \right) \prod_{i=1}^k x_i. \tag{28}$$

The proof of (28) is based on the following three lemmas. We adopt the conventions, $x_{<1} = 0$ and $x_{>k} = 1$.

Lemma 4.10. Suppose $1 \leq s \leq k$, then we have

$$\begin{aligned} x_s(1 + qx_s x_{s+1})P(x_{\neq s}, q) &\approx_2 x_{s+1}(1 + qx_{s-1} x_s)P(x_{\neq s}, q), \quad \text{if } s \geq 2, \\ x_1(1 + x_2 + qx_1 x_2)P(x_{\geq 2}, q) &\approx_2 x_2 P(x_{\geq 2}, q), \quad \text{if } s = 1. \end{aligned}$$

Proof. Similar to the proof of Lemma 4.4. \square

Lemma 4.11. Suppose $1 \leq s < l \leq k + 1$, then

$$(x_l - x_s) \prod_{i=s+1}^l x_i \prod_{i=l+1}^k x_i^2 \approx_2 q(x_{l-1} - x_{s-1}) \prod_{i=s}^{l-1} x_i \prod_{i=l}^k x_i^2, \quad \text{if } s \geq 2, \tag{29}$$

$$(x_l - x_1) \prod_{i=2}^l x_i \prod_{i=l+1}^k x_i^2 \approx_2 (1 + qx_{l-1}) \prod_{i=1}^{l-1} x_i \prod_{i=l}^k x_i^2, \quad \text{if } s = 1. \tag{30}$$

Proof. Let us prove (29). Using Lemma 4.10 we have

$$\begin{aligned} (1 + qx_{s-1} x_s) x_{s+1} \prod_{i=s+2}^{l-1} x_i \prod_{i=l}^k x_i^2 &\approx_2 x_s (1 + qx_s x_{s+1}) x_{s+2} \prod_{i=s+3}^{l-1} x_i \prod_{i=l}^k x_i^2 \\ &\approx_2 \dots \approx_2 x_s x_{s+1} \dots x_{l-1} (1 + qx_{l-1} x_l) x_l \prod_{i=l+1}^k x_i^2. \end{aligned}$$

Thus, (29) is proved. Eq. (30) can be proved similarly. \square

Lemma 4.12. For any $1 \leq s \leq k$ we have

$$(1 - x_s) \prod_{i=s+1}^k x_i \approx_2 q^{s-1} (1 + qx_{k-s+1}) \prod_{i=1}^{k-s+1} x_i \prod_{i=k-s+2}^k x_i^2.$$

Proof. By Lemma 4.11 we have

$$\begin{aligned} (1 - x_s) \prod_{i=s+1}^k x_i &\approx_2 q(x_k - x_{s-1}) \prod_{i=s}^k x_i \approx_2 q^2 (x_{k-1} - x_{s-2}) \left(\prod_{i=s-1}^{k-1} x_i \right) x_k^2 \\ &\approx_2 \dots \approx_2 q^{s-1} (x_{k-s+2} - x_1) \prod_{i=2}^{k-s+2} x_i \prod_{i=k-s+3}^k x_i^2 \\ &\approx_2 q^{s-1} (1 + qx_{k-s+1}) \prod_{i=1}^{k-s+1} x_i \prod_{i=k-s+2}^k x_i^2. \end{aligned}$$

The lemma is proved. \square

We are ready to prove (28). We have

$$\begin{aligned} \prod_{i=m+1}^k x_i &= \sum_{i=0}^{m-1} (1 - x_{m-i}) \prod_{i=m-i+1}^k x_i + \prod_{i=1}^k x_i \\ &\stackrel{\text{by Lemma 4.12}}{\approx_2} \left(\sum_{i=0}^{m-1} q^i (1 + qx_{k-i}) \prod_{j=k-i+1}^k x_j + 1 \right) \prod_{i=1}^k x_i \\ &= \left(2 \sum_{i=0}^{m-1} q^i \prod_{j=k-i+1}^k x_j + q^m \prod_{i=k-m+1}^k x_i \right) \prod_{i=1}^k x_i. \end{aligned}$$

This completes the proof of the proposition. \square

Proposition 4.13.

$$\sum_{\lambda_1 \geq \dots \geq \lambda_k} \frac{(-q)_{\lambda_1} q^{\frac{\lambda_1^2 + \lambda_1}{2} + \sum_{i=2}^k (\lambda_i^2 + \lambda_i)}}{(q)_{\lambda}} \left(1 + \sum_{i=0}^{m-1} q^{i \sum_{k-j+1}^i} \right) = \frac{(-q)_{\infty}}{(q)_{\infty}} (q^{m+1}, q^{2k-m+1}, q^{2k+2}; q^{2k+2})_{\infty}.$$

Proof. Consider the functions $\bar{\mathcal{E}}_{k+1,i}(a, q)$ from the paper [14]. It is proved there that

$$\bar{\mathcal{E}}_{k+1,m+1}(a, q) = \sum_{\lambda_1 \geq \dots \geq \lambda_k} \frac{q^{\frac{\lambda_1^2 + \lambda_1}{2} + \sum_{i=2}^k \lambda_i^2 + \sum_{i=m+1}^k \lambda_i} \left(-\frac{1}{a}\right)_{\lambda_1} a^{\lambda_1}}{(q)_{\lambda}}, \tag{31}$$

$$\bar{\mathcal{E}}_{k+1,m+1}\left(\frac{1}{q}, q\right) = \frac{(-q)_{\infty}}{(q)_{\infty}} ((q^{m+1}, q^{2k-m+1}, q^{2k+2}, q^{2k+2})_{\infty} + (q^m, q^{2k-m+2}, q^{2k+2}; q^{2k+2})_{\infty}). \tag{32}$$

Combining (31), (32) and Proposition 4.9 we get the proof of the proposition. \square

Propositions 4.8 and 4.13 complete the proof of the theorem in the case of even r .

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