MOSCOW MATHEMATICAL JOURNAL
Volume 13, Number 2, April–June 2013, Pages 233–265

PURSUING THE DOUBLE AFFINE GRASSMANNIAN III: CONVOLUTION WITH AFFINE ZASTAVA

ALEXANDER BRAVERMAN AND MICHAEL FINKELBERG

To the memory of Israel Moiseevich Gelfand

Abstract. This is the third paper of a series which describes a conjectural analog of the affine Grassmannian for affine Kac–Moody groups (also known as the double affine Grassmannian). The current paper is dedicated to describing a conjectural analog of the convolution diagram for the double affine Grassmannian and affine zastava.

2010 Math. Subj. Class. Primary: 19E08; Secondary: 22E65, 37K10.

Key words and phrases. Affine Grassmannian, zastava, chainsaw quiver, dented chainsaw quiver, rift quiver.

1. Introduction

1.1. The usual affine Grassmannian. Let G be a connected complex reductive group with a Cartan torus T, and let $\mathcal{K} = \mathbb{C}((s))$, $\mathcal{O} = \mathbb{C}[[s]]$. By the *affine Grassmannian* of G we shall mean the quotient $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$. It is known (cf. [2], [19]) that Gr_G is the set of $\mathbb{C}$-points of an ind-scheme over $\mathbb{C}$, which we will denote by the same symbol. Note that Gr_G is defined for any (not necessarily reductive) group G.

Let $\Lambda = \Lambda_G$ denote the coweight lattice of G and let $\Lambda^\vee$ denote the dual lattice (this is the weight lattice of G). We let $2\rho_G^\vee$ denote the sum of the positive roots of G.

The group-scheme $G(\mathcal{O})$ acts on Gr_G on the left and its orbits can be described as follows. One can identify the lattice Λ_G with the quotient $T(\mathcal{K})/T(\mathcal{O})$. Fix $\lambda \in \Lambda_G$ and let s^λ denote any lift of λ to $T(\mathcal{K})$. Let Gr_G^λ denote the $G(\mathcal{O})$-orbit of s^λ (this is clearly independent of the choice of s^λ). The following result is well-known:

Received October 10, 2011; in revised form October 21, 2012.

The second named author supported in part by the RFBR grants 12-01-33101, 12-01-00944, the National Research University Higher School of Economics' Academic Fund award No.12-09-0062 and the AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023. This study was carried out within the National Research University Higher School of Economics Academic Fund Program in 2012-2013, research grant No. 11-01-0017. This study comprises research findings from the "Representation Theory in Geometry and in Mathematical Physics" carried out within The National Research University Higher School of Economics' Academic Fund Program in 2012, grant No 12-05-0014.

Lemma 1.2. (1)

$$\mathrm{Gr}_G = \bigcup_{\lambda \in \Lambda_G} \mathrm{Gr}_G^\lambda.$$

(2) *We have* $\mathrm{Gr}_G^\lambda = \mathrm{Gr}_G^\mu$ *if an only if* λ *and* μ *belong to the same* W*-orbit on* Λ_G *(here* W *is the Weyl group of* G*). In particular,*

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda_G^+} \mathrm{Gr}_G^\lambda.$$

(3) *For every* $\lambda \in \Lambda^+$ *the orbit* Gr_G^λ *is finite-dimensional and its dimension is equal to* $\langle \lambda, 2\rho_G^\vee \rangle$.

Let $\overline{\mathrm{Gr}_G}^\lambda$ denote the closure of Gr_G^λ in Gr_G; this is an irreducible projective algebraic variety; one has $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G}^\lambda$ if and only if $\lambda - \mu$ is a sum of positive roots of the Langlands dual group $G^\vee$. We will denote by IC^λ the intersection cohomology complex on $\overline{\mathrm{Gr}_G}^\lambda$. Let $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ denote the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G. It is known that every object of this category is a direct sum of the IC^λ's.

1.3. Transversal slices. Consider the group $G[s^{-1}] \subset G((s))$; let us denote by $G[s^{-1}]_1$ the kernel of the natural ("evaluation at ∞") homomorphism $G[s^{-1}] \to G$. For any $\lambda \in \Lambda$ let $\mathrm{Gr}_{G,\lambda} = G[s^{-1}] \cdot s^\lambda$. Then it is easy to see that one has

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}_{G,\lambda}$$

Let also $\mathcal{W}_{G,\lambda}$ denote the $G[s^{-1}]_1$-orbit of s^λ. For any $\lambda, \mu \in \Lambda^+$, $\lambda \geqslant \mu$ set

$$\mathrm{Gr}_{G,\mu}^\lambda = \mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_{G,\mu}, \quad \overline{\mathrm{Gr}}_{G,\mu}^\lambda = \overline{\mathrm{Gr}}_G^\lambda \cap \mathrm{Gr}_{G,\mu}$$

and

$$\mathcal{W}_{G,\mu}^\lambda = \mathrm{Gr}_G^\lambda \cap \mathcal{W}_{G,\mu}, \quad \overline{\mathcal{W}}_{G,\mu}^\lambda = \overline{\mathrm{Gr}}_G^\lambda \cap \mathcal{W}_{G,\mu}.$$

Note that $\overline{\mathcal{W}}_{G,\mu}^\lambda$ contains the point s^μ in it. The variety $\overline{\mathcal{W}}_{G,\mu}^\lambda$ can be thought of as a transversal slice to Gr_G^μ inside $\overline{\mathrm{Gr}}_G^\lambda$ at the point s^μ (cf. [4, Lemma 2.9]).

1.4. The convolution. We can regard $G(\mathcal{K})$ as a total space of a $G(\mathcal{O})$-torsor over Gr_G. In particular, by viewing another copy of Gr_G as a $G(\mathcal{O})$-scheme, we can form the associated fibration

$$\mathrm{Gr}_G \star \mathrm{Gr}_G := G(\mathcal{K}) \underset{G(\mathcal{O})}{\times} \mathrm{Gr}_G = G(\mathcal{K}) \underset{G(\mathcal{O})}{\times} G(\mathcal{K})/G(\mathcal{O}).$$

One has the natural maps $p, m\colon \mathrm{Gr}_G \star \mathrm{Gr}_G \to \mathrm{Gr}_G$ defined as follows. Let $g \in G(\mathcal{K})$, $x \in \mathrm{Gr}_G$. Then

$$p(g \times x) = g \bmod G(\mathcal{O}); \quad m(g \times x) = g \cdot x.$$

For any $\lambda_1, \lambda_2 \in \Lambda_G^+$ let us set $\mathrm{Gr}_G^{\lambda_1} \star \mathrm{Gr}_G^{\lambda_2}$ to be the corresponding subscheme of $\mathrm{Gr}_G \star \mathrm{Gr}_G$; this is a fibration over $\mathrm{Gr}_G^{\lambda_1}$ with the typical fiber $\mathrm{Gr}_G^{\lambda_2}$. Its closure is $\overline{\mathrm{Gr}}^{\lambda_1} \star \overline{\mathrm{Gr}}^{\lambda_2}$. In addition, we define

$$(\mathrm{Gr}_G^{\lambda_1} \star \mathrm{Gr}_G^{\lambda_2})^{\lambda_3} = m^{-1}(\mathrm{Gr}_G^{\lambda_3}) \cap (\mathrm{Gr}_G^{\lambda_1} \star \mathrm{Gr}_G^{\lambda_2}).$$

It is known (cf. [18]) that

$$\dim((\mathrm{Gr}_G^{\lambda_1} \star \mathrm{Gr}_G^{\lambda_2})^{\lambda_3}) = \langle \lambda_1 + \lambda_2 + \lambda_3, \rho_G^\vee \rangle. \tag{1.1}$$

(It is easy to see that although $\rho_G^\vee \in \frac{1}{2}\Lambda_G^\vee$, the RHS of (1.1) is an integer whenever the above intersection is non-empty.)

Starting from any perverse sheaf $\mathcal{T}$ on Gr_G and a $G(\mathcal{O})$-equivariant perverse sheaf $\mathcal{S}$ on Gr_G, we can form their twisted external product $\mathcal{T}\widetilde{\boxtimes}\mathcal{S}$ (see e.g. Section 4 of [19]), which will be a perverse sheaf on $\mathrm{Gr}_G \star \mathrm{Gr}_G$. For two objects $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ we define their convolution

$$\mathcal{S}_1 \star \mathcal{S}_2 = m_!(\mathcal{S}_1 \widetilde{\boxtimes} \mathcal{S}_2).$$

The following theorem, which is a categorical version of the Satake equivalence, is a starting point for this paper, cf. [18], [16] and [19]. The best reference so far is [2, Sect. 5.3].

Theorem 1.5. (1) *Let* $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$. *Then* $\mathcal{S}_1 \star \mathcal{S}_2 \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$.

(2) *The convolution* $\star$ *extends to a tensor category structure on* $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$.

(3) *As a tensor category,* $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ *is equivalent to the category Rep*$(G^\vee)$. *Under this equivalence, the object* IC^λ *goes over to the irreducible representation* $L(\lambda)$ *of* $G^\vee$ *with highest weight* λ.

1.6. The equivalence $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathrm{Rep}(G^\vee)$ is given by a fiber functor [2], [19] of integration over semiinfinite orbits. Namely, let $N_- \subset G$ be the unipotent radical of the negative Borel subgroup, and let $\mathfrak{T}_\lambda \subset \mathrm{Gr}_G$ be the orbit of $N_-(\mathcal{K})$ through the point $s^\lambda \in \mathrm{Gr}_G$. Then the weight λ component of the fiber functor is given by the cohomology with supports in $\mathfrak{T}_\lambda$. Let us recall an equivalent construction of this fiber functor.

From now on we assume that G is almost simple and simply connected. We consider a smooth curve $\boldsymbol{C}$ of genus 0 with two marked points $0, \infty$. Let Bun_G (resp. Bun_B) stand for the moduli stack of G-bundles (resp. B-bundles) on $\boldsymbol{C}$. Here B is the positive Borel subgroup of G. The natural morphism $\mathrm{Bun}_B \to \mathrm{Bun}_G$ is *not* proper, and Drinfeld has discovered a natural relative compactification $\overline{\mathrm{Bun}}_B$ of Bun_B. It is the moduli stack of the following data:

(a) A G-bundle $\mathcal{F}_G$ on $\boldsymbol{C}$;

(b) For each dominant weight $\check{\lambda}$ of G, an invertible subsheaf $\mathcal{L}^{\check{\lambda}} \subset \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$. Here $V^{\check{\lambda}}$ stands for the irreducible G-module with highest weight $\check{\lambda}$, and $\mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$ stands for the associated vector bundle on $\boldsymbol{C}$.

The collection of invertible subsheaves $\mathcal{L}^{\check{\lambda}} \subset \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G}$ should satisfy the *Plücker relations*, that is, for any dominant weights $\check{\lambda}$ and $\check{\mu}$, the tensor product $\mathcal{L}^{\check{\lambda}} \otimes \mathcal{L}^{\check{\mu}} \subset \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G} \otimes \mathcal{V}^{\check{\mu}}_{\mathcal{F}_G}$ should coincide with $\mathcal{L}^{\check{\lambda}+\check{\mu}} \subset \mathcal{V}^{\check{\lambda}+\check{\mu}}_{\mathcal{F}_G}$ under the natural direct summand embedding $\mathcal{V}^{\check{\lambda}+\check{\mu}}_{\mathcal{F}_G} \hookrightarrow \mathcal{V}^{\check{\lambda}}_{\mathcal{F}_G} \otimes \mathcal{V}^{\check{\mu}}_{\mathcal{F}_G}$.

The connected components of $\overline{\mathrm{Bun}}_B$ are numbered by the coweights $\lambda \in \Lambda$: for $(\mathcal{L}^{\check{\lambda}}) \in \overline{\mathrm{Bun}}_B^{\lambda}$ we have $\deg \mathcal{L}^{\check{\lambda}} = -\langle \lambda, \check{\lambda} \rangle$.

We will denote by $\Lambda^{\mathrm{pos}} \subset \Lambda = \Lambda_G$ the cone of nonnegative linear combinations of positive coroots of G. For every $\alpha \in \Lambda^{\mathrm{pos}}$ we consider the closed embedding $i_\alpha\colon \overline{\mathrm{Bun}}_B \hookrightarrow \overline{\mathrm{Bun}}_B$ given by sending $(\mathcal{F}_G, \mathcal{L}^{\check\lambda} \subset \mathcal{V}^{\check\lambda}_{\mathcal{F}_G})$ to $(\mathcal{F}_G, \mathcal{L}^{\check\lambda}(-\langle\alpha, \check\lambda\rangle\cdot 0) \subset \mathcal{V}^{\check\lambda}_{\mathcal{F}_G})$.

Now let $\overline{\mathcal{H}}{}^\lambda_0 \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ stand for the Hecke correspondence at the point $0 \in \boldsymbol{C}$: the pairs of G-bundles $(\mathcal{F}_G, \mathcal{F}'_G)$ together with an isomorphism $\sigma\colon \mathcal{F}_G \to \mathcal{F}'_G$ off $0 \in \boldsymbol{C}$ whose pole at $0 \in \boldsymbol{C}$ has order less than or equal to λ. The fibers of the projection p_1 (resp. p_2) of $\overline{\mathcal{H}}{}^\lambda_0$ to the first (resp. second) copy of Bun_G are both isomorphic to $\overline{\mathrm{Gr}_G}^\lambda$.

We define the Hecke correspondence

$$(p, \phi)\colon \overline{\mathcal{G}}{}^\lambda_0 = \overline{\mathcal{H}}{}^\lambda_0 \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B \times \overline{\mathrm{Bun}}_B.$$

It is the moduli stack of the following data:

(a) a pair of G-bundles $\mathcal{F}_G$ and $\mathcal{F}'_G$ together with an isomorphism off $0 \in \boldsymbol{C}$ lying in $\overline{\mathcal{H}}{}^\lambda_0$;

(b) For each dominant weight $\check\lambda$ of G, an invertible subsheaf $\mathcal{L}^{\check\lambda} \subset \mathcal{V}^{\check\lambda}_{\mathcal{F}'_G}$ satisfying the Plücker relations.

Forgetting the datum of $\mathcal{F}_G$ defines the morphism $p\colon \overline{\mathcal{G}}{}^\lambda_0 \to \overline{\mathrm{Bun}}_B$. The morphism $\phi\colon \overline{\mathcal{G}}{}^\lambda_0 \to \overline{\mathrm{Bun}}_B$ is defined as follows. The condition $(\mathcal{F}_G, \mathcal{F}'_G) \in \overline{\mathcal{H}}{}^\lambda_0$ implies $\mathcal{V}^{\check\lambda}_{\mathcal{F}'_G} \subset \mathcal{V}^{\check\lambda}_{\mathcal{F}_G}(\langle -w_0\lambda, \check\lambda\rangle \cdot 0)$ for every dominant weight $\check\lambda$. Hence $\mathcal{L}^{\check\lambda}(\langle w_0\lambda, \check\lambda\rangle \cdot 0) \subset \mathcal{V}^{\check\lambda}_{\mathcal{F}_G}$, and we set $\phi(\mathcal{F}_G, \mathcal{F}'_G, \mathcal{L}^{\check\lambda} \subset \mathcal{V}^{\check\lambda}_{\mathcal{F}'_G}) := (\mathcal{F}_G, \mathcal{L}^{\check\lambda}(\langle w_0\lambda, \check\lambda\rangle \cdot 0) \subset \mathcal{V}^{\check\lambda}_{\mathcal{F}_G})$.

Finally, we are able to state a theorem (see [11, Theorem 3.1.4] and [14, Theorem 13.2]) providing a version of the fiber functor from the category $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ to $\mathrm{Rep}(G^\vee)$. For a finite-dimensional $G^\vee$-module V, and $\mu \in \Lambda_G$, we denote the μ-weight subspace of V by $V(\mu)$.

Theorem 1.7. $\phi_! \, \mathrm{IC}(\overline{\mathcal{G}}{}^\lambda_0) \simeq \bigoplus\limits_{\alpha \in \Lambda^{\mathrm{pos}}_G} i_{\alpha !}\, \mathrm{IC}(\overline{\mathrm{Bun}}_B) \otimes V^\lambda(w_0(\lambda) + \alpha).$

1.8. The goal of the present paper is to formulate an analogue of Theorem 1.7 for the double affine Grassmannian. However, as we have seen in [4], [6], the affine versions of the objects like $\overline{\mathrm{Gr}_G}^\lambda$ or $\overline{\mathrm{Bun}}_B$ are out of reach at the moment being "too global", and have to be replaced by certain transversal slices.

A transversal slice to the closed embedding $i_\alpha\colon \overline{\mathrm{Bun}}_B \hookrightarrow \overline{\mathrm{Bun}}_B$ is a well known *Drinfeld zastava* space Z^α (see [14], [9]). It is defined as the moduli scheme of collections of invertible subsheaves $\mathcal{L}^{\check\lambda} \subset \mathcal{O}_{\boldsymbol{C}} \otimes V^{\check\lambda}$ satisfying the Plücker relations, the degree conditions $\deg \mathcal{L}^{\check\lambda} = -\langle\alpha, \check\lambda\rangle$, and the conditions at $\infty \in \boldsymbol{C}$: each $\mathcal{L}^{\check\lambda} \subset \mathcal{O}_{\boldsymbol{C}} \otimes V^{\check\lambda}$ is a *line subbundle* near $\infty \in \boldsymbol{C}$, and the fiber $\mathcal{L}^{\check\lambda}|_\infty$ coincides with the highest line in $V^{\check\lambda}$.

By construction, we have a locally closed embedding $z_\alpha\colon Z^\alpha \hookrightarrow \overline{\mathrm{Bun}}_B$, and we define the scheme $\overline{\mathcal{G}Z}^{\lambda,\alpha}$ as the cartesian product of $z_\alpha\colon Z^\alpha \hookrightarrow \overline{\mathrm{Bun}}_B$ and $\phi\colon \overline{\mathcal{G}}{}^\lambda_0 \to \overline{\mathrm{Bun}}_B$.

For any $\beta \leqslant \alpha \in \Lambda^{\mathrm{pos}}_G$ we also have a closed embedding $i^\alpha_\beta\colon Z^{\alpha-\beta} \hookrightarrow Z^\alpha$ which sends a collection $(\mathcal{L}^{\check\lambda} \subset \mathcal{O}_{\boldsymbol{C}} \otimes V^{\check\lambda})$ to a collection $(\mathcal{L}^{\check\lambda}(-\langle\beta, \check\lambda\rangle \cdot 0) \subset \mathcal{O}_{\boldsymbol{C}} \otimes V^{\check\lambda})$.

Now Theorem 1.7 can be equivalently formulated as follows (see Theorem 13.2 of [14]):

Theorem 1.9. $\phi_! \mathrm{IC}(\overline{\mathcal{G}Z}^{\lambda,\alpha}) \simeq \bigoplus_{\beta \leqslant \alpha} i^{\alpha}_{\beta !} \mathrm{IC}(Z^{\alpha-\beta}) \otimes V^{\lambda}(\lambda - \beta)$.

The key observation underlying the proof of the theorem is that $\phi^{-1}(i^{\alpha}_{\alpha}(0)) \cong \overline{\mathfrak{T}}_{\lambda-\alpha} \cap \overline{\mathrm{Gr}}^{\lambda}_{G}$.

1.10. The group G_{aff}. To a connected reductive group G as above one can associate the corresponding affine Kac–Moody group G_{aff} in the following way. One can consider the polynomial loop group $G[t, t^{-1}]$ (this is an infinite-dimensional group ind-scheme)

It is well-known that $G[t, t^{-1}]$ possesses a canonical central extension $\widetilde{G}$ of $G[t, t^{-1}]$:

$$1 \to \mathbb{G}_m \to \widetilde{G} \to G[t, t^{-1}] \to 1.$$

Moreover, $\widetilde{G}$ has again a natural structure of a group ind-scheme.

The multiplicative group $\mathbb{G}_m$ acts naturally on $G[t, t^{-1}]$ and this action lifts to $\widetilde{G}$. We denote the corresponding semi-direct product by G_{aff}; we also let $\mathfrak{g}_{\mathrm{aff}}$ denote its Lie algebra.

The Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ is an untwisted affine Kac–Moody Lie algebra. In particular, it can be described by the corresponding affine root system. We denote by $\mathfrak{g}^{\vee}_{\mathrm{aff}}$ the *Langlands dual affine Lie algebra* (which corresponds to the dual affine root system) and by $G^{\vee}_{\mathrm{aff}}$ the corresponding dual affine Kac–Moody group, normalized by the property that it contains $G^{\vee}$ as a subgroup (cf. [4, Section 3.1] for more details).

We denote by $\Lambda_{\mathrm{aff}} = \mathbb{Z} \times \Lambda \times \mathbb{Z}$ the coweight lattice of G_{aff}; this is the same as the weight lattice of $G^{\vee}_{\mathrm{aff}}$. Here the first $\mathbb{Z}$-factor is responsible for the center of $G^{\vee}_{\mathrm{aff}}$ (or $\widehat{G}^{\vee}$); it can also be thought of as coming from the loop rotation in G_{aff}. The second $\mathbb{Z}$-factor is responsible for the loop rotation in $G^{\vee}_{\mathrm{aff}}$ it may also be thought of as coming from the center of G_{aff}). We also denote $\mathbb{Z} \times \Lambda \subset \mathbb{Z} \times \Lambda \times \mathbb{Z}$ by $\widehat{\Lambda}$, and we denote $k \times \Lambda \subset \mathbb{Z} \times \Lambda$ by $\widehat{\Lambda}_k \subset \widehat{\Lambda}$. We denote by $\Lambda^{+}_{\mathrm{aff}}$ the set of dominant weights of $G^{\vee}_{\mathrm{aff}}$ (which is the same as the set of dominant coweights of G_{aff}). We also denote by $\Lambda_{\mathrm{aff},k}$ the set of weights of $G^{\vee}_{\mathrm{aff}}$ of level k, i.e., all the weights of the form $(k, \overline{\lambda}, n)$. We put $\Lambda^{+}_{\mathrm{aff},k} = \Lambda^{+}_{\mathrm{aff}} \cap \Lambda_{\mathrm{aff},k}$.

Important notational convention. From now on we shall denote elements of Λ by $\overline{\lambda}, \overline{\mu}, \ldots$ (instead of just writing $\lambda, \mu, \ldots$ in order to distinguish them from the coweights of G_{aff} (= weights of $G^{\vee}_{\mathrm{aff}}$), which we shall just denote by $\lambda, \mu, \ldots$

Let $\Lambda^{+}_{k} \subset \Lambda$ denote the set of dominant coweights of G such that $\langle \overline{\lambda}, \alpha \rangle \leqslant k$ when α is the highest root of $\mathfrak{g}$. Then it is well-known that a weight $(k, \overline{\lambda}, n)$ of $G^{\vee}_{\mathrm{aff}}$ lies in $\Lambda^{+}_{\mathrm{aff},k}$ if and only if $\overline{\lambda} \in \Lambda^{+}_{k}$ (thus $\Lambda_{\mathrm{aff},k} = \Lambda^{+}_{k} \times \mathbb{Z}$).

Let also W_{aff} denote affine Weyl group of G, which is the semi-direct product of W and Λ. It acts on the lattice Λ_{aff} (resp. $\widehat{\Lambda}$) preserving each $\Lambda_{\mathrm{aff},k}$ (resp. each $\widehat{\Lambda}_k$). In order to describe this action explicitly it is convenient to set $W_{\mathrm{aff},k} = W \ltimes k\Lambda$, which naturally acts on Λ. Of course the groups $W_{\mathrm{aff},k}$ are canonically isomorphic to W_{aff} for all k. Then the restriction of the W_{aff}-action to $\Lambda_{\mathrm{aff},k} \simeq \Lambda \times \mathbb{Z}$ comes from the natural $W_{\mathrm{aff},k}$-action on the first multiple.

It is well known that every W_{aff}-orbit on $\Lambda_{\mathrm{aff},k}$ contains a unique element of $\Lambda^{+}_{\mathrm{aff},k}$. This is equivalent to saying that $\Lambda^{+}_{k} \simeq \Lambda / W_{\mathrm{aff},k}$.

1.11. Our main dream is to create an analog of the affine Grassmannian Gr_G and the above results about it in the case when G is replaced by the (infinite-dimensional) group G_{aff}. The first attempt to do so was made in [4]: namely, in *loc. cit.* we have constructed analogs of the varieties $\overline{\mathcal{W}}^{\lambda}_{G,\mu}$ in the case when G is replaced by G_{aff}. In [6], we constructed analogs of the varieties $m_n^{-1}(\overline{\mathcal{W}}^{\lambda}_{G,\mu}) \cap (\mathrm{Gr}^{\lambda_1}_G \star \cdots \star \mathrm{Gr}^{\lambda_n}_G)$ and $m_n^{-1}(\overline{\mathcal{W}}^{\lambda}_{G,\mu}) \cap (\overline{\mathrm{Gr}}^{\lambda_1}_G \star \cdots \star \overline{\mathrm{Gr}}^{\lambda_n}_G)$ (here $\lambda = \lambda_1 + \cdots + \lambda_n$) when G is replaced by G_{aff}. We have also constructed analogs of the corresponding pieces in the *Beilinson–Drinfeld Grassmannian* for G_{aff}.

We will denote by $\Lambda^{\mathrm{pos}}_{\mathrm{aff}}$ the cone of nonnegative linear combinations of positive roots of $G^{\vee}_{\mathrm{aff}}$. For $\alpha \in \Lambda^{\mathrm{pos}}_{\mathrm{aff}}$ the affine Drinfeld zastava space Z^{α} was constructed in [8]. It is a certain closure of the space of degree α based maps from $(\boldsymbol{C}, \infty)$ to the Kashiwara flag scheme of G_{aff}. We also have parabolic versions $Z^{\theta}_{G_{\mathrm{aff}},P}$ of $Z^{\alpha} = Z^{\alpha}_{G_{\mathrm{aff}},I}$ (I stands for the Iwahori subgroup of G_{aff}), which are certain closures of the spaces of based maps from $(\boldsymbol{C}, \infty)$ to the Kashiwara parabolic flag schemes. Among those, the Uhlenbeck space $\mathcal{U}^a_G(\mathbb{A}^2) = Z^a_{G_{\mathrm{aff}},G[t]}$ stands out: it corresponds to the maximal parabolic containing all the finite simple roots.

Unfortunately, the definition of zastava given in Section 1.8 produces in the affine case a scheme of *infinite type* $\boldsymbol{Z}^{\alpha}$. In the maximal parabolic case, the Uhlenbeck space $\mathcal{U}^a_G(\mathbb{A}^2)$ is a partial resolution of $\boldsymbol{Z}^a_{G_{\mathrm{aff}},G[t]}$. We have a natural forgetting morphism $\boldsymbol{Z}^{\alpha} \to \boldsymbol{Z}^a_{G_{\mathrm{aff}},G[t]}$, where a is the coefficient of the affine simple root in α, and Z^{α} is defined as the cartesian product of $\boldsymbol{Z}^{\alpha}$ and $\mathcal{U}^a_G(\mathbb{A}^2)$ over $\boldsymbol{Z}^a_{G_{\mathrm{aff}},G[t]}$. It is an affine scheme of finite type.

The disadvantage of the above definition is that Z^{α} does not solve any moduli problem, and hence is very cumbersome to work with. However, in the case $G = \mathrm{SL}(N)$, the zastava space Z^{α} possesses a semismall resolution of singularities $\mathcal{P}^{\alpha}$, an *affine Laumon space* [13], which is a moduli space of parabolic sheaves on $\boldsymbol{C} \times \mathbb{P}^1$. Moreover, according to [15], $\mathcal{P}^{\alpha}$ admits a realization as a quiver variety, i.e., as a certain GIT quotient. The corresponding categorical quotient $\mathfrak{Z}^{\alpha}$ is an affine reduced irreducible normal scheme isomorphic to Z^{α} (see [15], [7]).

1.12. The main result of the present paper is a construction of an affine version of the scheme $\overline{\mathcal{G}Z}^{\lambda,\alpha}$ equipped with a morphism ϕ to the affine zastava Z^{α}. It is constructed as a quiver variety in the case $G = \mathrm{SL}(N)$, and then for general G via the adjoint homomorphism $G \to \mathrm{SL}(\mathfrak{g})$. We conjecture that Theorem 1.9 holds true in the affine setting as well.

Although we cannot describe $\overline{\mathcal{G}Z}^{\lambda,\alpha}$ as a solution of a moduli problem, its open subscheme $\mathcal{G}Z^{\lambda,\alpha}$ does admit such a description. Let us first assume λ has level 1. We consider the projective plane $\mathbb{P}^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$ such that the line ℓ_∞ "at infinity" is given by the equation $z_0 = 0$, while $\boldsymbol{C} = \ell_0 \subset \mathbb{P}^2$ is given by the equation $z_2 = 0$. We consider the blowup $\widehat{\mathbb{P}}^2$ at the origin ($z_1 = z_2 = 0$), and keep the names ℓ_∞ and ℓ_0 for the proper transforms of ℓ_∞ and ℓ_0. Then $\mathcal{G}Z^{\lambda,\alpha}$ is the moduli space of G-bundles on $\widehat{\mathbb{P}}^2$ equipped with a reduction to B along ℓ_0 framed at ℓ_∞. Note that even with this modular definition, the construction of projection $\phi\colon \mathcal{G}Z^{\lambda,\alpha} \to Z^{\alpha}$ is rather nontrivial, cf. Section 3.12. For an explanation why the moduli space of G-bundles on the blowup appears in the convolution diagram of double affine Grassmannian and affine zastava, the interested reader may consult [12, Section 8].

Let us now assume λ has level k. We consider the blowup $\widehat{\mathbb{P}}^2_k$ of $\mathbb{P}^2$ at the origin, but not at the maximal ideal of the origin this time; rather at the ideal generated locally by (z_1^k, z_2). This blowup has an isolated singularity of Kleinian type A_{k-1} lying off the proper transforms of ℓ_∞ and ℓ_0. We consider the stacky resolution $\widehat{\mathcal{S}}^k$ of $\widehat{\mathbb{P}}^2_k$. Then again $\mathcal{G}Z^{\lambda,\alpha}$ is the moduli space of G-bundles on $\widehat{\mathcal{S}}^k$ equipped with a reduction to B along ℓ_0 framed at ℓ_∞. The projection $\phi\colon \mathcal{G}Z^{\lambda,\alpha} \to Z^\alpha$ is constructed in Section 4.7. Similarly to the finite dimensional case of Section 1.8, we have $\phi^{-1}(i^\alpha_\alpha(0)) \cong \mathfrak{T}_{\lambda-\alpha} \cap \mathrm{Gr}^\lambda_{G_{\mathrm{aff}}}$; for the definition of the RHS and the proof of the isomorphism, see [12, Section 8, especially Proposition 8.7].

In the special case when $\lambda - \alpha = \mu := (k, 0, 0)$, we have an intermediate open subspace $\mathcal{G}Z^{\lambda,\alpha} \subset \overline{\mathcal{W}}^\lambda_{G_{\mathrm{aff}},\mu} \subset \overline{\mathcal{G}Z}^{\lambda,\alpha}$. In Section 3.2 of [5] we have defined the *repellents* $\mathfrak{T}^e_\mu \subset \overline{\mathcal{W}}^\lambda_{G_{\mathrm{aff}},\mu}$; they were also considered in [22] under the name of *MV cycles*. We conjecture that the central fiber $\phi^{-1}(i^\alpha_\alpha(0)) \cap \overline{\mathcal{W}}^\lambda_{G_{\mathrm{aff}},\mu}$ coincides with $\mathfrak{T}^e_\mu$, and we prove the inclusion $\mathfrak{T}^e_\mu \subset \phi^{-1}(i^\alpha_\alpha(0)) \cap \overline{\mathcal{W}}^\lambda_{G_{\mathrm{aff}},\mu}$ in Proposition 5.4.

1.13. Structure of the paper. In Section 2 we recall the description of the affine zastava $Z^\alpha_{\mathrm{SL}(N)_{\mathrm{aff}}}$ in terms of representations of the *chainsaw quiver* of [15]. Contrary to the "global" approach of *loc. cit.*, we follow the classical ADHM approach on a 2-dimensional toric Deligne–Mumford stack $\mathcal{S}_N = \mathbb{P}^1/\underline{\mu}_N \times \mathbb{P}^1$. Here $\underline{\mu}_N$ is the group of N-th roots of unity, acting on $\mathbb{P}^1$ with fixed points 0, ∞, and the quotient is categorical near ∞, and stacky near 0. In Section 2.8–Section 2.11 we describe the irreducible components of the fixed point set $(Z^\alpha_{\mathrm{SL}(N)_{\mathrm{aff}}})^{\Gamma_k}$ of the action of a cyclic group Γ_k. In the central Section 3 we describe the parabolic torsion free sheaves on the blowup $\widehat{\mathbb{P}}^2$ in terms of the *dented chainsaw quiver* $\widehat{Q}$ (Section 3.1). The description is modeled on the one in [23] for torsion free sheaves on the blowup. The key Theorem 3.4 identifying the moduli space of parabolic torsion free sheaves on the blowup with a moduli space of $\widehat{Q}$-modules is due to A. Kuznetsov. We introduce the zastava space for the blowup as the moduli space of $\widehat{Q}$-modules with certain stability conditions (Section 3.2). In Section 4 we introduce the zastava space for the Kleinian blowup $\widehat{\mathcal{S}}^k$ (Section 1.12) via a trick identifying it with a Γ_k-fixed points component in the zastava space on the blowup $\widehat{\mathbb{P}}^2$. This allows us to describe it as a moduli space of representations of the *rift quiver* (Section 4.3, Theorem 4.5). Finally, in Section 5, for an arbitrary almost simple simply connected group G, we define the zastava space for the Kleinian blowup in terms of the one for $\mathrm{SL}(\mathfrak{g})$.

1.14. Acknowledgments. It is clear from the above that the paper owes its existence to A. Kuznetsov's generous explanations. We are grateful to him for the permission to reproduce his proof of the key Theorem 3.4. As our masters put it, "Il avait été d'abord prévu que A. Kuznetsov soit coauteur du présent article. Il a préféré s'en abstenir, pour ne pas être corresponsable des erreurs ou imprécisions qui s'y trouvent. Il n'en est pas moins responsable de bien des idées que nous exploitons." Thanks are due to the referee for the careful reading of the manuscript

and his valuable comments and suggestions. We are happy to thank the IAS at the Hebrew University of Jerusalem for the excellent working conditions.

2. Zastava as a Quiver Variety

2.1. Chainsaw. We recall some material from Section 2 of [15]. We consider the representations of the following *chainsaw quiver* Q

$$\begin{tikzcd}
\dots \arrow[r, "B_{-3}"] & V_{-2} \arrow[loop above, "A_{-2}"] \arrow[r, "B_{-2}"] \arrow[d, "q_{-2}"] & V_{-1} \arrow[loop above, "A_{-1}"] \arrow[r, "B_{-1}"] \arrow[d, "q_{-1}"] & V_0 \arrow[loop above, "A_0"] \arrow[r, "B_0"] \arrow[d, "q_0"] & V_1 \arrow[loop above, "A_1"] \arrow[r, "B_1"] \arrow[d, "q_1"] & V_2 \arrow[loop above, "A_2"] \arrow[r, "B_2"] \arrow[d, "q_2"] & \dots \\
\dots \arrow[ur, "p_{-2}"] & W_{-2} \arrow[ur, "p_{-1}"] & W_{-1} \arrow[ur, "p_0"] & W_0 \arrow[ur, "p_1"] & W_1 \arrow[ur, "p_2"] & W_2 \arrow[ur, "p_3"] & \dots
\end{tikzcd}$$

with relations $A_{l+1}B_l - B_lA_l + p_{l+1}q_l = 0\ \forall l$. Here the lower indices run through $\mathbb{Z}/N\mathbb{Z}$, and $\dim V_l = d_l$, $\dim W_l = 1$. We denote by $\underline{d}$ the collection of positive integers $(d_l)_{l\in\mathbb{Z}/N\mathbb{Z}}$. We denote by $\mathsf{M}_{\underline{d}}$ the scheme of representations of Q: a closed subscheme of

$$\bigoplus_{l\in\mathbb{Z}/N\mathbb{Z}} \mathrm{End}\,(V_l) \oplus \bigoplus_{l\in\mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}(V_l, V_{l+1}) \oplus \bigoplus_{l\in\mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}(W_{l-1}, V_l) \oplus \bigoplus_{l\in\mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}(V_l, W_l)$$

given by equations $A_{l+1}B_l - B_lA_l + p_{l+1}q_l = 0\ \forall l$. We denote by $G_{\underline{d}}$ the group $\prod_{l\in\mathbb{Z}/N\mathbb{Z}} \mathrm{GL}(V_l)$; it acts naturally on $\mathsf{M}_{\underline{d}}$. We denote by $\mathfrak{Z}_{\underline{d}}$ the categorical quotient $\mathsf{M}_{\underline{d}} /\!/ G_{\underline{d}}$. According to [15, Theorem 2.7] and [7, Theorem 3.5], $\mathfrak{Z}_{\underline{d}}$ is a reduced irreducible normal scheme isomorphic to the affine Drinfeld zastava space $Z^{\underline{d}}_{\mathrm{SL}(N)}$ introduced in [8].

Furthermore, we consider an open subscheme $\mathsf{M}^s_{\underline{d}} \subset \mathsf{M}_{\underline{d}}$ formed by all the *stable* representations of Q, i.e., those $(A_l, B_l, p_l, q_l)_{l\in\mathbb{Z}/N\mathbb{Z}} \in \mathsf{M}_{\underline{d}}$ such that there is no proper $\mathbb{Z}/N\mathbb{Z}$-graded subspace $V'_\bullet \subset V_\bullet$ stable under $A_\bullet$, $B_\bullet$ and containing $p(W_\bullet)$. Then the action of $G_{\underline{d}}$ on $\mathsf{M}^s_{\underline{d}}$ is free, and the GIT quotient $\mathfrak{M}_{\underline{d}} = \mathsf{M}^s_{\underline{d}}/G_{\underline{d}}$ is a semismall resolution of $\mathfrak{Z}_{\underline{d}}$. Moreover, according to Section 2.3 of [15], $\mathfrak{M}_{\underline{d}}$ is isomorphic to the moduli space $\mathcal{P}_{\underline{d}}$ of torsion free parabolic sheaves of degree $\underline{d}$ on a surface $\boldsymbol{S}$. Here $\boldsymbol{S}$ is the product of two projective lines $\boldsymbol{C}$ and $\boldsymbol{X}$ with marked points $0_{\boldsymbol{X}}, \infty_{\boldsymbol{X}} \in \boldsymbol{X}$ and $0_{\boldsymbol{C}}, \infty_{\boldsymbol{C}} \in \boldsymbol{C}$. The sheaves in question are equipped with a parabolic structure along a line $\boldsymbol{D}_0 := \boldsymbol{C} \times 0_{\boldsymbol{X}}$, and with a trivialization at "infinity" $\boldsymbol{D}_\infty := \boldsymbol{C} \times \infty_{\boldsymbol{X}} \cup \infty_{\boldsymbol{C}} \times \boldsymbol{X}$. The isomorphism $\mathfrak{M}_{\underline{d}} \simeq \mathcal{P}_{\underline{d}}$ is deduced in *loc. cit.* from the "parabolic vs. orbifold" correspondence of [3] by global considerations. We will rephrase the argument in more local terms in Section 2.4 and Section 2.5 after some preparation in Section 2.2 and Section 2.3.

2.2. ADHM. To warm up we recall the classical ADHM construction (see e.g. Section 2 of [20]) following the approach of Section 5 of [1]. To this end we introduce the homogeneous coordinates $(z : t)$ (resp. $(y : x)$) on $\boldsymbol{C}$ (resp. $\boldsymbol{X}$) such that $0_{\boldsymbol{C}}$ (resp. $0_{\boldsymbol{X}}$) is given by $z = 0$ (resp. $y = 0$), and $\infty_{\boldsymbol{C}}$ (resp. $\infty_{\boldsymbol{X}}$) is given by $t = 0$ (resp. $x = 0$). The ADHM construction goes as follows. We consider the vector spaces $V = \mathbb{C}^d$, $W = \mathbb{C}^N$, and the subscheme $\mathsf{M}_{N,d} \subset \mathrm{End}\,(V) \oplus$

$\mathrm{End}\,(V) \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W)$ cut out by the equation $AB - BA + pq = 0$ $(A, B \in \mathrm{End}\,(V),\ p \in \mathrm{Hom}(W, V),\ q \in \mathrm{Hom}(V, W))$. We consider an open subscheme $\mathsf{M}^s_{N,d} \subset \mathsf{M}_{N,d}$ formed by all the *stable* quadruples (A, B, p, q), i.e., such that V has no proper subspaces stable under A, B and containing $p(W)$. The group $\mathrm{GL}(V)$ acts naturally on $\mathsf{M}_{N,d}$; its action on $\mathsf{M}^s_{N,d}$ is free, and the GIT quotient $\mathsf{M}^s_{N,d}/\mathrm{GL}(V)$ is denoted by $\mathfrak{M}_{N,d}$. It is isomorphic to the moduli space $\mathcal{M}_{N,d}$ of torsion free sheaves of rank N and degree d on $\boldsymbol{S}$ trivialized at $\boldsymbol{D}_\infty$. Namely, $(A, B, p, q) \in \mathfrak{M}_{N,d}$ goes to the middle cohomology of the following monad of vector bundles on $\boldsymbol{S}$:

$$V \otimes \mathcal{O}_{\boldsymbol{S}}(-1, -1) \xrightarrow{C} \begin{pmatrix} V \otimes \mathcal{O}_{\boldsymbol{S}}(0, -1) \\ \oplus \\ V \otimes \mathcal{O}_{\boldsymbol{S}}(-1, 0) \\ \oplus \\ W \otimes \mathcal{O}_{\boldsymbol{S}} \end{pmatrix} \xrightarrow{D} V \otimes \mathcal{O}_{\boldsymbol{S}},$$

$C = (tA - z, xB - y, txq)$, $D = (-xB + y, tA - z, p)$, where we write $\mathcal{O}_{\boldsymbol{S}}(-1, -1)$ for $\mathcal{O}_{\boldsymbol{C}}(-1) \boxtimes \mathcal{O}_{\boldsymbol{X}}(-1)$, and $\mathcal{O}_{\boldsymbol{S}}(0, -1)$ for $\mathcal{O}_{\boldsymbol{C}} \boxtimes \mathcal{O}_{\boldsymbol{X}}(-1)$, etc., and we view x, y (resp. z, t) as a basis of $\Gamma(\boldsymbol{X}, \mathcal{O}_{\boldsymbol{X}}(1))$ (resp. $\Gamma(\boldsymbol{C}, \mathcal{O}_{\boldsymbol{C}}(1))$).

2.3. Stack $\mathcal{S}_{\boldsymbol{N}}$. We define a one-dimensional Deligne–Mumford stack $\mathcal{X}_N$ as follows. Let $\boldsymbol{X}_N \xrightarrow{\theta} \boldsymbol{X}$ denote the N-fold cyclic covering ramified over $0_{\boldsymbol{X}}$ and $\infty_{\boldsymbol{X}}$. It is equipped with the action of the Galois group $\Gamma_N \simeq \mathbb{Z}/N\mathbb{Z}$. The action of Γ_N on $\theta^{-1}(\boldsymbol{X} - 0_{\boldsymbol{X}} - \infty_{\boldsymbol{X}})$ is free, and the quotient is $\boldsymbol{X} - 0_{\boldsymbol{X}} - \infty_{\boldsymbol{X}}$. We glue the categorical quotient $\theta^{-1}(\boldsymbol{X} - 0_{\boldsymbol{X}})/\!/\Gamma_N = \boldsymbol{X} - 0_{\boldsymbol{X}}$ with the stack quotient $\theta^{-1}(\boldsymbol{X} - \infty_{\boldsymbol{X}})/\Gamma_N$ over the common open substack $\boldsymbol{X} - 0_{\boldsymbol{X}} - \infty_{\boldsymbol{X}}$ to obtain the desired stack $\mathcal{X}_N$. Note that $\mathcal{X}_N$ is equipped with a projection ϑ to $\boldsymbol{X}$ which is an isomorphism off $0_{\boldsymbol{X}}$. The unique point of $\mathcal{X}_N$ lying over $0_{\boldsymbol{X}}$ will be denoted by $0_{\mathcal{X}}$; its group of automorphisms is Γ_N. The unique point of $\mathcal{X}_N$ lying over $\infty_{\boldsymbol{X}}$ will be denoted by $\infty_{\mathcal{X}}$. Since N is fixed throughout the Section, we will omit the lower index N to simplify the notations.

We denote $\mathcal{O}_{\mathcal{X}}(\pm\infty_{\mathcal{X}})$ by $\mathcal{O}_{\mathcal{X}}(\pm N)$. For $0 \leqslant l \leqslant N$ we denote $\mathcal{O}_{\mathcal{X}}(-l \cdot 0_{\mathcal{X}})$ by $\mathcal{R}_l$. Note that $\mathcal{R}_N \simeq \mathcal{O}_{\mathcal{X}}(-N)$. We have the canonical embeddings

$$\mathcal{R}_0(-N) \simeq \mathcal{R}_N \xhookrightarrow{\xi_N} \mathcal{R}_{N-1} \xhookrightarrow{\xi_{N-1}} \ldots \xhookrightarrow{\xi_3} \mathcal{R}_2 \xhookrightarrow{\xi_2} \mathcal{R}_1 \xhookrightarrow{\xi_1} \mathcal{R}_0.$$

We define a 2-dimensional Deligne–Mumford stack $\mathcal{S}_N$ as $\boldsymbol{C} \times \mathcal{X}_N$; by an abuse of notation we will denote by ϑ its projection $\mathrm{id} \times \vartheta$ onto $\boldsymbol{S}$. We denote $\boldsymbol{C} \times 0_{\mathcal{X}}$ by $\mathcal{D}_0$, and we denote $\infty_{\boldsymbol{C}} \times \mathcal{X} \cup \infty_{\mathcal{X}} \times \boldsymbol{C}$ by $\mathcal{D}_\infty$. By an abuse of notation, we denote by $\mathcal{R}_l$ the line bundle $\mathcal{O}_{\boldsymbol{C}} \boxtimes \mathcal{R}_l$, and we denote by ξ_l the morphism $\mathrm{id} \boxtimes \xi_l$. According to [3], there is a one-to-one correspondence between the (torsion free, framed at $\mathcal{D}_\infty$) sheaves on $\mathcal{S}$, and the (torsion free, framed at $\boldsymbol{D}_\infty$) sheaves on $\boldsymbol{S}$ with parabolic structure along $\boldsymbol{D}_0$. Thus $\mathcal{P}_{\underline{d}}$ is the moduli space of torsion free sheaves of degree $\underline{d}$ on $\mathcal{S}$ framed at $\mathcal{D}_\infty$.

2.4. Monad for the stack $\mathcal{S}_{\boldsymbol{N}}$. Finally we are able to recall an ADHM-like construction of the isomorphism $\mathfrak{M}_{\underline{d}} \xrightarrow{\sim} \mathcal{P}_{\underline{d}}$. Note that ϑ_* establishes an isomorphism $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \simeq \Gamma(\boldsymbol{X}, \mathcal{O}_{\boldsymbol{X}}(1)) = \mathbb{C}\langle x, y\rangle$. The desired isomorphism $\mathfrak{M}_{\underline{d}} \xrightarrow{\sim} \mathcal{P}_{\underline{d}}$ sends a representative $(A_\bullet, B_\bullet, p_\bullet, q_\bullet)$ to the middle cohomology of the following monad of vector bundles on $\mathcal{S}$:

$$\bigoplus_{0<l\leqslant N} V_l\otimes\mathcal{R}_l\otimes\mathcal{O}_{\mathcal{S}}(-1,0) \xrightarrow{C} \begin{pmatrix} \bigoplus_{0<l\leqslant N} V_l\otimes\mathcal{R}_l \\ \oplus \\ \bigoplus_{0\leqslant l<N} V_{l+1}\otimes\mathcal{R}_l\otimes\mathcal{O}_{\mathcal{S}}(-1,0) \\ \oplus \\ \bigoplus_{0\leqslant l<N} W_l\otimes\mathcal{R}_l \end{pmatrix} \xrightarrow{D} \bigoplus_{0\leqslant l<N} V_{l+1}\otimes\mathcal{R}_l \tag{2.1}$$

Here the "matrix coefficients" of C, D are as follows: ${}_{ll}C^V_{ll} = tA_l - z$; ${}_{ll}C^V_{l,l-1} = -\xi_l$; ${}_{NN}C^V_{10} = xB_0$, and ${}_{ll}C^V_{l+1,l} = B_l$ for $0<l<N$; furthermore, ${}_{NN}C^W_{00} = txq_0$, and ${}_{ll}C^W_{ll} = tq_l$ for $0<l<N$. Furthermore, ${}_{l+1,l}D^V_{l+1,l} = tA_l - z$; ${}_{ll}D^V_{l,l-1} = \xi_l$; ${}_{NN}D^V_{10} = -xB_0$, and ${}_{ll}D^V_{l+1,l} = -B_l$ for $0<l<N$; furthermore, ${}_{ll}D^W_{l+1,l} = p_{l+1}$. We have used some evident shortcuts to simplify the notations, e.g. ${}_{NN}C^V_{10} = xB_0 := B_0\otimes x\otimes 1 \in \operatorname{Hom}(V_0, V_1)\otimes\operatorname{Hom}(\mathcal{R}_N, \mathcal{R}_0)\otimes\operatorname{Hom}_{\boldsymbol{C}}(\mathcal{O}_{\boldsymbol{C}}(-1), \mathcal{O}_{\boldsymbol{C}}(-1))$.

2.5. Inverse construction. Conversely, given a torsion free sheaf $\mathcal{F}$ on $\mathcal{S}_N$ framed at $\mathcal{D}_\infty$, and $0\leqslant l<N$, we have (cf. Section 5 of [1])

$$\begin{aligned} H^0(\mathcal{S}, \mathcal{R}^*_l\otimes\mathcal{F}(-1,0)) = H^0(\mathcal{S}, \mathcal{R}^*_l\otimes\mathcal{F}(0,-N)) = H^0(\mathcal{S}, \mathcal{F}(-1,-N)) = 0, \\ H^2(\mathcal{S}, \mathcal{R}^*_l\otimes\mathcal{F}(-1,0)) = H^2(\mathcal{S}, \mathcal{R}^*_l\otimes\mathcal{F}(0,-N)) = H^2(\mathcal{S}, \mathcal{F}(-1,-N)) = 0. \end{aligned}$$

Furthermore, for $0<l<N$, $H^1(\mathcal{S}, \mathcal{R}^*_l\otimes\mathcal{F}(-1,0)) \simeq H^1(\mathcal{S}, \mathcal{R}^*_l\otimes\mathcal{F}(0,-N)) \simeq V_l$, and $H^1(\mathcal{S}, \mathcal{F}(-1,0)) \simeq H^1(\mathcal{S}, \mathcal{F}(0,-N)) \simeq H^1(\mathcal{S}, \mathcal{F}(-1,-N)) \simeq V_0$. Furthermore, for $0\leqslant l<N-1$, we have a canonical exact sequence

$$0\to H^0(\mathcal{S}, \mathcal{R}^*_l\otimes\mathcal{F}) \to W_l \xrightarrow{p_{l+1}} V_{l+1} \to H^1(\mathcal{S}, \mathcal{R}^*_{l+1}\otimes\mathcal{F}) \to 0,$$

and also

$$0\to H^0(\mathcal{S}, \mathcal{R}^*_{N-1}\otimes\mathcal{F}) \to W_{N-1} \xrightarrow{p_0} V_0 \to H^1(\mathcal{S}, \mathcal{F}) \to 0,$$

The terms $(E_1^{i,j})_{i=-2,-1,0}$ of the Beilinson spectral sequence for $\mathcal{F}$ take the form

$$\begin{aligned} &\bigoplus_{0<l\leqslant N} \operatorname{Ext}^j(\mathcal{R}_l(1,0), \mathcal{F})\otimes\mathcal{R}_l(-1,0) \\ &\quad\to \bigoplus_{0<l\leqslant N} \operatorname{Ext}^j(\mathcal{R}_l, \mathcal{F})\otimes\mathcal{R}_l \oplus \bigoplus_{0\leqslant l<N} \operatorname{Ext}^j(\mathcal{R}_{l+1}(1,0), \mathcal{F})\otimes\mathcal{R}_l(-1,0) \\ &\qquad\qquad\to \bigoplus_{0\leqslant l<N} \operatorname{Ext}^j(\mathcal{R}_{l+1}, \mathcal{F})\otimes\mathcal{R}_l, \end{aligned}$$

that is,

$$\begin{tikzcd}
\bigoplus\limits_{0<l\leqslant N} V_l\otimes\mathcal{R}_l(-1,0) \arrow[r] \arrow[drr, dashed, "d_2^{-2,1}"'] & \begin{pmatrix} \bigoplus\limits_{0<l\leqslant N} V_l\otimes\mathcal{R}_l \\ \oplus \\ \bigoplus\limits_{0\leqslant l<N} V_{l+1}\otimes\mathcal{R}_l(-1,0) \end{pmatrix} \arrow[r] & \bigoplus\limits_{0\leqslant l<N} H^1(\mathcal{R}_{l+1}^*\otimes\mathcal{F})\otimes\mathcal{R}_l \\
 & & \bigoplus\limits_{0\leqslant l<N} H^0(\mathcal{R}_l^*\otimes\mathcal{F})\otimes\mathcal{R}_l
\end{tikzcd}$$

Finally, we can replace $H^1(\mathcal{R}_{l+1}^*\otimes\mathcal{F})=\operatorname{Coker}p_{l+1}$ (resp. $H^0(\mathcal{R}_l^*\otimes\mathcal{F})=\operatorname{Ker}p_{l+1}$) by V_{l+1} (resp. W_l), and lift the differential $d_2^{-2,1}\colon E_2^{-2,1}\to E_2^{0,0}$ to a morphism $V_l\otimes\mathcal{R}_l(-1,0)\to W_l\otimes\mathcal{R}_l$. Replacing the spectral sequence with the total complex we obtain the ADHM description (2.1) of $\mathcal{F}$.

2.6. Monad for the stack $\mathcal{S}'_N$. We also consider the following version of the above construction. Let $\mathcal{S}'=\mathcal{S}'_N$ be the stacky weighted projective plane $\mathbb{P}^2(N,N,1)$. More precisely, we consider the affine 3-space $\mathbb{A}^3$ with coordinates $(z_0,\,z_1,\,z_2)$, and with the action of $\mathbb{C}^*$ given by $c(z_0,\,z_1,\,z_2)=(c^Nz_0,\,c^Nz_1,\,cz_2)$. We define $\mathcal{S}':=(\mathbb{A}^3\setminus 0)/\mathbb{C}^*$. We define $\ell\subset\mathcal{S}'$ as the hyperplane $z_2=0$ (all the points of this line have automorphism group $\mathbb{Z}/N\mathbb{Z}$), and we define $\ell_\infty\subset\mathcal{S}'$ as the hyperplane $z_0=0$. Note that $\ell_\infty\simeq\mathcal{X}$. We denote $\mathcal{O}_{\mathcal{S}'}(l\ell)$ by $\mathcal{O}(l)$ for short; note that $\mathcal{O}_{\mathcal{S}'}(\ell_\infty)\simeq\mathcal{O}(N)$. Let $\mathcal{P}'_{\underline{d}}$ be the moduli space of torsion free sheaves of degree $\underline{d}$ on $\mathcal{S}'$ framed at ℓ_∞, i.e., such that $\mathcal{F}|_{\ell_\infty}=\mathcal{F}_\infty:=W_0\otimes\mathcal{O}_{\mathcal{X}}\oplus W_1\otimes\mathcal{O}_{\mathcal{X}}(-1)\oplus\ldots\oplus W_{N-1}\otimes\mathcal{O}_{\mathcal{X}}(1-N)$.

Since $\mathcal{S}-\mathcal{D}_\infty\simeq\mathcal{S}'-\ell_\infty$, and the framings at infinities match, we have an identification $\mathcal{P}_{\underline{d}}\simeq\mathcal{P}'_{\underline{d}}$. We describe the resulting isomorphism $\mathfrak{M}_{\underline{d}}\xrightarrow{\sim}\mathcal{P}'_{\underline{d}}$. It sends a representative $(A_\bullet,\,B_\bullet,\,p_\bullet,\,q_\bullet)$ to the middle cohomology of the following monad of vector bundles on $\mathcal{S}'$:

$$\bigoplus_{0<l\leqslant N} V_l(-l)\xrightarrow{C}\begin{pmatrix} \bigoplus\limits_{0<l\leqslant N} V_l(1-l) \\ \oplus \\ \bigoplus\limits_{0<l\leqslant N} V_l(N-l) \\ \oplus \\ \bigoplus\limits_{0\leqslant l<N} W_l(-l)\end{pmatrix}\xrightarrow{D}\bigoplus_{0<l\leqslant N} V_l(N+1-l). \tag{2.2}$$

Here the "matrix coefficients" of C, D are as follows:

$$-z_2\colon V_l(-l)\to V_l(1-l);\quad z_0B_0\colon V_N(-N)\to V_1(0);\quad B_l\colon V_l(-l)\to V_{l+1}(-l);$$
$$z_1-z_0A_l\colon V_l(-l)\to V_l(N-l);\quad z_0q_0\colon V_N(-N)\to W_0;\quad q_l\colon V_l(-l)\to W_l(-l);$$

furthermore,

$$z_1-z_0A_l\colon V_l(1-l)\to V_l(N+1-l);\quad z_2\colon V_l(N-l)\to V_l(N+1-l);$$
$$-z_0B_0\colon V_N(0)\to V_1(N);\quad -B_l\colon V_l(N-l)\to V_l(N-l)\to V_{l+1}(N-l);$$
$$z_0p_{l+1}\colon W_l(-l)\to V_{l+1}(N-l).$$

2.7. Rotation and the inverse construction. Conversely, given a torsion free sheaf $\mathcal{F}$ on $\mathcal{S}'$ framed at ℓ_∞, and $l = 1, \dots, N$, we have

$$H^0(\mathcal{S}', \mathcal{F}(l-N-1)) = H^2(\mathcal{S}', \mathcal{F}(l-N-1)) = 0,$$

and $V_l = H^1(\mathcal{S}', \mathcal{F}(l-N-1))$. The endomorphisms A_l arise from the action of $z_1 \in \Gamma(\mathcal{S}', \mathcal{O}(N))$, and B_l arises from the action of $z_2 \in \Gamma(\mathcal{S}', \mathcal{O}(1))$. More precisely, for $l \in \mathbb{Z}$, we have the morphism $z_2 \colon \mathcal{F}(l-N-1) \to \mathcal{F}(l-N)$, which induces $B_l \colon V_l \to V_{l+1}$ for $1 \leqslant l \leqslant N-1$, and also

$$z_2 \colon H^1(\mathcal{S}', \mathcal{F}(-N-1)) \to H^1(\mathcal{S}', \mathcal{F}(-N)) = V_1.$$

However, the short exact sequence

$$0 \to \mathcal{F}(-N-1) \xrightarrow{z_0} \mathcal{F}(-1) \to \mathcal{F}_\infty(-1) \to 0$$

gives rise to the long exact sequence of cohomology including

$$z_0 \colon H^1(\mathcal{S}', \mathcal{F}(-N-1)) \xrightarrow{\sim} H^1(\mathcal{S}', \mathcal{F}(-1)) = V_N.$$

So we define $B_0 \colon V_N \to V_1$ as the composition $z_2 z_0^{-1}$.

Furthermore, we define $A_0 \colon V_N \to V_N$ as the composition $z_1 z_0^{-1}$.

Furthermore, the short exact sequence

$$0 \to \mathcal{F}(-N) \xrightarrow{z_0} \mathcal{F} \to \mathcal{F}_\infty \to 0$$

gives rise to the long exact sequence of cohomology including

$$W_0 = H^0(\mathcal{X}, \mathcal{F}_\infty) \to H^1(\mathcal{S}', \mathcal{F}(-N)) = V_1.$$

We define p_1 as this latter map $W_0 \to V_1$.

Furthermore, for $0 \leqslant l < N$, the short exact sequence

$$0 \to \mathcal{F}(l-2N-1) \xrightarrow{z_0} \mathcal{F}(l-N-1) \to \mathcal{F}_\infty(l-N-1) \to 0$$

gives rise to the long exact sequence of cohomology including

$$H^1(\mathcal{S}', \mathcal{F}(l-N-1)) \to H^1(\mathcal{X}, \mathcal{F}_\infty(l-N-1)) = W_l \oplus W_{l+1} \oplus \ldots \oplus W_{N-1}.$$

For $0 < l < N$, we define $q_l \colon V_l = H^1(\mathcal{S}', \mathcal{F}(l-N-1)) \to W_l$ as the direct summand of the above morphism. For $l = 0$, we define $q_0 \colon V_N = H^1(\mathcal{S}', \mathcal{F}(-1)) \to W_0$ as the composition of the direct summand of the above morphism with

$$z_0^{-1} \colon V_N = H^1(\mathcal{S}', \mathcal{F}(-1)) \to H^1(\mathcal{S}', \mathcal{F}(-N-1)).$$

It remains to define A_l, $l \neq 0$, and p_l, $l \neq 1$. To this end, we define the *rotation* $\rho\underline{d}$ as follows: $\rho d_l := d_{l+1}$, $l \in \mathbb{Z}/N\mathbb{Z}$. We have a natural *rotation* isomorphism $R \colon \mathfrak{M}_{\underline{d}} \xrightarrow{\sim} \mathfrak{M}_{\rho\underline{d}}$, taking the quiver data $(V_\bullet, W_\bullet, A_\bullet, B_\bullet, p_\bullet, q_\bullet)$ to $(V_{\bullet-1}, W_{\bullet-1}, A_{\bullet-1}, B_{\bullet-1}, p_{\bullet-1}, q_{\bullet-1})$. We define the corresponding isomorphism $R \colon \mathcal{P}'_{\underline{d}} \xrightarrow{\sim} \mathcal{P}'_{\rho\underline{d}}$ presently.

Given a framed torsion free sheaf $\mathcal{F}$ on $\mathcal{S}'$, we define $\mathcal{G} = R(\mathcal{F})$ as the kernel of the natural projection $\mathcal{F}(1) \twoheadrightarrow \imath_* W_0(1)$. Here $\imath$ stands for the closed embedding $\mathcal{X} \simeq \ell_\infty \hookrightarrow \mathcal{S}'$. We have an exact sequence

$$0 \to W_0(1-N) \to \imath^*\mathcal{G} \xrightarrow{r} \imath^*\mathcal{F}(1) \to W_0(1) \to 0,$$

and the morphism r factors as the composition

$$\imath^*\mathcal{G} \to W_1 \oplus W_2(-1) \oplus \ldots \oplus W_{N-1}(2-N) \to \imath^*\mathcal{F}(1).$$

Since for any $l = 1, \ldots, N-1$ we have $\mathrm{Ext}^1_{\mathcal{X}}(W_l(1-l), W_0(1-N)) = 0$, we conclude that

$$\mathcal{G}|_{\ell_\infty} \simeq W_1 \otimes \mathcal{O}_{\mathcal{X}} \oplus W_2(-1) \oplus \ldots \oplus W_{N-1}(2-N) \oplus W_0(1-N).$$

Furthermore, the long exact cohomology sequence arising from the short exact sequence

$$0 \to \mathcal{G}(l-N-1) \to \mathcal{F}(l-N) \to \imath_* W_0(l-N) \to 0$$

implies $H^1(\mathcal{S}', \mathcal{G}(l-N-1)) = V_{l+1}$ for $0 < l < N$. Also, the long exact cohomology sequence arising from the short exact sequence

$$0 \to \mathcal{F}(-N) \to \mathcal{G}(-1) \to \imath_*(W_1(-1) \oplus \ldots \oplus W_{N-1}(1-N)) \to 0$$

implies $H^1(\mathcal{S}', \mathcal{G}(-1)) = H^1(\mathcal{S}', \mathcal{F}(-N)) = V_1$. Finally, it is clear that $R^N = \mathrm{Id}\colon \mathcal{P}'_{\underline{d}} \to \mathcal{P}'_{\underline{d}}$.

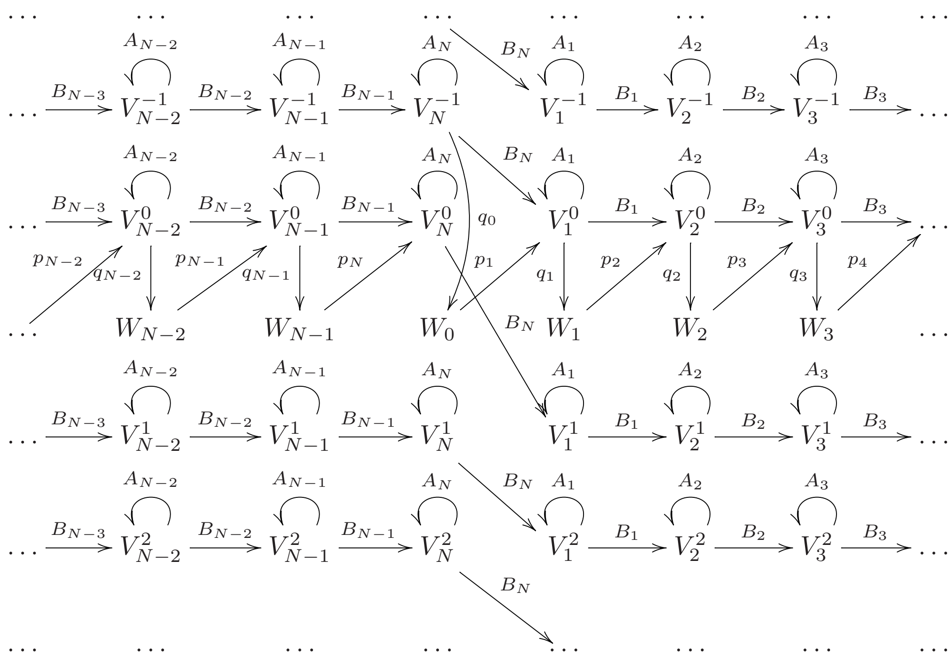
Returning to the definition of A_l, p_l, we set $A_l := R^{-l} A_0 R^l$, $p_l := R^{1-l} p_1 R^{l-1}$.

2.8. The action of $\boldsymbol{\Gamma_k}$. Let $\Gamma_k \simeq \mathbb{Z}/k\mathbb{Z}$ (resp. $\Gamma_{kN} \simeq \mathbb{Z}/kN\mathbb{Z}$) be the group of k-th (resp. kN-th) roots of unity, with generator ζ_k (resp. ζ_{kN}). We have a surjection $\Gamma_{kN} \twoheadrightarrow \Gamma_k,\ \zeta_{kN} \mapsto \zeta_k$. Recall the stack $\mathcal{X} = \mathcal{X}_N = (\mathbb{A}^2 \setminus 0)/\mathbb{C}^*$, where the action of $\mathbb{C}^*$ is given by $c(z_1, z_2) = (c^N z_1, cz_2)$. The group Γ_k acts on $\mathcal{X}_N$ as follows: $\zeta_k(z_1, z_2) = (z_1, \zeta_k z_2)$. The quotient stack $\mathcal{X}_N/\Gamma_k$ is denoted by ${}_k\mathcal{X}_N$. We also have a Γ_k-equivariant morphism $\nu\colon \mathcal{X}_N \to \mathcal{X}_N,\ \ \nu(z_1, z_2) = (z_1^k, z_2^k)$. It factors through $\mathcal{X}_N \to {}_k\mathcal{X}_N \xrightarrow{\Theta} \mathcal{X}_N$.

In the local coordinates of Section 2.3, $\mathcal{X}_N$ is glued from the affine line $(\mathbb{A}^1, y_1)$ with coordinate y_1, and $(\mathbb{A}^1, y_2)/\Gamma_N$: both $\mathbb{A}^1 \setminus \{y_1 = 0\}$ and $(\mathbb{A}^1 \setminus \{y_2 = 0\})/\Gamma_N$ coincide with $\mathbb{G}_m$, and we glue the charts with the help of $y_1 = y_2^{-N}$. Now ${}_k\mathcal{X}_N$ is glued from $(\mathbb{A}^1, y_1)/\Gamma_k$ and $(\mathbb{A}^1, y_2)/\Gamma_{kN}$ with the help of $y_1^k = y_2^{-kN}$. Note that the group Γ_k acts on the chart $(\mathbb{A}^1, y_2)/\Gamma_N$ as the quotient of Γ_{kN} by the normal subgroup $\Gamma_N \subset \Gamma_{kN}$.

The group Γ_k acts on $\mathcal{S}_N = \boldsymbol{C} \times \mathcal{X}_N$ via the second factor, and we denote $\mathcal{S}_N/\Gamma_k$ by ${}_k\mathcal{S}_N$. By an abuse of notation we denote by Θ the morphism $\mathrm{id} \times \Theta\colon {}_k\mathcal{S}_N \to \mathcal{S}_N$. The corresponding action of Γ_k on $\boldsymbol{S}$ is as follows: $\zeta_k(z : t,\ y : x) = (z : t,\ \zeta_k y : x)$. The corresponding action of Γ_k on $\mathcal{S}'$ is given by $\zeta_k(z_0, z_1, z_2) = (z_0, z_1, \zeta_k z_2)$.

The group Γ_k acts on the moduli space $\mathcal{P}_{\underline{d}}$ of parabolic sheaves trivialized at infinity via its action on $\boldsymbol{S}$ and the *trivial* action on the trivialization at infinity. The fixed point variety $\mathcal{P}_{\underline{d}}^{\Gamma_k}$ has various connected components, and we are going to describe them in quiver terms. To this end note that Γ_k acts on the moduli space $\mathcal{P}_{\underline{d}} = \mathfrak{M}_{\underline{d}}$ (resp. $\mathcal{P}'_{\underline{d}} = \mathfrak{M}_{\underline{d}}$) of torsion free sheaves on $\mathcal{S}$ (resp. $\mathcal{S}'$) framed at $\mathcal{D}_\infty$ (resp. ℓ_∞) via its action on $\mathcal{S}$ (resp. $\mathcal{S}'$) and the *trivial* action on the framing. We have $\mathcal{P}_{\underline{d}}^{\Gamma_k} = \mathfrak{M}_{\underline{d}}^{\Gamma_k} = (\mathcal{P}'_{\underline{d}})^{\Gamma_k}$. According to Section 2.4 (resp. Section 2.6) Γ_k acts on $\mathfrak{M}_{\underline{d}}$ as follows: $\zeta_k(A_\bullet, B_\bullet, p_\bullet, q_\bullet) = (A_\bullet, \zeta_k B_\bullet, \zeta_k p_\bullet, q_\bullet)$. To formulate the conclusion we consider the representations of the following quiver Q^k:



Here the lower indices of V run through $\mathbb{Z}/N\mathbb{Z}$, while the upper indices run through $\mathbb{Z}/k\mathbb{Z}$. The relations are as follows:

$$\begin{gathered} 0 = A_1B_N - B_NA_0 + p_1q_0\colon V_N^{-1} \to V_1^0; \\ 0 = A_{l+1}B_l - B_lA_l + p_{l+1}q_l\colon V_l^0 \to V_{l+1}^0 \quad \text{for } 1 \leqslant l \leqslant N-1, \text{ and} \\ 0 = A_1B_N - B_NA_0\colon V_N^r \to V_1^{r+1} \quad \text{for } r \neq -1, \text{ and} \\ 0 = A_{l+1}B_l - B_lA_l\colon V_l^r \to V_{l+1}^r \quad \text{in the remaining cases.} \end{gathered}$$

We set $d_l^r := \dim(V_l^r)$, and we denote by $\underline{\widetilde{d}}$ the collection of positive integers $(d_l^r)_{l\in\mathbb{Z}/N\mathbb{Z}}^{r\in\mathbb{Z}/k\mathbb{Z}}$. We set $\underline{d}(\underline{\widetilde{d}}) := (d_1, \ldots, d_N)$, where $d_l = \sum_{r\in\mathbb{Z}/k\mathbb{Z}} d_l^r$. We denote by $\mathsf{M}_{\underline{\widetilde{d}}}$ the scheme of representations of Q^k of dimension $\underline{\widetilde{d}}$. We denote by $G_{\underline{\widetilde{d}}}$ the group $\prod_{l\in\mathbb{Z}/N\mathbb{Z}}^{r\in\mathbb{Z}/k\mathbb{Z}} \mathrm{GL}(V_l^r)$; it acts naturally on $\mathsf{M}_{\underline{\widetilde{d}}}$. We denote by $\mathfrak{Z}_{\underline{\widetilde{d}}}$ the categorical quotient $\mathsf{M}_{\underline{\widetilde{d}}}/\!/G_{\underline{\widetilde{d}}}$. Furthermore, we consider an open subscheme $\mathsf{M}^s_{\underline{\widetilde{d}}} \subset \mathsf{M}_{\underline{\widetilde{d}}}$ formed by all the *stable* representations of Q^k, i.e., by the $(A_\bullet, B_\bullet, p_\bullet, q_\bullet) \in \mathsf{M}_{\underline{\widetilde{d}}}$ such that there is no proper graded subspace ${}'V_\bullet^\bullet \subset V_\bullet^\bullet$ stable under $A_\bullet$, $B_\bullet$ and containing $p(W_\bullet)$. The action of $G_{\underline{\widetilde{d}}}$ on $\mathsf{M}^s_{\underline{\widetilde{d}}}$ is free, and we consider the GIT quotient $\mathfrak{M}_{\underline{\widetilde{d}}} = \mathsf{M}^s_{\underline{\widetilde{d}}}/G_{\underline{\widetilde{d}}}$. Note that $\mathfrak{M}_{\underline{\widetilde{d}}}$ is nonempty if and only if

$$d_N^0 \geqslant d_1^1 \geqslant d_2^1 \geqslant \ldots \geqslant d_N^1 \geqslant d_1^2 \geqslant d_2^2 \geqslant \ldots \geqslant d_{N-2}^{-1} \geqslant d_{N-1}^{-1} \geqslant d_N^{-1}.$$

The above considerations imply the following

Proposition 2.9. *The fixed point variety $\mathcal{P}_{\underline{d}}^{\Gamma_k}$ is a union of connected components isomorphic to $\mathfrak{M}_{\underline{\widetilde{d}}}$ (to be denoted by $\mathcal{P}_{\underline{\widetilde{d}}}$), over all collections $\underline{\widetilde{d}}$ such that $\underline{d}(\underline{\widetilde{d}}) = \underline{d}$.*

2.10. Direct image. Given $\underline{d} = (d_1, \dots, d_N)$ we consider $\widetilde{\underline{d}} = \widetilde{\underline{d}}(\underline{d})$ such that $d_l^0 = d_l$ for any $1 \leqslant l \leqslant N$, and $d_l^r = d_N$ for any $1 \leqslant l \leqslant N$ and $r \neq 0$ (note that $\underline{d}(\widetilde{\underline{d}}(\underline{d})) = (d_1 + (k-1)d_N, \dots, d_N + (k-1)d_N) =: \underline{d} + (k-1)d_N$). Then it is easy to see that $\mathfrak{M}_{\widetilde{\underline{d}}} \simeq \mathfrak{M}_{\underline{d}}$. In effect, all the maps B_l except for $B_N : V_N^{-1} \to V_1^0$, and the ones in the 0th row, have to be isomorphisms intertwining the corresponding endomorphisms A_l and A_{l+1}.

Geometrically, the isomorphism $\mathcal{P}_{\widetilde{\underline{d}}} \xrightarrow{\sim} \mathcal{P}_{\underline{d}}$ has the following explanation. We have an evident projection $\psi\colon \boldsymbol{S} \to \boldsymbol{S}/\!/\Gamma_k \simeq \boldsymbol{S}$ (the categorical quotient). A Γ_k-fixed point of $\mathcal{P}_{\underline{d}+(k-1)d_N}$ is represented by a Γ_k-equivariant torsion free parabolic sheaf $\mathcal{F}_\bullet$ on $\boldsymbol{S}$. Then $\psi_*\mathcal{F}_\bullet$ carries a fiberwise action of Γ_k, and $(\psi_*\mathcal{F}_\bullet)^{\Gamma_k}$ is a torsion free parabolic sheaf on $\boldsymbol{S}$, trivialized at infinity. Its class in $\mathcal{P}_{\underline{d}}$ is the image of $\mathcal{F}_\bullet$ under the above isomorphism. For this reason, somewhat abusing notation, we will denote this isomorphism by $\psi_*^{\Gamma_k}$.

Alternatively, thinking of Γ_k-equivariant torsion free parabolic sheaves on $\boldsymbol{S}$ trivialized at infinity as of Γ_k-equivariant torsion free sheaves on $\mathcal{S}_N$ framed at infinity, that is, torsion free sheaves on ${}_k\mathcal{S}_N$ framed at infinity (i.e., whose restriction to ${}_k\mathcal{D}_\infty = \infty_{\boldsymbol{C}} \times {}_k\mathcal{X}_N \cup \boldsymbol{C} \times \infty_{{}_k\mathcal{X}_N}$ is equipped with an isomorphism to $\mathcal{O}_{{}_k\mathcal{X}_N} \oplus \mathcal{O}_{{}_k\mathcal{X}_N}(-1 \cdot 0_{{}_k\mathcal{X}_N}) \oplus \dots \oplus \mathcal{O}_{{}_k\mathcal{X}_N}((1-N) \cdot 0_{{}_k\mathcal{X}_N})$ on $\infty_{\boldsymbol{C}} \times {}_k\mathcal{X}_N$, and to $\mathcal{O}_{\boldsymbol{C}}^N$ on $\boldsymbol{C} \times \infty_{{}_k\mathcal{X}_N}$) we see that their isomorphism classes correspond bijectively to the isomorphism classes of stable representations of Q^k (an argument entirely similar to Section 2.4 and Section 2.5). Then $\psi_*^{\Gamma_k}$ is nothing but Θ_* (notation of Section 2.8).

For an arbitrary $\widetilde{\underline{d}}$, we consider $\underline{d} = (d_1, \dots, d_N) := (d_1^0, \dots, d_{N-1}^0, d_N^{-1})$. Then we still have a morphism $\Theta_* = \psi_*^{\Gamma_k} : \mathcal{P}_{\widetilde{\underline{d}}} \to \mathcal{P}_{\underline{d}}$, which is not necessarily an isomorphism. Going through the inverse constructions of Section 2.4 and Section 2.5 one arrives at the following description of $\Theta_* = \psi_*^{\Gamma_k} : \mathfrak{M}_{\widetilde{\underline{d}}} \to \mathfrak{M}_{\underline{d}}$ in quiver terms. We have $V_N' := V_N^{-1}$, $V_l' := V_l^0$ for $1 \leqslant l \leqslant N-1$. Furthermore, we have $A_l' := A_l$ for $1 \leqslant l \leqslant N$, and B_{N-1}' is the composition of all B's going from V_{N-1}^0 to V_N^0, then to V_1^1, and all the way through to V_N^{-1}; while all the other B_l' coincide with the corresponding B_l. Finally, q_l' coincides with the corresponding q_l, and for $1 \leqslant l \leqslant N-1$ the map p_l' coincides with the corresponding p_l; while p_N' is the composition of p_N with all the B's going from V_N^0 to V_1^1, and all the way through to V_N^{-1}.

The morphism $\psi_*^{\Gamma_k} : \mathfrak{M}_{\widetilde{\underline{d}}} \to \mathfrak{M}_{\underline{d}}$ induces the morphism $\psi^k : \mathfrak{Z}_{\widetilde{\underline{d}}} \to \mathfrak{Z}_{\underline{d}}$ from the affinization $\mathfrak{Z}_{\widetilde{\underline{d}}}$ of $\mathfrak{M}_{\widetilde{\underline{d}}}$ to the affinization $\mathfrak{Z}_{\underline{d}}$ of $\mathfrak{M}_{\underline{d}}$.

2.11. Defect. Let us give a geometric explanation of what is so special about the components $\mathcal{P}_{\widetilde{\underline{d}}}$, $\widetilde{\underline{d}} = \widetilde{\underline{d}}(\underline{d})$ considered in Section 2.10. Namely, they are the only components of the fixed point variety $\mathcal{P}_{\underline{d}}^{\Gamma_k}$ which contain the *nonempty* open subset formed by the Γ_k-equivariant *locally free* parabolic sheaves. Here *locally free* means locally free after forgetting the Γ_k-equivariant structure.

For an arbitrary Γ_k-equivariant torsion free parabolic sheaf $\mathcal{F}_\bullet$, there is a notion of the *saturation* $\widehat{\mathcal{F}}_\bullet$ (a locally free parabolic sheaf containing $\mathcal{F}_\bullet$, such that the quotient has a zero-dimensional support). The global sections of this quotient is a $\mathbb{Z}/N\mathbb{Z}$-graded Γ_k-module $\operatorname{def}(\mathcal{F}_\bullet)$, the *defect* of $\mathcal{F}_\bullet$. The class $[\operatorname{def}(\mathcal{F}_\bullet)]$ of $\operatorname{def}(\mathcal{F}_\bullet)$ in the K-group of $\mathbb{Z}/N\mathbb{Z}$-graded Γ_k-modules is represented by a collection $\widetilde{\underline{d}}$ of integers. The class $[\operatorname{def}(\mathcal{F}_\bullet)]$ may vary throughout a connected component of the fixed point variety $\mathcal{P}_{\underline{d}}^{\Gamma_k}$. However, its class $[\overline{\mathrm{def}}(\mathcal{F}_\bullet)]$ modulo the subgroup spanned by all the collections of the sort $\underline{\widetilde{d}}(\underline{d}')$, $\underline{d}' \in \mathbb{Z}^{\mathbb{Z}/N\mathbb{Z}}$, is constant throughout a connected component $\mathcal{P}_{\underline{\widetilde{d}}}$. Quite evidently, the class $[\overline{\mathrm{def}}(\mathcal{F}_\bullet)]$ for $\mathcal{F}_\bullet \in \mathcal{P}_{\underline{\widetilde{d}}}$, equals the class of $\underline{\widetilde{d}}$. In particular, in order to have a locally free parabolic sheaf $\mathcal{F}_\bullet$ (i.e., the one with zero defect) in a component $\mathcal{P}_{\underline{\widetilde{d}}}$ it is necessary and sufficient that $\underline{\widetilde{d}}$ be of the form $\underline{\widetilde{d}}(\underline{d})$ for some $\underline{d}$.

3. Zastava for Blown up Plane

The results of this section are strongly influenced by [23].

3.1. Dented chainsaw. We consider the representations of the following *dented chainsaw quiver* $\widehat{Q}$

$$\begin{tikzcd}
\cdots \arrow[r, "B_{N-3}"] & V_{N-2} \arrow[loop above, "A_{N-2}"] \arrow[r, "B_{N-2}"] \arrow[d, "q_{N-2}"] & V_{N-1} \arrow[loop above, "A_{N-1}"] \arrow[r, "B_{N-1}"] \arrow[d, "q_{N-1}"] & V_N \arrow[r, shift left, "e"'] & V_0 \arrow[l, shift left, "\delta"'] \arrow[r, "B_0"] \arrow[d, "q_0"] & V_1 \arrow[loop above, "A_1"] \arrow[r, "B_1"] \arrow[d, "q_1"] & V_2 \arrow[loop above, "A_2"] \arrow[r, "B_2"] \arrow[d, "q_2"] & \cdots \\
\cdots \arrow[ur, "p_{N-2}"] & W_{N-2} \arrow[ur, "p_{N-1}"] & W_{N-1} \arrow[ur, "p_N"] & & W_0 \arrow[ur, "p_1"] & W_1 \arrow[ur, "p_2"] & W_2 \arrow[ur, "p_3"] & \cdots
\end{tikzcd}$$

with relations $A_{l+1}B_l - B_lA_l + p_{l+1}q_l = 0$ for any $1 \leqslant l \leqslant N-2$; $A_1B_0 - B_0e\delta + p_1q_0 = 0$; $\delta eB_{N-1} - B_{N-1}A_{N-1} + p_Nq_{N-1} = 0$. Here $\dim W_l = 1$, $d_N := \dim V_N = d_0 := \dim V_0$, $\dim V_l = d_l$, $l = 1, \ldots, N-1$. We denote by $\widehat{\mathsf{M}}_{\underline{d}}$ the scheme of representations of $\widehat{Q}$. We denote by $\widehat{G}_{\underline{d}}$ the group $\prod_{0 \leqslant l \leqslant N} \mathrm{GL}(V_l)$; it acts naturally on $\widehat{\mathsf{M}}_{\underline{d}}$. Performing the celebrated Crawley–Boevey trick, we identify all the lines W_l with, say W_∞, so that W_∞ is the source of all p_l, and the target of all q_l. We will denote a typical representation of $\widehat{Q}$ by Y.

3.2. Stability conditions. Following [21, Section 4(ii)] we consider the enhanced dimension vectors $\underline{\widehat{d}} := (d_0, d_1, \ldots, d_{N-1}, d_N)$, and $\underline{\ddot{d}} := (d_0, d_1, \ldots, d_{N-1}, d_N, 1)$ with one extra coordinate equal to $\dim W_\infty = 1$. We consider a vector $\zeta^\bullet = (\zeta_0, \zeta_1, \ldots, \zeta_{N-1}, \zeta_N)$, where $\zeta_N = -1$, $\zeta_0 = 1$, $\zeta_l = 0$ for $l = 1, \ldots, N-1$. Also, for $0 < \varepsilon \ll 1$ we consider $\zeta^- := \zeta^\bullet - (\varepsilon, \ldots, \varepsilon)$. We set $\zeta_\infty^- := -\langle \zeta^-, \underline{\widehat{d}} \rangle$, and $\zeta_\infty^\bullet := -\langle \zeta^\bullet, \underline{\widehat{d}} \rangle$, where $\langle \cdot, \cdot \rangle$ stands for the sum of products of coordinates (the standard scalar product). Finally, we set $\widetilde{\zeta}^- := (\zeta^-, \zeta_\infty^-)$, and $\widetilde{\zeta}^\bullet := (\zeta^\bullet, \zeta_\infty^\bullet)$.

For a nonzero $\widehat{Q}$-submodule $Y' \subset Y$ of enhanced dimension $\underline{\ddot{d}}'$ (where the last coordinate may be either 1 or 0) we define the slope by

$$\theta^-(Y') := \frac{\langle \widetilde{\zeta}^-, \underline{\ddot{d}}' \rangle}{\langle (1, \ldots, 1), \underline{\ddot{d}}' \rangle}, \quad \theta^\bullet(Y') := \frac{\langle \widetilde{\zeta}^\bullet, \underline{\ddot{d}}' \rangle}{\langle (1, \ldots, 1), \underline{\ddot{d}}' \rangle}.$$

We say that a $\widehat{Q}$-module Y is ζ^--semistable (resp. $\zeta^\bullet$-semistable) if for any nonzero submodule $Y' \subset Y$ we have $\theta^-(Y') \leqslant \theta^-(Y)$ (resp. $\theta^\bullet(Y') \leqslant \theta^\bullet(Y)$). We say Y is ζ^--stable (resp. $\zeta^\bullet$-stable) if the inequality is strict unless $Y' = Y$. Note that ζ^--stability is equivalent to ζ^--semistability.

We define a scheme $\widehat{\mathfrak{M}}_{\underline{d}}$ as the moduli space of ζ^--semistable (equivalently, ζ^--stable) $\widehat{Q}$-modules. By GIT, $\widehat{\mathfrak{M}}_{\underline{d}}$ is the projective spectrum of the ring of $\widehat{G}_{\underline{d}}$-semiinvariants in $\mathbb{C}[\widehat{\mathsf{M}}_{\underline{d}}]$. Furthermore, we define a scheme $\widehat{\mathfrak{Z}}_{\underline{d}}$ as the moduli space of S-equivalence classes of $\zeta^\bullet$-semistable $\widehat{Q}$-modules. Since the stability condition $\zeta^\bullet$ lies on a wall of the chamber containing ζ^-, we have a projective morphism $\pi_{\zeta^\bullet,\zeta^-}\colon \widehat{\mathfrak{M}}_{\underline{d}} \to \widehat{\mathfrak{Z}}_{\underline{d}}$.

3.3. Parabolic sheaves on blow-up. We stick to the notations of [23]. Namely, $\mathbb{P}^2$ is the projective plane with homogeneous coordinates $[z_0 : z_1 : z_2]$, and $\ell_\infty \subset \mathbb{P}^2$ is the line "at infinity" given by the equation $z_0 = 0$. Furthermore, $\widehat{\mathbb{P}}^2$ is the blow-up of $\mathbb{P}^2$ "at the origin" (given by equations $z_1 = z_2 = 0$). It is the closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ defined by $\widehat{\mathbb{P}}^2 = \{([z_0 : z_1 : z_2], [z : w]) \colon z_1 w = z_2 z\}$. We denote by E the exceptional divisor in $\widehat{\mathbb{P}}^2$; we denote by $\ell_0 \subset \widehat{\mathbb{P}}^2$ the proper transform of the line $z_2 = 0$ in $\mathbb{P}^2$; finally, by an abuse of notation, we denote by $\ell_\infty \subset \widehat{\mathbb{P}}^2$ the proper transform of the line $\ell_\infty \subset \mathbb{P}^2$.

We set $W := W_1 \oplus W_2 \oplus \ldots \oplus W_{N-1} \oplus W_0$. Given an N-tuple of nonnegative integers $\underline{d} = (d_0, \ldots, d_{N-1})$ we say that a *parabolic sheaf* $\mathcal{F}_\bullet$ of degree $\underline{d}$ is an infinite flag of torsion free coherent sheaves of rank N on $\widehat{\mathbb{P}}^2$: $\ldots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ such that

(a) $\mathcal{F}_{k+N} = \mathcal{F}_k(\ell_0)$ for any $k \in \mathbb{Z}$;
(b) $\mathrm{ch}_1(\mathcal{F}_k) = k[\ell_0]$ for any $k \in \mathbb{Z}$: the first Chern classes are proportional to the fundamental class of ℓ_0;
(c) $\mathrm{ch}_2(\mathcal{F}_k) = d_i$ for $i \equiv k \pmod N$;
(d) $\mathcal{F}_0$ is locally free at ℓ_∞ and trivialized at ℓ_∞: $\mathcal{F}_0|_{\ell_\infty} = W \otimes \mathcal{O}_{\ell_\infty}$;
(e) For $-N \leqslant k \leqslant 0$ the sheaf $\mathcal{F}_k$ is locally free at ℓ_∞, and the quotient sheaves $\mathcal{F}_k/\mathcal{F}_{-N}$, $\mathcal{F}_0/\mathcal{F}_k$ (both supported at $\ell_0 \subset \widehat{\mathbb{P}}^2$) are locally free at the point $\ell_0 \cap \ell_\infty$; moreover, the local sections of $\mathcal{F}_k|_{\ell_\infty}$ are those sections of $\mathcal{F}_0|_{\ell_\infty} = W \otimes \mathcal{O}_{\ell_\infty}$ which take value in $W_1 \oplus \ldots \oplus W_{k+N} \subset W$ at $\ell_0 \cap \ell_\infty$.

One can show that the fine moduli space $\widehat{\mathcal{P}}_{\underline{d}}$ of degree $\underline{d}$ parabolic sheaves exists, and is a smooth connected quasiprojective variety of dimension $2d_0 + \ldots + 2d_{N-1}$.

Theorem 3.4 (A. Kuznetsov). *There is an isomorphism* $\Xi\colon \widehat{\mathfrak{M}}_{\underline{d}} \xrightarrow{\sim} \widehat{\mathcal{P}}_{\underline{d}}$.

The proof occupies Sections 3.5–3.9.

3.5. Stack $\widehat{\mathcal{S}}_N$. We denote by $\ell' \subset \widehat{\mathbb{P}}^2$ the proper transform of the line $z_1 = 0$ in $\mathbb{P}^2$. We consider the open subvarieties $U' := \widehat{\mathbb{P}}^2 - \ell_0 \simeq \ell' \times \mathbb{A}^1$ with coordinate $z = z_1 z_2^{-1}$ along $\mathbb{A}^1$, and $U_0 := \widehat{\mathbb{P}}^2 - \ell' \simeq \ell_0 \times \mathbb{A}^1$ with coordinate $w = z_2 z_1^{-1}$ along $\mathbb{A}^1$. Note that ℓ_0 (resp. ℓ') in $\widehat{\mathbb{P}}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1$ is cut out by the equation $w = 0$ (resp. $z = 0$). We consider the ramified Galois covering $\theta\colon \mathbb{A}^1 \to \mathbb{A}^1$, $w = s^N$ with Galois group Γ_N. We denote by $\theta\colon \widetilde{U}_0 \to U_0$ the base change of this covering under $U_0 \to \mathbb{A}^1$. The action of Γ_N on $\theta^{-1}(U_0 \cap U')$ is free, and $\theta^{-1}(U_0 \cap U')/\Gamma_N = U_0 \cap U'$. We define a 2-dimensional Deligne–Mumford stack $\widehat{\mathcal{S}}_N$ as the result of gluing U' and $\widetilde{U}_0/\Gamma_N$ over the common open $U_0 \cap U'$. Note that $\widehat{\mathcal{S}}_N$ is equipped with a projection ϑ to $\widehat{\mathbb{P}}^2$ which is an isomorphism off ℓ_0. The line in $\widehat{\mathcal{S}}_N$ lying over ℓ_0 will be denoted by $\ell \subset \widehat{\mathcal{S}}_N$; its automorphism group is Γ_N. Since N is fixed throughout the Section, we will often omit the lower index N to simplify the notations.

We also have a smooth morphism π (a $\mathbb{P}^1$-bundle) from $\widehat{\mathcal{S}}_N$ to the 1-dimensional stack $\mathcal{X}_N$ of Section 2.3 such that $\pi^{-1}(\infty_{\mathcal{X}}) = \ell'$, and $\pi^{-1}(0_{\mathcal{X}}) = \ell$. A section of π sending $\mathcal{X}$ to $\ell_\infty \subset \widehat{\mathcal{S}}$ will be denoted by $\imath$. We choose a section y_2 of $\mathcal{O}_{\mathcal{X}}(1)$ with a simple zero at $0_{\mathcal{X}}$, and a section y_1 of $\mathcal{O}_{\mathcal{X}}(N)$ with a simple zero at $\infty_{\mathcal{X}}$ (in notations of Section 2.2 and Section 2.3 we have $y_2^N = y,\ \ y_1 = x$). We keep the same names for the corresponding sections of $\mathcal{O}_{\widehat{\mathcal{S}}}(\ell)$ and $\mathcal{O}_{\widehat{\mathcal{S}}}(N\ell)$ constant along the fibers of π. Finally, we choose a section x_2 of $\mathcal{O}_{\widehat{\mathcal{S}}}(\ell_\infty - N\ell) = \mathcal{O}_{\widehat{\mathcal{S}}}(\ell_\infty - \ell')$ with a simple zero at E.

Here is an alternative toric description of $\widehat{\mathcal{S}}$. We consider $\mathbb{A}^4$ with coordinates $x_1,\ x_2,\ y_1,\ y_2$, and with an open subset $\widetilde{U} \subset \mathbb{A}^4$ obtained by removing two planes: $L_1 = \{x_1,\ x_2,\ 0,\ 0\}$ and $L_2 = \{0,\ 0,\ y_1,\ y_2\}$. The torus $T_2 = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\widetilde{U}$ as follows: $(c_1,\ c_2)\cdot(x_1,\ x_2,\ y_1,\ y_2) = (c_1c_2^N x_1,\ c_1x_2,\ c_2^N y_1,\ c_2y_2)$. We have $\widehat{\mathcal{S}} = \widetilde{U}/T_2$. Note that $x_1 \in \Gamma(\widehat{\mathcal{S}},\ \mathcal{O}(\ell_\infty))$ is an equation of ℓ_∞; $x_2 \in \Gamma(\widehat{\mathcal{S}},\ \mathcal{O}(E))$ is an equation of E; $y_1 \in \Gamma(\widehat{\mathcal{S}},\ \pi^*\mathcal{O}_{\mathcal{X}}(N))$ is an equation of $\pi^{-1}(\infty_{\mathcal{X}})$; $y_2 \in \Gamma(\widehat{\mathcal{S}},\ \pi^*\mathcal{O}_{\mathcal{X}}(1))$ is an equation of ℓ.

According to [3], there is a one-to-one correspondence between the (torsion free, framed at ℓ_∞) sheaves on $\widehat{\mathcal{S}}$, and the (torsion free, framed at ℓ_∞) sheaves on $\widehat{\mathbb{P}}^2$ with parabolic structure along ℓ_0. Thus $\widehat{\mathcal{P}}_{\underline{d}}$ is the moduli space of torsion free sheaves of degree $\underline{d}$ on $\widehat{\mathcal{S}}$ framed at ℓ_∞. More precisely, the framing at ℓ_∞ is an isomorphism $\imath^*\mathcal{F} \simeq \mathcal{F}_\infty := W_0 \otimes \mathcal{O}_{\mathcal{X}} \oplus W_1 \otimes \mathcal{O}_{\mathcal{X}}(-1) \oplus \ldots \oplus W_{N-1} \otimes \mathcal{O}_{\mathcal{X}}(-N+1)$. For technical reasons, it will be more convenient for us to view $\widehat{\mathcal{P}}_{\underline{d}}$ as the moduli space of twisted sheaves $\mathcal{G} := \mathcal{F}(-\ell)$ with framing at ℓ_∞: $\imath^*\mathcal{G} \simeq \mathcal{G}_\infty := W_0 \otimes \mathcal{O}_{\mathcal{X}}(-1) \oplus W_1 \otimes \mathcal{O}_{\mathcal{X}}(-2) \oplus \ldots \oplus W_{N-1} \otimes \mathcal{O}_{\mathcal{X}}(-N)$.

3.6. Plan of the proof. An exceptional collection of line bundles $\{\mathcal{O}_{\mathcal{X}},\ \mathcal{O}_{\mathcal{X}}(1),\ \ldots,\ \mathcal{O}_{\mathcal{X}}(N)\}$ on $\mathcal{X}$ gives rise to an equivalence of the derived category $D(\mathcal{X})$ of coherent sheaves on $\mathcal{X}$ and the derived category $D(K)$ of representations of the following quiver K:

$$\begin{tikzcd} V_0 \arrow[r, "B_0"] \arrow[rr, bend right, "\delta"'] & \ldots \arrow[r, "B_{N-1}"] & V_N. \end{tikzcd} \tag{3.1}$$

Since $\widehat{\mathcal{S}}$ is a $\mathbb{P}^1$-bundle over $\mathcal{X}$, it possesses the following exceptional collection of line bundles:

$$\{\pi^*\mathcal{O}_{\mathcal{X}}(-N-1),\ \ldots,\ \pi^*\mathcal{O}_{\mathcal{X}}(-1),\ (\pi^*\mathcal{O}_{\mathcal{X}})(-\ell_\infty),\ \ldots,\ (\pi^*\mathcal{O}_{\mathcal{X}}(N))(-\ell_\infty)\}.$$

The derived category $D(\widehat{\mathcal{S}})$ is equivalent to the category of diagrams

$$\mathcal{G}'(-N) \xleftarrow{\mu_0} \mathcal{G}'' \xrightarrow{\mu_\infty} \mathcal{G}'$$

where $\mathcal{G}',\ \mathcal{G}'' \in D(\mathcal{X}) \simeq D(K)$. For $\mathcal{G} \in D(\widehat{\mathcal{S}})$ we set $\mathcal{G}' = \pi_*\mathcal{G}$, $\mathcal{G}'' = \pi_*\mathcal{G}(-\ell_\infty)$. We have an exact triangle

$$\ldots \to \pi^*\mathcal{G}''(-\ell_\infty + N\ell) \xrightarrow{\mu} \pi^*\mathcal{G}' \to \mathcal{G} \to \pi^*\mathcal{G}''(-\ell_\infty + N\ell)[1] \to \ldots. \tag{3.2}$$

By the adjointness and the projection formula, the morphism μ is the same as the morphism $\mu_\infty \oplus \mu_0$ from $\mathcal{G}''$ to $\mathcal{G}' \otimes \pi_* \mathcal{O}_{\widehat{\mathcal{S}}}(\ell_\infty - N\ell) = \mathcal{G}' \oplus \mathcal{G}'(-N)$. The framing of $\mathcal{G}$ at ℓ_∞ implies the existence of the following exact triangle:

$$\ldots \to \mathcal{G}'' \xrightarrow{\mu_\infty} \mathcal{G}' \to \mathcal{G}_\infty \xrightarrow{\beta} \mathcal{G}''[1] \to \ldots.$$

Now the dual exceptional collection of $\{\mathcal{O}_\mathcal{X}, \mathcal{O}_\mathcal{X}(1), \ldots, \mathcal{O}_\mathcal{X}(N)\} \in D(\mathcal{X})$ is

$$\{\mathcal{O}_\mathcal{X}(-1-N), \mathrm{Coker}(\mathcal{O}_\mathcal{X}(-2-N) \hookrightarrow \mathcal{O}_\mathcal{X}(-1-N)), \ldots,$$
$$\mathrm{Coker}(\mathcal{O}_\mathcal{X}(-2N) \hookrightarrow \mathcal{O}_\mathcal{X}(1-2N)), \mathcal{O}_\mathcal{X}(-N)\}.$$

Decomposing $\mathcal{G}''$ with respect to this exceptional collection we obtain a representation of the quiver (3.1). The additional data $p_\bullet$ of the dented chainsaw quiver correspond to the morphism β (equivalently, μ_∞) above, while the additional data $(e, A_\bullet, q_\bullet)$ correspond to the morphism μ_0 above. The property of $V_\bullet$ being vector spaces (as opposed to complexes of vector spaces) is equivalent to the property of $\mathcal{G}$ being a perverse coherent sheaf with torsion supported at the exceptional divisor E and in codimension 2 (i.e., at finitely many points off E). Now $\zeta^\bullet$-semistability is equivalent to the vanishing of torsion of $\mathcal{G}$ at the generic point of E, while the ζ^--semistability is equivalent to the latter vanishing plus vanishing of the first cohomology of $\mathcal{G}$, that is $\mathcal{G}$ being a torsion free sheaf.

3.7. Monad for the stack $\widehat{\mathcal{S}}_N$. Given a $\widehat{Q}$-module Y, we construct $\mathcal{G}$ as the following monad of vector bundles on $\widehat{\mathcal{S}}$ (in cohomological degrees -1, 0, 1):

$$\bigoplus_{l=0}^{N} V_l(-\ell_\infty - (l+1)\ell) \xrightarrow{C} \begin{pmatrix} \bigoplus_{l=1}^{N} V_l(-\ell_\infty - l\ell) \\ \oplus \\ V_N(-\ell_\infty - \ell) \\ \oplus \\ \bigoplus_{l=0}^{N-1} W_l((-l-1)\ell) \\ \oplus \\ V_0((-N-1)\ell) \\ \oplus \\ \bigoplus_{l=1}^{N-1} V_l((-l-1)\ell) \end{pmatrix} \xrightarrow{D} \bigoplus_{l=1}^{N} V_l(-l\ell). \tag{3.3}$$

The morphisms C, D are described as follows.

We introduce the complex $\mathcal{G}''$ in the derived coherent category of $\mathcal{X}$:

$$\bigoplus_{l=0}^{N} V_l(-N-l-1) \xrightarrow{\gamma''} \bigoplus_{l=1}^{N} V_l(-N-l) \oplus V_N(-N-1). \tag{3.4}$$

Here the "matrix coefficients" of γ'' are as follows:

$$y_2 \colon V_l(-N-l-1) \to V_l(-N-l); \quad B_l \colon V_l(-N-l-1) \to V_{l+1}(-N-l-1);$$
$$-y_1 \colon V_N(-2N-1) \to V_N(-N-1); \quad \delta \colon V_0(-N-1) \to V_N(-N-1).$$

Alternatively, $\mathcal{G}''$ is canonically quasiisomorphic to another complex

$$V_0(-N-1) \oplus \bigoplus_{l=1}^{N-1} V_l(-l-1) \xrightarrow{\gamma'} \bigoplus_{l=1}^{N} V_l(-l), \tag{3.5}$$

where the "matrix coefficients" of γ' are as follows:

$$y_2\colon V_l(-l-1) \to V_l(-l); \quad B_l\colon V_l(-l-1) \to V_{l+1}(-l-1);$$
$$y_1 B_0\colon V_0(-N-1) \to V_1(-1); \quad y_2\delta\colon V_0(-N-1) \to V_N(-N).$$

The quasiisomorphism is given by

$$\begin{tikzcd}
\bigoplus_{l=0}^{N} V_l(-N-l-1) \arrow[r, "\gamma"] \arrow[d, "\upsilon"] & \bigoplus_{l=1}^{N} V_l(-N-l) \oplus V_N(-N-1) \arrow[d, "\upsilon'"] \\
V_0(-N-1) \oplus \bigoplus_{l=1}^{N-1} V_l(-l-1) \arrow[r, "\gamma'"] & \bigoplus_{l=1}^{N} V_l(-l),
\end{tikzcd} \tag{3.6}$$

where the nonzero components of υ, υ' are as follows: $\mathrm{id}\colon V_0(-N-1) \to V_0(-N-1)$; $-y_1\colon V_l(-N-l-1) \to V_l(-l-1)$ (resp. $y_1\colon V_l(-N-l) \to V_l(-l)$) for $1 \leqslant l \leqslant N-1$; $y_1\colon V_N(-2N) \to V_N(-N)$; $y_2\colon V_N(-N-1) \to V_N(-N)$.

Finally, we are able to describe the morphisms C, D of (3.3). We have $C = C_1 + C_2 + C_3$, where C_1 is

$$\pi^*\gamma''(-\ell_\infty + N\ell)\colon \bigoplus_{l=0}^{N} V_l(-\ell_\infty - (l+1)\ell) \to \bigoplus_{l=1}^{N} V_l(-\ell_\infty - l\ell) \oplus V_N(-\ell_\infty - \ell),$$

the matrix elements of

$$C_2\colon \bigoplus_{l=0}^{N} V_l(-\ell_\infty - (l+1)\ell) \to \bigoplus_{l=0}^{N-1} W_l((-l-1)\ell)$$

are $x_1 q_l \in \mathrm{Hom}(V_l(-\ell_\infty - (l+1)\ell), W_l((-l-1)\ell))$, i.e., they correspond by the adjointness and projection formula to $q_l \in \mathrm{Hom}_{\mathfrak{X}}(V_l(-l-1), W_l(-l-1))$. Now

$$C_3\colon \bigoplus_{l=0}^{N} V_l(-\ell_\infty - (l+1)\ell) \to V_0((-N-1)\ell) \oplus \bigoplus_{l=1}^{N-1} V_l((-l-1)\ell)$$

corresponds by adjointness and projection formula to υ of (3.6) plus

$$(0,\, A_1(-2),\, \ldots,\, A_{N-1}(-N),\, e(-N-1))\colon$$
$$\bigoplus_{l=0}^{N} V_l(-l-1) \to V_0(-N-1) \oplus \bigoplus_{l=1}^{N-1} V_l(-l-1).$$

In other words, $C_3 = x_2\upsilon + x_1 e + x_1 \sum_{l=1}^{N-1} A_l$.

We have $D = D_1 + D_2 + D_3$, where

$$D_1\colon \bigoplus_{l=1}^{N} V_l(-\ell_\infty - l\ell) \oplus V_N(-\ell_\infty - \ell) \to \bigoplus_{l=1}^{N} V_l(-l\ell)$$

corresponds by adjointness and projection formula to υ' of (3.6) plus

$$(-A_1(-1),\, \ldots,\, -A_{N-1}(-N+1),\, -\delta e(-N),\, B_0 e(-1))\colon \bigoplus_{l=1}^{N} V_l(-l) \oplus V_N(-1) \to \bigoplus_{l=1}^{N} V_l(-l).$$

In other words,

$$D_1 = x_2\upsilon' - x_1 \sum_{l=1}^{N-1} A_l - x_1\delta e + x_1 B_0 e.$$

Now

$$D_2\colon \bigoplus_{l=0}^{N-1} W_l((-l-1)\ell) \to \bigoplus_{l=1}^{N} V_l(-l\ell)$$

is

$$-\pi^*\beta\colon \bigoplus_{l=0}^{N-1} W_l((-l-1)\ell) \to \bigoplus_{l=1}^{N} V_l(-l\ell).$$

The "matrix elements" of $\beta \in \operatorname{Hom}(\mathcal{G}_\infty, \bigoplus_{l=1}^{N} V_l(-l))$ are $p_{l+1}\colon W_l(-l-1) \to V_{l+1}(-l-1)$. Finally,

$$D_3\colon V_0((-N-1)\ell) \oplus \bigoplus_{l=1}^{N-1} V_l((-l-1)\ell) \to \bigoplus_{l=1}^{N} V_l(-l\ell)$$

is $\pi^*\gamma'$.

3.8. Inverse construction. Conversely, given a torsion free sheaf $\mathcal{G}$ on $\widehat{\mathcal{S}}$ with a framing $\imath^*\mathcal{G} \simeq \mathcal{G}_\infty$ we have $H^0(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + l\ell)) = H^2(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + l\ell)) = 0$ for $0 \leqslant l \leqslant N$, and we set $V_l := H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + l\ell))$. Furthermore, we set $\mathcal{G}'' := \pi_*(\mathcal{G}(-\ell_\infty))$, and $\mathcal{G}' := \pi_*\mathcal{G}$, so that $V_l = H^1(\mathcal{X}, \mathcal{G}''(l))$.

Let $\Delta\colon \widehat{\mathcal{S}} \hookrightarrow \widehat{\mathcal{S}} \times \widehat{\mathcal{S}}$ stand for the diagonal embedding. We have the following exact triangles in the derived category of coherent sheaves on $\widehat{\mathcal{S}}$:

$$\begin{aligned}\ldots \to \mathcal{O}_{\widehat{\mathcal{S}}\times_{\mathcal{X}}\widehat{\mathcal{S}}}(-\ell_\infty, -\ell_\infty + N\ell) \xrightarrow{\xi} \mathcal{O}_{\widehat{\mathcal{S}}\times_{\mathcal{X}}\widehat{\mathcal{S}}} \to \Delta_*\mathcal{O}_{\widehat{\mathcal{S}}} \\ \to \mathcal{O}_{\widehat{\mathcal{S}}\times_{\mathcal{X}}\widehat{\mathcal{S}}}(-\ell_\infty, -\ell_\infty + N\ell)[1] \to \ldots,\end{aligned} \tag{3.7}$$

$$\ldots \to \mathcal{O}_{\widehat{\mathcal{S}}\times_{\mathcal{X}}\widehat{\mathcal{S}}}(-\ell_\infty, 0) \to \mathcal{O}_{\widehat{\mathcal{S}}\times_{\mathcal{X}}\widehat{\mathcal{S}}} \xrightarrow{\eta} \mathcal{O}_{\ell_\infty\times_{\mathcal{X}}\widehat{\mathcal{S}}} \xrightarrow{\rho} \mathcal{O}_{\widehat{\mathcal{S}}\times_{\mathcal{X}}\widehat{\mathcal{S}}}(-\ell_\infty, 0)[1] \to \ldots. \tag{3.8}$$

It follows that $\Delta_*\mathcal{O}_{\widehat{\mathcal{S}}}$ is the convolution of the following compex of objects of the derived category of coherent sheaves on $\widehat{\mathcal{S}}$:

$$\mathcal{O}_{\widehat{\mathcal{S}}\times_{\mathcal{X}}\widehat{\mathcal{S}}}(-\ell_\infty, -\ell_\infty + N\ell) \xrightarrow{\eta\circ\xi} \mathcal{O}_{\ell_\infty\times_{\mathcal{X}}\widehat{\mathcal{S}}} \xrightarrow{\rho} \mathcal{O}_{\widehat{\mathcal{S}}\times_{\mathcal{X}}\widehat{\mathcal{S}}}(-\ell_\infty, 0)[1]. \tag{3.9}$$

Now since $\mathcal{G} \simeq \mathrm{pr}_{2*}(\Delta_*\mathcal{O}_{\widehat{\mathcal{S}}} \otimes \mathrm{pr}_1^*\,\mathcal{G})$, we see that $\mathcal{G}$ is the convolution of the following complex of objects of the derived coherent category of $\widehat{\mathcal{S}}$:

$$\pi^*\mathcal{G}''(-\ell_\infty + N\ell) \xrightarrow{\alpha} \pi^*\mathcal{G}_\infty \xrightarrow{\pi^*\beta} \pi^*\mathcal{G}''[1]. \tag{3.10}$$

Here β enters the exact triangle

$$\ldots \to \mathcal{G}'' \to \mathcal{G}' \to \imath^*\mathcal{G} \xrightarrow{\beta} \mathcal{G}''[1] \to \ldots \tag{3.11}$$

while α by adjointness and projection formula is the same as the direct sum of two morphisms $\alpha' \colon \mathcal{G}'' \to \mathcal{G}_\infty$, and $\alpha'' \colon \mathcal{G}''(N) \to \mathcal{G}_\infty$. The condition $\pi^*\beta \circ \alpha = 0$ implies $\alpha' = 0$.

Let now $\Delta^{\mathcal{X}}$ stand for the diagonal embedding $\mathcal{X} \hookrightarrow \mathcal{X} \times \mathcal{X}$. Then we have the exact sequences of coherent sheaves on $\mathcal{X} \times \mathcal{X}$:

$$0 \to \mathcal{O}(0, -N) \oplus \bigoplus_{l=0}^{N-1} \mathcal{O}(l, -l-1) \to \bigoplus_{l=0}^{N} \mathcal{O}(l, -l) \to \Delta^{\mathcal{X}}_* \mathcal{O}_{\mathcal{X}} \to 0, \tag{3.12}$$

$$0 \to \mathcal{O}(0, -N-1) \oplus \bigoplus_{l=1}^{N-1} \mathcal{O}(l, -l-1) \to \bigoplus_{l=1}^{N} \mathcal{O}(l, -l) \to \Delta^{\mathcal{X}}_* \mathcal{O}_{\mathcal{X}} \to 0, \tag{3.13}$$

which yield the resolutions (3.4) and (3.5) for $\mathcal{G}'' \simeq \mathrm{pr}_{2*}(\Delta^{\mathcal{X}}_* \mathcal{O}_{\mathcal{X}} \otimes \mathrm{pr}_1^* \mathcal{G}'')$. In particular, $B_l \colon V_l \to V_{l+1}$ is induced by

$$y_2 \colon H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + l\ell)) \to H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + (l+1)\ell)),$$

and $\delta \colon V_0 \to V_N$ is induced by

$$y_1 \colon H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty)) \to H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + N\ell)).$$

Furthermore, the morphism $\alpha''(l-N) \colon \mathcal{G}''(l) \to \mathcal{G}_\infty(l-N)$ induces the morphism

$$V_l = H^1(\mathcal{X}, \mathcal{G}''(l)) \to H^1(\mathcal{X}, \mathcal{G}_\infty(l-N) = W_l \oplus \ldots \oplus W_{N-1}$$

with components $q_l,\ q_{l+1}B_l,\ \ldots,\ q_{N-1}B_{N-2}\ldots B_l$. The morphism $\beta(l) \colon \mathcal{G}_\infty(l) \to \mathcal{G}''(l)[1]$ induces the morphism

$$W_0 \oplus \ldots \oplus W_{l-1} = H^0(\mathcal{X}, \mathcal{G}_\infty(l)) \to H^1(\mathcal{X}, \mathcal{G}''(l)) = V_l$$

whose last component is $p_l \colon W_{l-1} \to V_l$.

The exact triangle $\ldots \to \mathcal{G}'' \to \mathcal{G}' \to \mathcal{G}_\infty \to \mathcal{G}''[1] \to \ldots$ along with the acyclicity of $\mathcal{G}_\infty$ yields an isomorphism $V_0 = H^1(\mathcal{X}, \mathcal{G}'') \simeq H^1(\mathcal{X}, \mathcal{G}')$. Now $e \colon V_N \to V_0$ is induced by

$$\times_2 \colon V_N = H^1(\mathcal{X}, \mathcal{G}''(N)) = H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + N\ell)) \to H^1(\widehat{\mathcal{S}}, \mathcal{G}) = H^1(\mathcal{X}, \mathcal{G}') = V_0.$$

Finally, the exact triangle

$$\ldots \to \mathcal{G}''(l-N) \to \mathcal{G}'(l-N) \to \mathcal{G}_\infty(l-N) \to \mathcal{G}''(l-N)[1] \to \ldots$$

yields the long exact sequence

$$\begin{aligned} 0 \to H^1(\mathcal{X}, \mathcal{G}''(l-N)) \to H^1(\mathcal{X}, \mathcal{G}'(l-N)) \to H^1(\mathcal{X}, \mathcal{G}_\infty(l-N)) \\ \to H^2(\mathcal{X}, \mathcal{G}''(l-N)) \to \ldots. \end{aligned} \tag{3.14}$$

A resolution

$0 \to \mathcal{G}''(l-N) \to \mathcal{G}''\oplus\mathcal{G}''(l)\oplus\mathcal{G}''(l+1)\oplus\ldots\oplus\mathcal{G}''(N-1) \to \mathcal{G}''(l+1)\oplus\ldots\oplus\mathcal{G}''(N) \to 0$

implies $H^1(\mathcal{X},\, \mathcal{G}''(l-N)) \simeq \operatorname{Ker}\Big(V_0 \oplus \bigoplus_{m=l}^{N-1} V_m \to \bigoplus_{m=l+1}^{N} V_m\Big)$, which, together with (3.14), yields an isomorphism

$$H^1(\mathcal{X},\, \mathcal{G}'(l-N)) \simeq \operatorname{Ker}\Big(V_0 \oplus \bigoplus_{m=l}^{N-1} V_m \oplus \bigoplus_{m=l}^{N-1} W_m \xrightarrow{\varrho} \bigoplus_{m=l+1}^{N} V_m\Big).$$

Here the "matrix coefficients" of ϱ are as follows: $\delta\colon V_0 \to V_N$; $B_m\colon V_m \to V_{m+1}$; $\operatorname{Id}\colon V_m \to V_m$; $p_{m+1}\colon W_m \to V_{m+1}$. In particular, we have a morphism

$$\varpi\colon H^1(\mathcal{X},\, \mathcal{G}'(l-N)) \to V_l.$$

Now the morphism $x_2((l-N)\ell)\colon \mathcal{G}(-\ell_\infty + l\ell) \to \mathcal{G}((l-N)\ell)$ gives rise to the morphism $V_l = H^1(\widehat{\mathcal{S}},\, \mathcal{G}(-\ell_\infty + l\ell)) \to H^1(\widehat{\mathcal{S}},\, \mathcal{G}((l-N)\ell)) = H^1(\mathcal{X},\, \mathcal{G}'(l-N))$. Composing it with ϖ we obtain the morphism $A_l\colon V_l \to V_l$.

3.9. Analysis of stability. Clearly, at ℓ_∞, C is injective, and D is surjective. Hence C is injective (as a morphism of sheaves) everywhere, i.e., H^{-1} of the monad (3.3) is 0, and the support of H^1 of the same monad does not intersect ℓ_∞. Hence this support lies in the union of the exceptional divisor E and finitely many points. Now H^0 of (3.3) cannot have torsion in codimension 2 (as the middle cohomology of a 3-term complex of vector bundles), so it can only have torsion at curves not intersecting ℓ_∞, i.e., at E. If this H^0 does have torsion at E, then the fibers of C at E (i.e., $x_2 = 0$) have nontrivial kernels. We can set $x_1 = 0$, and then the kernel of C at a point $[y_1 : y_2] \in E$ consists of collections of $v_l \in V_l$, $0 \leqslant l \leqslant N$ such that $q_l v_l = 0$, $0 \leqslant l \leqslant N-1$; $ev_N = 0$; $B_{N-1}v_{N-1} + y_2 v_N = 0$; $\delta v_0 - y_1 v_N = 0$; $B_{l-1}v_{l-1} + y_2 v_l = 0 = A_l v_l$, $1 \leqslant l \leqslant N-1$. Note that if $v_0 = 0$ then in case $y_1 \neq 0$ we obtain $v_N = 0$, while in case $y_2 \neq 0$ we obtain $v_1 = 0 \Rightarrow v_2 = 0 \Rightarrow \ldots \Rightarrow v_N = 0$. Moreover, in case $y_2 \neq 0 \neq y_1$ we have $v_1 = y_2^{-1}B_0 v_0$, $v_2 = y_2^{-2}B_1B_0v_0$, $\ldots$, $v_N = y_2^{-N}B_{N-1}\ldots B_1B_0v_0 = y_1^{-1}\delta v_0$. The existence of a nontrivial solution of the equation $y_2^{-N}B_{N-1}\ldots B_1B_0v_0 = y_1^{-1}\delta v_0$ for any y_1, y_2 implies that $\operatorname{Ker}(B_{N-1}\ldots B_1B_0) \cap \operatorname{Ker}(\delta) \neq 0$. We take $0 \neq v_0 \in \operatorname{Ker}(B_{N-1}\ldots B_1B_0) \cap \operatorname{Ker}(\delta)$, and the corresponding $v_1, \ldots, v_{N-1}, v_N = 0$. We set $S_0 := \mathbb{C}v_0 \subset V_0, \ldots, S_{N-1} := \mathbb{C}v_{N-1} \subset V_{N-1}$, $S_N := 0 \subset V_N$. Then $S_\bullet \subset V_\bullet$ violates the $\zeta^\bullet$-semistability since $\dim S_0 > \dim S_N$ (see [21, Definition 1.1.(1) and Section 4(ii)]). Conversely, the argument like [23, Lemma 7.2.(2)] proves that the fiberwise injectivity of C at the generic point of E implies $\zeta^\bullet$-semistability.

The vanishing of H^1(3.3) is equivalent to the fiberwise surjectivity of D everywhere, that is, to the fiberwise injectivity of the adjoint morphism D^* everywhere. Since D^* is clearly injective at ℓ_∞ we can study the points off ℓ_∞, i.e., we can set $x_1 = 1$. The kernel of D^* at a point $(x_2, [y_1 : y_2])$ consists of collections $v_l^* \in V_l^*$, $0 \leqslant l \leqslant N$ such that $p_l^* v_l^* = 0$, $1 \leqslant l \leqslant N$; $y_1B_0^*v_1^* + y_2\delta^*v_N^* = 0$; $(e^*\delta^* - x_2y_1)v_N^* = 0$; $e^*B_0^*v_1^* + x_2y_2v_N^* = 0$; $B_l^*v_{l+1}^* + y_2v_l^* = 0 = A_l^*v_l^* - x_2y_1v_l^*$, $1 \leqslant l \leqslant N-1$. Note that v_0^* does not enter these equations, and we can set $v_0 = 0$. Given a nontrivial solution of these equations we set $S_0^* := 0 \subset V_0^*$, $S_1^* := \mathbb{C}v_1^* \subset V_1^*, \ldots, S_N^* := \mathbb{C}v_N^* \subset V_N^*$. We set $T_l := (S_l^*)^\perp \subset V_l$, $0 \leqslant l \leqslant N$. We have $0 = \dim S_0^* = \operatorname{codim} T_0 \leqslant \operatorname{codim} T_N = \dim S_N^*$, which violates the ζ^--semistability (see [21, Definition 1.1.(2) and Section 4(ii)]). Conversely, the argument like [23, Lemma 5.1] proves that the fiberwise surjectivity of D implies ζ^--semistability.

Theorem 3.4 is proved.

3.10. Parabolic sheaves trivial along ℓ_0. We say that a parabolic sheaf $\mathcal{F}_\bullet$ is *trivial along* ℓ_0 if $\mathcal{F}_0$ is locally free at ℓ_0, and $\mathcal{F}_0|_{\ell_0}$ is trivial; moreover, for $-N \leqslant k \leqslant 0$ the sheaf $\mathcal{F}_k$ is locally free at ℓ_0, and the quotient sheaves $\mathcal{F}_k/\mathcal{F}_{-N}$, $\mathcal{F}_0/\mathcal{F}_k$ are both locally free and trivial (as vector bundles) at ℓ_0. It is easy to see that the moduli space $\widehat{\mathcal{P}}_{\underline{d},\mathrm{triv}} \subset \widehat{\mathcal{P}}_{\underline{d}}$ of parabolic sheaves trivial along ℓ_0 is an open subset of $\widehat{\mathcal{P}}_{\underline{d}}$, nonempty if and only if $d_0 = d_1 = \ldots = d_N$.

We also consider an open subset $\widehat{\mathfrak{M}}_{\underline{d},\mathrm{iso}} \subset \widehat{\mathfrak{M}}_{\underline{d}}$ formed by the representations of $\widehat{Q}$ such that $B_0, \ldots, B_{N-1}$ are all isomorphisms (evidently, $\widehat{\mathfrak{M}}_{\underline{d},\mathrm{iso}}$ is nonempty if and only if $d_0 = d_1 = \ldots = d_N$). The proof of Theorem 3.4 admits the following

Corollary 3.11. $\Xi(\widehat{\mathfrak{M}}_{\underline{d},\mathrm{iso}}) = \widehat{\mathcal{P}}_{\underline{d},\mathrm{triv}}$.

Proof. The complex $\mathcal{G}'' \in D(\mathcal{X})$ corresponds to the representation (3.1) in $D(K)$. It is easy to check that $B_0, \ldots, B_{N-1}$ in (3.1) are all isomorphisms if and only if $\mathrm{Ext}^\bullet_{\mathcal{X}}(\mathcal{G}'', \mathcal{H}) = 0$ for any skyscraper sheaf $\mathcal{H}$ supported at $0_{\mathcal{X}}$. More precisely, B_l is an isomorphism if and only if $\mathrm{Ext}^\bullet_{\mathcal{X}}(\mathcal{G}'', \mathrm{Coker}(\mathcal{O}_{\mathcal{X}}(-l-1) \hookrightarrow \mathcal{O}_{\mathcal{X}}(-l))) = 0$. Thus all B_l are isomorphisms if and only if $\mathcal{G}''$ has a finite support on $\mathcal{X}$ disjoint from $0_{\mathcal{X}}$. From the exact triangle (3.11), $\mathcal{G}'$ is isomorphic to $\imath^*\mathcal{G} = \mathcal{G}_\infty$ near $0_{\mathcal{X}}$. From the exact triangle (3.2), $\mathcal{G}$ is isomorphic to $\pi^*\mathcal{G}_\infty$ near ℓ, that is, $\mathcal{F}_\bullet$ is trivial along ℓ_0. □

3.12. Blowdown. The blowdown morphism $\widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ *does not* lift to a morphism of the stacks $\widehat{\mathcal{S}} \to \mathcal{S}'$ (see Section 2.6). It only gives rise to a correspondence $\widehat{\mathcal{S}} \xleftarrow{\tilde{\mu}} \mathcal{W} \xrightarrow{\tilde{\nu}} \mathcal{S}'$. Since both $\widehat{\mathcal{S}}$ and $\mathcal{S}'$ are toric stacks, this correspondence can be described in toric terms. Namely, $\widehat{\mathcal{S}}$ is given by a fan $\hat{F}$ formed by the vectors $(1, 0)$; $(0, 1)$; $(-1, 0)$; $(-N, -N)$ in $\mathbb{Z}^2$, while $\mathcal{S}'$ is given by a fan F' formed by the vectors $(1, 0)$; $(0, 1)$; $(-N, -N)$, and $\mathcal{W}$ is given by a fan $\tilde{F}$ formed by the vectors $(1, 0)$; $(0, 1)$; $(-N, 0)$; $(-N, -N)$. The evident embedding $F' \subset \tilde{F}$ corresponds to our $\tilde{\nu}$. Since $\tilde{F}$ is obtained from $\hat{F}$ by dilating the vector $(-1, 0)$, we obtain the desired morphism $\tilde{\mu}\colon \mathcal{W} \to \widehat{\mathcal{S}}$. We set $\Pi := \tilde{\nu}_*\tilde{\mu}^*\colon D^b\,\mathrm{Coh}(\widehat{\mathcal{S}}) \to D^b\,\mathrm{Coh}(\mathcal{S}')$. Note that the assumptions of Theorem 4.2(2) of [17] are satisfied in our situation.

Given $\mathcal{F} = \mathcal{G}(\ell) \in \widehat{\mathcal{P}}_{\underline{d}}$ the complex $\Pi\mathcal{F}$ is not necessarily a torsion free sheaf: it can have the first cohomology (a torsion sheaf at the origin); it is rather a *perverse coherent sheaf.* Its class is well defined in the zastava space $\mathfrak{Z}_{\underline{d}}$ (see Section 2.1). Thus we obtain the morphism $\widehat{\mathcal{P}}_{\underline{d}} \to \mathfrak{Z}_{\underline{d}}$, which factors through the morphism $\widehat{\mathfrak{Z}}_{\underline{d}} \to \mathfrak{Z}_{\underline{d}}$ (since $\mathfrak{Z}_{\underline{d}}$ is affine, $\widehat{\mathfrak{Z}}_{\underline{d}}$ is normal, and $\widehat{\mathcal{P}}_{\underline{d}} \to \widehat{\mathfrak{Z}}_{\underline{d}}$ is proper) to be denoted by Π.

Conjecture 3.13. *In quiver terms,* $\Pi\colon \widehat{\mathfrak{Z}}_{\underline{d}} \to \mathfrak{Z}_{\underline{d}}$ *sends* $(V_\bullet, A_\bullet, B_\bullet, \delta, e, p_\bullet, q_\bullet)$ *to* $(V'_\bullet, A'_\bullet, B'_\bullet, p'_\bullet, q'_\bullet)$*, where* $V'_l := V_l$ *for* $l = 1, \ldots, N$*, and* $A'_l := A_l$ *for* $l =$

$1, \ldots, N-1$, *while* $A'_0 := \delta e$. *Furthermore,* $B'_l := B_l$ *for* $l = 1, \ldots, N-1$, *while* $B'_0 := B_0 e$. *Furthermore,* $p'_l = p_l$ *for* $l = 1, \ldots, N$, *and* $q'_l := q_l$ *for* $l = 1, \ldots, N-1$, *while* $q_0 := q_0 e$.

4. Zastava for Kleinian Blowup

4.1. Kleinian Blowup. We consider $\mathcal{S}'_1 = \mathbb{P}^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$. We blow up the sheaf of ideals I supported at the origin, and generated locally by (z_1^k, z_2). This blowup is a singular toric surface lying in the projectivization $\mathrm{Proj}(\mathcal{O}(-k) \oplus (\mathcal{O}(-1))$ of the vector bundle $\mathcal{O}(-k) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^2$. In fact, it has a unique singular point, lying in a chart $\overline{U}^2$ with coordinates z_1, z_2, z satisfying $z_2 z = z_1^k$. We define a smooth toric stack $\widehat{\mathcal{S}}_1^k$ as the stacky resolution of our blowup at the singular point. The neighbourhood U^2 of the stacky point (the preimage of $\overline{U}^2$) is isomorphic to $\mathbb{A}^2/\Gamma_k$ (with hyperbolic action). The stack $\widehat{\mathcal{S}}_1^k$ is given by a fan $\hat{F}^k$ formed by the vectors $(1, 0); (0, 1); (-1, k-1); (-1, -1)$ in $\mathbb{Z}^2$.

Let $\ell_0 \subset \widehat{\mathcal{S}}_1^k$ be the proper transform of the line $\ell_0 \subset \mathcal{S}'_1$ given by the equation $z_2 = 0$. It lies in the union of two charts U^1 with coordinates z_1, w and U^∞ with coordinates z_0, z_2. We consider the ramified Galois coverings U_N^1 with coordinates $z_1, \sqrt[N]{w}$, and U_N^∞ with coordinates $z_0, \sqrt[N]{z_2}$, with Galois group Γ_N. Gluing the stacky quotients U_N^1/Γ_N, and U_N^∞/Γ_N with U^2, we obtain the smooth toric stack $\widehat{\mathcal{S}}_N^k$. It is given by a fan $\hat{F}_N^k$ formed by the vectors $(1, 0)$; $(0, 1)$; $(-1, k-1)$; $(-N, -N)$ in $\mathbb{Z}^2$. The preimage of $\ell_0 \subset \widehat{\mathcal{S}}_1^k$ is denoted by $\ell \subset \widehat{\mathcal{S}}_N^k$. Its automorphism group is Γ_N.

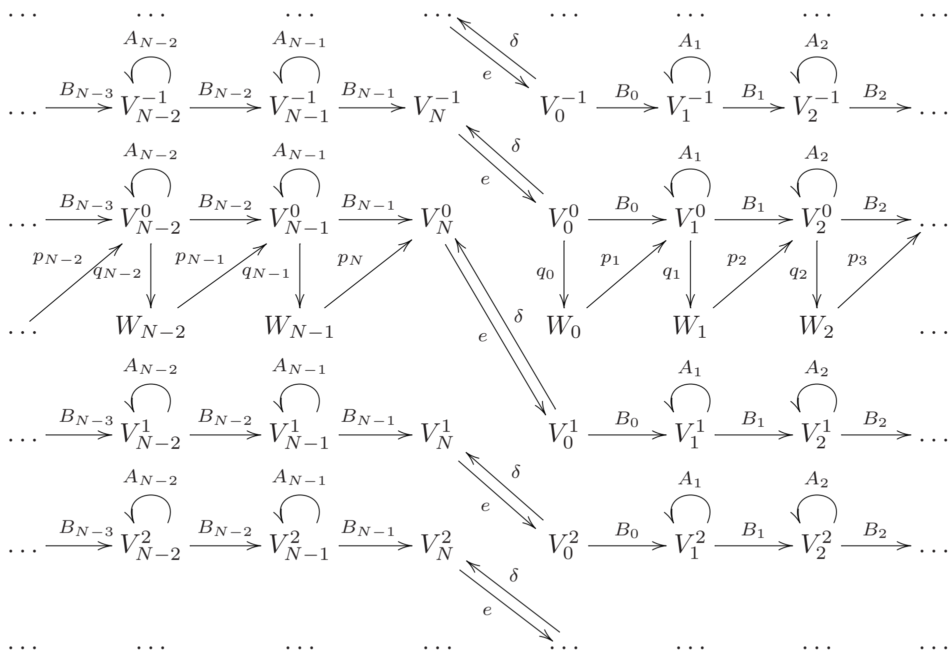
We have a correspondence $\widehat{\mathcal{S}}_N^k \xleftarrow{\tilde{\mu}_k} \mathcal{W}^k \xrightarrow{\tilde{\nu}^k} \mathcal{S}'_N$ (cf. Section 3.12) where $\mathcal{W}^k$ is given by a fan $\tilde{F}^k$ formed by the vectors $(1, 0)$; $(0, 1)$; $(-N, N(k-1))$; $(-N, -N)$. The evident embedding $F' \subset \tilde{F}^k$ corresponds to our $\tilde{\nu}^k$. Since $\tilde{F}^k$ is obtained from $\hat{F}_N^k$ by dilating the vector $(-1, k-1)$, we obtain the desired morphism $\tilde{\mu}_k \colon \mathcal{W}^k \to \widehat{\mathcal{S}}_N^k$.

4.2. Parabolic sheaves on Kleinian Blowup. Let $\widehat{\mathcal{P}}^k$ be the moduli space of rank N torsion free parabolic sheaves on $\widehat{\mathcal{S}}_1^k$ (with parabolic structure along ℓ_0, and with trivial first Chern class) trivialized at infinity; equivalently, $\widehat{\mathcal{P}}^k$ is the moduli space of rank N torsion free sheaves on $\widehat{\mathcal{S}}_N^k$ framed at

$$\ell_\infty \colon \imath^* \mathcal{F} \simeq \mathcal{F}_\infty := W_0 \otimes \mathcal{O}_\mathfrak{X} \oplus W_1 \otimes \mathcal{O}_\mathfrak{X}(-1) \oplus \ldots \oplus W_{N-1} \otimes \mathcal{O}_\mathfrak{X}(N-1)$$

(and with first Chern class trivial off ℓ_∞). Given $\mathcal{F} \in \widehat{\mathcal{P}}^k$ the complex $\Pi^k \mathcal{F} := \tilde{\nu}_*^k \tilde{\mu}_k^* \mathcal{F}$ is not necessarily a torsion free sheaf on $\mathcal{S}'_N$: it can have the first cohomology (a torsion sheaf at the origin); it is rather a perverse coherent sheaf. Its class is well defined in the zastava space $\mathfrak{Z}$ (the direct limit of all $\mathfrak{Z}_{\underline{d}}$ with respect to the natural embeddings $\mathfrak{Z}_{\underline{d}} \hookrightarrow \mathfrak{Z}_{\underline{d}'}$, $\underline{d}' \geqslant \underline{d}$ componentwise, adding defect at the origin). Thus we obtain the morphism $\widehat{\mathcal{P}}^k \to \mathfrak{Z}$. Our goal is to describe the moduli space $\widehat{\mathcal{P}}^k$ (in particular, to number its connected components) in quiver terms, as well as the morphism $\widehat{\mathcal{P}}^k \to \mathfrak{Z}$. A connected component of $\widehat{\mathcal{P}}^k$ will be called *good* if it contains a *nonempty* open subset formed by *locally free* parabolic sheaves.

4.3. Rift. We consider the representations of the following *rift* quiver $\widehat{Q}^k$:



Here the upper indices of V run through $\mathbb{Z}/k\mathbb{Z}$. The dimension of V_l^r is denoted by d_l^r. We consider the dimension vector $\widehat{\underline{d}} := (d_l^r)_{0\leqslant l\leqslant N}^{r\in\mathbb{Z}/k\mathbb{Z}}$. Furthermore, $\dim W_l = 1$, and all these lines are identified with, say W_∞, so that W_∞ is the source of all p_l and the target of all q_l.

Relations:

$$
\begin{aligned}
&0 = \delta e B_{N-1} - B_{N-1}A_{N-1} + p_N q_{N-1} \colon V_{N-1}^0 \to V_N^0.\\
&0 = A_1 B_0 - B_0 e\delta + p_1 q_0 \colon V_0^0 \to V_1^0.\\
&0 = A_{l+1}B_l - B_l A_l + p_{l+1}q_l \colon V_l^0 \to V_{l+1}^0 \quad \text{for } l = 1, \ldots, N-2.\\
&0 = \delta e B_{N-1} - B_{N-1}A_{N-1} \colon V_{N-1}^r \to V_N^r \quad \text{for } r \neq 0.\\
&0 = A_1 B_0 - B_0 e\delta \colon V_0^r \to V_1^r \quad \text{for } r \neq 0.\\
&0 = A_{l+1}B_l - B_l A_l \colon V_l^r \to V_{l+1}^r \quad \text{for } l = 1, \ldots, N-2,\ r \neq 0.
\end{aligned}
$$

We denote a typical representation of $\widehat{Q}^k$ by Y. We denote by $\mathsf{M}_{\widehat{\underline{d}}}$ the scheme of representations of $\widehat{Q}^k$ of dimension $\widehat{\underline{d}}$. We denote by $G_{\widehat{\underline{d}}}$ the group $\prod_{0\leqslant l\leqslant N}^{r\in\mathbb{Z}/k\mathbb{Z}} \mathrm{GL}(V_l^r)$; it acts naturally on $\mathsf{M}_{\widehat{\underline{d}}}$.

4.4. Stability conditions. Following [21, Section 4(ii)] we consider the enhanced dimension vector $\ddot{\underline{d}} := (d_l^r, 1)$ with one extra coordinate equal to $\dim W_\infty = 1$. We consider a vector $\zeta^\bullet = (\zeta_l^r)$ where $\zeta_N^r = -1$, $\zeta_0^r = 1$, $\zeta_l^r = 0$ for $1 \leqslant l \leqslant N-1$. Also, for $0 < \varepsilon \ll 1$ we consider $\zeta^- := \zeta^\bullet - (\varepsilon, \ldots, \varepsilon)$. We set $\zeta_\infty^- := -\langle \zeta^-, \widehat{\underline{d}}\rangle$, and $\zeta^\bullet_\infty := -\langle \zeta^\bullet, \widehat{\underline{d}}\rangle$, where $\langle\cdot,\cdot\rangle$ stands for the sum of products of coordinates (the standard scalar product). Finally, we set $\widetilde{\zeta}^- := (\zeta^-, \zeta^-_\infty)$, and $\widetilde{\zeta}^\bullet := (\zeta^\bullet, \zeta^\bullet_\infty)$.

For a nonzero $\widehat{Q}^k$-submodule $Y' \subset Y$ of enhanced dimension $\ddot{\underline{d}}'$ (where the last coordinate may be either 1 or 0) we define the slope by

$$\theta^-(Y') := \frac{\langle \widetilde{\zeta}^-, \ddot{\underline{d}}'\rangle}{\langle (1,\ldots,1), \ddot{\underline{d}}'\rangle}, \quad \theta^\bullet(Y') := \frac{\langle \widetilde{\zeta}^\bullet, \ddot{\underline{d}}'\rangle}{\langle (1,\ldots,1), \ddot{\underline{d}}'\rangle}.$$

We say that a $\widehat{Q}^k$-module Y is ζ^--semistable (resp. $\zeta^\bullet$-semistable) if for any nonzero submodule $Y' \subset Y$ we have $\theta^-(Y') \leqslant \theta^-(Y)$ (resp. $\theta^\bullet(Y') \leqslant \theta^\bullet(Y)$). We say Y is ζ^--stable (resp. $\zeta^\bullet$-stable) if the inequality is strict unless $Y' = Y$. Note that ζ^--stability is equivalent to ζ^--semistability.

We define a scheme $\widehat{\mathfrak{M}}^k_{\widehat{\underline{d}}}$ as the moduli space of ζ^--semistable (equivalently, ζ^--stable) $\widehat{Q}^k$-modules. By GIT, $\widehat{\mathfrak{M}}^k_{\widehat{\underline{d}}}$ is the projective spectrum of the ring of $\widehat{G}_{\widehat{\underline{d}}}$-semiinvariants in $\mathbb{C}[\widehat{\mathsf{M}}_{\widehat{\underline{d}}}]$. Furthermore, we define a scheme $\widehat{\mathfrak{Z}}^k_{\widehat{\underline{d}}}$ as the moduli space of S-equivalence classes of $\zeta^\bullet$-semistable $\widehat{Q}^k$-modules. Since the stability condition $\zeta^\bullet$ lies on a wall of the chamber containing ζ^-, we have a projective morphism $\pi_{\zeta^\bullet,\zeta^-}\colon \widehat{\mathfrak{M}}^k_{\widehat{\underline{d}}} \to \widehat{\mathfrak{Z}}^k_{\widehat{\underline{d}}}$.

Theorem 4.5. *A good connected component of $\widehat{\mathcal{P}}^k$ is isomorphic to $\widehat{\mathfrak{M}}_{\widehat{\underline{d}}}$ for a dimension vector $\widehat{\underline{d}}$ such that $d^0_0 = d^0_N$, and for $r \neq 0$, $d^r_l = d^r_m$ $\forall l, m = 1,\ldots,N$.*

The proof is given in the next Subsection.

4.6. The action of $\boldsymbol{\Gamma_k}$. The action of Γ_k on $\mathcal{S}'_1$ (see Section 2.8) lifts to the action of Γ_k on the blowup $\widehat{\mathcal{S}}_1$, and also lifts to the action of Γ_k on $\widehat{\mathcal{S}}_N$. Hence Γ_k acts on the moduli space $\widehat{\mathcal{P}}_{\underline{d}}$ of parabolic sheaves on $\widehat{\mathcal{S}}_1$ trivialized at infinity via its action on $\widehat{\mathcal{S}}_1$ and the *trivial* action on the trivialization at infinity.

The fixed point variety $\widehat{\mathcal{P}}^{\Gamma_k}_{\underline{d}} = \widehat{\mathfrak{M}}^{\Gamma_k}_{\underline{d}}$ can be described in quiver terms as well. Namely, the construction of quiver in Section 3.8 implies that the action of the generator ζ_k of Γ_k on the quiver components works as follows:

$$\zeta_{kN}(A_\bullet, B_\bullet, e, \delta, p_\bullet, q_\bullet) = (A_\bullet, \zeta_k B_\bullet, \zeta_k^{-2}e, \zeta_k^2\delta, \zeta_k p_\bullet, q_\bullet).$$

It follows that the various connected components of $\widehat{\mathfrak{M}}^{\Gamma_k}_{\underline{d}}$ are isomorphic to $\widehat{\mathfrak{M}}^k_{\widehat{\underline{d}}}$ for various dimension vectors $\widehat{\underline{d}}$ such that

$$\underline{d} = \Big(\sum_{r\in\mathbb{Z}/k\mathbb{Z}} d^r_0, \sum_{r\in\mathbb{Z}/k\mathbb{Z}} d^r_1, \ldots, \sum_{r\in\mathbb{Z}/k\mathbb{Z}} d^r_N\Big).$$

Among these connected components we single out the ones classifying the parabolic sheaves with Γ_k-equivariantly trivial determinant. This is the condition $d^r_0 = d^r_N$ for any $r \in \mathbb{Z}/k\mathbb{Z}$. Also, we single out the components classifying the Γ_k-equivariant parabolic sheaves with the *trivial* defect class $[\overline{\mathrm{def}}(\mathcal{F}_\bullet)]$ (see Section 2.11). This is the condition that for $r \neq 0$, $d^r_l = d^r_m$ $\forall l, m = 1,\ldots,N$. We will refer to such connected components (satisfying both of the above conditions) as the *admissible* ones.

Now let us consider the stacks $\widehat{\mathcal{S}}_1/\Gamma_k$, and $\widehat{\mathcal{S}}_1/|\Gamma_k$. The latter stands for the coarse (categorical) quotient in a neighbourhood of ℓ_0 (but the stacky quotient elsewhere). We have an evident projection $\phi\colon \widehat{\mathcal{S}}_1/\Gamma_k \to \widehat{\mathcal{S}}_1/|\Gamma_k$. Given a Γ_k-equivariant parabolic sheaf $\mathcal{F}$ on $\widehat{\mathcal{S}}_1$ lying in an admissible connected component $\widehat{\mathfrak{M}}^k_{\widehat{\underline{d}}}$, the parabolic sheaf $\phi_*\mathcal{F}$ is trivialized at infinity. Similarly to Section 2.10, ϕ_* induces an isomorphism of an admissible connected component $\widehat{\mathfrak{M}}^k_{\widehat{\underline{d}}}$ with a connected component of the moduli space of torsion free parabolic sheaves on $\widehat{\mathcal{S}}_1/|\Gamma_k$ (with the trivial action of Γ_k at the trivialization at infinity). However, the stacks $\widehat{\mathcal{S}}_1/|\Gamma_k$ and $\widehat{\mathcal{S}}_1^k$ are isomorphic off infinity, so the latter connected component is nothing else than a connected component of $\widehat{\mathcal{P}}^k$.

This completes the proof of Theorem 4.5.

4.7. The morphism Π^k. For an admissible dimension vector $\widehat{\underline{d}}$ we denote the corresponding connected component of $\widehat{\mathcal{P}}^k$ by $\widehat{\mathcal{P}}^k_{\widehat{\underline{d}}} \simeq \widehat{\mathfrak{M}}^k_{\widehat{\underline{d}}}$. We are going to describe in quiver terms the morphism $\widehat{\mathcal{P}}^k_{\widehat{\underline{d}}} \to \mathfrak{Z}$ of Section 4.2. Note that since $\mathfrak{Z}$ is affine, $\pi_{\zeta^\bullet,\zeta^-}\colon \widehat{\mathfrak{M}}^k_{\widehat{\underline{d}}} \to \widehat{\mathfrak{Z}}^k_{\widehat{\underline{d}}}$ is proper, and $\widehat{\mathfrak{Z}}^k_{\widehat{\underline{d}}}$ is normal, the morphism $\widehat{\mathcal{P}}^k_{\widehat{\underline{d}}} \to \mathfrak{Z}$ factors as the composition of $\pi_{\zeta^\bullet,\zeta^-}$ and a certain morphism $\widehat{\mathfrak{Z}}^k_{\widehat{\underline{d}}} \to \mathfrak{Z}$ to be denoted by Π^k.

The comparison of constructions of Section 4.6, Section 2.10, Section 3.12, and Conjecture 3.13 implies that $\Pi^k = \psi^k \circ \Psi^k$, where $\Psi^k\colon \widehat{\mathfrak{Z}}^k_{\widehat{\underline{d}}} \to \mathfrak{Z}_{\widetilde{\underline{d}}}$ is defined as follows. First, $\widetilde{\underline{d}} := (d_l^r)^{r\in\mathbb{Z}/k\mathbb{Z}}_{1\leqslant l\leqslant N}$ is obtained from the vector $\widehat{\underline{d}}$ just by erasing the coordinates d_0^r, $r \in \mathbb{Z}/k\mathbb{Z}$. Second, Ψ^k acts on the quiver data as follows: $'V_l^r := V_l^r$ for $1 \leqslant l \leqslant N$, $r \in \mathbb{Z}/k\mathbb{Z}$. Furthermore, $'B_N := B_0 e\colon V_N^r \to V_1^{r+1}$ and $'A_N := \delta e\colon V_N^r \to V_N^r$ for $r \in \mathbb{Z}/k\mathbb{Z}$. Furthermore, $'q_0 := q_0 e\colon V_N^{-1} \to W_0$, and all the other primed letters are equal to the corresponding letters without primes.

In particular, we see that $\Pi^k(\widehat{\mathfrak{Z}}^k_{\widehat{\underline{d}}})$ lands into the connected component $\mathfrak{Z}_{\underline{d}}$, where $\underline{d} = (d_1^0, \ldots, d_N^0)$.

4.8. An open piece. We consider the following (admissible) dimension vector: $d_l^r = v_r$ for any $l = 0, \ldots, N$. Let $\widehat{\mathfrak{M}}^k_{\widehat{\underline{d}},\mathrm{iso}} \subset \widehat{\mathfrak{M}}^k_{\widehat{\underline{d}}}$ be the open subset given by the condition that $B_{N-1}B_{N-2}\ldots B_1B_0\colon V_0^r \to V_N^r$ is an isomorphism for any $r \in \mathbb{Z}/k\mathbb{Z}$ (equivalently, all the B_l are isomorphisms). It follows from Corollary 3.11 that this open subset classifies the parabolic sheaves on $\widehat{\mathcal{S}}_1^k$ *trivial along* ℓ_0. Then the trivialization at infinity extends through ℓ_0 as well, and we are left with a torsion free sheaf on the open set U^2 (notations of Section 4.1) trivialized at infinity. According to [20, 4.2], the moduli space of torsion free sheaves on U^2 trivialized at infinity is the classical Nakajima quiver variety $\mathfrak{M}(v, w)$ of type $\widetilde{A}_{k-1}$, where $w = (N, 0, \ldots, 0)$, and $v = (v_0, \ldots, v_{k-1})$.

The isomorphism $\Phi\colon \widehat{\mathfrak{M}}^k_{\widehat{\underline{d}},\mathrm{iso}} \xrightarrow{\sim} \mathfrak{M}(v, w)$ in quiver terms is given by $W_0'' := W_0 \oplus \ldots \oplus W_{N-1}$, and $W_r'' := 0$ for $r \neq 0$. Furthermore, $V_r'' := V_0^r$. Furhermore,

$$B_r'' := eB_{N-1}B_{N-2}\ldots B_1B_0\colon V_r'' \to V_{r+1}'',$$

while

$$A''_r := (B_{N-1}B_{N-2}\dots B_1B_0)^{-1}\delta\colon V''_r \to V''_{r-1}.$$

Finally, $p''_0 := \bigoplus_{1\leqslant l\leqslant N} p^l\colon W''_0 \to V''_0$, and $q''_0 := \bigoplus_{0\leqslant l\leqslant N-1} q^l\colon V''_0 \to W''_0$, where

$$p^l := B_0^{-1}B_1^{-1}\dots B_{l-2}^{-1}B_{l-1}^{-1}p_l\colon W_l \to V_0^0,$$

and

$$q^l := q_lB_{l-1}B_{l-2}\dots B_1B_0\colon V_0^0 \to W_l.$$

4.9. The image of Π^k. Note that while Theorem 4.5 provides the necessary (admissibility) conditions for a component $\widehat{\mathcal{P}}^k_{\underline{\hat{d}}}$ to be good, it does not give the sufficient conditions. Let us give such sufficient conditions in the setup of Section 4.8. Thus we restrict ourselves to the components $\widehat{\mathfrak{M}}^k_{\underline{\hat{d}}}$ which contain the *nonempty* open subset formed by the *locally free* parabolic sheaves. Equivalently, we are interested in the components $\widehat{\mathfrak{M}}^k_{\underline{\hat{d}},\mathrm{iso}} \simeq \mathfrak{M}(v,\, w)$ which contain the *nonempty* open subset formed by the vector bundles on U^2. The corresponding dimension vectors (d_l^r) (equivalently, vectors v), will be called *good*. The well-known Nakajima criterion states that v is good if and only if the $\widehat{\mathfrak{sl}(k)}$-weight $w - Cv$ is dominant and has nonzero multiplicity in the level N vacuum integrable module $L(w)$. Here C is the affine Cartan matrix of $\tilde{A}_{k-1}$. More explicitly, the dominance condition reads as follows: $v_0 + v_2 \geqslant 2v_1,\, \dots,\, v_{k-2} + v_0 \geqslant 2v_{k-1},\, v_{k-1} + v_1 + N \geqslant 2v_0$. In particular, $v_0 \geqslant v_i$ $\forall i \in \mathbb{Z}/k\mathbb{Z}$.

Let us note that the type of our bundles on U^2 at the hyperbolic point is $\overline{\lambda} = {}^t(w - Cv)$, where the transposition is extensively discussed in [4], [6]. In notations of *loc. cit.* (especially Section 7 of [4]), we have $\mathfrak{M}^{\mathrm{reg}}(v,\, w) \simeq \mathrm{Bun}^{\lambda}_{\mathrm{SL}(N),\mu}(\mathbb{A}^2/\Gamma_k)$, where λ, μ are the following integrable $\mathfrak{sl}(N)_{\mathrm{aff}}$-weights:

$$\lambda = \left(k,\, \overline{\lambda},\, \frac{a + \frac{(\overline{\mu},\overline{\mu})}{2} - \frac{(\overline{\lambda},\overline{\lambda})}{2}}{k}\right),$$

$\mu = (k,\, 0,\, 0)$. Here a stands for the second Chern class of the Γ_k-equivariant vector bundles on a compactification of $\mathbb{A}^2$. According to [4], the Nakajima criterion can be equivalently reformulated as follows: v is good if and only if μ has nonzero multiplicity in the level k integrable $\mathfrak{sl}(N)_{\mathrm{aff}}$-module $L(\lambda)$. In particular, $\lambda \geqslant \mu$, i.e., the difference $\alpha := \lambda - \mu$ is a linear combination of simple roots of $\mathfrak{sl}(N)_{\mathrm{aff}}$ with coefficients in $\mathbb{N}$.

We view α as a vector with coordinates $(\alpha_1,\, \dots,\, \alpha_N)$. It is easy to see that $(\alpha_1,\, \dots,\, \alpha_N) \leqslant (v_0,\, \dots,\, v_0)$ componentwise, and hence we have an embedding $\mathfrak{Z}_\alpha \hookrightarrow \mathfrak{Z}_{(v_0,\dots,v_0)}$ adding the defect of the complementary degree at the origin.

From now on let us write $\widehat{\mathfrak{Z}}^\lambda_\mu$ for $\widehat{\mathfrak{Z}}^k_{\underline{\hat{d}}}$.

Conjecture 4.10. *Consider a good component* $\widehat{\mathfrak{Z}}^\lambda_\mu = \widehat{\mathfrak{Z}}^k_{\underline{\hat{d}}} \xrightarrow{\Pi^k} \mathfrak{Z}_{v_0,\dots,v_0}$.

(a) *The image of* Π^k *is contained in* $\mathfrak{Z}_\alpha \subset \mathfrak{Z}_{v_0,\dots,v_0}$, *so we may and will view* Π^k *as a morphism* $\widehat{\mathfrak{Z}}^\lambda_\mu \to \mathfrak{Z}_\alpha$.

(b) *The morphism* $\Pi^k\colon \widehat{\mathfrak{Z}}^\lambda_\mu \to \mathfrak{Z}_\alpha$ *is birational and stratified semismall, so that the direct image* $\Pi^k_* \operatorname{IC}(\widehat{\mathfrak{Z}}^\lambda_\mu)$ *is a direct sum of IC-sheaves of certain strata of* $\mathfrak{Z}_\alpha$ *with certain multiplicities.*

(c) *For* $\beta \leqslant \alpha$*, and the corresponding stratum* $\mathfrak{Z}_\beta \subset \mathfrak{Z}_\alpha$*, the multiplicity* m_β *of* $\operatorname{IC}(\mathfrak{Z}_\beta)$ *in* $\Pi^k_* \operatorname{IC}(\widehat{\mathfrak{Z}}^\lambda_\mu)$ *equals the weight multiplicity* $L^\lambda(\lambda-\alpha+\beta)$ *of the integrable* $\mathfrak{sl}(N)_{\mathrm{aff}}$*-module* L^λ*.*

5. General G

5.1. Arbitrary groups. Let G be an almost simple simply connected group with the Lie algebra $\mathfrak{g}$. We have the adjoint representation $G \to \mathrm{SL}(\mathfrak{g}) = \mathrm{SL}(N)$, where $N = \dim \mathfrak{g}$. We choose a Borel subgoup B_N of $\mathrm{SL}(N)$ containing the image of the positive Borel subgoup $B \subset G$.

We consider the moduli space of G-bundles on $\widehat{\mathcal{S}}_1/|\Gamma_k$ (see Section 4.6) equipped with a reduction to B along ℓ_0 and framing at ℓ_∞. The component of this moduli space having an open piece $\mathrm{Bun}^\lambda_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ (cf. Section 4.9) will be denoted by $\widetilde{Z}^\lambda_{G,\mu}$. Here $\lambda,\, \mu \in \Lambda^+_{\mathrm{aff},k}$, $\lambda \geqslant \mu = (k,\,0,\,0)$. The adjoint homomorphism $\mathrm{ad}\colon (G,\,B) \to (\mathrm{SL}(N),\,B_N)$ induces a closed embedding $\mathrm{ad}\colon \widetilde{Z}^\lambda_{G,\mu} \hookrightarrow \widetilde{Z}^{\mathrm{ad}_*\lambda}_{\mathrm{SL}(N),\mathrm{ad}_*\mu}$. We define $\overline{\mathcal{G}Z}^{\lambda,\alpha}_{G_{\mathrm{aff}}}$ as the closure of $\mathrm{ad}(\widetilde{Z}^\lambda_{G,\mu})$ in $\widehat{\mathfrak{Z}}^{\mathrm{ad}_*\lambda}_{\mathrm{ad}_*\mu} \supset \widetilde{Z}^{\mathrm{ad}_*\lambda}_{\mathrm{SL}(N),\mathrm{ad}_*\mu}$; here $\alpha := \lambda - \mu$. Then Π^k restricts to the proper morphism $\phi\colon \overline{\mathcal{G}Z}^{\lambda,\alpha}_{G_{\mathrm{aff}}} \to Z^\alpha_{G_{\mathrm{aff}}}$.

Conjecture 5.2. (a) *The morphism* $\phi\colon \overline{\mathcal{G}Z}^{\lambda,\alpha}_{G_{\mathrm{aff}}} \to Z^\alpha_{G_{\mathrm{aff}}}$ *is birational and stratified semismall, so that the direct image* $\phi_* \operatorname{IC}(\overline{\mathcal{G}Z}^{\lambda,\alpha}_{G_{\mathrm{aff}}})$ *is a direct sum of IC-sheaves of certain strata of* $Z^\alpha_{G_{\mathrm{aff}}}$ *with certain multiplicities.*

(b) *For* $\beta \leqslant \alpha$*, and the corresponding stratum* $Z^\beta_{G_{\mathrm{aff}}} \subset Z^\alpha_{G_{\mathrm{aff}}}$*, the multiplicity* m_β *of* $\operatorname{IC}(Z^\beta_{G_{\mathrm{aff}}})$ *in* $\phi_* \operatorname{IC}(\overline{\mathcal{G}Z}^{\lambda,\alpha}_{G_{\mathrm{aff}}})$ *equals the weight multiplicity* $L^\lambda(\lambda-\alpha+\beta)$ *of the integrable* $G^\vee_{\mathrm{aff}}$*-module* L^λ*.*

5.3. Repellents. As we have mentioned in the Introduction, Conjecture 5.2 is an affine analogue of the (known) statement about the convolution between the (classical) affine Grassmannian and zastava of G. This classical statement is deduced from the description of fibers of the convolution morphism as the intersections of Schubert varieties in the affine Grassmannian with *semiinfinite orbits*. We expect a similar description applies in the affine situation. We will define an open subset of $\overline{\mathcal{G}Z}^{\lambda,\alpha}_{G_{\mathrm{aff}}}$, which identifies with a transversal slice in the double affine Grassmannian of G, and prove that its intersection with an affine analogue of a semiinfinite orbit lies in the central fiber of ϕ.

We have an intermediate open subset

$$\widetilde{Z}^{\mathrm{ad}_*\lambda}_{\mathrm{SL}(N),\mathrm{ad}_*\mu} \subset \check{Z}^{\mathrm{ad}_*\lambda}_{\mathrm{SL}(N),\mathrm{ad}_*\mu} \subset \widehat{\mathfrak{Z}}^{\mathrm{ad}_*\lambda}_{\mathrm{ad}_*\mu}$$

specified in quiver terms by the condition that the composition

$$B_{N-1}B_{N-2}\ldots B_1 B_0\colon V^r_0 \to V^r_N$$

is an isomorphism for any $r \in \mathbb{Z}/k\mathbb{Z}$, cf. Section 4.8. It is nothing else than the Uhlenbeck space $\mathfrak{U}^{\mathrm{ad}_*\lambda}_{\mathrm{SL}(N),\mathrm{ad}_*\mu}(\mathbb{A}^2/\Gamma_k)$. The closure of $\mathrm{ad}(\widetilde{Z}^\lambda_{G,\mu})$ in

$$\mathfrak{U}^{\mathrm{ad}_*\lambda}_{\mathrm{SL}(N),\mathrm{ad}_*\mu}(\mathbb{A}^2/\Gamma_k) = \check{Z}^{\mathrm{ad}_*\lambda}_{\mathrm{SL}(N),\mathrm{ad}_*\mu} \supset \widetilde{Z}^{\mathrm{ad}_*\lambda}_{\mathrm{SL}(N),\mathrm{ad}_*\mu}$$

is nothing else than the Uhlenbeck space $\mathfrak{U}^\lambda_{G,\mu}(\mathbb{A}^2/\Gamma_k) = \overline{\mathcal{W}}^\lambda_{G_{\mathrm{aff}},\mu}$. In Section 3.2 of [5] we have introduced the locally closed subvariety $\mathfrak{T}^e_\mu \subset \mathfrak{U}^\lambda_{G,\mu}(\mathbb{A}^2/\Gamma_k) = \overline{\mathcal{W}}^\lambda_{G_{\mathrm{aff}},\mu}$ as the repellent of a certain $\mathbb{C}^*$-action (e stands for the neutral element of the affine Weyl group). (In type A these repellents were introduced in [22] under the name of *MV cycles*.) We conjecture that the central fiber $\phi^{-1}(i^\alpha_\alpha(0)) \cap \mathfrak{U}^\lambda_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ coincides with the repellent $\mathfrak{T}^e_\mu \subset \mathfrak{U}^\lambda_{G,\mu}(\mathbb{A}^2/\Gamma_k)$. We can prove only one inclusion.

First, we recall the definition of the $\mathbb{C}^*$-action for the reader's convenience. We choose a Cartan torus $T \subset B \subset G$ with the coweight lattice Λ. The 2-dimensional torus $\mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{A}^2$ naturally: $(a, b)\cdot(z, t) = (az, bt)$, and on $\mathfrak{U}^\lambda_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ by the transport of structure. Let $\mathbb{C}^*_{\mathrm{hyp}} := \{(c, c^{-1})\} \subset \mathbb{C}^* \times \mathbb{C}^*$ stand for the antidiagonal, alias hyperbolic, subgroup. Let us denote the torus $\mathbb{C}^*_{\mathrm{hyp}} \times T$ by $\widehat{T}$. This is a Cartan torus of $\widetilde{G}$ with the coweight lattice $\widehat{\Lambda}$. Thus, the slices $\overline{\mathcal{W}}^\lambda_\mu = \mathfrak{U}^\lambda_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ are equipped with the action of the torus $\widehat{T}$. Let I (resp. $I_{\mathrm{aff}} = I \sqcup i_0$) stand for the set of vertices of the Dynkin diagram of G (resp. of G_{aff}). For $i \in I$ we denote by $\overline{\omega}_i \in \Lambda$ the corresponding fundamental coweight of G, and we denote by $a_i \in \mathbb{N}$ the corresponding label of the Dynkin diagram of G_{aff}. Then $\omega_{i_0} := (1, 0) \in \mathbb{Z} \times \Lambda = \widehat{\Lambda}$, $\omega_i := (a_i, \overline{\omega}_i) \in \widehat{\Lambda}$, $i \in I$, are the fundamental coweights of G_{aff}. We set $\rho := \sum_{i \in I_{\mathrm{aff}}} \omega_i$ (not to be confused with the halfsum $\bar{\rho}$ of positive coroots of G). The torus $\widehat{T}$ acts on $\overline{\mathcal{W}}^\lambda_\mu$ with the only fixed point, to be denoted abusively by μ, and we define $\mathfrak{T}^e_\mu \subset \overline{\mathcal{W}}^\lambda_\mu$ as the repellent $\mathfrak{T}^e_\mu := \{g \in \overline{\mathcal{W}}^\lambda_\mu \colon \lim_{c\to\infty} \rho(c) g = \mu\}$.

Proposition 5.4. $\mathfrak{T}^e_\mu \subset \phi^{-1}(i^\alpha_\alpha(0)) \cap \mathfrak{U}^\lambda_{G,\mu}(\mathbb{A}^2/\Gamma_k)$.

Proof. If a point a lies in $\mathfrak{T}^e_\mu$, then $\phi(a)$ is repelled from $i^\alpha_\alpha(0) \in Z^\alpha_{G_{\mathrm{aff}}}$ under the following action of $\mathbb{C}^*$ on $Z^\alpha_{G_{\mathrm{aff}}}$. Recall that $Z^\alpha_{G_{\mathrm{aff}}}$ is a certain closure of the moduli space of G-bundles on $\mathbb{P}^2$ (with homogeneous coordinates $[z_0 : z_1 : z_2]$) trivialized at ℓ_∞ (given by $z_0 = 0$) and equipped with a reduction to B along ℓ_0 (given by $z_2 = 0$). The Cartan torus $T \subset B \subset G$ acts on $Z^\alpha_{G_{\mathrm{aff}}}$ via trivialization at ℓ_∞, while $\mathbb{C}^*_{\mathrm{vert}}$ acts on $\mathbb{P}^2$ by $c[z_0 : z_1 : z_2] = [z_0 : z_1 : cz_2]$, and hence on $Z^\alpha_{G_{\mathrm{aff}}}$ by transport of structure. Note that the action of $\mathbb{C}^*_{\mathrm{vert}}$ lifts to the action on $\widehat{\mathbb{P}}^2_k$ with the following property: if f is a non $\mathbb{C}^*_{\mathrm{vert}}$-fixed point on the exceptional divisor, then as $c \in \mathbb{C}^*_{\mathrm{vert}}$ tends to infinity, $c \cdot f$ tends to the singular point of the exceptional divisor. Moreover, this action of $\mathbb{C}^*_{\mathrm{vert}}$ on $\widehat{\mathbb{P}}^2_k$ in the chart $\overline{U}^2$ (notations of Section 4.1) coincides with the action of $\mathbb{C}^*_{\mathrm{hyp}}$ on $\mathbb{A}^2/\!/\Gamma_k$. We consider the one-parametric subgroup $\mathbb{C}^* \to \mathbb{C}^*_{\mathrm{vert}} \times T \colon c \mapsto (c^h, \bar{\rho}(c))$, where $\bar{\rho}$ is the halfsum of positive coroots of G viewed as a cocharacter of T, while h is the Coxeter number of G. The desired action of $\mathbb{C}^*$ on $Z^\alpha_{G_{\mathrm{aff}}}$ is the action of this one-parametric subgroup.

Finally, the only points of $Z^\alpha_{G_{\mathrm{aff}}}$ repelled from anything at all are the T-fixed points $\mathbb{A}^\alpha \subset Z^\alpha_{G_{\mathrm{aff}}}$. In effect, we have the projection $\varrho : Z^\alpha_{G_{\mathrm{aff}}} \to \mathbb{A}^\alpha$ (see [8, Section 9]) equivariant with respect to the $\mathbb{C}^*$-action. According to the factorization principle [8, Corollary 9.4], it suffices to prove the desired statement for the points of $Z^\alpha_{G_{\mathrm{aff}}}$ lying in the central fiber $\overline{\mathfrak{F}}^\alpha := \varrho^{-1}(\alpha \cdot 0)$. This is clear from the stratification [10, (7) in Section 4.1 (or (4.2) in Section 4.1 of arXiv:0912.5132)] of $\overline{\mathfrak{F}}^\alpha$.

It follows $\phi(a) = i^\alpha_\alpha(0)$. □

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Department of Mathematics, Brown University, 151 Thayer St., Providence RI 02912, USA

E-mail address: `braval@math.brown.edu`

IMU, IITP and National Research University Higher School of Economics, Department of Mathematics, 20 Myasnitskaya st, Moscow 101000 Russia

E-mail address: `fnklberg@gmail.com`