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Selections of bounded variation under the excess restrictions ☆

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Abstract

Let X be a metric space with metric d, $\mathrm{c}(X)$ denote the family of all nonempty compact subsets of X and, given $F, G \in \mathrm{c}(X)$, let $\mathrm{e}(F, G) = \sup_{x\in F} \inf_{y\in G} d(x, y)$ be the Hausdorff excess of F over G. The excess variation of a multifunction $F : [a, b] \to \mathrm{c}(X)$, which generalizes the ordinary variation V of single-valued functions, is defined by $V_+(F, [a, b]) = \sup_\pi \sum_{i=1}^{m} \mathrm{e}(F(t_{i-1}), F(t_i))$ where the supremum is taken over all partitions $\pi = \{t_i\}_{i=0}^{m}$ of the interval $[a, b]$. The main result of the paper is the following selection theorem: *If $F : [a, b] \to \mathrm{c}(X)$, $V_+(F, [a, b]) < \infty$, $t_0 \in [a, b]$ and $x_0 \in F(t_0)$, then there exists a single-valued function $f : [a, b] \to X$ of bounded variation such that $f(t) \in F(t)$ for all $t \in [a, b]$, $f(t_0) = x_0$, $V(f, [a, t_0]) \leqslant V_+(F, [a, t_0])$ and $V(f, [t_0, b]) \leqslant V_+(F, [t_0, b])$.* We exhibit examples showing that the conclusions in this theorem are sharp, and that it produces new selections of bounded variation as compared with [V.V. Chistyakov, Selections of bounded variation, J. Appl. Anal. 10 (1) (2004) 1–82]. In contrast to this, a multifunction F satisfying $\mathrm{e}(F(s), F(t)) \leqslant C(t - s)$ for some constant $C \geqslant 0$ and all $s, t \in [a, b]$ with $s \leqslant t$ (Lipschitz continuity with respect to $\mathrm{e}(\cdot,\cdot)$) admits a Lipschitz selection with a Lipschitz constant not exceeding C if $t_0 = a$ and may have only discontinuous selections of bounded variation if $a < t_0 \leqslant b$. The same situation holds for continuous selections of $F : [a, b] \to \mathrm{c}(X)$ when it is excess continuous in the sense that $\mathrm{e}(F(s), F(t)) \to 0$ as $s \to t - 0$ for all $t \in (a, b]$ and $\mathrm{e}(F(t), F(s)) \to 0$ as $s \to t + 0$ for all $t \in [a, b)$ simultaneously.

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1. The main result

We begin by reviewing certain preliminary definitions and facts needed for our results. Throughout the paper X will denote a metric space with metric d.

A function $f : T \to X$ on a nonempty set $T \subset \mathbb{R}$ is said to be *of bounded variation* if its total Jordan variation $V(f,T)$ given by

$$V(f,T) \equiv V_d(f,T) = \sup_{\pi} \sum_{i=1}^{m} d\big(f(t_i), f(t_{i-1})\big) \quad \big(V(f,\emptyset)=0\big)$$

is finite, the supremum being taken over all partitions $\pi = \{t_i\}_{i=0}^{m}$ of the set T, i.e., $m \in \mathbb{N}$ and $\{t_i\}_{i=0}^{m} \subset T$ such that $t_{i-1} \leqslant t_i$ for all $i \in \{1,\dots,m\}$. The two well-known properties of the variation V (e.g., [5]) are the *additivity* in the second argument: $V(f,T) = V(f,(-\infty,t]\cap T) + V(f,[t,\infty)\cap T)$ for all $t \in T$, and the *sequential lower semicontinuity* in the first argument: if a sequence of functions $\{f_n\}_{n=1}^{\infty}$ mapping T into X converges pointwise on T to a function $f : T \to X$ (i.e., $\lim_{n\to\infty} d(f_n(t), f(t)) = 0$ for all $t \in T$), then $V(f,T) \leqslant \liminf_{n\to\infty} V(f_n,T)$.

Given two nonempty sets $F, G \subset X$, the *Hausdorff excess of F over G* is defined by (see, e.g., [2, Chapter II]):

$$\mathrm{e}(F,G) \equiv \mathrm{e}_d(F,G) = \sup_{x\in F} \mathrm{dist}(x,G), \quad \text{where } \mathrm{dist}(x,G) = \inf_{y\in G} d(x,y).$$

The following properties of the excess function $\mathrm{e}(\cdot,\cdot)$ are well known: if F, G and H are nonempty subsets of X, then (i) $\mathrm{e}(F,G)=0$ if and only if $F \subset \overline{G}$ where $\overline{G}$ is the closure of G in X; (ii) $\mathrm{e}(F,G) \leqslant \mathrm{e}(F,H) + \mathrm{e}(H,G)$; (iii) the value $\mathrm{e}(F,G)$ is finite if F and G are bounded and, in particular, closed and bounded, or compact.

Another, more intuitive, definition of $\mathrm{e}(F,G)$ can be given as follows. If $B_\varepsilon(x) = \{y \in X\colon d(y,x) < \varepsilon\}$ is the open ball of radius $\varepsilon > 0$ centered at $x \in X$ and $\mathcal{O}_\varepsilon(G) = \{x \in X\colon \mathrm{dist}(x,G) < \varepsilon\} = \bigcup_{x\in G} B_\varepsilon(x)$ is the open ε-neighbourhood of G, then $\mathrm{e}(F,G) = \inf\{\varepsilon > 0\colon F \subset \mathcal{O}_\varepsilon(G)\}$.

The *Hausdorff distance* between nonempty sets F and G from X is defined as follows (e.g., [2, Chapter II]):

$$D(F,G) = \max\big\{\mathrm{e}(F,G), \mathrm{e}(G,F)\big\} = \inf\big\{\varepsilon > 0\colon F \subset \mathcal{O}_\varepsilon(G) \text{ and } G \subset \mathcal{O}_\varepsilon(F)\big\}.$$

The function $D(\cdot,\cdot)$ is a metric, called the *Hausdorff metric*, on the family of all nonempty closed bounded subsets of X and, in particular, on the family $\mathrm{c}(X)$ of all nonempty compact subsets of X.

By a *multifunction* from T into X we mean a rule F assigning to each point t from T a nonempty subset $F(t) \subset X$. We will mostly be interested in multifunctions of the form $F : T \to \mathrm{c}(X)$. Such a multifunction is said to be *of bounded variation* (with respect to D) if its total Jordan variation is finite:

$$V_D(F,T) = \sup_{\pi} \sum_{i=1}^{m} D\big(F(t_i), F(t_{i-1})\big) < \infty.$$

A (single-valued) function $f : T \to X$ is said to be a *selection of F on T* provided $f(t) \in F(t)$ for all $t \in T$.

The following theorem on the existence of selections of bounded variation is given in [6, Theorem 5.1] (the previous special cases of this theorem are contained in [1,4,5,10,11]):

Theorem A. *If $F:T\to \mathrm{c}(X)$, $V_D(F,T)<\infty$, $t_0\in T$ and $x_0\in F(t_0)$, then there exists a selection f of F of bounded variation on T such that $f(t_0)=x_0$ and $V(f,T)\leqslant V_D(F,T)$. Moreover, if F is continuous with respect to D, then in addition a selection f of F may be chosen to be continuous on T.*

The aim of this paper is to remove the assumption $V_D(F,T)<\infty$ from Theorem A and replace it by a weaker one, $V_{\mathrm{e}}(F,T)<\infty$ (for more precise condition see below), which, as we will show, still preserves the existence of selections of F of bounded variation. In order to achieve this, we introduce the following definition.

The *excess variation to the right* $V_+(F,T)$ of a multifunction $F:T\to \mathrm{c}(X)$ is

$$V_+(F,T)=\sup_{\pi}\sum_{i=1}^{m}\mathrm{e}\big(F(t_{i-1}),F(t_i)\big)\quad \big(V_+(F,\varnothing)=0\big), \tag{1}$$

where the supremum is taken over all partitions $\pi=\{t_i\}_{i=0}^m$ of T. Analogously, the *excess variation to the left* of F is given by

$$V_-(F,T)=\sup_{\pi}\sum_{i=1}^{m}\mathrm{e}\big(F(t_i),F(t_{i-1})\big)\quad \big(V_-(F,\varnothing)=0\big).$$

Note that both V_+ and V_- are generalizations of the ordinary variation $V=V_d$ for single-valued functions f. Also, the value $V_D(F,T)$ is finite if and only if both values $V_+(F,T)$ and $V_-(F,T)$ are finite.

To simplify the matters and make the ideas involved more clear in the rest of the paper (except Theorem B on p. 878 and Theorem C on p. 883) we assume that $T=[a,b\rangle$, with $a\in\mathbb{R}$ and $a<b$, is either the closed interval $[a,b]$ with $b\in\mathbb{R}$ or the half-closed interval $[a,b)$ with $b\in\mathbb{R}\cup\{\infty\}$. A similar convention applies to the interval $T=\langle a,b]$. In their full generality our results are valid for any nonempty set $T\subset\mathbb{R}$ with $\inf T\in T$ or $\sup T\in T$ corresponding to $[a,b\rangle$ or $\langle a,b]$ under consideration, respectively (cf. [6, Section 5]).

Our main result, an extension of Theorem A to be proved in Section 2, is as follows.

Theorem 1. *Suppose that $F:T\to \mathrm{c}(X)$, $t_0\in T$ and $x_0\in F(t_0)$. We have:*

(a) *if $T=[a,b\rangle$ and $V_+(F,T)<\infty$, then there exists a selection of bounded variation f of F on T such that $f(t_0)=x_0$,*

$$V\big(f,[a,t_0)\big)\leqslant V_+\big(F,[a,t_0)\big),\qquad V\big(f,[t_0,b\rangle\big)\leqslant V_+\big(F,[t_0,b\rangle\big),\quad \textit{and}$$
$$V\big(f,[a,b\rangle\big)-\lim_{s\to t_0-0}d\big(f(s),x_0\big)\leqslant V_+\big(F,[a,t_0)\big)+V_+\big(F,[t_0,b\rangle\big)\leqslant V_+\big(F,[a,b\rangle\big);$$

(b) *if $T=\langle a,b]$ and $V_-(F,T)<\infty$, then there exists a selection of bounded variation f of F on T such that $f(t_0)=x_0$,*

$$V\big(f,\langle a,t_0]\big)\leqslant V_-\big(F,\langle a,t_0]\big),\qquad V\big(f,(t_0,b]\big)\leqslant V_-\big(F,(t_0,b]\big),\quad \textit{and}$$
$$V\big(f,\langle a,b]\big)-\lim_{s\to t_0+0}d\big(f(s),x_0\big)\leqslant V_-\big(F,\langle a,t_0]\big)+V_-\big(F,(t_0,b]\big)\leqslant V_-\big(F,\langle a,b]\big).$$

The case when the multifunction F additionally admits continuous selections of bounded variation is treated in Section 4 (Theorem 3).

In order to see how Theorem 1 implies Theorem A, assume that $T = \langle a, b\rangle$ is an interval, which is either open, closed, half-closed, bounded or not, $t_0 \in T$, $V_-(F, \langle a, t_0])$ and $V_+(F, [t_0, b\rangle)$ are finite (this is the case when $V_D(F, T) < \infty$) and $x_0 \in F(t_0)$. Applying Theorem 1 we find a selection f_- of F on $\langle a, t_0]$ such that $f_-(t_0) = x_0$ and $V(f_-, \langle a, t_0]) \leqslant V_-(F, \langle a, t_0])$ and a selection f_+ of F on $[t_0, b\rangle$ such that $f_+(t_0) = x_0$ and $V(f_+, [t_0, b\rangle) \leqslant V_+(F, [t_0, b\rangle)$. Defining $f : \langle a, b\rangle \to X$ by $f(t) = f_-(t)$ if $t \in \langle a, t_0]$ and $f(t) = f_+(t)$ if $t \in [t_0, b\rangle$ we obtain a desired selection of F satisfying $f(t_0) = x_0$ and, by virtue of the additivity property of V in the second variable,

$$V\big(f, \langle a, b\rangle\big) = V\big(f_-, \langle a, t_0]\big) + V\big(f_+, [t_0, b\rangle\big) \leqslant V_-\big(F, \langle a, t_0]\big) + V_+\big(F, [t_0, b\rangle\big),$$

which is estimated by $V_D(F, \langle a, t_0]) + V_D(F, [t_0, b\rangle) = V_D(F, \langle a, b\rangle)$ if the last quantity is finite. These arguments also apply to obtain Lipschitz and continuous selections of bounded variation of F on $\langle a, b\rangle$ (see Section 4).

For more motivation, historical comments and possible applications of the results of this paper we refer to [1,4–6,10].

The paper is organized as follows. In Section 2 we study properties of the excess variation V_+ and prove Theorem 1. In Section 3 we present an example of a multifunction, for which Theorem 1 is applicable while Theorem A is not, and show that the conclusions of Theorem 1 are sharp. Section 4 is devoted to the existence and non-existence of Lipschitz and continuous selections of bounded variation.

2. Proof of the main result

Since assertions (a) and (b) in Theorem 1 are completely similar, we concentrate on (a). In the proof of this theorem we will need Lemmas 1 and 2 and Theorem B presented below in this section.

In the next two lemmas we gather several properties of the excess variation V_+ (the properties of the excess variation V_- are similar).

Lemma 1. *Let $F : [a, b\rangle \to \mathrm{c}(X)$ and $V_+(F, [a, b\rangle) < \infty$. We have*:

(a) $V_+(F, [a, b\rangle) = 0$ *if and only if* $F(s) \subset F(t)$ *for all* $s, t \in [a, b\rangle$, $s \leqslant t$.
(b) *If* $s, t \in [a, b\rangle$, $s \leqslant t$, *then* $V_+(F, [a, s]) + V_+(F, [s, t]) = V_+(F, [a, t])$.
(c) $\lim_{s\to t-0} V_+(F, [a, s]) = V_+(F, [a, t))$ *for each* $t \in (a, b\rangle$.

Proof. (a) This is a consequence of the definition of V_+ and property (i) of the excess function $\mathrm{e}(\cdot,\cdot)$ from Section 1 on closed or compact subsets of X.

(b) First, note that if a new point is inserted into a given partition $\pi = \{t_i\}_{i=0}^m$ of T, the sum under the supremum sign in (1) will not decrease: in fact, suppose $s \in T$ and $t_{k-1} < s < t_k$ for some $k \in \{1, \dots, m\}$, then applying property (ii) of $\mathrm{e}(\cdot,\cdot)$ from Section 1, we get

$$\mathrm{e}\big(F(t_{k-1}), F(t_k)\big) \leqslant \mathrm{e}\big(F(t_{k-1}), F(s)\big) + \mathrm{e}\big(F(s), F(t_k)\big), \tag{2}$$

and the assertion for the sums follows. This observation implies that in order to calculate the value $V_+(F, T)$ from (1), instead of all partitions of T we may consider only those that contain an a priori fixed finite number of points from T.

So, let $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = s$ be a partition of $[a, s]$ and $s = t_m < t_{m+1} < \cdots < t_{n-1} < t_n = t$ be a partition of $[s, t]$. We have:

$$\sum_{i=1}^{m} \mathrm{e}\big(F(t_{i-1}), F(t_i)\big) + \sum_{j=m+1}^{n} \mathrm{e}\big(F(t_{j-1}), F(t_j)\big) \leqslant V_+\big(F, [a, t]\big).$$

Taking the supremum over all partitions of $[a, s]$ and $[s, t]$, we arrive at the inequality $V_+(F, [a, s]) + V_+(F, [s, t]) \leqslant V_+(F, [a, t])$.

Now, let $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ be a partition of $[a, t]$ and assume that $t_{k-1} \leqslant s \leqslant t_k$ for some $k \in \{1, \dots, m\}$. By virtue of (2), we find

$$\sum_{i=1}^{m} \mathrm{e}\big(F(t_{i-1}), F(t_i)\big) \leqslant V_+\big(F, [a, s]\big) + V_+\big(F, [s, t]\big),$$

and it remains to take the supremum over all partitions of $[a, t]$.

(c) The definition of V_+ implies that, given $\varepsilon > 0$, there exists a partition $a = \tau_0 < \tau_1 < \cdots < \tau_m < t$ of $[a, t)$ (depending on ε) such that

$$V_+\big(F, [a, t)\big) - \varepsilon \leqslant \sum_{i=1}^{m} \mathrm{e}\big(F(\tau_{i-1}), F(\tau_i)\big) \leqslant V_+\big(F, [a, \tau_m]\big).$$

It follows that for any $\tau_m \leqslant s < t$ we get:

$$V_+\big(F, [a, t)\big) - \varepsilon \leqslant V_+\big(F, [a, \tau_m]\big) \leqslant V_+\big(F, [a, s]\big) \leqslant V_+\big(F, [a, t)\big),$$

which proves (c) and completes the proof of our lemma. □

Lemma 2. *Let $F : [a, b\rangle \to \mathrm{c}(X)$ and $V_+(F, [a, b\rangle) < \infty$. Define the V_+-variation function $v : [a, b\rangle \to [0, \infty)$ by $v(t) = V_+(F, [a, t])$ for $t \in [a, b\rangle$. Then*

$$\lim_{s \to t-0} \mathrm{e}\big(F(s), F(t)\big) = v(t) - v(t-0) \quad \text{for all } t \in (a, b\rangle \tag{3}$$

and

$$\lim_{s \to t+0} \mathrm{e}\big(F(t), F(s)\big) = v(t+0) - v(t) \quad \text{for all } t \in [a, b), \tag{4}$$

where $v(t-0)$ and $v(t+0)$ are the left and right limits of v at t, respectively.

Proof. After the property of Lemma 1(b) has been proved, this lemma might be considered as a consequence of [5, Lemma 4.2]. However, in that reference functions under consideration were assumed to take their values in a metric space where the distance function is symmetric. In our case the excess function $\mathrm{e}(\cdot, \cdot)$ is not symmetric (for $\mathrm{e}(F, G) \neq \mathrm{e}(G, F)$ in general), and so, we have to take care of that. For the reader's convenience we reproduce the proof from the above reference in a somewhat shortened form.

By virtue of Lemma 1(b), the function v is nondecreasing and, hence, regulated, i.e., it has the left limit $v(t-0)$ at all points $t \in (a, b\rangle$ and the right limit $v(t+0)$ at all points $t \in [a, b)$. The existence of the limits at the left-hand sides of (3) and (4) can be proved in exactly the same way as in [5, Lemma 4.1] by using the Cauchy criterion if we take into account property (ii) of the excess function from Section 1.

Proof of (3). By Lemma 1(b), for $t \in (a, b\rangle$ and $s \in [a, t)$ we have:

$$e\big(F(s), F(t)\big) \leqslant V_+\big(F, [s, t]\big) = v(t) - v(s),$$

and so, as $s \to t-0$, $\lim_{s\to t-0} e(F(s), F(t)) \leqslant v(t) - v(t-0)$. To prove the reverse inequality, by the definition of $V_+(F, [a, t])$ for any $\varepsilon > 0$ we choose a partition $\{t_i\}_{i=0}^m \cup \{t\}$ of $[a, t]$ with $t_m < t$ such that

$$V_+\big(F, [a, t]\big) \leqslant \sum_{i=1}^m e\big(F(t_{i-1}), F(t_i)\big) + e\big(F(t_m), F(t)\big) + \varepsilon.$$

If $s \in [t_m, t)$, noting that $e(F(t_m), F(t)) \leqslant e(F(t_m), F(s)) + e(F(s), F(t))$, we get:

$$V_+\big(F, [a, t]\big) \leqslant V_+\big(F, [a, s]\big) + e\big(F(s), F(t)\big) + \varepsilon,$$

which implies $v(t) - v(s) \leqslant e(F(s), F(t)) + \varepsilon$, and it remains to pass to the limit as $s \to t-0$ and take into account the arbitrariness of $\varepsilon > 0$. □

Proof of (4). Given $t \in [a, b)$ and $s \in (t, b)$, we have:

$$e\big(F(t), F(s)\big) \leqslant V_+\big(F, [t, s]\big) = v(s) - v(t),$$

and so, $\lim_{s\to t+0} e(F(t), F(s)) \leqslant v(t+0) - v(t)$. The reverse inequality will follow if we show that for any $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon) \in (t, b)$ such that

$$v(s) - v(t) \leqslant e\big(F(t), F(s)\big) + \varepsilon \quad \text{for all } t < s \leqslant t_0, \tag{5}$$

then let s go to $t+0$ and note that $\varepsilon > 0$ is arbitrary. To prove (5), we note that $V_+(F, [t, b\rangle) \leqslant V_+(F, [a, b\rangle) < \infty$, and so, there exists a partition $\{t\} \cup \{t_i\}_{i=0}^m$ (depending on ε) of $[t, b\rangle$ with $t < t_0$ such that

$$V_+\big(F, [t, t_m]\big) \leqslant V_+\big(F, [t, b\rangle\big) \leqslant e\big(F(t), F(t_0)\big) + \sum_{i=1}^m e\big(F(t_{i-1}), F(t_i)\big) + \varepsilon.$$

If $t < s \leqslant t_0$, we have $e(F(t), F(t_0)) \leqslant e(F(t), F(s)) + e(F(s), F(t_0))$, and so,

$$V_+\big(F, [t, t_m]\big) \leqslant e\big(F(t), F(s)\big) + V_+\big(F, [s, t_m]\big) + \varepsilon,$$

implying, by Lemma 1(b),

$$V_+\big(F, [a, s]\big) - V_+\big(F, [a, t]\big) = V_+\big(F, [t, t_m]\big) - V_+\big(F, [s, t_m]\big) \leqslant e\big(F(t), F(s)\big) + \varepsilon,$$

which is precisely (5) according to the definition of v. □

In order to formulate Theorem B, we recall the notion of the *modulus of variation* of a function $f : T \to X$ due to Chanturiya [3] (see also [9, Section 11.3]): this is the sequence of the form $\{\nu(k, f, T)\}_{k=1}^\infty$ where $\nu(k, f, T) = \sup \sum_{i=1}^k d(f(b_i), f(a_i))$ and the supremum is taken over all collections $a_1, \dots, a_k, b_1, \dots, b_k$ of $2k$ numbers from T such that $a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_k \leqslant b_k$. The following theorem is a *pointwise selection principle* in terms of the modulus of variation [7, Theorem 1]:

Theorem B. *Suppose that a sequence of functions $\{f_n\}_{n=1}^\infty$ mapping T into X is such that* (a) $\lim_{k\to\infty}(\limsup_{n\to\infty} \nu(k, f_n, T)/k) = 0$, *and* (b) *the closure of the set $\{f_n(t)\}_{n=1}^\infty$ in X is compact for each $t \in T$. Then there exists a subsequence of $\{f_n\}_{n=1}^\infty$, which converges pointwise on T to a function $f : T \to X$ satisfying* $\lim_{k\to\infty} \nu(k, f, T)/k = 0$.

Now we are in a position to prove our main result. In the proof we employ several ideas from [1,5] and [6, Section 5].

Proof of Theorem 1(a). For the sake of clarity we divide the proof into four steps. In the first two steps we prove the theorem for $T=[a,b]$ and $t_0=a$, in the third step—for $T=[a,b)$ and $t_0=a$, and in the fourth step—for $T=[a,b\rangle$ and $t_0\in[a,b\rangle$ with $t_0>a$.

Step 1. Suppose that $T=[a,b]$ and $t_0=a$, so that $x_0\in F(a)$ by the assumption. Since the V_+-variation function $v:[a,b]\to[0,\infty)$ from Lemma 2 is regulated, the set of its discontinuities is at most countable. Putting

$$T_v=\Big\{t\in(a,b]\colon\ v(t-0)\equiv\lim_{s\to t-0}v(s)=v(t)\Big\}$$

and

$$T_F=\Big\{t\in(a,b]\colon\ \lim_{s\to t-0}\mathrm{e}\big(F(s),F(t)\big)=0\Big\},$$

we have, by virtue of Lemma 2, $T_F=T_v$, and so, the set $[a,b]\setminus T_F=[a,b]\setminus T_v$ is at most countable. We set

$$S=\{a,b\}\cup\big(\mathbb{Q}\cap[a,b]\big)\cup\big([a,b]\setminus T_F\big),$$

where $\mathbb{Q}$ is the set of all rational numbers, and note that S is dense in $[a,b]$ and at most countable. We enumerate the points in S arbitrarily and, with no loss of generality, suppose that S is countable, say, $S=\{t_i\}_{i=0}^{\infty}$ with $t_0=a$. Then for any $n\in\mathbb{N}$ the set $\pi_n=\{t_i\}_{i=0}^{n-1}\cup\{b\}$ is a partition of $[a,b]$. Ordering the points in π_n in strictly ascending order and denoting them by $\pi_n=\{t_i^n\}_{i=0}^{n}$, we find

$$a=t_0^n<t_1^n<\cdots<t_{n-1}^n<t_n^n=b,\quad\text{and}\tag{6}$$

$$\forall t\in S\ \exists n_0=n_0(t)\in\mathbb{N}\quad\text{such that}\quad t\in\pi_n\quad\text{for all } n\geqslant n_0.\tag{7}$$

We now construct an approximating sequence for the desired selection. Given $n\in\mathbb{N}$, we first define elements $x_i^n\in F(t_i^n)$ for $i\in\{0,1,\dots,n\}$ inductively as follows:

(i) we set $x_0^n=x_0$, and
(ii) if $i\in\{1,\dots,n\}$ and $x_{i-1}^n\in F(t_{i-1}^n)$ is already chosen, we pick $x_i^n\in F(t_i^n)$ such that $d(x_{i-1}^n,x_i^n)=\operatorname{dist}(x_{i-1}^n,F(t_i^n))$.

For each $n\in\mathbb{N}$ we define a function $f_n:[a,b]\to X$ by setting

$$f_n(t)=\begin{cases}x_i^n & \text{if } t=t_i^n \text{ and } i\in\{0,1,\dots,n\},\\ x_{i-1}^n & \text{if } t\in(t_{i-1}^n,t_i^n) \text{ and } i\in\{1,\dots,n\}.\end{cases}\tag{8}$$

Observe that $f_n(a)=f_n(t_0^n)=x_0^n=x_0$ for all $n\in\mathbb{N}$.

Step 2. Now we show that the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the assumptions of Theorem B. Condition (a) in that theorem is a consequence of the additivity of V, definitions (8) and (ii), the excess and V_+:

$$\nu\big(k,f_n,[a,b]\big)\leqslant V\big(f_n,[a,b]\big)=\sum_{i=1}^{n}V\big(f_n,\big[t_{i-1}^n,t_i^n\big]\big)=\sum_{i=1}^{n}d\big(x_{i-1}^n,x_i^n\big)$$

$$= \sum_{i=1}^{n} \operatorname{dist}\bigl(x_{i-1}^n, F\bigl(t_i^n\bigr)\bigr) \leqslant \sum_{i=1}^{n} \mathrm{e}\bigl(F\bigl(t_{i-1}^n\bigr), F\bigl(t_i^n\bigr)\bigr)$$
$$\leqslant V_+\bigl(F, [a,b]\bigr) \quad \text{for all } k, n \in \mathbb{N}, \tag{9}$$

which implies

$$\limsup_{n\to\infty} \nu\bigl(k, f_n, [a,b]\bigr) \leqslant V_+\bigl(F, [a,b]\bigr) \quad \text{for all } k \in \mathbb{N}.$$

Let us verify condition (b) of Theorem B. We consider two possibilities: (I) $t \in S$, and (II) $t \in [a,b] \setminus S$.

(I) Suppose that $t \in S$. By virtue of (7), there exists $n_0 = n_0(t) \in \mathbb{N}$ such that $t \in \pi_n$ for all $n \geqslant n_0$, and so, for each $n \geqslant n_0$ there exists $i = i(n,t) \in \{0, 1, \dots, n\}$ such that $t = t_i^n$. It follows from (8), (i) and (ii) that

$$f_n(t) = f_n\bigl(t_i^n\bigr) = x_i^n \in F\bigl(t_i^n\bigr) = F(t) \quad \text{for all } n \geqslant n_0, \tag{10}$$

and it suffices to take into account the compactness of $F(t)$.

(II) Let $t \in [a,b] \setminus S$. Then $t \in (a,b) \cap T_F$ is irrational and, in particular, by the definition of T_F we have:

$$\mathrm{e}\bigl(F(s), F(t)\bigr) \to 0 \quad \text{as } (a,b) \ni s \to t-0. \tag{11}$$

Due to the density of S in $[a,b]$, there exists a sequence of points $\{s_k\}_{k=1}^\infty \subset S \cap (a,t)$ such that $s_k \to t$ as $k \to \infty$. Since $s_k \in S$ for each $k \in \mathbb{N}$, we can find, by (7), a number $n(k) \in \mathbb{N}$ (depending also on t) such that $s_k \in \pi_{n(k)}$ and, therefore, $s_k = t_{j(k)}^{n(k)}$ for some $j(k) \in \{0, 1, \dots, n(k)-1\}$. Again, thanks to property (7), we may assume with no loss of generality that the sequence $\{n(k)\}_{k=1}^\infty$ is strictly increasing. Since $s_k < t$, it follows from (6) that there exists a unique number $i(k) \in \{j(k), \dots, n(k)-1\}$ such that

$$s_k = t_{j(k)}^{n(k)} \leqslant t_{i(k)}^{n(k)} < t < t_{i(k)+1}^{n(k)} \quad \text{for all } k \in \mathbb{N}. \tag{12}$$

Now this and the property that $s_k \to t$ as $k \to \infty$ give:

$$t_{i(k)}^{n(k)} \to t \quad \text{as } k \to \infty. \tag{13}$$

By the second line of definition (8) and (12), we have

$$f_{n(k)}(t) = x_{i(k)}^{n(k)} \in F\bigl(t_{i(k)}^{n(k)}\bigr) \quad \text{for all } k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$ pick an element $x_t^k \in F(t)$ such that

$$d\bigl(x_{i(k)}^{n(k)}, x_t^k\bigr) = \operatorname{dist}\bigl(x_{i(k)}^{n(k)}, F(t)\bigr).$$

Then (11) and (13) imply

$$d\bigl(f_{n(k)}(t), x_t^k\bigr) \leqslant \mathrm{e}\bigl(F\bigl(t_{i(k)}^{n(k)}\bigr), F(t)\bigr) \to 0 \quad \text{as } k \to \infty.$$

Since the set $F(t)$ is compact and $\{x_t^k\}_{k=1}^\infty \subset F(t)$, there exists a subsequence of $\{x_t^k\}_{k=1}^\infty$, again denoted by $\{x_t^k\}_{k=1}^\infty$, and an element $x_t \in F(t)$ such that $d(x_t^k, x_t) \to 0$ as $k \to \infty$, and so,

$$d\bigl(f_{n(k)}(t), x_t\bigr) \leqslant d\bigl(f_{n(k)}(t), x_t^k\bigr) + d\bigl(x_t^k, x_t\bigr) \to 0 \quad \text{as } k \to \infty. \tag{14}$$

This proves that the closure of the sequence $\{f_n(t)\}_{n=1}^\infty$ in X is compact for all $t \in [a,b]$.

By Theorem B, there exists a subsequence of $\{f_n\}_{n=1}^\infty$, which we again denote by $\{f_{n(k)}\}_{k=1}^\infty$, and a function $f : [a,b] \to X$ such that $d(f_{n(k)}(t), f(t)) \to 0$ as $k \to \infty$ for all $t \in [a,b]$.

Clearly, $f(a)=x_0$. The inclusion $f(t)\in F(t)$ for all $t\in[a,b]$ is a consequence of the closedness of $F(t)$, (10) and (14). Finally, the lower semicontinuity of the Jordan variation V and inequality (9) ensure that

$$V\big(f,[a,b]\big)\leqslant \liminf_{k\to\infty} V\big(f_{n(k)},[a,b]\big)\leqslant V_+\big(F,[a,b]\big). \tag{15}$$

Thus, our theorem is proved for $T=[a,b]$ and $t_0=a$.

Step 3. Assume now that $T=[a,b)$ with $b\in\mathbb{R}\cup\{\infty\}$ and $t_0=a$. Choose an increasing sequence $\{t_n\}_{n=1}^{\infty}\subset[a,b)$ such that $t_n\to b$ as $n\to\infty$. Since $V_+(F,[a,t_1])\leqslant V_+(F,[a,b))<\infty$, applying steps 1–2 we get a function $f_0:[a,t_1]\to X$ such that $f_0(t)\in F(t)$ for all $t\in[a,t_1]$, $f_0(a)=x_0$ and $V(f_0,[a,t_1])\leqslant V_+(F,[a,t_1])$. Inductively, if $n\in\mathbb{N}$ and a selection f_{n-1} of F on $[t_{n-1},t_n]$ is already chosen, we note that $V_+(F,[t_n,t_{n+1}])\leqslant V_+(F,[a,b))<\infty$ and apply again steps 1–2 to obtain a selection f_n of F on $[t_n,t_{n+1}]$ such that $f_n(t_n)=f_{n-1}(t_n)$ and $V(f_n,[t_n,t_{n+1}])\leqslant V_+(F,[t_n,t_{n+1}])$. Given $t\in[a,b)$, so that $t\in[t_{n-1},t_n]$ for some $n\in\mathbb{N}$, we set $f(t)=f_{n-1}(t)$. Then the function $f:[a,b)\to X$ is a selection of F on $[a,b)$, $f(t_0)=f_0(a)=x_0$ and, by virtue of Lemma 1(b) and (c) we have:

$$\begin{aligned}
V\big(f,[a,b)\big)&=\lim_{k\to\infty} V\big(f,[a,t_k]\big)=\lim_{k\to\infty}\sum_{n=1}^{k} V\big(f_{n-1},[t_{n-1},t_n]\big)\\
&\leqslant \lim_{k\to\infty}\sum_{n=1}^{k} V_+\big(F,[t_{n-1},t_n]\big)=\lim_{k\to\infty} V_+\big(F,[a,t_k]\big)=V_+\big(F,[a,b)\big).
\end{aligned}$$

Step 4. Now suppose that $T=[a,b\rangle$ and $t_0\in(a,b\rangle$. Noting that $V_+(F,[a,t_0))$ and $V_+(F,[t_0,b\rangle)$ do not exceed $V_+(F,[a,b\rangle)$ and $x_0\in F(t_0)$, we apply steps 1–3 twice: to F on $[t_0,b\rangle$ in order to find a selection f_1 of F on $[t_0,b\rangle$ such that $f_1(t_0)=x_0$ and $V(f_1,[t_0,b\rangle)\leqslant V_+(F,[t_0,b\rangle)$, and to F on $[a,t_0)$ with arbitrary $y_0\in F(a)$ to obtain a selection f_2 of F on $[a,t_0)$ such that $f_2(a)=y_0$ and $V(f_2,[a,t_0))\leqslant V_+(F,[a,t_0))$. We set $f(t)=f_2(t)$ for $t\in[a,t_0)$ and $f(t)=f_1(t)$ if $t\in[t_0,b\rangle$. Clearly, f is a selection of F of bounded variation on $[a,b\rangle$ with the desired properties and such that (cf. the jump relations for functions of bounded variation in [5, Theorem 4.6(a)])

$$\begin{aligned}
V\big(f,[a,b\rangle\big)&=V\big(f,[a,t_0]\big)+V\big(f,[t_0,b\rangle\big)\\
&=V\big(f_2,[a,t_0)\big)+\lim_{s\to t_0-0} d\big(f(s),f(t_0)\big)+V\big(f_1,[t_0,b\rangle\big)\\
&\leqslant V_+\big(F,[a,t_0)\big)+\lim_{s\to t_0-0} d\big(f(s),x_0\big)+V_+\big(F,[t_0,b\rangle\big)\\
&\leqslant V_+\big(F,[a,b\rangle\big)+\lim_{s\to t_0-0} d\big(f(s),x_0\big)<\infty,
\end{aligned}$$

where the existence of the limit follows from the fact that $f=f_2$ on $[a,t_0)$ is of bounded variation and the Cauchy criterion: if $a\leqslant s_1\leqslant s_2<t_0$, we have:

$$\begin{aligned}
&\big|d\big(f_2(s_1),x_0\big)-d\big(f_2(s_2),x_0\big)\big|\\
&\quad\leqslant d\big(f_2(s_1),f_2(s_2)\big)\leqslant V\big(f_2,[s_1,s_2]\big)\\
&\quad=V\big(f_2,[a,s_2]\big)-V\big(f_2,[a,s_1]\big)\to V\big(f_2,[a,t_0)\big)-V\big(f_2,[a,t_0)\big)=0
\end{aligned}$$

as $s_1,s_2\to t_0-0$.

This completes the proof of Theorem 1. □

3. Examples

Example 3.1. In this section we present an example of a multifunction F such that $V_+(F,[a,b])$ is finite, and so Theorem 1 applies, giving selections of bounded variation of F, whereas $V_-(F,[a,b])$ is infinite, and Theorem A is thus inapplicable.

Let $X=\ell^1(\mathbb{N})$ be the Banach space of all summable sequences $x:\mathbb{N}\to\mathbb{R}$, written as $x=\{x_i\}_{i=1}^\infty$, equipped with the norm $\|x\|=\sum_{i=1}^\infty |x_i|$, and let the unit vector $u_n=\{x_i\}_{i=1}^\infty$ in X be defined as usual by $x_i=0$ if $i\neq n$ and $x_n=1$. Given $k\in\mathbb{N}\cup\{\infty\}$, we set $F_k=\{0\}\cup\{c_n u_n\}_{n=1}^k$, where $\{c_n\}_{n=1}^\infty$ is a decreasing sequence of positive numbers such that

$$c_n\to 0 \quad \text{as } n\to\infty \quad \text{and} \quad \sum_{n=1}^\infty c_n=\infty \tag{16}$$

(e.g., $c_n=1/n$). Clearly, $F_k\in \mathrm{c}(X)$ for all $k\in\mathbb{N}$, and the first condition in (16) implies $F_\infty\in \mathrm{c}(X)$ as well. We define a multifunction $F:[0,1]\to \mathrm{c}(X)$ as follows:

$$F(t)=F_k \quad \text{if } \frac{k-1}{k}\leqslant t<\frac{k}{k+1} \text{ for } k\in\mathbb{N} \quad \text{and} \quad F(1)=F_\infty.$$

Since $F_k\subset F_{k+1}\subset F_\infty$ for all $k\in\mathbb{N}$, then condition $0\leqslant s\leqslant t\leqslant 1$ implies $F(s)\subset F(t)$, and so, by Lemma 1(a), $V_+(F,[0,1])=0$. In order to show that $V_-(F,[0,1])=\infty$, we first observe that if $k\in\mathbb{N}$, then

$$\mathrm{e}(F_{k+1},F_k)=\sup_{x\in F_{k+1}}\inf_{y\in F_k}\|x-y\|=c_{k+1}+\inf_{1\leqslant n\leqslant k}c_n=c_{k+1}+c_k$$

and

$$\mathrm{e}(F_\infty,F_k)=\sup_{n\geqslant k+1}\Bigl(c_n+\inf_{1\leqslant i\leqslant k}c_i\Bigr)=\sup_{n\geqslant k+1}c_n+\inf_{1\leqslant i\leqslant k}c_i=c_{k+1}+c_k.$$

Now for an arbitrary $m\in\mathbb{N}$ and for the partition π_m of $[0,1]$ of the form $\pi_m=\{(k-1)/k\}_{k=1}^m\cup\{1\}$ we have:

$$\begin{aligned}
V_-\bigl(F,[0,1]\bigr)&\geqslant\sum_{k=1}^{m-1}\mathrm{e}\biggl(F\Bigl(\frac{k}{k+1}\Bigr),F\Bigl(\frac{k-1}{k}\Bigr)\biggr)+\mathrm{e}\biggl(F(1),F\Bigl(\frac{m-1}{m}\Bigr)\biggr)\\
&=\sum_{k=1}^{m-1}\mathrm{e}(F_{k+1},F_k)+\mathrm{e}(F_\infty,F_m)\\
&=-c_1+c_{m+1}+2\sum_{k=1}^m c_k\to\infty \quad \text{as } m\to\infty.
\end{aligned}$$

Example 3.2. Multifunction F from Example 3.1 has two constant selections $f(t)\equiv 0$ and $f(t)\equiv c_1u_1$ guaranteed by Theorem 1 and satisfying initial conditions $f(0)=0$ and $f(0)=c_1u_1$, respectively, and $V(f,[0,1])\leqslant V_+(F,[0,1])=0$. However, if we assume in Theorem 1 that $x_0\in F(t_0)$ with $a<t_0\leqslant b$, then condition $V(f,[a,b])\leqslant V_+(F,[a,b])$ may be violated for any selection f of F such that $f(t_0)=x_0$. To see this, we assume in the previous example that $t_0=1/2$ and $x_0=c_2u_2$. Clearly, $x_0\in F(t_0)=F_2$. If $f:[0,1]\to X$ is any selection of F such that $f(1/2)=c_2u_2$, then since $f(0)\in F(0)=F_1=\{0,c_1u_1\}$, we have either $f(0)=0$ or $f(0)=c_1u_1$, and so,

$$V\bigl(f,[0,1]\bigr)\geqslant\bigl\|f(1/2)-f(0)\bigr\|\geqslant c_2>0=V_+\bigl(F,[0,1]\bigr). \tag{17}$$

The first inequality in Theorem 1(a) states that $V(f,[a,t_0)) \leqslant V_+(F,[a,t_0))$. In general it cannot be replaced by the inequality $V(f,[a,t_0]) \leqslant V_+(F,[a,t_0])$ if $f(t_0)=x_0$ with $t_0>a$; it suffices to argue as in (17):

$$V\big(f,[0,1/2]\big) \geqslant \big\|f(1/2)-f(0)\big\| \geqslant c_2 > 0 = V_+\big(F,[0,1/2]\big).$$

This observation also shows that the limit from the left in the third inequality of Theorem 1(a) is indispensable.

Example 3.3. We note that the inequality $V(f,[t_0,b\rangle) \leqslant V_+(F,[t_0,b\rangle)$ from Theorem 1 may fail even for $[t_0,b\rangle=[a,b]$ if at least one value $F(t)$ of F is only closed and bounded but not compact. The corresponding example was constructed in [6, Example 5.2].

4. Lipschitz and continuous selections

Recall that a multifunction $F:T\to \mathrm{c}(X)$ is said to be *Lipschitz* (with respect to the Hausdorff metric D) if its minimal Lipschitz constant given by

$$L_D(F,T)=\sup\big\{D\big(F(t),F(s)\big)/|t-s|:\ s,t\in T,\ s\neq t\big\}$$

is finite. If $f:T\to X$ is a single-valued function, we denote its minimal Lipschitz constant by $L(f,T)\equiv L_d(f,T)$.

The following theorem on the existence of Lipschitz selections of Lipschitz multifunctions is valid [6, Section 6] (for particular cases see [1,4,5,8,10], [11, Section Supplement 1], [12, Part C, Theorem (7.14)], [13]):

Theorem C. *If $F:T\to \mathrm{c}(X)$, $L_D(F,T)<\infty$, $t_0\in T$ and $x_0\in F(t_0)$, then there exists a Lipschitz selection f of F on T such that $f(t_0)=x_0$, $L(f,T)\leqslant L_D(F,T)$ and $V(f,T)\leqslant V_D(F,T)$.*

Note that if in Theorem C the set T is unbounded, it may happen that $V_D(F,T)$ is infinite; if this is the case, the last condition in this theorem is superfluous.

In order to obtain a version of Theorem C with respect to the excess function, we introduce the following definition which is parallel to (1).

A multifunction $F:T\to \mathrm{c}(X)$ is said to be *excess Lipschitz to the right* (or Lip_+, for short) if its *minimal excess Lipschitz to the right constant* defined by

$$L_+(F,T)=\sup\big\{\mathrm{e}\big(F(s),F(t)\big)/(t-s):\ s,t\in T,\ s<t\big\}$$

is finite. In a similar manner we define $L_-(F,T)$ (as well as Lip_-) by replacing the value $\mathrm{e}(F(s),F(t))$ in the definition of $L_+(F,T)$ by $\mathrm{e}(F(t),F(s))$. Clearly, if T is bounded, then $V_+(F,T)\leqslant L_+(F,T)\cdot(\sup T-\inf T)$, and if $F=f$ is single-valued, then $L_+(f,T)=L_-(f,T)=L(f,T)$. Multifunction F from Example 3.1 is Lip_+ on [0, 1].

We have the following counterpart of Theorem C:

Theorem 2. *If $F:T=[a,b\rangle\to \mathrm{c}(X)$, $L_+(F,T)<\infty$, $t_0=a$ and $x_0\in F(t_0)$, then there exists a Lipschitz selection f of F on T such that $f(t_0)=x_0$, $L(f,T)\leqslant L_+(F,T)$ and $V(f,T)\leqslant V_+(F,T)$. A similar assertion holds if we replace $T=[a,b\rangle$ by $T=\langle a,b]$, $L_+(F,T)$—by $L_-(F,T)$, $t_0=a$—by $t_0=b$ and $V_+(F,T)$—by $V_-(F,T)$.*

Taking into account Theorem 1, the proof of Theorem 2 follows the same lines with obvious modifications as those in the proof of Theorem 6.1(a) from [6], and so, it is omitted. We note that, in contrast to Theorem C, Theorem 2 does not hold if $t_0 \in [a,b\rangle$ and $t_0 > a$, that is, F may have no continuous selections at all. This can be seen from Example 3.2 (cf. (17)) rewritten as

$$\big\| f(1/2) - f(s) \big\| \geqslant c_2 > 0 \quad \text{for all } 0 \leqslant s < 1/2.$$

In order to cope with continuous selections, we introduce the following definition of continuity for a multifunction $F:[a,b\rangle \to \mathrm{c}(X)$: it is said to be *excess continuous to the right on* $[a,b\rangle$ (or, briefly, C_+) if

$$\lim_{s\to t-0} \mathrm{e}\big(F(s), F(t)\big) = 0 \quad \text{for all } t \in (a,b\rangle \tag{18}$$

and

$$\lim_{s\to t+0} \mathrm{e}\big(F(t), F(s)\big) = 0 \quad \text{for all } t \in [a,b) \tag{19}$$

simultaneously. Note that if F is Lip_+ on $[a,b\rangle$, then it is also C_+. An example of a multifunction $F:[0,1] \to \mathrm{c}(X)$, which is C_+, but not continuous with respect to the Hausdorff metric D, is constructed in Example 3.1: in fact, since $F(s) \subset F(t)$ for all $0 \leqslant s \leqslant t \leqslant 1$, conditions (18) and (19) are satisfied. On the other hand, given $k \in \mathbb{N}$, we have, for $t_k = k/(k+1)$,

$$\lim_{s\to t_k-0} \mathrm{e}\big(F(t_k), F(s)\big) = \mathrm{e}(F_{k+1}, F_k) = c_{k+1} + c_k > 0.$$

The notion of the *excess continuity to the left* (or C_-) for $F:\langle a,b] \to \mathrm{c}(X)$ is introduced similarly to (18) and (19): $\mathrm{e}(F(t), F(s)) \to 0$ as $s \to t-0$ for all $t \in (a,b]$ and $\mathrm{e}(F(s), F(t)) \to 0$ as $s \to t+0$ for all $t \in \langle a,b)$ simultaneously.

We point out that condition (18) (as well as (19)) is very weak as compared with the condition $\lim_{s\to t-0} D(F(s), F(t)) = 0$ and, taking into account the second definition of the excess from Section 1, it amounts to the following: for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $s \in [t-\delta, t)$ and $x \in F(s)$ there exists $y \in F(t)$ with $d(x,y) < \varepsilon$.

Now we have the following extension of the second part of Theorem A from Section 1 (note at once that Theorem 3 below does not hold if $t_0 > a$ as the observation following Theorem 2 shows):

Theorem 3. *Let $F: T = [a,b\rangle \to \mathrm{c}(X)$ be C_+, $V_+(F,T) < \infty$, $t_0 = a$ and $x_0 \in F(t_0)$. Then there exists a continuous selection of bounded variation f of F on T such that $f(t_0) = x_0$ and $V(f,T) \leqslant V_+(F,T)$. A similar assertion holds if we replace $T = [a,b\rangle$ by $T = \langle a,b]$, C_+—by C_-, $V_+(F,T)$—by $V_-(F,T)$ and $t_0 = a$—by $t_0 = b$.*

Proof. The idea of the proof comes from the factorization procedure for metric space valued functions of bounded variation [4], [5, Section 3]. So, by employing a suitable "change of variables" we reduce Theorem 3 to Theorem 2.

We set $\ell = V_+(F, [a,b\rangle)$. Since F is C_+, the V_+-variation function v maps $[a,b\rangle$ onto $[0,\ell\rangle$ continuously by Lemma 2. Given $s \in [0,\ell\rangle$, we denote by $v^{-1}(s) = \{t \in [a,b\rangle:\ v(t) = s\}$ the inverse image of the singleton $\{s\}$ and let $\mu(s) = \min v^{-1}(s)$, so that $v(\mu(s)) = s$, and the function $\mu:[0,\ell\rangle \to [a,b\rangle$ is continuous and nondecreasing.

We define a multifunction $G:[0,\ell\rangle \to \mathrm{c}(X)$ as follows:

$$G(s) = \bigcap_{t\in v^{-1}(s)} F(t) \quad \text{for all } s \in [0,\ell\rangle. \tag{20}$$

That G is well defined, i.e., that $G(s) \neq \emptyset$ (the compactness is immediate) for all values of s, can be seen from the following: given $t_1, t_2 \in v^{-1}(s)$, $t_1 \leqslant t_2$, we have by Lemma 1(b) that

$$\mathrm{e}\big(F(t_1), F(t_2)\big) \leqslant V_+\big(F, [t_1, t_2]\big) = v(t_2) - v(t_1) = s - s = 0,$$

and so, $F(t_1) \subset F(t_2)$. It follows that $G(s) = F(\mu(s))$ for all $s \in [0, \ell\rangle$. Also, since $t \in v^{-1}(v(t))$, (20) implies $G(v(t)) \subset F(t)$ for all $t \in [a, b\rangle$. Clearly, $\mu(0) = a$, and so, $x_0 \in F(a) = F(\mu(a)) = G(0)$. Moreover, G is Lip_+ on $[0, \ell\rangle$: indeed, for $s_1, s_2 \in [0, \ell\rangle$ with $s_1 < s_2$ we have, by Lemma 1(b):

$$\begin{aligned}
\mathrm{e}\big(G(s_1), G(s_2)\big) &= \mathrm{e}\big(F\big(\mu(s_1)\big), F\big(\mu(s_2)\big)\big) \leqslant V_+\big(F, \big[\mu(s_1), \mu(s_2)\big]\big) \\
&= V_+\big(F, \big[a, \mu(s_2)\big]\big) - V_+\big(F, \big[a, \mu(s_1)\big]\big) \\
&= v\big(\mu(s_2)\big) - v\big(\mu(s_1)\big) = s_2 - s_1.
\end{aligned}$$

By Theorem 2, there exists a Lipschitz selection g of G on $[0, \ell\rangle$ such that $g(0) = x_0$ and $L(g, [0, \ell\rangle) \leqslant L_+(G, [0, \ell\rangle) \leqslant 1$. The desired selection f of F is defined as the composed function $f = g \circ v$. It is clear that $f : [a, b\rangle \to X$ is continuous as the composition of two continuous functions, $f(a) = g(v(a)) = g(0) = x_0$,

$$f(t) = g\big(v(t)\big) \in G\big(v(t)\big) \subset F(t) \quad \text{for all } t \in [a, b\rangle$$

and, since $L(g, [0, \ell\rangle) \leqslant 1$, we have $V(f, [a, b\rangle) \leqslant V_+(F, [a, b\rangle)$. □

In Example 3.1 we have $v(t) = V_+(F, [a, t]) \equiv 0$, $G : \{0\} \to \mathrm{c}(X)$ and $G(0) = F(\mu(0)) = F(0) = F_1$, and so, we obtain as a continuous selection of F only $f(t) \equiv 0$ if $f(0) = 0$ or $f(t) \equiv c_1 u_1$ if $f(0) = c_1 u_1$.

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