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Creating a Set of Optimal Objects using Partial Preference Information

V. V. Podinovski

Abstract—A strict formulation of the problem of creating a set that includes a determined number of preferred components from a prescribed finite aggregate of objects, where a partial quasiorder is defined, is given. The properties of the optimal and nondominated sets that involve objects are examined.

Keywords: selection of an optimal object subset, partial preference relation, optimal sets, nondominated sets, l-optimal objects, l-nondominated objects

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INTRODUCTION

Mathematical decision theory is developing rapidly and is gaining greater practical importance. The theory focuses for the most part on *single-choice problems*, i.e., the problems of selecting one of the best or an optimal object [for example, 1 and 2] from a prescribed aggregate of objects (choices, plans, strategies, or alternatives). If the preferences are simulated with a disconnected binary relation, a strict partial order, then the applicants for the optimal object are nondominated objects. This is typical, for example, of multiobjective problems when the preferences of the person making decisions (PMD) are not clearly recognized [3].

However, other formulations of the decision problems are often observed. These are the problems of the selection of a few (the chosen number is $l > 1$) optimal objects from a prescribed finite aggregate. These problems can be divided into two groups. One of them is the problems in which initially some optimal objects should be selected and then one of the best should be chosen from the selected objects. The problems of creating complicated systems serve as examples: at the first design stage two or three of the best choices are selected from some system alternatives and then after more careful consideration one of the best, which is subsequently designed with a required completeness criterion, is chosen between them. These can be called *two-stage single-choice problems*. Here only nondominated objects can pretend to be the best.

The other group is *multiple-choice problems* or *subset-choice problems* where the application of all l-selected optimal objects, not excepting any, is supposed. Examples of these problems include competitions and tenders of all types, member group formation, etc.

If a preference relation is given by the aggregate of all objects, it is necessary to expand the relation into the aggregate of all l-selected sets to solve the problem of l-selected optimal objects. A number of works [4–6] have been devoted to the problem of the preference relation expansion into a finite aggregate to the relation over a power set and its value function construction. However, they supposed that a complete ordering should be expanded into a power set, as well, a number of additional sufficiently strong assumptions, which were formulated as axioms concerning some properties of a desired connected relation over the power set, were taken. Our objective is to expand the partial ordering and only into an aggregate of the sets of l-objects where any additional assumptions are not taken.

In the literature two approaches to selecting applicants for the best objects in these problems (for example, [7]) were considered. According to the former all nondominated objects of “the first level” should be taken. If the objects are less than l, the second “level” formed by nondominated objects in the aggregate remaining after a deletion of the first “level” objects from an initial aggregate should be added to them. If all objects are less than l, the third “level” objects should be added to them and etc. until for the first time the number of all chosen objects are not less l. The latter approach regards that all l of such “levels” should be taken. But in [8] it was shown that the former approach narrows unreasonably the number of applicants for the best objects and the latter in contrast, expands it unreasonably. A corresponding example will be given in this paper.

In [8], the concept generalization of a nondominated object was introduced for multiple-choice problems: an object is called *l-nondominated* if no more than $l - 1$ objects are more preferred than it in accordance with a strict order relation. Also, it was found that only *l-nondominated* objects can pretend to be l-optimal. In [9], a number of the main properties of *l-nondominated* objects was presented, the formulas in order to calculate the performance evaluations of the decision rules in the multiple-choice problems was given and the Pareto's decision rule performance was evaluated. In [10], the performance evaluations of the decision rules using information on the importance of the homogeneous tests were obtained [11]. In [12, 13] the authors give the collection of both previously obtained and recent results in the performance evaluations of some decision rules as applied to the various decision problem formulations, including the multiple-choice problems and the rule using information on the relative importance of the homogeneous tests [8, 14]. In [15], it was shown that the development level of the theory under discussion is clearly inadequate to meet demands in practice and the direction for future development was defined.

In this paper the problem of selecting a few optimal objects is examined within the framework of the general selection methodology when the preference relations are partial, the procedure for forming the set of l-best objects is defined and proved. The main results of the paper have been already presented in [16].

1. THE OPTIMAL AND NONDOMINATED SETS OF OBJECTS

We consider the problem of selecting (a final selection) $l > 1$ the optimal objects from the finite set of the objects X. We take $|X| = n$, i.e., the total of the objects is n, then $n > 1$. The preference-indifference relation of PMD R is given over X: xRy means that an object x is not less preferred than y. The relation R results in the relations of the strict preference P, indifference I, and incongruence N: xIy is correct when it is true xRy and yRx; xPy is performed then xRy is correct but yRx is incorrect; xNy takes place when either xRy or yRx is not correct.

The relation R is a quasiorder, it is reflexive and transitive; in addition the preference relation P is a strict partial order (it is irreflexive and transitive), and the indifference relation I is an equivalency (it is reflexive, symmetric, and transitive). As well, P is transitive in I: from xPy and yIz, and also xPz results from xPy and yPz.

Let us consider the power set L (sets) consisting of l-objects. This set is finite as well: such sets are only $C_n^l = n!/l!(n-l)!$. The designations such as $A = \{a^1, ..., a^l\}$ and $B = \{b^1, ..., b^l\}$ will be used for the sets of L. Let Π be a set of set permutations $\{1, ..., l\}$. The sequence (ordered set) $A = \{a^1, ..., a^l\}$ is meant by the set permutation (aggregate) $\pi \in \Pi$ corresponding to the permutation $\pi(A) = \langle a^{\pi(1)}, ..., a^{\pi(l)}\rangle$. For example, if $l = 3$ and $\pi = (2, 1, 3)$, then $\pi(A) = \langle a^2, a^1, a^3\rangle$.

Since in accordance with the context of the problem of selecting l-optimal objects under consideration the order of the objects in the chosen set is immaterial, then the preference—indifference relation R^l over the set L is given as follows.

Definition 1. The relation AR^lB is performed if and only if these permutations π, $\rho \in \Pi$ exist that are true:

$$a^{\pi(i)} R b^{\rho(i)}, \quad i = 1, ..., l. \tag{1}$$

It is easy to understand that Definition 1 is equal to either of the two following definitions:

$$\begin{aligned} AR^lB &\Leftrightarrow \exists\rho \in \Pi: a^i R b^{\rho(i)}, i = 1, ..., l; \\ AR^lB &\Leftrightarrow \exists\pi \in \Pi: a^{\pi(i)} R b^i, i = 1, ..., l. \end{aligned} \tag{2}$$

It should be noted that Definition 1 in its own way is relative to the definition of the preference—indifference relation of the multiobject problems with equally important homogeneous tests [18].

The relation R^l, as is easy to understand, is a quasiorder. It leads to the strict preference relation, the strict partial order P^l and the indifference relation, the equivalency I^l in L. It is clear that AI^lB is performed when in (1) and (2) R can be substituted by I for each i and AP^lB is correct if R can be replaced by P only for a single i in (1) and (2).

Definition 2. The set of the objects A from the aggregate L is called *strictly optimal* (accordingly *optimal, nondominated*), if for any set $B \in L$ that is different from A, AP^lB is correct (accordingly if for any set $B \in L$ AR^lB is correct, BP^lA is incorrect).

A set that is not optimal (nondominated) is called *nonoptimal* (accordingly *dominated*).

Let $P^l(L)$, $R^l(L)$, and $\bar{P}^l(L)$ be the sets of strictly optimal, optimal, and nondominated sets respectively. By Definitions 1 and 2 the relations

$$P^l(L) \subseteq R^l(L) \subseteq \bar{P}^l(L), \tag{3}$$

are true. If there is an optimal set then it exhausts the set $P^l(L)$ obviously [and $R^l(L) = P^l(L)$,] and the desired l of the best objects are defined univalently — these are all objects from the set. As well, in (3) both $\subseteq$ are performed as the equalities =. Suppose that this set does not exist but there is an optimal set. Even if it is not single, all optimal sets are equivalent to the relation I^l. Therefore, any of them can have "equal right" to pretend to be the best and its forming objects can be regarded as l-best. In addition, the second side (right side) $\subseteq$ in 3 is performed as =. Finally, if there are not any optimal objects, by finiteness of the set L, nondominated sets will be certain to exist. Moreover, the set of all sets will be outwardly stable: for any set $B \in L\backslash\bar{P}^l(L)$ the set $A \in \bar{P}^l(L)$ will be found so that AP^lB is correct (see Theorem 2, given below). Therefore, the applicants for the optimal set can be only nondominated sets and the objects from these sets can pretend to be l-best.

Unfortunately, in the last case the set $\bar{P}^l(L)$ will contain noncompared sets in R^l. Such cases can be

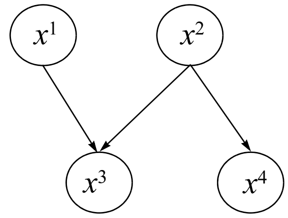

Fig. 1. Graph of the preference relation P in a set of the objects.

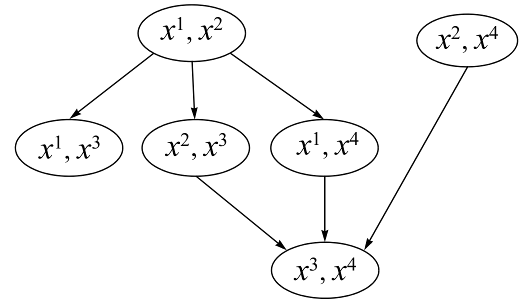

Fig. 2. Graph (the transitive frame) of the preference relation P^2 in a set of two objects.

found more often in practice if the preference information of PMD and/or indefinite factors are partial so the relation R is only partial and there are "a lot of" objects that are not compared in R for an interpreted and proved selection of a single set from the set of all nondominated is necessary to extend the relation R^l. This can be performed only on account of the extension of the relation R, for that it should involve additional information about the preferences of the PMD (if it is possible) and/or take additional assumptions about the properties of its preferences and check their performance.

Example 1. Suppose that the preference relation P is a strictly partial order, (its graph is shown in Fig. 1) to be defined by the set X consisting of four objects. The indifference relation I is an equivalency relation. The total of the object pairs is $C_4^2 = 6$. Their partial ordering in accordance with the relation P^2 is presented in Fig. 2. There is no optimal set (and especially strict optimal) here. There are two nondominated objects: these are $\{x^1, x^2\}$ and $\{x^2, x^4\}$. Note that the first upper level is $X^1 = \{x^1, x^2\}$ and it does not have the object x^4. The second upper level is $X^2 = \{x^3, x^4\}$ so that integrating two levels $X^1 \cup X^2 = X$ contains the "superfluous" object x^3.

The quasiorder $\hat{R}$ is said *to continue consistently* the quasiorder R [17 and 19], or R is a *subrelation of the relation* $\hat{R}$, and $\hat{R} \models R$ is written if the relations are correct

$$\hat{R} \supseteq R,\ \hat{I} \supseteq I,\ \hat{P} \supseteq P,$$

(the latter is due to the former). It follows from these relations that

$$P^l(L) \subseteq \hat{P}^l(L), \quad R^l(L) \subseteq \hat{R}^l(L), \quad \bar{\hat{P}}^l(L) \subseteq \bar{P}^l(L). \quad (4)$$

The formulas (4) show that, when the extension of the preference–indifference relations R is consistent, the sets $P^l(L)$ and $R^l(L)$, in general, are extending but the set $\bar{P}^l(L)$ is restricting. If $\hat{R}$ is a connected quasiorder (i.e., there are no incomparable objects), then $\hat{R}^l(L)$ will not be empty. Therefore, the preference–indifference relation $\hat{R}$ is "sufficiently connected," the equation $\hat{R}^l(L) = \bar{\hat{P}}^l(L)$ is performed, and we will be able to regard any set of $\hat{R}^l(L)$ as a solution to the problem of the selecting l-best objects under consideration (Example 4 is shown below).

2. STRICTLY L-OPTIMAL, L-OPTIMAL, AND L-NONDOMINATED OBJECTS

Since it is too burdensome to select optimal and nondominated sets directly based on Definition 1, using (1) or (2), even with a "not very wide" set of X objects, this raises the question of the search for the constructive solution methods of an arising problem. For this purpose the following concepts can be useful.

Definition 3. The object x is called *strictly l-optimal* (corresponding to *l-optimal* and *l-nondominated*) if xPy is correct for all objects y that are different from x, but some number of them may be less than l (corresponding to the case where xRy is true for all objects y, but some number of them is less than l if the number of the objects y, where yPx is correct, is less than l).

An object that is not l-nondominated is called *l-dominated*.

We note that if the equivalency I is an equivalent relation, the notions of strictly l –optimal and l-optimal objects are the same. But when $l = 1$ the definitions of l-optimal and l-nondominated objects seem to be the definitions of the optimal and dominated objects that are widely used in optimization problems (the selection of the one best object).

Suppose that $P_l(X)$, $R_l(X)$, and $\bar{P}_l(X)$, are the sets of strictly l-optimal, l-optimal, and l-nondominated objects, respectively. From Definition 3 it directly follows that the propositions are true:

$$P_l(X) \subseteq R_l(X) \subseteq \bar{P}_l(X). \quad (5)$$

If $k < l$, then $P_k(X) \subseteq P_l(X)$,

$$R_k(X) \subseteq R_l(X), \quad \bar{P}_k(X) \subseteq \bar{P}_l(X). \quad (6)$$

Theorem 1. The following propositions are true:

1.1 The sets $P_l(X)$, $R_l(X)$, and $\bar{P}_l(X)$ are closed above by R if $x \in P_l(X)$, (accordingly $x \in R_l(X)$, $x \in \bar{P}_l(X)$), and yRx, then $y \in P_l(X)$ (accordingly $y \in R_l(X)$, $y \in \bar{P}_l(X)$).

1.2 The number of strictly l-optimal objects cannot exceed l; the conditions $|\bar{P}_l(X)| = l$, $|P_l(X)| = l$, and $\bar{P}_l(X) = P_l(x)$ are equivalent.

1.3 The sets of l-nondominated objects are not empty and outwardly l-stable:

–if $x \notin \bar{P}_l(X)$, then in $\bar{P}_l(X)$ there are not less than l objects y, such as yPx and

–if $x^1, ..., x_s \notin \bar{P}_l(X)$, $x^{s+1}, ..., x^l \in \bar{P}_l(X)$, where $s \le l$, then in $\bar{P}_l(X)$ there are s objects $x^{01}, ..., x^{0s}$ that are different from $x^{s+1}, ..., x_l$ and such as $x^{01}Px^1, ..., x^{0s}Px^s$.

1.4. Suppose that the equivalency I is an equal relation; if $|R_l(X)| = l$, then the equality

$$R_l(X) = \bar{P}_l(X) \quad (7)$$

is fair.

1.5. If the quasiorder R is connected, then the equality (7) is fair.

Note that if the equivalency I is not an equal relationship, then the realization of the condition $|R_l(X)| \ge l$ does not guarantee that the equality (7) is fair.

Proof of Theorem 1. The propositions concerning the sets $R_l(X)$ and $\bar{P}_l(X)$ are found in [9]. Let us prove the propositions, regarding the set $P_l(X)$.

1.1 If $x \in P_l(X)$, then, according to Definition 3, the number of the objects z is greater than $n - l$ when xPz is true. But, yPz follows from yRx and xPz. Thus, it proves to be more than $n - l$ of the objects y when yPz is true. Therefore, $y \in P_l(X)$.

1.2 Let us assume that $|P_l(X)| = s > l$. Then, by Definition 3, for any fixed object x^1 from the set $P_l(X)$ we find the object $x^2 \ne x^1$ in the same set when x^1Px^2. For the object x^2 in its turn, the object $x^3 \ne x^2$ is found in $P_l(X)$, when x^2Px^3. Moreover, x^1Px^3 and $x^3 \ne x^1$ by virtue of the fact that P is transitive and irreflexive. By similarly reasoning, we make sure that for every $x^i \in P_l(X)$, $i = 3, ..., s - 1$ there is the object $x^{i+1} \in P_l(X)$, when x^iPx^{i+1} with all objects $x^1, ..., x^s$ are different in pairs. But, then x^iPx_1, $i = 2, ..., s$ will be performed, which seems contrary to the assumption $x^1 \in P_l(X)$.

Let $|\bar{P}_l(X)| = l$ be. For $x \notin \bar{P}_l(X)$ and every $y \in \bar{P}_l(X)$, by 1.1, yPx is true. This means that yPz cannot be performed, at most, for $l - 1$ of the objects z (all of them are from the set $P_l(x)$), which are different from y. Hence, by Definition 3, $y \in P_l(X)$), so that $\bar{P}_l(X) \subseteq P_l(X)$. In terms of (5) we obtain the equality $\bar{P}_l(X) = P_l(X)$.

Now let $|P_l(X)| = l$, $x \in P_l(X)$ and $y \in X \backslash P_l(X)$. First, we assume that x is a minimum object by P in $P_l(X)$, i.e., there is no object $\nu \in P_l(X)$ so that $xP\nu$. As altogether there are only l objects in the composition $P_l(X)$, then by Definition 3, xPy should be performed. Now assume that x is not a minimum object by P. Then a minimum object $\nu \in P_l(X)$ by P is found in the set $P_l(X)$, so that $xP\nu$. As soon as it has been found, νPy is true for this object. Owing to the transitivity of the relation P xPy is performed, as well. Thus, xPy is true for either of l-objects $x \in P_l(X)$ and any object $y \in X \backslash P_l(X)$. Consequently, $y \notin \bar{P}_l(X)$. Hence, $\bar{P}_l(X) = P_l(X)$.

Finally, we suppose that the equality $\bar{P}_l(X) = P_l(X)$ is fair. As we have already established, $|P_l(X)| \le l$. For any $x \notin \bar{P}_l(X)$ by 1.3, there are not less than l objects y to be found in $\bar{P}_l(X)$ when yPx, so that $|\bar{P}_l(X)| \ge l$. Thus, $|\bar{P}_l(X)| = |P_l(X)| = l$.

1.3 The proposition was proved in (9).

1.4 This proposition is a simple consequence of Proposition 1.2, because in this case the notions of strictly l-optimal and l-optimal objects are the same.

1.5. With regard for (5) it is enough to show that $R_l(X) \supseteq \bar{P}_l(X)$ is correct. Let $x \in \bar{P}_l(X)$. Then, by Definition 3, the number of the objects y is greater than $n - l$ when yPx is incorrect. This means that by virtue of the connection R the number of the objects y is greater than $n - l$ when xRy is correct. Therefore, by Definition 3, $x \in R_l(X)$).

The proof of Theorem 1 is completed.

Example 2. Under the conditions of Example 1 we have

$$P_2(X) = R_2(X) = \{x^2\} \subset \bar{P}_2(X) = \{x^1, x^2, x^4\} \subset X.$$

3. THE USE OF L-OPTIMAL AND L-NONDOMINATED OBJECTS TO CREATE THE OPTIMAL AND NONDOMINATED SETS

We will give the conditions determining the existence and characterizing some properties of the optimal and nondominated objects.

Theorem 2. The following propositions are fair:

2.1. The existence and properties of a strictly optimal set:

- a strictly optimal set can consist only of strictly l-optimal objects and

- a strictly optimal set is true if and only if $|P_l(X)| = l$ with $P^l(L) = \{P_l(X)\}$ (i.e., strictly optimal set is single).

2.2. The existence and properties of optimal sets:

—the optimal set can only include just l-optimal objects; the optimal set is involved in all strictly l-optimal objects;

—if $|R_l(X)| = l$, then the optimal set with $R^l(L) = \{R_l(X)\} = P^l(L)$ is true if and only if the equation (7) is performed; when $|R_l(X)| > l$, the optimal set is true if and only if the equation (7) is performed and in $R_l(X)$ there is a minimum object by R; and

—if $k < l$, $R_k(X) \subset R_l(X)$ and the object $x \in R_k(X)$ is not involved in the set $A \in L$, then this set is not optimal.

2.3. The existence and properties of the nondominated sets:

—each nondominated set consists of only l-nondominated objects and includes all strictly l-optimal objects;

—the set of the nondominated sets is not empty and outwardly stable in any n, l and R; and

—if $x \notin A$, $y \in A$ and xPy is true then the set A is dominated.

Proof of Theorem 2.

We need the following auxiliary proposition.

Lemma. Assume that the set B is obtained from the set A by changing from a^j to $x \notin A$. Then:

$$xRa^j \Leftrightarrow AR^lB, \quad xPa^j \Leftrightarrow AP^lB, \quad xIa^j \Leftrightarrow AI^lB,$$
$$xNa^j \Leftrightarrow AN^lB.$$

The Lemma is almost evident. But the proof is unwieldy and we will not show it.

2.1. Let us have the set $A \in P^l(L)$ and the object $x \in A$. By Definition 1 for the set B, resulted from changing the object x to any object $y \notin A$, AP^lB is true; whence it follows that xPy is true. Thus, xPy cannot be performed only for $y \in A$, i.e., at most for $l - 1$ objects which are different from x. However, by Definition 2, $x \in P_l(X)$.

According to 1.2, $|P_l(X)| \le l$. If this inequality is strict, then there is no strictly optimal set. Let $|P_l(X)| = l$. We show that the set $A = P_l(X)$ is strictly optimal. It is easy to be sure that xPy is true for any $x \in P_l(X)$ and $y \notin P_l(X)$. Therefore, AP^lB is fair for any $B \ne A$.

2.2. Let us have the set $A \in L$. By Definition 1, AR^lB is true for the set B that results from changing the object $x \in A$ to the object $y \notin A$; whence it follows that xRy is true. Thus, xRy cannot be performed only for $y \in A$, i.e., at most for $l - 1$ objects. Because of this, $x \in R^l(X)$. As the optimal set is nondominated, it includes all strictly l-optimal objects by the proposition from 2.3, which will be proved below (its proof does not hold for 2.2).

Let $|R_l(X)| = l$. Assume that (7) is performed. Then by 1.2 $R_l(X) = P_l(X)$ and $R_l(X)$ is a strictly optimal set (see 2.1.). Now suppose there is an optimal set A. By (5), $R_l(X) \subseteq \bar{P}_l(X)$. It remains to prove that $R_l(X) \supseteq \bar{P}_l(X)$ is true. Let us assume that the latter is false, so that $x \in \bar{P}_l(X) \backslash R_l(X)$ exists. As shown by Proposition 1.1, the relation xRy cannot be performed for any $y \in R_l(X)$. Let B be a set obtained from A by the substitution of the arbitrary object y into x. Because A is optimal, AR^lB is true by the Lemma, thus it follows that xRy must be performed. The contradiction is obtained, so that (7) should be performed. The proof of the corresponding proposition is rather unwieldy for the case $|R_l(X)| > l$ and so we drop it. We notice merely that it applies the following auxiliary proposition. Let $|R_l(X)| = m$, where m is the number of I minimal equivalence classes by R objects in $R_l(X)$ and q is the number of this class objects in the least number; then $q \ge m - l + 1$.

Let $k < l$, $R_k(X) \subset R_l(X)$ and the object $x \in R_k(X)$ is not inserted in the set $A \in L$. Since $|A| = l$ and $|R_k(X)| \le l - 1$, then the object $a \notin R_k(X)$ is found in the set A. Let B be a set obtained from A by the substitution of the object a into x. Because xNa or xPa are true, so that aRx is false, then AR^lB is false by the Lemma. Therefore, the set A is not optimal.

2.3. Let A be a nondominated set, so that for the set B that results from the substitution of the object $x \in A$ into $y \notin A$, BP^lA is incorrect, thus it follows that yPx is incorrect by the Lemma. Hence yPx can be performed only for the object $y \in A$, i.e., at most for the objects $l - 1$. Therefore, the object x is l-nondominated.

Let us suppose that the nondominated object A does not contain the object $x \in P_l(X)$, so that in the set A we found the object y when xPy is fair. But then for the set B, obtained from A by the substitution of the object y into the object x, BP^lA should be performed, which is impossible.

The proposition of external stability follows from the general result of the external stability of the maximal element set for a finite set where the strict partial order is given [17].

Suppose that the objects $x \notin A$, $y \in A$ and xPy are true. For the set B obtained from the set A by the substitution y into x, BP^lA is fair. Consequently, the set A is dominated.

The proof of Theorem 2 is completed.

Example 3. Under conditions of Example 1 there are only two nondominated sets: these are $\{x^1, x^2\}$ and $\{x^2, x^4\}$. Each of these sets consists of two nondominated objects and contains an object x^2, which is strictly two-optimal.

When optimal or nondominated objects are formed one should lean on Theorem 2 and take account of the properties of the l-optimal and l-nondominated objects. First and foremost, by Theorem 2, all l-dominated objects must be excluded from the consideration as candidates for l-optimal objects.

If strictly l-optimal objects are equal to l, then these make up a strictly optimal set and only they must be regarded as l optimal objects (Proposition 2.1). Also,

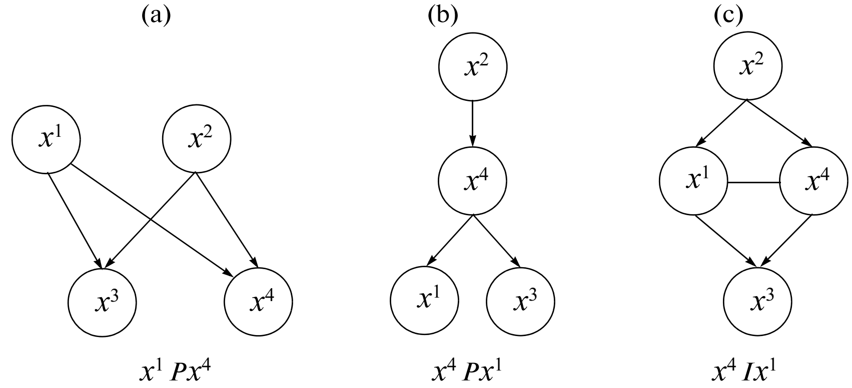

Fig. 3. Graphs (transitive bases without loops) of the quasiorders obtained by the consistent expansion of the initial relation R at the expense of additional information and the results of the preference comparison of the pair of objects x^1 and x^4.

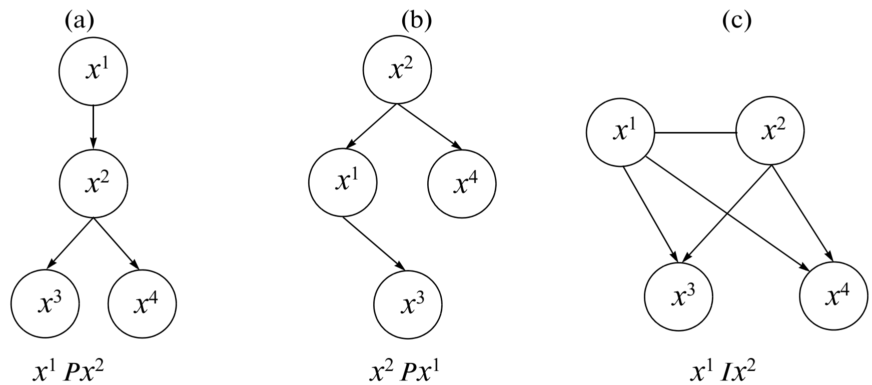

Fig. 4. Graphs (transitive bases without loops) of the quasiorders obtained by the consistent expansion of the initial relation R at the expense of additional information and the results of the preference comparison of the pair of objects x^1 and x^2.

the corresponding proposition from 2.2 indicates the conditions where a strictly optimal set exists.

If the objects in $R_l(X)$ are greater than l and the conditions assumed in Proposition 2.2 are performed (equation (7) is fair and the smallest object by R exists in the set $R_l(X)$), then the optimal set should be formed as follows. At first, all objects, which are not the smallest (these are no more than $l - 1$), should be taken from the set $R_l(X)$, and then be added the number of any least objects required to make up l (all the smallest objects are indistinguishable from each other). But if the conditions that were assumed in Proposition 2.2, are not performed, then the optimal set is not found.

If the optimal set is not found, then from the set of all l-nondominated objects we may select the objects that must be certainly included in the number of l optimal ones, i.e., that are included in the strictly optimal and optimal sets. These are all objects from the set $P_l(X) \cup R_k(X)$, where k is the greatest number $k < l$ so that $R_k(X) \subset R_l(X)$. These objects are not greater than $l - 1$. The remaining l-nondominated objects are only candidates to be l-optimal.

In order to create a strictly optimal set on a reasonable basis we should extend the relation R consistently, as was already mentioned above, when we obtain supplementary information and check its consistency and/or take additional assumptions of the PMD's preferences and make sure they are performed.

Example 4. Assume that under the conditions of Example 1 additional information on the preference relationship of the objects x^1 and x^4 is obtained. If it has been found that x^1Px^4, then the object x^1 becomes strictly dual-optimal and the only strictly optimal object will be $\{x^1, x^2\}$ (Fig. 3a). If it has proved, on the contrary, to be x^4Px^1, then the object x^4 becomes strictly dual-optimal and only the object $\{x^2, x^4\}$ will be strictly optimal (Fig. 3b). Finally, if it has become evident that x^1Ix^4, then either objects x^1 and x^4 become dual-optimal, both of the sets $\{x^1, x^2\}$ and $\{x^2, x^4\}$ will be optimal and any of these sets can be taken as a problem solution for selecting two optimal objects (Fig. 3 c).

Now let us assume that the additional information refers to the objects x^1 and x^2. If x^1Px^2, then the object x^1 becomes strictly dual-optimal and the only strictly optimal set will be $\{x^1, x^2\}$ (Fig. 4a). If x^2Px^1, then the selection uncertainty is not reduced: either sets $\{x^1, x^2\}$ and $\{x^2, x^4\}$ become nondominated and incomparable (Fig. 4b). Finally, if it has become evident that x^1Ix^2, then the object x^1 becomes strictly dual-optimal and the only strictly optimal set will be $\{x^1, x^2\}$ (Fig. 4c).

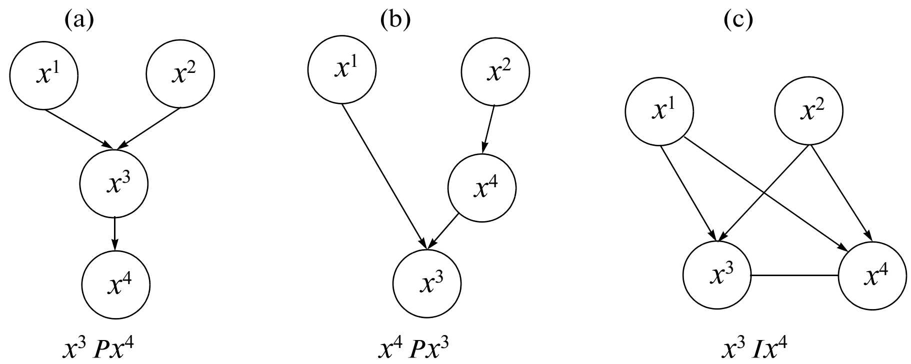

Fig. 5. Graphs (transitive bases without loops) of the quasiorders obtained by the consistent expansion of the initial relation R at the expense of additional information and the results of the preference comparison of the pair of objects x^3 and x^4.

Now let us assume that the additional information refers to the dual-dominated object x^3 and the dual-dominated object x^4. If x^3Px^4, then the object x^1 becomes strictly dual-optimal and the only strictly optimal set will be $\{x^1, x^2\}$ (Fig. 5a). If x^4Px^3, then the selection uncertainty is not reduced: either sets $\{x^1, x^2\}$ and $\{x^2, x^4\}$ become nondominated and incomparable (Fig. 4b). Finally, if x^3Ix^4, then the object x^1 becomes strictly dual-optimal and the only strictly optimal set will be $\{x^1, x^2\}$ (Fig. 5c).

CONCLUSIONS

In this paper the development of multiple-choice theory is given for the problems where a set from a prescribed number of the optimal objects should be formed and also when their preference ordering is not necessary but the preference relationship over the set of all objects is a partial quasiorder. These results should be taken into account in computer decision-support systems for selecting a few optimal objects.

The current tendency for the development of the theory is to study *multiple-choice and ordering problems* using partial preference information when it is necessary to form a set of a given number of optimal objects and to rank all of these objects in preference (or only the chosen number of the most optimal between them).

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