Theoretical and Mathematical Physics, **181**(2): 1418–1427 (2014)

ENERGY SPLITTING IN DYNAMICAL TUNNELING

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We propose an operator method for calculating the semiclassical asymptotic form of the energy splitting value in the general case of tunneling between symmetric orbits in the phase space. We use this approach in the case of a particle on a circle to obtain the asymptotic form of the energy tunneling splitting related to the over-barrier reflection from the potential. As an example, we consider the quantum pendulum in the rotor regime.

Keywords: dynamical tunneling, tunneling splitting, Schrödinger equation, semiclassical approximation, over-barrier reflection

1. Introduction

Tunneling is one of the basic quantum effects and is of interest in many fields of contemporary physics [1], [2]. The basic manifestation of tunneling is particle penetration through a potential barrier separating two domains of classical motion in the configuration space. It is well known that tunneling leads to energy level splitting [3]–[7] in the case of a mirror-symmetric double-well potential, and a similar effect is also possible in the absence of symmetry [8]–[11]. Another manifestation of the tunneling effect is particle reflection in motion above the potential barrier, which can also be treated as tunneling through the classically forbidden region (barrier) in the momentum representation [12], [13].

Tunneling can generally occur between two different trajectories in the phase space, i.e., this is the so-called dynamical tunneling, which can arise in various quantum mechanical models and has been actively studied in recent years [14]–[16]. Very interesting "dynamical" cases arise in the presence of a magnetic field (see [17]–[19]). An important problem is to calculate the tunneling splitting of energies in the case of dynamical tunneling. The influence of tunneling on the spectral structure was also investigated using the methods of adiabatic approximation for the Schrödinger operator with a rapidly oscillating potential (see [20], [21]).

The simplest example of dynamical tunneling is given by the problem of particle motion in a circle under the action of a potential field V in the case where its energy is greater than the barrier $\max V(x)$ (the rotor regime). A classical particle in such a situation rotates (runs through the entire circle) in one of the two possible directions. In quantum mechanics, there can be a pair of states such that the particle "rotates" simultaneously in two directions in each of these states, and the tunneling splitting of the energy level corresponding to this pair thus arises.

Calculating the tunneling splitting of the discrete spectrum of a particle on a circle is equivalent to calculating the widths of gaps, i.e., of distances between zones in the Bloch spectrum of the Schrödinger operator on the straight line with a periodic potential (see, e.g., [22] or [23]). The case $V(x) = \cos x$ corresponds to the quantum pendulum [24], and the corresponding Schrödinger equation is equivalent to

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 181, No. 2, pp. 337–348, November, 2014. Original article submitted July 17, 2014.

the Mathieu equation. An asymptotic form of the gap width in the case where the potential is analytic and the topology of Stokes lines has the same form as in the case of $V(x) = \cos x$ was obtained by Dykhne [25], and the complete rigorous proof and an analysis of several additional cases were given by Simonyan [26] (also see [27]). A general description of the spectral structure in this situation can be found in [17], and a deep interpretation of the Dykhne–Simonyan formula in geometric terms of the complexified phase space was given in [28].

There are several different asymptotic regimes in the problem of estimating the gap width (see the corresponding remarks in [29], [30]). The gap number is a large parameter in many works, and the energy (the gap centrum) then tends to infinity. An appropriate renormalization allows seeing that this regime is semiclassical [4] but includes an additional assumption that the potential is small compared with the total energy. It can be shown that under such an assumption, the Dykhne–Simonyan formula [25], [26] implies the well-known Harrell–Avron–Simon formula [29], [30] for the width of instability regions of the Mathieu equation. There are several deep results (see, e.g., [31]) about the relation between the smoothness of the potential and the rate of decrease in the gap width as the energy increases. It is known that the gap width decreases exponentially if the potential is analytic [32] and decreases according to a power law if the potential has a finite smoothness [33].

Here, we propose a unique approach for calculating the energy splitting (see Sec. 2) in the general case of tunneling between two symmetric orbits in the phase space; this approach generalizes the formulas for a symmetric double well obtained in [4], [5]. We use this approach to obtain an asymptotic formula for the energy splitting of the Schrödinger operator on a circle in the case of an arbitrary sufficiently smooth potential (Theorem 1). In this problem of dynamical tunneling, the obtained formula can be regarded as a momentum (integro-difference) analogue of the well-known (integro-differential) Lifshitz–Herring formula [4], [34], which is used in problems of coordinate tunneling. The obtained formula gives an alternative proof of the Dykhne–Simonyan formula in the case of an analytic potential (see Sec. 3) and permits obtaining the estimates proposed in [33] in the case of a finitely smooth potential.

2. Tunneling splitting of energies

We consider the one-dimensional stationary Schrödinger equation on a circle

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \tag{1}$$

$$\psi(x + 2\pi) = \psi(x), \tag{2}$$

where $\hbar$ is a small parameter of the semiclassical approximation and $V(x) \in C^2(\mathbb{R})$ is a periodic potential such that $V(x + 2\pi) = V(x)$. The corresponding Schrödinger operator has the form

$$\widehat{H} = \frac{\hat{p}^2}{2} + V(x).$$

The WKB method permits obtaining an asymptotic form of the fundamental system of solutions of Eq. (1) for energies E greater than the maximum of the potential $V(x)$ (see, e.g., [17], [27], [28]):

$$\begin{aligned} \varphi_1(x) &= \sqrt{\frac{\omega}{p(x)}}\exp\left(\frac{i}{\hbar}\int_{x_0}^{x} p(x)\,dx\right)[1 + O(\hbar)], \\ \varphi_2(x) &= \sqrt{\frac{\omega}{p(x)}}\exp\left(-\frac{i}{\hbar}\int_{x_0}^{x} p(x)\,dx\right)[1 + O(\hbar)], \end{aligned} \tag{3}$$

where $p(x) = \sqrt{2(E - V(x))}$ is the classical momentum, $p(x) > 0$, and ω is the classical oscillation frequency such that

$$\omega^{-1} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dx}{p(x)}.$$

Asymptotic estimates (3) are uniform in x and can be differentiated. The states $\varphi_{1,2}$ are normalized, the state φ_1 corresponds to the particle counterclockwise motion in a circle (in the positive direction of the x axis), and φ_2 corresponds to clockwise motion.

We use periodicity condition (2) to obtain an analogue of the Planck–Bohr–Sommerfeld discretization rule for a particle on a circle,

$$\frac{1}{2\pi} \int_0^{2\pi} \sqrt{2(E - V(x))}\, dx = n\hbar + O(\hbar^2), \tag{4}$$

where $n \sim 1/\hbar$ is an integer such that the corresponding energy E is significantly greater than the maximum of the potential. The over-barrier reflection from the potential (tunneling in the momentum representation) generally leads to a small splitting of the energies satisfying rule (4). For the energies significantly greater than the potential maximum, the spectrum of $\widehat{H}$ therefore consists of pairs of nearby points, and the distance between neighboring pairs is of the order of $\hbar$. The corresponding eigenfunctions are linear combinations of the functions φ_1 and φ_2.

Let E_1 and E_2 be a pair of eigenvalues of the operator $\widehat{H}$, and let ψ_1 and ψ_2 be the corresponding eigenfunctions:

$$\widehat{H}\psi_1 = E_1\psi_1, \qquad \widehat{H}\psi_2 = E_2\psi_2.$$

Let $\hat{\sigma}$ be a self-adjoint operator. We then take the scalar product of the first equality by $\hat{\sigma}\psi_2$ from the right and the scalar product of the second equality by $\hat{\sigma}\psi_1$ from the left, use the self-adjointness of the operators $\widehat{H}$ and $\hat{\sigma}$, and obtain

$$\langle \hat{\sigma}\widehat{H}\psi_1, \psi_2\rangle = E_1\langle \hat{\sigma}\psi_1, \psi_2\rangle, \qquad \langle \widehat{H}\hat{\sigma}\psi_1, \psi_2\rangle = E_2\langle \hat{\sigma}\psi_1, \psi_2\rangle.$$

If the operator $\hat{\sigma}$ is chosen such that $\langle \hat{\sigma}\psi_1, \psi_2\rangle \neq 0$, then

$$\Delta = E_2 - E_1 = \frac{\langle [\widehat{H}, \hat{\sigma}]\psi_1, \psi_2\rangle}{\langle \hat{\sigma}\psi_1, \psi_2\rangle}, \tag{5}$$

where $[\widehat{H}, \hat{\sigma}]$ is the commutator of the operators $\widehat{H}$ and $\hat{\sigma}$.

Formula (5) is a generalization of classical formulas for calculating the tunneling splitting in different problems of quantum mechanics. For example, formula (5) permits obtaining the Herring formula [34] for the multidimensional symmetric double-well potential if we take $\hat{\sigma} = \sigma(\hat{x})$, where the function $\sigma(x)$ is equal to unity on one side of the plane of symmetry and to zero on the other side. We can similarly obtain the classical formula for the splitting value in the one-dimensional symmetric double-well potential [4] and the formula for the widths of the energy bands of the periodic Schrödinger operator with an energy below the maximum of the potential (problem 3 in Chap. 6, 55, in [35]).

In the general case of tunneling between different domains of the phase space, the operator $\hat{\sigma}$ must separate the states localized in one domain of the phase space from the states localized in the other symmetric domain. We then can easily show that the denominator in (5) satisfies the approximate formula

$$\langle \hat{\sigma}\psi_1, \psi_2\rangle \approx \frac{\alpha}{2}, \quad |\alpha| = 1,$$

where the phase multiplier α is determined by the choice of the global phase factors of the states ψ_i. The expression in the numerator in (5) can be simplified using the known formulas for the commutator of pseudodifferential operators [36].

As an example, we consider the problem of splitting in the symmetric double-well potential $V(x) = V(-x)$ on the straight line [4]. If we take $\hat{\sigma} = \theta(\hat{x})$, where $\theta(x)$ is the Heaviside function, then

$$[\widehat{H}, \hat{\sigma}] = -\frac{\hbar^2}{2}\left[\frac{d^2}{dx^2}, \theta(x)\right] = -\frac{\hbar^2}{2}\left(\delta'(x) + 2\delta(x)\frac{d}{dx}\right),$$

where $\delta(x) = \theta'(x)$ is the Dirac delta function. Therefore,

$$\langle[\widehat{H}, \hat{\sigma}]\psi_1, \psi_2\rangle = -\frac{\hbar^2}{2}(\psi_1'\bar{\psi}_2 - \psi_1\bar{\psi}_2')\big|_{x=0}.$$

If $\psi_{1,2}(x)$ are real functions such that $\psi_1(x)$ is even and $\psi_2(x)$ is odd, then

$$\langle[\widehat{H}, \hat{\sigma}]\psi_1, \psi_2\rangle = \frac{\hbar^2}{2}\psi_1(0)\psi_2'(0), \qquad \langle\hat{\sigma}\psi_1, \psi_2\rangle \approx \frac{1}{2}.$$

We use expression (5) to obtain the formula for the splitting value

$$\Delta = \hbar^2\psi_1(0)\psi_2'(0).$$

If we substitute the WKB asymptotic form of the states $\psi_{1,2}$ in this formula, then we obtain the classical formula [4] for the splitting value in the symmetric double-well potential

$$\Delta = \frac{\omega\hbar}{\pi}\exp\left(-\frac{1}{\hbar}\int_{-a}^{a}|p|\,dx\right),$$

where $a > 0$ is the turning point and ω is the frequency of classical oscillations.

Following the proposed method, we take $\hat{\sigma} = \sigma(\hat{p})$, where

$$\sigma(p) = \begin{cases} 1, & p > 0, \\ 1/2, & p = 0, \\ 0, & p < 0, \end{cases}$$

in the considered problem for a particle on a circle. We substitute $\hat{\sigma}$ in formula (5), perform necessary simplifying transformations, and obtain the following theorem.

Theorem 1. *If $E_{1,2}$ form a pair of close eigenvalues of the operator $\widehat{H}$ corresponding to the rotor regime and the corresponding eigenfunctions $\psi_{1,2}$ are chosen real and are normalized such that*

$$\begin{aligned} \psi_1(x) &= \sqrt{\frac{2\omega}{p(x)}}\cos\left(\frac{1}{\hbar}\int_{x_0}^{x}p(x)\,dx\right) + O(\hbar), \\ \psi_2(x) &= \sqrt{\frac{2\omega}{p(x)}}\sin\left(\frac{1}{\hbar}\int_{x_0}^{x}p(x)\,dx\right) + O(\hbar), \end{aligned} \tag{6}$$

then the energy splitting satisfies the asymptotic formula

$$\Delta = \frac{1 + O(\hbar)}{(2\pi)^2}\int_0^{2\pi}dx_1\int_0^{2\pi}\big(V(x_2) - V(x_1)\big)\cot\left(\frac{x_2 - x_1}{2}\right)\psi_1(x_1)\psi_2(x_2)\,dx_2. \tag{7}$$

Proof. Let π_k be the operator of projection on the state with momentum $p = k\hbar$,

$$\pi_k \psi(x) = e^{ikx} \frac{1}{2\pi} \int_0^{2\pi} \psi(x) e^{-ikx}\, dx.$$

Therefore,

$$\hat{\sigma} = \sum_{k=1}^{+\infty} \pi_k + \frac{\pi_0}{2}.$$

The commutation relations

$$[e^{ix}, \pi_k] = e^{ix}(\pi_k - \pi_{k-1}),$$

$$[e^{ix}, \hat{\sigma}] = -e^{ix}\pi_0 + \frac{1}{2} e^{ix}(\pi_0 - \pi_{-1}) = -\frac{1}{2} e^{ix}(\pi_0 + \pi_{-1})$$

obviously hold. Because the potential $V(x)$ is a periodic function, we can assume that $V(x) = v(e^{ix})$. We use the well-known formula for the commutator to obtain

$$[\widehat{H}, \hat{\sigma}] = [V(x), \hat{\sigma}] = [v(e^{ix}), \hat{\sigma}] = -\frac{1}{2} \overset{3}{e^{ix}} \cdot (\pi_0 \overset{2}{+} \pi_{-1}) \cdot \delta v(\overset{1}{e^{ix}}, \overset{4}{e^{ix}}),$$

where $\delta v(z_1, z_2)$ is the difference derivative of the function $v(z)$,

$$\delta v(z_1, z_2) = \frac{v(z_1) - v(z_2)}{z_1 - z_2},$$

and the digits over symbols denote the order of action of the operators.

Because

$$\langle \pi_k \psi, \varphi \rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi} dx_1 \int_0^{2\pi} e^{-ikx_1} e^{ikx_2} \psi(x_1) \overline{\varphi(x_2)}\, dx_2,$$

we obtain

$$\langle [\widehat{H}, \hat{\sigma}] \psi_1, \psi_2 \rangle = -\frac{1}{2(2\pi)^2} \int_0^{2\pi} dx_1 \int_0^{2\pi} e^{ix_2}(1 + e^{ix_1} e^{-ix_2}) \delta v(e^{ix_1}, e^{ix_2}) \psi_1(x_1) \psi_2(x_2)\, dx_2.$$

We use the trigonometric formula

$$\cot \frac{x_2 - x_1}{2} = i \frac{e^{ix_2} + e^{ix_1}}{e^{ix_2} - e^{ix_1}}$$

and the relation

$$\delta v(e^{ix_1}, e^{ix_2}) = \frac{V(x_1) - V(x_2)}{e^{ix_1} - e^{ix_2}}$$

to obtain

$$\langle [\widehat{H}, \hat{\sigma}] \psi_1, \psi_2 \rangle = \frac{i}{2} \frac{1}{(2\pi)^2} \int_0^{2\pi} dx_1 \int_0^{2\pi} \big(V(x_2) - V(x_1)\big) \cot\left(\frac{x_2 - x_1}{2}\right) \psi_1(x_1) \psi_2(x_2)\, dx_2.$$

With formula (5) taken into account, to conclude the proof, it remains to show that

$$\langle \hat{\sigma} \psi_1, \psi_2 \rangle = \frac{i}{2} + O(\hbar).$$

This estimate readily follows from (6). The theorem is proved.

It is obvious that the WKB asymptotic expansions of form (6) are insufficiently exact for calculating the integral in formula (7), but they can be used to characterize the integrand. The integral in (7) is an integral of a rapidly oscillating function without stationary points. Therefore, if the potential $V(x)$ is smooth, then the splitting tends to zero faster than any power of $\hbar$.

If the potential $V(x)$ is an analytic function, then the asymptotic expansion of integral (7) can be obtained by the saddle-point method [37]. The stationary points are determined by the relation $p(x_1) = p(x_2) = 0$, i.e., coincide with the complex turning points of Eq. (1). The main contribution to the splitting value therefore comes from the complex turning points, which agrees well with the general theory of over-barrier reflection [4], [38]–[40].

We can pass from the x-representation to the p-representation in formula (7). We then obtain

$$\Delta = 2[1+O(\hbar)]\,\mathrm{Im}\sum_{n=1}^{\infty}\overline{\widetilde{V}(n)}\sum_{k=1}^{n}\big(\tilde{\psi}_1(n-k+1)\tilde{\psi}_2(k-1)+\tilde{\psi}_1(n-k)\tilde{\psi}_2(k)\big), \tag{8}$$

where $\tilde{f}(k)$ is the kth coefficient in the expansion of $f(x)$ in the Fourier series

$$\tilde{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\,dx.$$

Formula (8) implies that if the potential $V(x)$ is a trigonometric polynomial or if the coefficients $\tilde{V}(n)$ decrease sufficiently fast, then the main contribution to the splitting value comes from the terms with finite n and correspondingly with the momenta $p = n\hbar$ close to zero. We note that the momentum $p = 0$ is the center of the classically forbidden region in the p-representation.

3. Quantum pendulum

As an example of applying Theorem 1, we consider the quantum pendulum. The Hamiltonian of the quantum pendulum has the form

$$\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + \gamma\cos x.$$

The quantum pendulum is one of the classic models in quantum mechanics [24], [41], [42], and stationary Schrödinger equation (1) is equivalent to the Mathieu equation. In this section, we show how to obtain the semiclassical Dykhne–Simonyan formula (see formula (19) below) from the general formula for the splitting (see Theorem 1).

We consider a pair of close eigenvalues $E_{1,2}$ and assume that the corresponding eigenfunctions $\psi_{1,2}$ are chosen as in Theorem 1, i.e., the functions $\psi_{1,2}$ are real, $\psi_1(x)$ is an even function, and $\psi_2(x)$ is an odd function. We then substitute the potential $V(x) = \gamma\cos x$ in formula (8) and obtain

$$\Delta = \gamma\,\mathrm{Im}\{\tilde{\psi}_1(1)\tilde{\psi}_2(0) + \tilde{\psi}_1(0)\tilde{\psi}_2(1)\}[1+O(\hbar)], \tag{9}$$

where

$$\tilde{\psi}_j(n) = \frac{1}{2\pi}\int_0^{2\pi}\psi_j(x)e^{-inx}\,dx, \quad j = 1,2. \tag{10}$$

A splitting formula similar to formula (9) was used in [30] under the additional assumption that the potential is small compared with the total energy, i.e., $\gamma \sim \hbar$.

Stationary Schrödinger equation (1) in the p-representation becomes

$$\frac{p^2}{2}\tilde{\psi}(n) + \frac{\gamma}{2}\big(\tilde{\psi}(n+1)+\tilde{\psi}(n-1)\big) = E\tilde{\psi}(n), \tag{11}$$

where $p = n\hbar$. Equation (11) is a second-order recurrence relation with slowly varying coefficients. Formal asymptotic solutions of this equation of the form

$$\frac{1}{\sqrt{V'(x(p))}} \exp\left(\frac{i}{\hbar}\int_{p_0}^{p} x(p)\,dp\right).$$

can be found using the discrete WKB method [43], [44] or operator methods [36]. With the explicit form of the potential taken into account, we obtain

$$u_\pm(n) = \left(\left(\frac{E}{\gamma} - \frac{(n\hbar)^2}{2\gamma}\right)^2 - 1\right)^{-1/4} \exp\left(\pm\frac{1}{\hbar}\int_0^{n\hbar} \operatorname{arccosh}\left(\frac{E}{\gamma} - \frac{p^2}{2\gamma}\right)dp\right)[1 + O(\hbar)], \tag{12}$$

where $u_+(n) = u_-(-n)$ and $\operatorname{arccosh} z = \log(z + \sqrt{z^2 - 1}) > 0$ for $z > 1$. Such asymptotic expressions were rigorously justified in [44], [45]. Asymptotic formulas (12) hold in the region of classically forbidden values of the momentum between two turning points, i.e., for n such that $|n\hbar| \le p_0 - \varepsilon$, where $\varepsilon > 0$ is a fixed number, $\pm p_0$ are turning points, and $p_0 = \sqrt{2(E - \gamma)}$.

The linearly independent solutions $u_\pm(n)$ form a fundamental system of solutions of Eq. (11) for a given energy $E > \gamma$. We expand the state $\tilde\psi_{1,2}$ with respect to this system.

Proposition 1. *The states $\psi_{1,2}$ in the p-representation become*

$$\tilde\psi_1(n) = C_1\big(u_+(n) + u_-(n)\big), \qquad \tilde\psi_2(n) = iC_2\big(u_+(n) - u_-(n)\big), \tag{13}$$

where C_i are real normalization constants,

$$\begin{aligned} C_1 &= \sqrt{\frac{\omega\hbar}{4\pi\gamma}} \exp\left(-\frac{1}{\hbar}\int_0^{p_0} \operatorname{arccosh}\left(\frac{E}{\gamma} - \frac{p^2}{2\gamma}\right)dp\right)[1 + O(\hbar)], \\ C_2 &= -\sqrt{\frac{\omega\hbar}{4\pi\gamma}} \exp\left(-\frac{1}{\hbar}\int_0^{p_0} \operatorname{arccosh}\left(\frac{E}{\gamma} - \frac{p^2}{2\gamma}\right)dp\right)[1 + O(\hbar)]. \end{aligned} \tag{14}$$

Remark 1. Because the fundamental system of solutions $u_\pm(n)$ depends on the energy E, the energy E is equal to E_j in decomposition (13) for $\tilde\psi_j(n)$. We can set $E = (E_1 + E_2)/2$ in asymptotic formulas (12) and (14).

Proof. Because the functions $\psi_{1,2}(x)$ are chosen real such that $\psi_1(x)$ is even and $\psi_2(x)$ is odd, their Fourier coefficients satisfy the relations

$$\tilde\psi_1(n) = \tilde\psi_1(-n) = \overline{\tilde\psi_1(n)}, \qquad \tilde\psi_2(n) = -\tilde\psi_2(-n) = -\overline{\tilde\psi_2(n)}.$$

Decomposition (13) therefore holds.

The normalization constants C_i can be determined by several methods. For example, we can construct global asymptotic expressions in the p-representation using the rules for passing through a simple turning point proposed in [43]. On the other hand, the normalization constants can be obtained by passing from the x-representation to the p-representation. The states $\psi_{1,2}$ in the x-representation are normalized and have the form

$$\psi_1(x) = \frac{1}{\sqrt2}\big(\varphi_1(x) + \varphi_2(x)\big), \qquad \psi_2(x) = \frac{1}{i\sqrt2}\big(\varphi_1(x) - \varphi_2(x)\big). \tag{15}$$

We pass to the p-representation and obtain

$$\tilde{\psi}_1(n) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\sqrt{2}}\big(\varphi_1(x)+\varphi_2(x)\big)e^{-inx}\,dx,$$
$$\tilde{\psi}_2(n) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{i\sqrt{2}}\big(\varphi_1(x)-\varphi_2(x)\big)e^{-inx}\,dx. \tag{16}$$

We use formulas (3) for $\varphi_{1,2}(x)$ to obtain the asymptotic behavior of integrals (16). The integrand in (16) oscillates rapidly, and the stationary points of the phase become complex if the momentum $p = n\hbar$ corresponds to the classically forbidden region. We use the saddle-point method [37] to obtain

$$\tilde{\psi}_1(n) = \sqrt{\frac{\omega\hbar}{4\pi\gamma}}\left(\left(\frac{E}{\gamma}-\frac{(n\hbar)^2}{2\gamma}\right)^2 - 1\right)^{-1/4} \times$$
$$\times \exp\left(-\frac{1}{\hbar}\int_{n\hbar}^{p_0} \operatorname{arccosh}\left(\frac{E}{\gamma}-\frac{p^2}{2\gamma}\right)dp\right)[1+O(\hbar)]. \tag{17}$$

Because the complex integration contour arising in the saddle-point method must lie in the domain where asymptotic formulas (3) hold, asymptotic formula (17) holds for n such that

$$\varepsilon < \hbar n < \sqrt{2(E-\gamma)} - \varepsilon.$$

Matching asymptotic formulas (13) and (17), we obtain the expression for C_1. The formula for C_2 can be obtained similarly. The proposition is proved.

We substitute the asymptotic expressions for the states $\tilde{\psi}_{1,2}$ in (9) to obtain the final formula for the value of tunneling splitting of the energies of the quantum pendulum

$$\Delta = -\frac{\omega\hbar}{\pi}\exp\left(-\frac{1}{\hbar}\int_{-p_0}^{p_0} \operatorname{arccosh}\left(\frac{E}{\gamma}-\frac{p^2}{2\gamma}\right)dp\right)[1+O(\hbar)]. \tag{18}$$

The minus sign in this formula shows that the even state $\psi_1(x)$ has a greater energy than the odd state $\psi_2(x)$, which agrees well with the general theory of the Hill equation with an even potential.

The integral in the exponent in (18) can be written in a different form. For example, if we pass from $x\,dp$ to $p\,dx$, we obtain

$$\Delta = -\frac{\omega\hbar}{\pi}\exp\left(-\frac{1}{\hbar}\int_{-\operatorname{arccosh}(E/\gamma)}^{\operatorname{arccosh}(E/\gamma)} \sqrt{2(E-\gamma\cosh x)}\,dx\right)[1+O(\hbar)], \tag{19}$$

where $\sqrt{2(E-\gamma\cosh x)}$ is the tunneling momentum, i.e., the classical momentum

$$p = \sqrt{2\big(E - V(x)\big)}$$

for purely imaginary values of the coordinate x. The integral in (18) is thus taken over an instanton, i.e., a path is in the complexified Lagrangian manifold $p^2/2 + V(x) = E$ and connects two classical trajectories of motion. The points $\pm p_0$ are the beginning and the end of the instanton and are on classical trajectories of motion with the respective positive and negative momenta (see [28]).

Acknowledgments. The author thanks M. V. Karasev for setting the problem, for the fruitful discussions, and for the help in preparing the paper.

This work is supported by the Russian Foundation for Basic Research (Grant No. 12-01-00627_a).

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