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Contribution to the Symplectic Structure in the Quantization Rule Due to Noncommutativity of Adiabatic Parameters

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Abstract. A geometric construction of the àla Planck action integral (quantization rule) determining adiabatic terms for fast-slow systems is considered. We demonstrate that in the first (after zero) adiabatic approximation order, this geometric rule is represented by a deformed fast symplectic 2-form. The deformation is controlled by the noncommutativity of the slow adiabatic parameters. In the case of one fast degree of freedom, the deformed symplectic form incorporates the contraction of the slow Poisson tensor with the adiabatic curvature.

The same deformed fast symplectic structure is used to represent the improved adiabatic invariant in a geometric form.

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1. INTRODUCTION

Contemporary quantum mechanics effectively exploits the operator theory and algebra, but each time when it is possible, it is very interesting to establish its links with approaches based on geometry. As for the semiclassical approximation theory, the dual way of thinking both from geometric and algebraic standpoints is natural and necessary to construct intuitional bridges as well as simple computational algorithms.

In this note, we deal with a model quantum system that can be described by a Hamiltonian of the following type:

$$\hat{H} = H(\hat{r}, \hat{k}; \hat{x}) \tag{1.1}$$

with canonical commutation relations between basic generators:

$$[\hat{r}, \hat{k}] = i\hbar \tag{1.2}$$

and between the "parameters":

$$[\hat{x}^j, \hat{x}^k] = -i\varepsilon\hbar J^{jk}. \tag{1.3}$$

Here J^{ik} are the components of the Darboux block tensor $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ which has an even dimension. All other mutual commutators are assumed to be zero. In (1.1) and below, we use the Weyl symmetrization rule for defining functions in noncommuting elements.

Let us suppose that both $\hbar$ and ε in (1.2) and (1.3) are small: $\hbar \to 0$, $\varepsilon \to 0$, and thus, we have a combined semiclassical plus adiabatic regime. The mutual relation between these two small parameters is assumed to be as follows:

$$\hbar = O(\varepsilon).$$

The generators $\hat{r}$, $\hat{k}$ are referred to as *fast* ones, and $\hat{x}$, as *slow* ones. We shall assume that for frozen slow parameters x, the energy levels of the Hamiltonian $H(r, k; x)$ in the fast r, k-space are compact (closed connected curves).

There is ordinary way to compute the Born–Oppenheimer adiabatic "terms" for the Hamiltonian (1.1) just by applying the semiclassical quantization rule in the fast phase space:

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$$\frac{1}{2\pi}\int_{H=E} k\,dr = \hbar(n+\mu/4), \qquad n \in \mathbb{Z}.$$

Here n is an integer "quantum number" and μ is the Maslov index. This rule can be also rewritten as

$$\frac{1}{2\pi}\int_{\Sigma_E} dk\wedge dr = s_n, \qquad s_n \overset{\text{def}}{=} \hbar(n+\mu/4), \tag{1.4}$$

where $\Sigma_E = \Sigma_E(x)$ is a membrane with boundary $\partial\Sigma_E \subset \{H = E\}$ in the fast fiber $\{x = \text{const}\}$.

From equation (1.4), we obtain the Hamiltonians $E = E_n(x)$, the so-called adiabatic terms. Then it is possible to approximately replace the original Hamiltonian $\hat{H}$ (1.1) by the *effective slow Hamiltonians*

$$\hat{E}_n + \varepsilon\hat{M}_n + O(\hbar^2+\varepsilon^2). \tag{1.5}$$

The remainder $O(\hbar^2)$ here appears, because the geometric rule (1.4), in general, does not generate the exact quantum spectrum.

The correcting summand εM_n and the higher remainders $O(\varepsilon^2)$ in (1.5) appear due to slow x-dependence of the nth eigensubspace of the Hamiltonian $H(\hat{r},\hat{k};x)$ in fast directions. Since the slow semiclassical parameter in (1.5) is $\varepsilon\hbar$, the correction εM_n generates oscillations of the type $\exp\{\frac{i}{\hbar}phase\}$ in wave functions. And such a type of oscillations has to be related to some additional symplectic geometry of the phase space. What is this geometry?

Analytical formulas for the correcting ε-term in the effective slow Hamiltonian were derived first, of course, in classical mechanics (see, for instance, in [1, 2]) but without any analysis of its correlation with symplectic geometry. In the quantum framework, the ε-term in (1.5) is related to geometry as follows. The function M_n is represented as a sum of two summands $M_n = M'_n + M''_n$ the first of which can be written as

$$M'_n = DE_n\cdot J\alpha_n, \tag{1.6}$$

where the 1-form α_n is Berry's geometric vector-potential [3, 4] in the bundle of the nth eigensubspaces in the fast directions, and $D = \partial/\partial x$ (see, for instance, in [5]).

This is the strength 2-form β_n corresponding to the vector-potential α_n that determines an additional contribution to the slow phase space symplectic structure $\omega = \frac{1}{2}J^{-1}\,dx\wedge dx$. The total slow symplectic structure becomes

$$\omega + \varepsilon\beta_n + O(\varepsilon^2) \tag{1.7}$$

(see, for instance, in [5, 6]).

The formula for the second component M''_n derived in [5] (see also below in Section 3) had no such a clear symplectic interpretation. This problem was discussed for instance in [7, 8], and in [6], it was observed that M''_n can be represented in a geometric way by using a certain connection in a bundle over the slow space involving fast eigensubspaces with all quantum numbers $n' \neq n$.

In the present note, we prove that, for a Hamiltonian of type (1.1), the contribution of M''_n is equivalent to a deformation of the fast symplectic structure in the quantization rule (1.4). The deformed quantization rule looks as follows:

$$\frac{1}{2\pi}\int_{\Sigma_E(x)} (a+\varepsilon b) = s_n, \tag{1.8}$$

where $a = dk\wedge dr$ and

$$b = -\frac{1}{2}\mathrm{Tr}(\beta J)\,dk\wedge dr. \tag{1.9}$$

Here β is the curvature of the Hamiltonian connection θ corresponding to the $U(1)$-action by angular rotations in fast fibers [9] (the restriction of β onto the nth fast eigensubspace coincides with the strength 2-form β_n which appears in (1.7)).

If one determines the deformed "terms" $E = \mathcal{E}_n(x)$ as solutions of (1.8), then the effective slow Hamiltonian (1.5) can be represented as

$$\mathcal{E}_n(\hat{\mathcal{X}}_n) + O(\varepsilon^2+\hbar^2). \tag{1.10}$$

Here $\mathcal{X} = x + \varepsilon J\theta + O(\varepsilon^2)$ and $\hat{\mathcal{X}}_n = \hat{\mathcal{X}}\big|_{n\text{th eigensubspace}}$ are new slow generators with deformed commutation relations corresponding to the slow symplectic structure (1.7):

$$[\hat{\mathcal{X}}_n \overset{\otimes}{,} \hat{\mathcal{X}}_n] = -i\varepsilon\hbar\Psi_n(\hat{\mathcal{X}}_n) + O(\varepsilon^3\hbar), \tag{1.11}$$

where, on the right-hand side, one uses the inverse tensor of the 2-form (1.7):

$$\Psi_n = (\omega + \varepsilon\beta_n + O(\varepsilon^2))^{-1} = J - \varepsilon J\beta_n J + O(\varepsilon^2).$$

Thus the *slow effective Hamiltonians* (1.10) *turn out to be completely symplectic*: they are determined by two deformed symplectic structures (1.7) and (1.8).

Note that the presence of the contraction of the slow Poisson tensor J with the curvature β in (1.9) implies that the deforming form b in (1.8) can appear only due to the noncommutativity of slow parameters for which the curvature does not vanish.

In the "geometric phase" and "Hanny angle" effects accompanying the long-time excursion of slow parameters [3, 4, 9–13], the noncommutativity of the slow space plays no rule (only the first component M_n' (1.6) contributes to these effects but not M_n''; the slow noncommutativity does not affect the ε-shift of the fast frequency up to $O(\varepsilon^2)$).

As we see from (1.8) and (1.9), the slow noncommutativity, i.e., the slow Poisson tensor J, controls the deformation of the fast geometry as a whole by forcing the introduction of new deformed action-angle coordinates in the phase space.

The deformed action function

$$\mathfrak{S}_\varepsilon \overset{\text{def}}{=} \frac{1}{2\pi}\int_{\Sigma_H(x)} (a + \varepsilon b) \tag{1.12}$$

which we use in the quantization condition (1.8) has a very simple physical meaning in classical mechanics. Let us recall the notion of *improved adiabatic invariant* [2, 14]: it is an integral of motion for the given adiabatic system with accuracy $O(\varepsilon^\infty)$ which has 2π-periodic flow and whose ε-power expansion starts from the classical adiabatic invariant. The statement is:

$$\textit{Improved adiabatic invariant} = \mathfrak{S}_\varepsilon + \varepsilon\{\textit{varying with zero average}\}\big|_{\frac{1}{\varepsilon}\text{-long time}} + O(\varepsilon^2). \tag{1.13}$$

Thus the function $\mathfrak{S}_\varepsilon$ (1.12) is a *semi-invariant*, i.e., it is not a constant of motion up to $O(\varepsilon^2)$ but an oscillating quantity with constant long-time average.

Since $\mathfrak{S}_\varepsilon$ does not depend on the fast angle (does not feel $U(1)$-gyrations in the fast space), it can be regarded as a $U(1)$-invariant. But from the dynamical point of view, this geometric $U(1)$-invariant is a semi-invariant only. Therefore, the geometric formulation of the adiabatic quantization rule in (1.8) is only "semi"-consistent with the usual Ehrenfest concept of adiabatic invariants in quantum theory [15, 16].

In other words, although fast geometric $U(1)$-invariants are not dynamic invariants in higher order of ε, nevertheless, they could be useful for developing the adiabatic version of the geometric and deformation quantization theories. Such a quantum "fast geometry" is preserved in average under a long-time evolution.

The consistency of the quantization rule (1.8) with the Ehrenfest adiabatic principle is restored if one notes that the varying second summand in (1.13) can be eliminated simply by replacing the old slow coordinates x by the new ones $\mathcal{X}$ in (1.12):

$$\textit{Improved adiabatic invariant} = \frac{1}{2\pi}\int_{\Sigma_H(\mathcal{X})} (a + \varepsilon b) + O(\varepsilon^2). \tag{1.14}$$

That is why it does not matter to relate the fast quantum number n to the geometric $U(1)$-invariant $\mathfrak{S}_\varepsilon$, as in (1.8) or to the improved adiabatic invariant.

Below in Section 2, we prove the statements (1.13), (1.14), and in Section 3, prove formulas (1.8), (1.10). At the end of the paper, in the Appendix, we describe (in the classical framework) a modified version of our scheme [17, 18] for computing all higher ε-corrections in the improved adiabatic invariant and adiabatic "terms."

2. ADIABATIC SEMI-INVARIANT AND IMPROVED INVARIANT

In this section, we prove the statements (1.13), (1.14).

Dealing with classical mechanics, one must replace the commutation relations (1.2), (1.3) by the Poisson brackets between fast coordinates r, k as follows:

$$\{r, k\} = -1, \tag{2.1}$$

as well as between the slow coordinates x as follows:

$$\{x \overset{\otimes}{,} x\} = \varepsilon J, \tag{2.2}$$

and then introduce the direct sum of brackets on the total fast-slow phase space:

$$\{\cdot\,,\cdot\}_\varepsilon \overset{\text{def}}{=} \{\cdot\,,\cdot\}_0 + \varepsilon\{\cdot\,,\cdot\}. \tag{2.3}$$

For the given Hamiltonian $H = H(r, k; x)$, one defines the classical action function

$$S_0 \overset{\text{def}}{=} \frac{1}{2\pi}\int_{\Sigma_H} a, \qquad a = dk\wedge dr, \tag{2.4}$$

and expresses the Hamiltonian via the action

$$H(r, k; x) = f_0\big(S_0(r, k; x), x\big) \tag{2.5}$$

by using some function $f_0 = f_0(s, x)$.

The dependence of the action S_0 on the slow parameters x generates the notion of Hamiltonian connection [9]. The connection 1-form $\theta = \sum \theta_j\, dx^j$ is determined by

$$DS_0 = \{S_0, \theta\}_0. \tag{2.6}$$

Here we denote by D the derivative with respect to x, i.e., $D_j \equiv \partial/\partial x^j$. The solution of (2.6) is given by

$$\theta = (DS_0)^{\#} \quad \text{or} \quad \theta(y; x) = \frac{1}{2\pi}\int_0^{2\pi} DS_0\big(Y^\tau(y; x); x\big)(\tau - \pi)\, d\tau, \tag{2.7}$$

where $Y^\tau(\cdot; x)$ is the flow of the action $S_0(\cdot; x)$ on the fast fiber.

The connection θ has the curvature 2-form

$$\beta = \frac{1}{2}\sum \beta_{jk}\, dx^j\wedge dx^k \quad \text{with} \quad \beta_{jk} \overset{\text{def}}{=} D_j\theta_k - D_k\theta_j + \{\theta_j, \theta_k\}_0. \tag{2.8}$$

It follows from (2.6) that this curvature is in involution with the action S_0, and thus, it can be represented as a function in the action:

$$\beta_{jk} = \beta_{jk}(S_0; x). \tag{2.9}$$

The Bianchi identities for the curvature implies that the 2-form $\frac{1}{2}\sum \beta_{jk}(s; x)\, dx^j\wedge dx^k$ is closed for any fixed s, and thus, there exists a primitive 1-form $\alpha = \sum \alpha_j(s, x)\, dx^j$ such that

$$D_j\alpha_k - D_k\alpha_j = \beta_{jk}. \tag{2.10}$$

The 1-form α is referred to as the Berry's vector-potential. In contrast to θ, the potential α does not depend on the fast angle and is preferable for computations; see the discussion concerning this point in [5]. For instance, it is possible calculate the holonomy of the θ-connection simply by integrating α along a closed loop in the slow space, keeping the action level constant (or keeping one and the same nth eigenvalue as in (1.6)). The integral of α along a path is referred to as the geometric phase.

Now let us see how these objects generate the improved (or complete) adiabatic invariant S_ε, which is the ε-power expansion

$$S_\varepsilon = S_0 + \varepsilon S' + \varepsilon^2 S'' + \cdots \tag{2.11}$$

such that

$$\{H, S_\varepsilon\}_\varepsilon = 0 \qquad \mathrm{mod}\ O(\varepsilon^\infty). \tag{2.12}$$

There are known algorithms for computing the corrections $S', S'', \dots$ (see in [2, 19, 20]). Here we use the formula for S_0 in a special form (required below) derived by a general method [17]. This method is briefly described in Appendix.

The formula for the first correction S' in (2.11) follows from formula (A.9) in Appendix:

$$S' = \{S_0, \mathcal{B}_0\}_0 - \frac{1}{2}\mathcal{A}_{0j} J^{jk} \{S_0, \mathcal{A}_{0k}\}_0, \tag{2.13}$$

where $\mathcal{B}_0$ is taken from (A.10):

$$\mathcal{B}_0 = \frac{1}{\omega_0}\mathcal{A}^{\#}_{0j} J^{jk} D_k f_0 + \frac{1}{2\omega_0}\big(\mathcal{A}_{0j} J^{jk}\{H, \mathcal{A}_{0k}\}_0\big)^{\#}. \tag{2.14}$$

Here and below, the summation by repeated indices is assumed and the operation # is defined as in (2.7) or (A.7) at $\varepsilon = 0$.

In (2.13), (2.14), we use the zero curvature connection $\mathcal{A}_0$ defined as follows:

$$\mathcal{A}_0 = \theta - \alpha, \tag{2.15}$$

where $\theta = (DS_0)^{\#}$ and α are determined by (2.7), (2.10).

From (2.14) we derive

$$\begin{aligned}\{S_0, \mathcal{B}_0\}_0 &= \frac{1}{\omega_0}(\mathcal{A}_{0j} - \langle \mathcal{A}_{0j}\rangle) J^{jk} D_k f_0 + \frac{1}{2\omega_0}(\mathcal{A}_{0j} J^{jk}\{H, \mathcal{A}_{0k}\}_0 - \langle \mathcal{A}_{0j} J^{jk}\{H, \mathcal{A}_{0j}\}\rangle) \\ &= \frac{1}{\omega_0}\theta_j J^{jk} D_k f_0 + \frac{1}{2}\big((\theta_j - \alpha_j) J^{jk}\{S_0, \theta_k\}_0 - \langle \theta_j J^{jk}\{S_0, \theta_k\}_0\rangle\big).\end{aligned}$$

Here we used the properties of the operations # and $\langle \dots \rangle$ similar to (A.8) and the fact that $\langle \theta \rangle = 0$.

Then formula (2.13) becomes the following:

$$S' = \frac{1}{\omega_0}\theta_j J^{jk} D_k f_0 - \frac{1}{2}\langle \theta_j J^{jk}\{S_0, \theta_k\}_0\rangle. \tag{2.16}$$

Here and in the above formulas we write $\omega_0 = \partial f_0/\partial s$, $D_k f_0 = \partial f_0/\partial x^k$; these derivatives of the function f_0 are taken at the value $s = S_0$. It is easy to prove that (2.16) is equivalent to the formula for S' obtained in [20] (see Lemma 5.1 therein).

If we introduce the action-angle coordinates s, τ such that $a = dk \wedge dr = ds \wedge d\tau$, then the averaging operation $\langle \dots \rangle$ in (2.16) can be represented by the integration over the angle:

$$\langle \dots \rangle = \frac{1}{2\pi}\int_0^{2\pi} \dots \, d\tau.$$

At the same time, the bracket operation with the action function S_0 can be written explicitly as $\{S_0, \dots\}_0 = \frac{\partial}{\partial \tau}(\dots)$. Thus the average at the second summand in (2.16) reads

$$\langle \theta_j J^{jk}\{S_0, \theta_k\}\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(\theta_j J^{jk}\frac{\partial \theta_k}{\partial \tau}\right) d\tau = \frac{1}{2\pi}\int_{\partial \Sigma_H} \theta_j J^{jk} \partial\theta_k = \frac{1}{2\pi}\int_{\Sigma_H} \partial\theta_j \wedge J^{jk}\partial\theta_k.$$

Here ∂ denotes the differential with respect to the fast coordinates.

Hence, we obtain the following geometric representation of the first correction S' in the improved adiabatic invariant (2.11)

$$S' = \frac{1}{\omega_0}\theta J D f_0 + \frac{1}{2\pi}\int_{\Sigma_H} B, \qquad B \stackrel{\mathrm{def}}{=} -\frac{1}{2}\partial\theta_j \wedge J^{jk}\partial\theta_k. \tag{2.17}$$

Now recall that the fast space has dimension 2, then we can compute the form B as follows:

$$B = -\frac{1}{2}J^{jk}\{\theta_j, \theta_k\}_0 \cdot a, \qquad a = dk \wedge dr. \tag{2.18}$$

The brackets of components of θ are related to the curvature tensor:

$$\{\theta_j, \theta_k\}_0 = \beta_{kj} - (\tilde{\nabla}_k\theta_j - \tilde{\nabla}_j\theta_k). \tag{2.19}$$

Here $\tilde{\nabla} = D - \{\theta, \dots\}_0$ is the covariant derivative preserving S_0 (see Theorem A.1, note that $\tilde{\nabla}$ differs from ∇_0 by the bracket operation $\{\alpha, \dots\}_0$ which annihilates S_0).

Thus, any average $\langle\tilde{\nabla}G\rangle = 0$ vanishes, and therefore, all summands $\tilde{\beta}_{kj} = \tilde{\nabla}_k\theta_j - \tilde{\nabla}_j\theta_k$ in (2.19) have zero average $\langle\tilde{\beta}_{kj}\rangle = 0$. Hence, (2.18) reads

$$B = -\frac{1}{2}\mathrm{Tr}\,(J\beta)\cdot a + \tilde{B}, \tag{2.20}$$

where

$$\tilde{B} = -\frac{1}{2}\mathrm{Tr}\,(J\tilde{\beta})\cdot a \quad \text{and} \quad \int_{\Sigma_H}\tilde{B} = 0. \tag{2.21}$$

So we obtain the following formula for S' instead of (2.17):

$$S' = \theta J v + \frac{1}{2\pi}\int_{\Sigma_H} b, \qquad v \stackrel{\mathrm{def}}{=} \frac{Df_0}{\partial f_0}\Big|_{s=S_0}, \tag{2.22}$$

where the 2-form b is defined in (1.9).

Theorem 2.1. (i) *The first ε-correction S' in the improved adiabatic invariant* (2.11) *is given by the formula* (2.22), *where the function $f_0 = f_0(s;x)$ is taken from* (2.5), *$Df_0 = \partial f_0/\partial x$, $\partial f_0 = \partial f_0/\partial s$, and Σ_H are the same membranes as in the definition* (2.4) *of the classical adiabatic invariant $S_0 = \frac{1}{2\pi}\int_{\Sigma_H} a$.*

(ii) *Formula* (1.13) *for the improved adiabatic invariant reads*

$$S_\varepsilon = \mathfrak{S}_\varepsilon + \varepsilon\theta J v + O(\varepsilon^2). \tag{2.23}$$

The average of the summand $\theta J v$ under a $1/\varepsilon$-long time evolution is zero up to $O(\varepsilon)$.

(iii) *The summand $\theta J v$ can be removed from the right-hand side of* (2.23) *by changing the argument of the semi-invariant $\mathfrak{S}_\varepsilon$* (1.12) *from x to $\mathcal{X} = x + \varepsilon J\theta + O(\varepsilon^2)$ which generates the integral representation* (1.14) *for the improved adiabatic invariant, namely*:

$$S_\varepsilon = \frac{1}{2\pi}\int_{\Sigma_H(\mathcal{X})}(a + \varepsilon b) \;+\; O(\varepsilon^2). \tag{2.24}$$

Remark 2.1. The correcting 2-form b in integral formulas (1.8), (1.12), (1.14), (2.24) looks so simple as in (1.9) only in the case of two-dimensional fast phase space. In a more general case, formulas (2.18)–(2.21) do not work; one needs to use everywhere the general 2-form B as in (2.17).

3. QUANTIZATION RULE FOR ADIABATIC TERMS

Let us compute the first correction f' in the expansion $f_\varepsilon = f_0 + \varepsilon f' + \varepsilon^2 f'' + \cdots$ of the adiabatic term in the representation (A.3); see in Appendix.

This correction is given just by the second equation (A.9) at $\varepsilon = 0$. Namely,

$$f' = \langle \mathcal{A}_0 \rangle JDf_0 + \frac{1}{2}\langle \mathcal{A}_0 J\{H, \mathcal{A}_0\}_0\rangle = -\alpha JDf_0 + \frac{\omega_0}{2}\langle \theta J\{S_0, \theta\}_0\rangle. \tag{3.1}$$

Here we use the decomposition (2.15) and the properties of α and θ discussed in Section 2.

In the second equation in (3.1), the last summand is similar to the one appearing in (2.16). Applying the results of the previous calculations, we obtain the following statement.

Theorem 3.1. *The first ε-correction in the expansion of the adiabatic term* (4.11) *is given by*

$$f' = Df_0 \cdot J\alpha - \frac{\omega_0}{2\pi}\int_{\Sigma_H} b, \tag{3.2}$$

where the 2-form b is defined by (1.9).

Now let us consider the quantum framework. Let us stress that all formulas bellow are written without taking into account the $O(\varepsilon^\infty)$ corrections. The quantum version of (A.3) is

$$\hat{H} = f_\varepsilon(\hat{S}_\varepsilon; \hat{X}_\varepsilon) + O(\hbar^2). \tag{3.3}$$

The remainder $O(\hbar^2)$ appears as the correction from the quantum composite function theorem.

In view of (A.2), all operators $\hat{X}_\varepsilon$ in (3.3) commute with the operator $\hat{S}_\varepsilon$ up to $O(\hbar^3)$. Thus, in the semiclassical approximation, we can introduce the eigenprojectors Π_n of the operator $\hat{S}_\varepsilon$ and obtain from (3.3):

$$\hat{H}\Pi_n = \Pi_n f_\varepsilon(s_n; \hat{X}_\varepsilon) + O(\hbar^2), \tag{3.4}$$

where the number s_n is given by (1.4).

The generators $\hat{X}^j_\varepsilon$ in (3.4) satisfy the canonical commutation relations

$$[\hat{X}_\varepsilon \overset{\otimes}{,} \hat{X}_\varepsilon] = -i\varepsilon\hbar J^{jk} + O(\hbar^3). \tag{3.5}$$

By using (A.4), we can transform these generators into the following other generators

$$\mathcal{X} \overset{\text{def}}{=} X_\varepsilon + \varepsilon J\alpha \quad \text{or} \quad \mathcal{X} = x + \varepsilon J\theta + O(\varepsilon^2). \tag{3.6}$$

The operators $\hat{\mathcal{X}}^j$ obey commutation relations of the type (1.11)

$$[\hat{\mathcal{X}} \overset{\otimes}{,} \hat{\mathcal{X}}] = -i\varepsilon\hbar\Psi(\hat{\mathcal{X}}) + O(\varepsilon^3\hbar), \tag{3.7}$$

where $\Psi = J - \varepsilon J\beta J + O(\varepsilon^2)$. Then, from (3.1) and (2.6), we see that (3.4) takes the form

$$\hat{H}\Pi_n = \Pi_n(E_n(\hat{\mathcal{X}}) - \varepsilon\hat{\sigma}_n) + O(\varepsilon^2 + \hbar^2), \tag{3.8}$$

where $E_n(x) = f_0(s_n; x)$ and the functions $\sigma_n = \sigma_n(x)$ are given by

$$\sigma_n = \frac{\omega_0}{2}\langle \theta \cdot \mathrm{ad}(S_0)\rangle\Big|_{s=s_n} = \left(\frac{\omega_0}{2\pi}\int_{\Sigma_H} b\right)\Big|_{s=s_n}. \tag{3.9}$$

In the first formula (3.9), we denote by $\mathrm{ad}(\dots)$ the Hamiltonian field in the slow phase space corresponding to the given function. This first expression for σ_n is exactly the one derived in [5] (see formulas (24), (24a) there).

The second expression for σ_n in (3.9) uses a new geometric object: the 2-form b (1.9) which comes from Theorem 3.1.

Note that, in the introduction, we used the notation M_n'' for the summand $-\sigma_n$ in the Hamiltonian on the right-hand side of (3.8). Now we explain how this summand can be included into a geometric quantization condition.

Let us consider the equation

$$\frac{1}{2\pi}\int_{\Sigma_H(x)}(a+\varepsilon b)=s \tag{3.10}$$

with respect to the value H. We denote the solution of (3.10) by $H=\mathcal{H}_\varepsilon(s;x)$. In particular, at $\varepsilon=0$, we have $\mathcal{H}_0\equiv f_0$, i.e., the function used above in (2.5).

Also denote $\sigma=\frac{\omega_0}{2\pi}\int_{\Sigma_H}b$. Then

$$\mathcal{H}_\varepsilon(s;x)=\mathcal{H}_0(s-\varepsilon\sigma/\omega_0;x)=f_0(s;x)-\varepsilon\sigma(s;x)+O(\varepsilon^2).$$

For the value $s=s_n$, this relation reads

$$\mathcal{E}_n(x)\equiv\mathcal{H}_\varepsilon(s_n;x)=f_0(s_n;x)-\varepsilon\sigma(s_n;x)+O(\varepsilon^2)=E_n(x)-\varepsilon\sigma_n(x)+O(\varepsilon^2).$$

Therefore, (3.8) becomes

$$\hat{H}\Pi_n=\Pi_n\mathcal{E}_n(\hat{\mathcal{X}})+O(\varepsilon^2+\hbar^2). \tag{3.11}$$

In addition, the operators $\hat{\mathcal{X}}$ commute with $\hat{S}_\varepsilon$ up to $O(\varepsilon^2\hbar)$ and one can restrict them onto the nth eigensubspace to obtain operators $\hat{\mathcal{X}}_n$ with relations (1.11).

Theorem 3.2. (i) *The quantization condition* (1.8) *for the energy E values or equation* (3.10) *with $s=s_n$ for the Hamiltonian H values determine the functions $E=\mathcal{E}_n(x)=\mathcal{H}_\varepsilon(s_n,x)$ obeying* (3.11)*.*

The effective slow Hamiltonians $\mathcal{E}_n(\mathcal{X})$ in (3.11) *can also be obtained by solving the relation* (2.24) *for the improved adiabatic invariant with respect to H at the level $S_\varepsilon=s_n$.*

(ii) *The original Hamiltonian is expressed via the improved adiabatic invariant and new slow coordinates $\mathcal{X}=x+\varepsilon J\theta+O(\varepsilon^2)$ by formula inverse to* (2.24),

$$H=\mathcal{H}_\varepsilon(S_\varepsilon,\mathcal{X})+O(\varepsilon^2),\quad \hat{H}=\mathcal{H}_\varepsilon(\hat{S}_\varepsilon,\hat{\mathcal{X}})+O(\varepsilon^2+\hbar^2). \tag{3.12}$$

The operators $\hat{\mathcal{X}}$ obey commutation relations (3.7) *and commute with the operator $\hat{S}_\varepsilon$ up to to $O(\hbar\varepsilon^2)$.*

Remark. One can derive another type of integral geometric formulas for the improved adiabatic invariant and adiabatic quantization rule. Indeed, about the Hamiltonian S_ε we know that it has 2π-periodic flow and know that its complete set of integrals of motion (up to $O(\varepsilon^\infty)$) consists of functions H and $\mathcal{X}_\varepsilon=\mathcal{X}+O(\varepsilon^2)$. Therefore its periodic trajectories are given just by intersections of surfaces $\{H=const\}$ and $\{\mathcal{X}_\varepsilon=const\}$ in the total slow-fast phase space. If denote by $\Sigma_{H\wedge\mathcal{X}_\varepsilon}$ membranes whose boundaries coincide with these intersections then the corresponding action function is given by the integral $\frac{1}{2\pi}\int_{\Sigma_{H\wedge\mathcal{X}_\varepsilon}}(a+\frac{1}{\varepsilon}\omega)$. And this action coincides with the function S_ε itself.

Now by using the representation

$$\mathcal{X}_\varepsilon=x+\varepsilon J\theta_\varepsilon \tag{3.13}$$

similar to (3.6), one obtains the following integral formula for the improved adiabatic invariant:

$$S_\varepsilon=\frac{1}{2\pi}\int_{\Sigma_{H\wedge\mathcal{X}_\varepsilon}}(a-\varepsilon B_\varepsilon)$$

with the correcting 2-form $B_\varepsilon \stackrel{def}{=} -\frac{1}{2}\partial\theta_\varepsilon \wedge J\partial\theta_\varepsilon$ similar to (2.17).

At the level $S_\varepsilon = s_n$ formula (3.14) becomes the equation

$$\frac{1}{2\pi}\int_{\Sigma_{H\wedge\mathcal{X}_\varepsilon}} (a - \varepsilon B_\varepsilon) = s_n. \tag{3.15}$$

whose solutions with respect to H determine the adiabatic terms as functions in $\mathcal{X}_\varepsilon$ up to $O(\varepsilon^\infty)$.

The main working object here is the set (3.13) of "guiding center" coordinates $\mathcal{X}_\varepsilon$ related to a higher order analog θ_ε of the Hamiltonian connection 1-form. This analog is obtained from the solution of (A.5),(A.9),(A.10) by splitting $\mathcal{A}_\varepsilon = \theta_\varepsilon - \alpha_\varepsilon$ similarly to (2.15) (see details in [17], formula (B.9)).

The critical distinction of this construction from above results (2.24), (1.8) is that the membranes $\Sigma_{H\wedge\mathcal{X}_\varepsilon}$ do not belong to "vertical" fast fibers and thus there is no complete separation of fast and slow variables in formulas (3.14), (3.15). Nevertheless they are useful, for instance, in studying the extension of (2.24), (1.8) to higher adiabatic orders.

4. CONCLUSION

We proved that, in the adiabatic system (1.1)–(1.3), the fast component can be separated, i.e., the degrees of freedom can be reduced, at least up to $O(\varepsilon^2)$ in a purely geometric way by deforming the fast and slow symplectic structures without transforming the Hamiltonian. These geometric deformations are generated by the Poisson tensor of the slow space (noncommutatity of adiabatic parameters) in combination with the adiabatic Berry-type connection.

We also found out that, by using the same geometric deformations, the improved adiabatic invariant can be represented, at least up to $O(\varepsilon^2)$, in a form similar to the usual action integral on fast fibers.

Our hypothesis is that these statements are true up to $O(\varepsilon^\infty)$. This would mean that, for generic classical and semiclassical adiabatic systems with one fast degree of freedom, their reduction in all higher ε-orders can be described via deformation of symplectic structures only.

The main consequence from our study is that is reveals a "symplectic force" which synchronizes the fast gyrations in the system and arises due to inconsistency between the slow symplectic structure and the adiabatic curvature. A consequence of this synchronization is that the effective slow Hamiltonians (the adiabatic terms) in the first and probably, in higher ε-orders, are given by the direct geometric formula a'la Planck quantization rule over the fast space. The reduced slow dynamics in the new slow coordinates is then governed by this geometric Hamiltonian only, without needing additional ε-corrections.

The results obtained here can also be interpreted in the fashion of generalized "magneto-torsion field," "Peierls substitution," and "guiding center" [21–25] in the slow phase space.

APPENDIX: "DYNAMICS" BY ADIABATIC PARAMETER FOR INTEGRALS OF MOTION AND CONNECTION

In the paper [17], we derived a system of equations for adiabatic integrals of motion and for the connection (which control the hodograph parallel translations) by using the small parameter ε in the role of the time variable, starting from the trivial integrable situation at $\varepsilon = 0$. In the classical case, the details were described in [18]. In this appendix, we demonstrate how to modify this system to make it more explicit.

Note that there is an operator scheme for considering higher adiabatic terms based on "diagonalization" and on the use of derivations with respect to ε [21, 22], as well computations of a similar type in the classical framework [19, 20]. But these approaches do not exploit the phase space structure and zero curvature equations, which are our central objects in [17, 18], and does not allow us to transfer computations easily from quantum to classical level and back.

Recall that, in [18], we have fixed a point $\underline{x}$ in the slow space, wrote $\underline{S}_0(r,k) \stackrel{\text{def}}{=} S_0(r,k;\underline{x})$, and considered a phase space transformation g_0 preserving the brackets $\{\cdot\,,\cdot\}_0$ and such that $S_0 = g_0^*\underline{S}_0$ and $g_0^*(x^j) = x^j$. In Section 3 of [18], we have explained how such a transformation acting along the fast fibers can be constructed.

At the second step, we introduced the canonical transformation g_ε of the total fast-slow phase space equipped with Poisson brackets $\{\cdot,\cdot\}_\varepsilon$ (2.3) in such a way that the function $S_\varepsilon \overset{def}{=} g_\varepsilon^* \underline{S}_0$ is the asymptotic integral of motion for the Hamiltonian H (1.1), that is, as in (2.12),

$$\{H, S_\varepsilon\}_\varepsilon = 0 \quad \text{up to} \quad O(\varepsilon^\infty). \tag{A.1}$$

In the terminology of [2, 14], S_ε is called the improved adiabatic invariant (in [18] we also referred to it as the complete adiabatic invariant).

The main point of the work [18] was that the transformation g_ε is constructed not by the usual step-by-step increase of the accuracy of approximations from ε^N to ε^{N+1}, but by deriving a differential equation with respect to ε.

If the transformed slow coordinates are denoted by $X_\varepsilon^j = g_\varepsilon^* x^j$, then in addition to (A.1) up to $O(\varepsilon^\infty)$, we have the relations

$$\{S_\varepsilon, X_\varepsilon^j\}_\varepsilon = 0, \qquad \{X_\varepsilon \overset{\otimes}{,} X_\varepsilon\}_\varepsilon = \varepsilon J. \tag{A.2}$$

The original Hamiltonian H is a function in S_ε and in the new slow coordinate X_ε, i.e.,

$$H = f_\varepsilon(S_\varepsilon; X_\varepsilon). \tag{A.3}$$

An explicit computation of the adiabatic "term" f_ε was also done in [17, 18] by deriving some differential equation with respect to ε.

In [17, 18], we used auxiliary functions $\underline{\mathcal{A}}_{\varepsilon j}$ and $\underline{\mathcal{B}}_\varepsilon$ which take part in the construction of the "dynamical" system for g_ε and $f_\varepsilon, S_\varepsilon$ with respect to the adiabatic parameter ε. Now we obtain a simpler system of equations by exploiting the functions $\mathcal{A}_{\varepsilon j} = g_\varepsilon^* \underline{\mathcal{A}}_{\varepsilon j}$ and $\mathcal{B}_\varepsilon = g_\varepsilon^* \underline{\mathcal{B}}_\varepsilon$.

The covector $\mathcal{A}_\varepsilon$ is related to the new slow variable X_ε as follows:

$$X_\varepsilon = x + \varepsilon J \mathcal{A}_\varepsilon. \tag{A.4}$$

The function $\mathcal{B}_\varepsilon$ together with $\mathcal{A}_\varepsilon$ obey a kind of zero curvature condition

$$D\mathcal{B}_\varepsilon + \{\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon\}_\varepsilon = \frac{\partial \mathcal{A}_\varepsilon}{\partial \varepsilon} + \frac{1}{2}\mathcal{A}_{\varepsilon j} J^{jk} D_k \mathcal{A}_\varepsilon \tag{A.5}$$

(here summation by repeated indices is assumed). We refer to [17, 18] for detailed explanations about this zero curvature framework. The functions $\mathcal{B}_\varepsilon, \mathcal{A}_{\varepsilon j}$ play the role of connection coefficients in parallel transport equations for $g_\varepsilon, f_\varepsilon, S_\varepsilon$ in the extended fast-slow phase space with additional ε-direction as a "time" component of the space.

Note that the function S_ε keeps the basic property of the classical action S_0 to have a 2π-periodic Hamiltonian flow. Let us denote by Y_ε^t the flow generated by S_ε and define the corresponding averaging operation on phase-space functions

$$\langle G \rangle \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} Y_\varepsilon^{t*} G \, dt \tag{A.6}$$

as well as the integrating operation

$$G^\# \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} (t-\pi) Y_\varepsilon^{t*} G \, dt. \tag{A.7}$$

The standard properties of these operations are

$$\langle\langle G \rangle\rangle = \langle G \rangle, \quad \langle G^\# \rangle = 0, \quad \{S_\varepsilon, G^\#\}_\varepsilon = G - \langle G \rangle, \quad \{S_\varepsilon, \langle G \rangle\}_\varepsilon = 0. \tag{A.8}$$

The covector field $\mathcal{A}_\varepsilon$ determines the Hamiltonian connection $\nabla_\varepsilon \overset{\text{def}}{=} D + ad_\varepsilon(\mathcal{A}_\varepsilon)$, where by $ad_\varepsilon(\cdot)$ we denote the Hamiltonian field corresponding to the given function on the total fast-slow phase space with respect to the brackets (2.3). This Hamiltonian connection preserves S_ε and commutes with operations (A.6) and (A.7). The problem is that we do not know S_ε for $\varepsilon > 0$ and, therefore, cannot determine $\mathcal{A}_\varepsilon$ starting from S_ε as we did in Section 2 at $\varepsilon = 0$. One needs some system of equations for simultaneous (joint) computation of all these objects.

The basic derivation of the desired system was done in [18] in terms of $\underline{\mathcal{A}}_\varepsilon$, $\underline{\mathcal{B}}_\varepsilon$. Now we use another basic functions $\mathcal{A}_\varepsilon$, $\mathcal{B}_\varepsilon$. From the results of [18], it follows that the system becomes

$$\frac{\partial}{\partial\varepsilon}S_\varepsilon = \{S_\varepsilon, \mathcal{B}_\varepsilon\}_\varepsilon - \frac{1}{2}\mathcal{A}_{\varepsilon j}J^{jk}\{S_\varepsilon, \mathcal{A}_{\varepsilon k}\}_\varepsilon, \quad \Big(\frac{\partial}{\partial\varepsilon} - \langle\mathcal{A}_\varepsilon\rangle JD\Big)f_\varepsilon = \frac{1}{2}\langle\mathcal{A}_{\varepsilon j}J^{jk}\{H, \mathcal{A}_{\varepsilon k}\}_\varepsilon\rangle. \tag{A.9}$$

These are differential equations with respect to the adiabatic parameter ε and they can be solved starting from the "initial data" at $\varepsilon = 0$, i.e., from the functions S_0 and f_0 that we know.

Of course, we need also equations for $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ taking part in (A.9). The first set of such equations is (A.5). The number of equations in (A.5) is equal to the dimension of the slow space, or to the dimension of the covector $\mathcal{A}_\varepsilon$. Thus there must be one more equation determining the ε-"dynamics" of $\mathcal{B}_\varepsilon$.

This last equation does not contain the derivative with respect to ε but just gives the explicit formula

$$\begin{aligned}\mathcal{B}_\varepsilon =& \frac{\varepsilon}{\omega_\varepsilon}\Big(\frac{\partial\mathcal{A}_{\varepsilon j}}{\partial\varepsilon} + \frac{1}{2}\mathcal{A}_{\varepsilon l}J^{lm}D_m\mathcal{A}_{\varepsilon j}\Big)^{\#} J^{jk}D_k f_\varepsilon(S_\varepsilon, X_\varepsilon) \\ &+ \frac{1}{\omega_\varepsilon}\mathcal{A}^{\#}_{\varepsilon j}J^{jk}D_k f_\varepsilon(S_\varepsilon, X_\varepsilon) + \frac{1}{2\omega_\varepsilon}\big(\mathcal{A}_{\varepsilon j}J^{jk}\{H, \mathcal{A}_{\varepsilon k}\}_\varepsilon\big)^{\#}.\end{aligned} \tag{A.10}$$

Here X_ε is determined by $\mathcal{A}_\varepsilon$ from (A.4) and $\omega_\varepsilon = \partial f_\varepsilon(S_\varepsilon, X_\varepsilon)$.

Now by using (A.10), the function $\mathcal{B}_\varepsilon$ can be expressed via $\mathcal{A}_\varepsilon$, S_ε in (A.5), (A.9), and then this system can easily be solved in the class of formal series in ε:

$$S_\varepsilon = S_0 + \varepsilon S' + \dots, \quad f_\varepsilon = f_0 + \varepsilon f' + \dots, \quad \mathcal{A}_\varepsilon = \mathcal{A}_0 + \varepsilon\mathcal{A}' + \dots. \tag{A.11}$$

Note that the solution S_ε, $\mathcal{A}_\varepsilon$ will automatically satisfy the homological equation

$$DS_\varepsilon = \{S_\varepsilon, \mathcal{A}_\varepsilon\}_\varepsilon, \tag{A.12}$$

as well as the zero curvature equation

$$D_j\mathcal{A}_{\varepsilon k} - D_k\mathcal{A}_{\varepsilon j} + \{\mathcal{A}_{\varepsilon j}, \mathcal{A}_{\varepsilon k}\}_\varepsilon = 0 \tag{A.13}$$

if and only if both of these conditions hold at $\varepsilon = 0$ for the covector field $\mathcal{A}_0$ and the classical action S_0. The latter is guaranteed just by the definition of $\mathcal{A}_0 = \theta - \alpha$ described in Section 2.

Equations (A.12), (A.13) are equivalent to relations (A.2) with X_ε given by (A.4).

Theorem A.1. *The "dynamical" system of equations* (A.9), (A.5) *with the adiabatic parameters ε as a "time" variable* (plus formula (A.10)), together with the "Cauchy data" S_0, f_0, $\mathcal{A}_0$ at $\varepsilon = 0$, uniquely determine the formal power series S_ε, f_ε, $\mathcal{A}_\varepsilon$ (A.11) such that the Hamiltonian H up to $O(\varepsilon^\infty)$ can be represented in the form (A.3) with new Darboux coordinates X_ε (A.4) obeying relations (A.2). The integral of motion S_ε has the 2π-periodic Hamiltonian flow and satisfies the relation (A.12).

The representation of the Hamiltonian (A.3) allows us to explicitly perform the restriction onto the levels $\{S_\varepsilon = s\}$ of the integral of motion and compute up to $O(\varepsilon^\infty)$ the effective slow Hamiltonians $f_\varepsilon(s; X_\varepsilon)$ in new Darboux coordinates obeying relations (A.2). In the quantum case, one can explicitly restrict the Hamiltonian $\hat{H}$, as well the operators $\hat{X}_\varepsilon$, onto nth eigensubspace of $\hat{S}_\varepsilon$, corresponding to the eigenvalue $s_n + O(\hbar^2)$. This can be done with accuracy $O(\varepsilon^\infty)$.

Note that, in Sections 2 and 3, by working with accuracy $O(\varepsilon^2)$, we use the new slow coordinates $\mathcal{X}$ which are related to the above Darboux coordinates as follows: $X_\varepsilon = \mathcal{X} - \varepsilon J\alpha + O(\varepsilon^2)$ where the Berry's vector-potential α is determined in (2.10).

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