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MATHEMATICS

Selection Principles for Maps of Several Variables[1]

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Abstract—We show that a pointwise precompact sequence of maps from the n-dimensional rectangle into a metric semigroup, whose total variations in the sense of Vitali, Hardy and Krause are uniformly bounded, contains a pointwise convergent subsequence. We present a variant of this result for maps with values in a reflexive separable Banach space with respect to the weak pointwise convergence of maps.

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The Helly selection principle [10] asserts that a uniformly bounded sequence of real functions on a closed interval $[a, b]$, whose Jordan variations are uniformly bounded, contains a pointwise convergent subsequence. This classical theorem has numerous applications in the theory of functions, theory of Fourier series, stochastic analysis, complex analysis, theory of multifunctions [6, 11, 15]. For functions of one variable the Helly selection principle has been generalized in different directions usually connected with removing the assumptions that functions are real valued and that their (generalized) variations are uniformly bounded. The most general results in this case and the up to date references on the recent extensions of Helly's theorem are presented in [8].

For functions of several real variables Helly type theorems are known in the literature not so well, which is naturally connected with certain difficulties. For instance, note that the notion of the Jordan variation can be extended for functions of several variable in a far nonunique way [9, 11]. In this respect under the restrictions on certain types of variations (in particular, in the framework of the theory of distributions [1]) one can obtain generalizations of the Helly selection principle for functions of two or more variables only with respect to the almost everywhere convergence [11, 14]. Nonetheless, if one understands the variation of functions of several variables in the sense of Vitali, Hardy, and Krause [7, 13], then, as it was shown in [2, 11–13], one can obtain a generalization of Helly's theorem for real valued functions of several variables in the classical formulation with respect to the everywhere convergence.

The purpose of this paper is to present two variants of the Helly type pointwise selection principle with respect to the everywhere convergence for maps of n variables with values in a metric semigroup. For these maps, to which, in particular, some classes of multifunctions [6, Section 12] belong, we introduce and study the notions of Vitali type mixed "differences" and the total variation. Theorem 1 is formulated quite classically, and it is based on the Helly theorem for totally monotone functions of several variables [11, 13]. In Theorem 2 we present a generalization of Helly's theorem for maps with values in a reflexive Banach space.

1. MIXED DIFFERENCES AND VARIATIONS

We begin with definitions, notations and auxiliary facts. Given two points $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ from $\mathbb{R}^n$ such that $a < b$ componentwise, we denote by $I_a^b = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ the n-dimensional rectangle and by $\mathcal{F}$—the set of all maps $f: I_a^b \to M$ from I_a^b into a metric semigroup $(M, d, +)$, where (M, d) is a metric space with metric d, $(M, +)$ is an Abelian semigroup with respect to the addition operation $+$, and d is translation invariant: $d(u, v) = d(u + w, v + w)$ for all $u, v, w \in M$. An example of a metric semigroup is the family of all nonempty closed bounded convex subsets of a real normed space equipped with the Hausdorff metric [5]. As usual, Greek letters denote (n-dimensional) multiindices, and if $\theta = (\theta_1, \theta_2, ..., \theta_n)$, then $|\theta| = \theta_1 + \theta_2 + ... + \theta_n$ is the length of the multiindex θ. In addition, 0 and 1 in this context will denote the multiindices $(0, ..., 0)$ and

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(1, ..., 1), respectively, and inequalities of the form $0 \le \theta \le \alpha$ or $\sigma \le \kappa$ are understood componentwise.

Given $x, y \in I_a^b$, with $x < y$, we define the Vitali type n-th mixed "difference" of a map $f \in \mathcal{F}$ on the subrectangle $I_x^y \subset I_a^b$ by

$$\mathrm{md}_n(f, I_x^y)$$

$$= d\left(\sum_{\theta \in \mathcal{E}} f(x + \theta(y - x)), \sum_{\eta \in \mathcal{O}} f(x + \eta(y - x))\right),$$

where $\mathcal{E}$ is the set of all multiindices $\theta \le 1$, for which $|\theta|$ is even, $x + \theta(y - x) = (x_1 + \theta_1(y_1 - x_1), \ldots, x_n + \theta_n(y_n - x_n))$, and $\mathcal{O}$ is the set of all multiindices $\eta \le 1$ with odd length $|\eta|$. By the n-th Vitali variation of the map $f \in \mathcal{F}$ on the rectangle I_a^b we understand the quantity (in the case $M = \mathbb{R}$ see [7, 13])

$$V_n(f, I_a^b) = \sup_{\mathcal{P}} \sum_{1 \le \sigma \le \kappa} \mathrm{md}_n(f, I_{x[\sigma-1]}^{x[\sigma]}), \tag{1}$$

where the supremum is taken over all multiindices κ and all net partitions $\mathcal{P} = \{x[\sigma]: \sigma \le \kappa\}$ of the rectangle I_a^b with a collection of points of the form $x[\sigma] = x[\sigma_1, \sigma_2, \ldots, \sigma_n] = (x_1(\sigma_1), x_2(\sigma_2), \ldots, x_n(\sigma_n)) \in I_a^b$ such that $x[0] = a$, $x[\kappa] = b$, and $x[\sigma - 1] < x[\sigma]$ for all $1 \le \sigma \le \kappa$ (in other words, $\mathcal{P}$ is the Cartesian product of ordinary partitions of intervals $[a_i, b_i]$, $i = 1, 2, \ldots, n$). Note that the sums of the mixed differences from (1) are monotone in the sense that their values do not decrease if new points are inserted into the partition $\mathcal{P}$, and I_a^b is the union over all $1 \le \sigma \le \kappa$ of nonoverlapping rectangles $I_{x[\sigma-1]}^{x[\sigma]} \subset I_a^b$ with sides parallel to the coordinate axes.

We need also the notion of the variation of a map of order less than n. Let $0 \ne \alpha \le 1$ and $f \in \mathcal{F}$. Following [7], we define the truncation of a vector $x \in \mathbb{R}^n$ by the multiindex α by

$$x\lfloor\alpha = (x_i: i \in \{1, 2, \ldots, n\}, \alpha_i = 1) \in \mathbb{R}^{|\alpha|}.$$

Note that if $x \in I_a^b$, then $x\lfloor\alpha \in I_a^b\lfloor\alpha = I_{a\lfloor\alpha}^{b\lfloor\alpha} \subset \mathbb{R}^{|\alpha|}$. Given $z \in I_a^b$, we define the map $f_\alpha^z: I_a^b\lfloor\alpha \to M$, truncated by the multiindex α, according to the formula $f_\alpha^z(x\lfloor\alpha) = f(z + \alpha(x - z))$ for all $x \in I_a^b$, so that f_α^z depends only on $|\alpha|$ variables $x_i \in [a_i, b_i]$, for which $\alpha_i = 1$, and the other variables are fixed and equal to z_i (if $\alpha_i = 0$). Now, if we replace n by $|\alpha|$, f—by f_α^z with $z = a$ and I_a^b—by $I_a^b\lfloor\alpha$ in the definition (1), then we get the definition of the $|\alpha|$-th variation of the map $f \in \mathcal{F}$ in the modification of Hardy and Krause, which will be denoted by $V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha)$.

The total variation of the map $f \in \mathcal{F}$ in the sense of Vitali, Hardy and Krause [7, 11, 13] is the quantity

$$\mathrm{TV}(f, I_a^b) = \sum_{0 \ne \alpha \le 1} V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha),$$

and the set $\mathrm{BV}(I_a^b; M) = \{f \in \mathcal{F}: \mathrm{TV}(f, I_a^b) < \infty\}$ is the space of all maps of bounded total variation.

In the following three lemmas we collect the main properties of the mixed differences and variations of all orders, which on the one hand generalize the well known properties of Jordan's variation for functions of one variable and on the other hand are used in the proofs of the main results of the paper, Theorems 1 and 2.

Lemma 1. *If $f \in \mathcal{F}$, $x, y \in I_a^b$, $x \le y$, $z \in I_a^b$ and $0 \ne \alpha \le 1$, then*

$$\mathrm{md}_{|\alpha|}(f_\alpha^z, I_x^y\lfloor\alpha) = d\left(\sum_{\theta \in \mathcal{E}, \theta \le \alpha} f(z + \alpha(x - z) + \theta(y - x)),\right.$$

$$\left.\sum_{\eta \in \mathcal{O}, \eta \le \alpha} f(z + \alpha(x - z) + \eta(y - x))\right).$$

In particular, if $z = a$ and $z = x$, then we have, respectively:

$$\mathrm{md}_{|\alpha|}(f_\alpha^a, I_x^y\lfloor\alpha) = \mathrm{md}_{|\alpha|}(f_\alpha^{a + \alpha(x - a)}, I_{a + \alpha(x - a)}^y\lfloor\alpha),$$

$$\mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y\lfloor\alpha)$$

$$= d\left(\sum_{\theta \in \mathcal{E}, \theta \le \alpha} f(x + \theta(y - x)), \sum_{\eta \in \mathcal{O}, \eta \le \alpha} f(x + \eta(y - x))\right).$$

Lemma 2. *If $f \in \mathcal{F}$, $x, y \in I_a^b$ and $x < y$, then*

$$d(f(x + \gamma(y - x)), f(x))$$

$$\le \sum_{0 \ne \alpha \le \gamma} \mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y\lfloor\alpha) \text{ for all } 0 \ne \gamma \le 1,$$

$$\mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y\lfloor\alpha)$$

$$\le \sum_{\alpha \le \beta \le 1} \mathrm{md}_{|\beta|}(f_\beta^x, I_{a + \alpha(x - a)}^{x + \alpha(y - x)}\lfloor\beta) \text{ for all } 0 \ne \alpha \le 1.$$

Lemma 3. (a) *If $f \in \mathcal{F}$, $0 \ne \alpha \le 1$ and $\mathcal{P} = \{x[\sigma]: \sigma \le \kappa\}$ is a net partition of I_a^b, then $V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha)$ is equal to the sum of expressions $V_{|\alpha|}(f_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]}\lfloor\alpha)$ over all multiindices $1 \le \sigma \le \kappa$ (additivity of the $|\alpha|$-th variation).*

(b) *If a sequence of maps* $\{f_j\} = \{f_j\}_{j=1}^{\infty}$ *from* $\mathcal{F}$ *converges pointwise on* I_a^b *to a map* $f \in \mathcal{F}$ *as* $j \to \infty$, *then* $\mathrm{TV}(f, I_a^b) \le \liminf_{j\to\infty} \mathrm{TV}(f_j, I_a^b)$ (*lower semicontinuity of* TV).

(c) *If* $f \in \mathcal{F}$, $x, y \in I_a^b$ *and* $x \le y$, *then the following inequalities hold:*

$$d(f(y), f(x)) \le \sum_{0 \ne \alpha \le 1} \mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y \llcorner \alpha) \le \mathrm{TV}(f, I_x^y);$$

$$\sum_{0 \ne \alpha \le \gamma} V_{|\alpha|}(f_\alpha^x, I_x^y \llcorner \alpha) = \mathrm{TV}(f, I_x^{x+\gamma(y-x)})$$
$$\le \mathrm{TV}(f, I_a^{x+\gamma(y-x)}) - \mathrm{TV}(f, I_a^x)$$

for all $0 \ne \gamma \le 1$.

(d) *If* $f \in \mathrm{BV}(I_a^b; M)$, *then the function* $\nu\colon I_a^b \to \mathbb{R}$, *defined by the rule* $\nu(x) = \mathrm{TV}(f, I_a^x)$ *for all* $x \in I_a^b$, *is totally monotone, i.e.*,

$$(-1)^{|\alpha|} \sum_{0 \le \theta \le \alpha} (-1)^{|\theta|} \nu(x + \theta(y - x)) \ge 0$$

for all $x, y \in I_a^b$, $x \le y$, *and* $0 \ne \alpha \le 1$, *and* $\mathrm{TV}(\nu, I_a^b) = \mathrm{TV}(f, I_a^b)$.

We note that for $M = \mathbb{R}$ Lemma 1 and the second inequalities in Lemmas 2 and 3c were established in [7], the first inequalities in Lemmas 2 and 3c and Lemma 3d were obtained in [13], and properties (a) and (b) from Lemma 3 are known from [11].

2. HELLY TYPE THEOREM

We say that a sequence $\{f_j\} \subset \mathcal{F}$ is pointwise precompact (on I_a^b) if, for each $x \in I_a^b$, the closure in M of the sequence $\{f_j(x)\}$ is compact. The following Helly type selection principle holds for maps of n variables with values in a metric semigroup $(M, d, +)$.

Theorem 1. *If a sequence of maps* $\{f_j\} \subset \mathcal{F}$ *is pointwise precompact on* I_a^b *and satisfies the condition*

$$\sup_{j \in \mathbb{N}} \mathrm{TV}(f_j, I_a^b) < \infty, \qquad (2)$$

then $\{f_j\}$ *contains a subsequence, which converges pointwise on* I_a^b *as* $j \to \infty$ *to a map* $f \in \mathrm{BV}(I_a^b; M)$.

This theorem generalizes the results of [4] ($n = 1$ and M is a metric space), [11, 12] ($n = 2$ and $M = \mathbb{R}$), [13] ($n \in \mathbb{N}$ and $M = \mathbb{R}$) and [2] ($n = 2$ and M is a metric semigroup).

3. WEAK POINTWISE SELECTION PRINCIPLE

In this Section we present a variant of Theorem 1, connected with the weak pointwise convergence, in the case when the values of maps lie in a reflexive separable Banach space.

Let $(M, \|\cdot\|)$ be a linear normed space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and M^* be its dual, i.e., the space of all continuous linear functionals on M. It is well known that M^* is a Banach space with respect to the norm $\|u^*\|^* = \sup\{|u^*(u)|: u \in M \text{ and } \|u\| \le 1\}$, $u^* \in M^*$. Recall that a sequence $\{u_j\}$ of elements from M converges weakly in M to an element $u \in M$ (in short, $u_j \xrightarrow{w} u$ in M) if $u^*(u_j) \to u^*(u)$ in $\mathbb{K}$ as $j \to \infty$ for all $u^* \in M^*$; moreover, in this case the following inequality holds: $\|u\| \le \liminf_{j\to\infty} \|u_j\|$.

Since a linear normed space $(M, \|\cdot\|)$ is a metric semigroup, the notions of the Vitali n-th variation, $|\alpha|$-th variation for $0 \ne \alpha \le 1$ and the total variation of a map $f\colon I_a^b \to M$ are introduced as above with respect to the induced metric $d(u, v) = \|u - v\|$, $u, v \in M$.

Theorem 2. *Let* $(M, \|\cdot\|)$ *be a reflexive separable Banach space, whose dual space* $(M^*, \|\cdot\|^*)$ *is also separable, and let* $\{f_j\} \subset \mathcal{F}$ *be a sequence of maps. If* $\{f_j\}$ *satisfies condition* (2) *and*

$$\sup_{j \in \mathbb{N}} \|f_j(x)\| < \infty \textit{ for all } x \in I_a^b,$$

then there exist a subsequence of $\{f_j\}$, *denoted as the original sequence by* $\{f_j\}$, *and a map* $f \in BV(I_a^b; M)$ *such that* $f_j(x) \xrightarrow{w} f(x)$ *in* M *as* $j \to \infty$ *for all* $x \in I_a^b$.

This theorem is an extension to maps of several variables of the weak selection principle from [3, Chapter 1, Theorem 3.5] given for maps of bounded Jordan variation of one variable.

By examples one can show that all the assumptions in Theorems 1 and 2 are essential for their validity.

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