On the Fourier Transform of the Characteristic Functions of Domains with $C^1$ Boundary

V. V. Lebedev

Received May 20, 2011

Abstract. We consider domains $D \subseteq \mathbb{R}^n$ with $C^1$ boundary and study the following question: For what domains $D$ does the Fourier transform $\hat{1}_D$ of the characteristic function $1_D$ belong to $L^p(\mathbb{R}^n)$?

Key words: harmonic analysis, domain with smooth boundary, Fourier transform of a characteristic function.

Introduction

Let $D$ be a bounded domain (an open connected set) in $\mathbb{R}^n$, $n \geq 2$. Consider its characteristic function $1_D$, i.e., the function such that $1_D(t) = 1$ if $t \in D$ and $1_D(t) = 0$ if $t \notin D$. Consider the Fourier transform $\hat{1}_D$ of this function. In the present work we study the following question: For what domains do we have $\hat{1}_D \in L^p(\mathbb{R}^n)$? Only the case when $1 < p < 2$ is interesting.

It will be convenient for us to deal with the spaces $A_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, of tempered distributions $f$ on $\mathbb{R}^n$ such that the Fourier transform $\hat{f}$ belongs to $L^p(\mathbb{R}^n)$. The norm on $A_p(\mathbb{R}^n)$ is defined in a natural way as

$$||f||_{A_p(\mathbb{R}^n)} = ||\hat{f}||_{L^p(\mathbb{R}^n)}.$$  

Recall that (see, e.g., [1, Chap. V, Sec. 1]), for $1 \leq p \leq 2$, the Fourier transform (as well as its inverse) is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1/p + 1/q = 1$; thus, each distribution in $A_p(\mathbb{R}^n)$, $1 \leq p \leq 2$, is a function in $L^q(\mathbb{R}^n)$.

Direct calculation shows that if $D$ is a cube in $\mathbb{R}^n$, then $1_D \in A_p(\mathbb{R}^n)$ for all $p > 1$. The same is true in the case when $D$ is a polytope (i.e., a finite union of simplices). On the other hand, using the well-known asymptotics for Bessel functions, one can verify that if $D \subseteq \mathbb{R}^n$ is a ball, then $1_D \in A_p(\mathbb{R}^n)$ for $p > 2n/(n+1)$ and $1_D \notin A_p(\mathbb{R}^n)$ for $p \leq 2n/(n+1)$. The same result holds in the general case of bounded domains with $C^2$ boundary. (This follows from Theorems 1 and 2 of the present work; see Corollary 2.) Thus, for (bounded) domains with $C^2$ boundary, $2n/(n+1)$ is the critical exponent of integrability for the Fourier transform of the characteristic function.

In the present work we shall obtain a series of results on the behavior of the Fourier transform of the characteristic functions of bounded domains with $C^1$ boundary. Generally speaking, this case is essentially different from the $C^2$ case, as is shown by the example of a domain $D \subseteq \mathbb{R}^2$ such that its boundary is $C^1$ and, at the same time, $1_D \in A_p(\mathbb{R}^2)$ for all $p > 1$ (see Section 3). (The critical value for planar domains with $C^2$ boundary is $4/3$.)

We note that various questions on the rate of decrease at infinity of the Fourier transform of characteristic functions of domains, as well as closely related questions on the behavior of the Fourier transform of (smooth) measures supported on surfaces, were investigated by many authors and represent a classical topic in harmonic analysis; see Stein’s survey [2], which contains an extensive bibliography, and his book [3] (Chap. VIII). The basic tools to obtain asymptotic estimates in these investigations are the stationary phase method and the van der Corput lemma. The use of these tools requires considerable smoothness of the boundary of the domain under examination. The order of smoothness should be at least 2 even in the planar case. The crucial role in this approach is played by the curvature of the surface (the boundary of the domain). Our approach does not use any arguments related to curvature and makes it possible to consider domains with $C^1$ boundary.
We denote the boundary of a domain \( D \subseteq \mathbb{R}^n \) by \( \partial D \). Saying that the boundary of \( D \) is \( C^1 \) or \( C^2 \), we mean that each of its points has a neighborhood in which the boundary \( \partial D \) is the graph of a certain (real-valued) function of class \( C^1 \) or \( C^2 \), respectively (that is, of a function all of whose partial derivatives of the first or the second order, respectively, are continuous).

For each domain \( D \subseteq \mathbb{R}^n \) with \( C^1 \) boundary, let \( \nu_D(x) \) be the outer unit normal vector to \( \partial D \) at a point \( x \in \partial D \). The corresponding map \( \nu_D: \partial D \to S^{n-1} \) of the boundary of \( D \) into the unit sphere \( S^{n-1} \) centered at the origin is called the normal map. By \( \omega(\nu_D, \delta) \) we denote the modulus of continuity of \( \nu_D \):

\[
\omega(\nu_D, \delta) = \sup_{x,y \in \partial D; |x-y| \leq \delta} |\nu_D(x) - \nu_D(y)|, \quad \delta \geq 0,
\]

where \( |u| \) is the length of a vector \( u \in \mathbb{R}^n \). Next, let \( \omega(\delta) \) be an arbitrary nondecreasing continuous function on \([0, \infty)\) with \( \omega(0) = 0 \). In the case when \( \omega(\nu_D, \delta) = O(\omega(\delta)), \delta \to +0 \), we say that the boundary \( \partial D \) is \( C^{1,\omega} \). For bounded domains, this condition is equivalent to the condition that each point on the boundary of \( D \) has a neighborhood in which \( \partial D \) is the graph of a certain function of class \( C^{1,\omega} \). In other words, for each point \( x \in \partial D \), one can find a neighborhood \( B \) of \( x \) and a domain \( V \subseteq \mathbb{R}^{n-1} \) such that \( B \cap \partial D \) is the graph of some (real-valued) function \( \varphi \in C^{1,\omega}(V) \), i.e., of a function with \( \omega(\partial V, \nabla \varphi, \delta) = O(\omega(\delta)), \delta \to +0 \), where

\[
\omega(\partial V, \nabla \varphi, \delta) = \sup_{x,y \in \partial V; |x-y| \leq \delta} |\nabla \varphi(x) - \nabla \varphi(y)|, \quad \delta \geq 0,
\]

is the modulus of continuity of the gradient \( \nabla \varphi \) of \( \varphi \).

If the boundary \( \partial D \) of a domain \( D \) is \( C^1 \), \( C^2 \), or \( C^{1,\omega} \), then we write \( \partial D \in C^1 \), \( \partial D \in C^2 \), or \( \partial D \in C^{1,\omega} \), respectively.

If \( \omega(\delta) = \delta^\alpha \), \( 0 < \alpha \leq 1 \), then we write simply \( C^{1,\alpha} \) instead of \( C^{1,\delta^\alpha} \).

In Section 1 we give a simple proof of the relation \( 1_D \in A_p(\mathbb{R}^n) \), which holds for all \( p > 2n/(n+1) \) whenever \( D \subseteq \mathbb{R}^n \) is a bounded domain with \( C^1 \) boundary (see Theorem 1). For convex domains (without smoothness assumptions on the boundary), such an assertion was earlier proved by Herz [4].

In Section 2 we obtain the main result of the present work. Namely, we show (see Theorem 2) that if \( \partial D \in C^{1,\omega} \) and

\[
\int_0^1 \frac{\delta^{n(p-1)-1}}{\omega(\delta)^{n-p}} d\delta = \infty,
\]

then \( 1_D \notin A_p(\mathbb{R}^n) \). In particular (see Corollary 1), if \( \partial D \in C^{1,\alpha} \), then \( 1_D \notin A_p(\mathbb{R}^n) \) for

\[
p \leq 1 + \frac{(n-1)\alpha}{n+\alpha}.
\]

Putting \( \alpha = 1 \) here and taking into account the preceding result, we obtain the assertion on the critical exponent for domains with \( C^2 \) boundary mentioned at the beginning of the introduction (see Corollary 2).

In Section 3 we consider planar domains. According to the result mentioned above, if, for a domain \( D \subseteq \mathbb{R}^2 \), we have \( \partial D \in C^{1,\omega} \) and

\[
\int_0^1 \frac{\delta^{2p-3}}{\omega(\delta)^{2-p}} d\delta = \infty,
\]

then \( 1_D \notin A_p(\mathbb{R}^2) \). In particular, if \( \partial D \in C^{1,\alpha} \), then \( 1_D \notin A_p(\mathbb{R}^2) \) for \( p \leq 1 + \alpha/(2 + \alpha) \). We show (see Theorem 3) that this result is sharp, namely, for each class \( C^{1,\omega} \) (under a certain simple condition imposed on \( \omega \)), there exists a bounded domain \( D \subseteq \mathbb{R}^2 \) such that \( \partial D \in C^{1,\omega} \) and, for all \( p > 1 \) satisfying

\[
\int_0^1 \frac{\delta^{2p-3}}{\omega(\delta)^{2-p}} d\delta < \infty,
\]

we have \( 1_D \in A_p(\mathbb{R}^2) \). In particular (see Corollary 3), if \( 0 < \alpha < 1 \), then there exists a planar domain \( D \) with \( C^{1,\alpha} \) boundary such that \( 1_D \in A_p(\mathbb{R}^2) \) for all \( p > 1 + \alpha/(2 + \alpha) \). It also follows
that (see Corollary 4) there exists a planar domain $D$ with $C^1$ boundary such that $1_D \in A_p(\mathbb{R}^2)$ for all $p > 1$ (it suffices to take $\omega(\delta)$ decreasing to zero slower than any power of $\delta$, i.e., so that $\lim_{\delta \to 0} \omega(\delta)/\delta^\varepsilon = \infty$ for all $\varepsilon > 0$).

The results of the present work are essentially based on the results obtained in [5] and [6], where, for real-valued $C^1$ functions $\varphi$, the author studied the growth of the $A_p$ norms of the exponential functions $e^{i\lambda \varphi}$. The question on the growth of the norms of these functions naturally arises in relation to the well-known Beurling–Helson theorem (see the history of the question in [5]). Simple arguments (see Lemma 1 of the present work) reduce the study of characteristic functions to the study of the behavior of exponential functions.

By $|E|$ we denote the Lebesgue measure of a (measurable) set $E \subseteq \mathbb{R}^n$ and by $|E|_{S^{n-1}}$, the spherical measure of a set $E \subseteq S^{n-1}$. We use $(x, y)$ to denote the inner product of vectors $x$ and $y$ in $\mathbb{R}^n$. If $E \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}^n$, then we put $E + t = \{x + t : x \in E\}$. Various positive constants are denoted by $c$, $c_p$, and $c_{p,n}$.

The results of this work were presented in part at the 11th and 14th Summer St. Petersburg Meetings in Mathematical Analysis [7], [8] and completely at the III International Conference “Harmonic Analysis and Approximation” in Tsahkadzor (Armenia) [9].

The author is grateful to E. A. Gorin, who read the introduction and made useful remarks.

1. The General Case of Domains with $C^1$ Boundary

**Theorem 1.** Let $D$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $\partial D \in C^1$. Then $1_D \in A_p(\mathbb{R}^n)$ for all $p > 2n/(n + 1)$.

**Proof.** For $s > 0$, consider the Sobolev spaces $W^s_2(\mathbb{R}^n)$ of functions $f \in L^2(\mathbb{R}^n)$ satisfying

$$
\|f\|_{W^s_2(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (|\xi|^{2s} + 1)|\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty.
$$

It is easy to verify that

$$
W^2_2(\mathbb{R}^n) \subseteq A_p(\mathbb{R}^n)
$$

if $2n/(n + 2s) < p < 2$. Indeed, for $p^* = 2/p$, $1/p^* + 1/q^* = 1$, we have $spq^* > n$. Using the Hölder inequality, we obtain

$$
\|f\|^p_{A_p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^p d\xi = \int_{\mathbb{R}^n} (|\hat{f}(\xi)|||\xi|^s + 1)|^{p}\frac{1}{(|\xi|^s + 1)^p} d\xi
$$

$$
\leq \left( \int_{\mathbb{R}^n} (|\hat{f}(\xi)|||\xi|^s + 1)^{pp^*} d\xi \right)^{1/p^*} \left( \int_{\mathbb{R}^n} \frac{1}{(|\xi|^s + 1)^{pq^*}} d\xi \right)^{1/q^*} \leq c_{p,s}\|f\|_{W^s_2(\mathbb{R}^n)}.
$$

To prove the theorem, it remains to take into account the fact that, for each bounded domain $D \subseteq \mathbb{R}^n$ with $C^1$ boundary, we have $1_D \in W^s_2(\mathbb{R}^n)$ for all $s < 1/2$. This is a trivial consequence of the theorem on (pointwise) multipliers in Sobolev spaces [10, Sec. 5].

We shall give an independent short and simple proof of the relation $1_D \in W^s_2$, $s < 1/2$. It is well known that, for $0 < s < 1$, the norm $\|f\|_{W^s_2(\mathbb{R}^n)}$ and the norm

$$
\|f\| = \|f\|_{L^2(\mathbb{R}^n)} + \left( \int_{\mathbb{R}^n} \frac{1}{|t|^{n+s}} \left( \int_{\mathbb{R}^n} |f(x + t) - f(x)|^2 dx \right) dt \right)^{1/2}
$$

(1)

are equivalent (see, e.g., [11, Chap. V, Sec. 3.5]). Now, note that, for each $t \in \mathbb{R}^n$, the symmetric difference

$$
((D - t) \setminus D) \cup (D \setminus (D - t))
$$

of the sets $D - t$ and $D$ is contained in the (closed) $|t|$-neighborhood of the boundary $\partial D$ of $D$, and hence its (Lebesgue) measure is at most $c|t|$. It is also clear that the measure of this symmetric difference is at most $2|D|$. Thus,

$$
\int_{\mathbb{R}^n} |1_D(x + t) - 1_D(x)|^2 dx \leq \min(c|t|, 2|D|), \quad t \in \mathbb{R}^n,
$$

with
and it remains to use the equivalence of the norm \( \| \cdot \|_{W^2_2(\mathbb{R}^n)} \) and the norm defined by (1). The theorem is proved.*

Remark 1. The method used in the proof of Theorem 1 can be applied to arbitrary sets (not only domains). Recall that the upper Minkowski dimension \( \dim_M F \) of a bounded set \( F \subseteq \mathbb{R}^n \) is defined by
\[
\dim_M F = \inf \{ 0 \leq \gamma \leq n : |(F)_\delta| = O(\delta^{n-\gamma}), \delta \to +0 \},
\]
where \( (F)_\delta \) is the \( \delta \)-neighborhood of \( F \) [13]. Let \( E \subseteq \mathbb{R}^n, n \geq 1, \) be a bounded set of positive measure, and let \( a \) be the upper Minkowski dimension of its boundary \( \partial E \). Suppose that \( a < n \). Repeating, with obvious modifications, the arguments used above, we see that \( 1_E \in W^2_2(\mathbb{R}^n) \) for all \( s < (n-a)/2 \). Hence \( 1_E \in A_p(\mathbb{R}^n) \) for all \( p > 2n/(2n-a) \). Note that, for \( s < (n-a)/2 \), the use of norm (1) is justified, since \( a \geq n-1 \). Indeed, let us verify that if a set \( E \subseteq \mathbb{R}^n \) is bounded and has positive measure, then \( \dim_M \partial E \geq n-1 \). Assuming that \( E \setminus \partial E \neq \emptyset \) (otherwise there is nothing to prove), fix a point \( x_0 \in E \setminus \partial E \). There exists an open ball \( B \) centered at \( x_0 \) that does not contain points of the boundary \( \partial E \) and, moreover, is at a positive distance from \( \partial E \). Let \( S \) denote the boundary sphere of the ball \( B \). We define a map \( \theta : \mathbb{R}^n \setminus B \to S \) as follows. Take a point \( x \in \mathbb{R}^n \setminus B \) and consider the ray from \( x_0 \) that passes trough \( x \). Let \( \theta(x) \) denote the point of intersection of this ray with the sphere \( S \). Clearly, the map \( \theta \) is Lipschitz (moreover, it is nonexpanding, i.e., \(|\theta(x_1) - \theta(x_2)| \leq |x_1 - x_2| \) for all \( x_1, x_2 \in \mathbb{R}^n \setminus B \)). It is easy to see that the image of the boundary of the set \( E \) under the map \( \theta \) is the whole sphere \( S \). At the same time, it is known [13, Chap. 7] that Lipschitz maps do not increase the dimension of a set. Thus,
\[
n - 1 = \dim_M S = \dim_M \theta(\partial E) \leq \dim_M \partial E.
\]

2. Domains with \( C^{1,\omega} \) Boundary

**Theorem 2.** Let \( D \) be a bounded domain in \( \mathbb{R}^n, n \geq 2, \) with \( \partial D \in C^{1,\omega} \). If
\[
\int_0^1 \frac{\delta^{n(p-1)-1}}{(\omega(\delta))^{n-p}} \, d\delta = \infty,
\]
then \( 1_D \notin A_p(\mathbb{R}^n) \).

From Theorem 2 we immediately obtain the following corollary.

**Corollary 1.** Let \( 0 < \alpha \leq 1, \) and let \( D \) be a bounded domain in \( \mathbb{R}^n, n \geq 2, \) with \( \partial D \in C^{1,\alpha} \). If
\[
p \leq 1 + \frac{(n-1)\alpha}{n+\alpha},
\]
then \( 1_D \notin A_p(\mathbb{R}^n) \).

We particularly note the case of domains with \( C^2 \) boundary and the even more general \( C^{1,1} \) case. Namely, using Corollary 1 and Theorem 1, we obtain the following corollary.

**Corollary 2.** Let \( D \) be a bounded domain in \( \mathbb{R}^n, n \geq 2, \) with \( \partial D \in C^{1,1} \). Then \( 1_D \in A_p(\mathbb{R}^n) \) for \( p > 2n/(n+1) \) and \( 1_D \notin A_p(\mathbb{R}^n) \) for \( p \leq 2n/(n+1) \). In particular, this holds for bounded domains with \( C^2 \) boundary.

Before proving the theorem, we give some preliminaries and prove a lemma.

Recall (see, e.g., [11, Chap. IV, Sec. 3.1]) that a function \( m \in L^\infty(\mathbb{R}^n) \) is called an \( L^p \)-Fourier multiplier \((1 \leq p \leq \infty)\) if the operator \( Q \) given by
\[
\hat{Qf} = m \hat{f}, \quad f \in L^p \cap L^2(\mathbb{R}^n),
\]
is a bounded operator from \( L^p(\mathbb{R}^n) \) to itself. The space \( M_p(\mathbb{R}^n) \) of all such multipliers endowed with the norm
\[
\|m\|_{M_p(\mathbb{R}^n)} = \|Q\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}
\]
*Note that if \( E \subseteq \mathbb{R}^n, n \geq 1, \) is a set of positive measure, then \( 1_E \notin W^{1/2}_2(\mathbb{R}^n) \) [12, Corollary 2.2].
is a Banach algebra (with respect to the usual multiplication of functions). It is well known that the characteristic function of an arbitrary parallelepiped is a multiplier for all $p$, $1 < p < \infty$ (see, e.g., [11, Chap. IV, Sec. 4.1] and also [14, Chap. I, Sec. 1.3]).

To be an arbitrary function, and let $f_k \in A_p(\mathbb{R}^n)$ be arbitrary, and let $f_k \in A_p \cap L^2(\mathbb{R}^n)$, $k = 1, 2, \ldots$, be a sequence that converges to $f$ in $A_p(\mathbb{R}^n)$. We have

$$\|mf_j - mf_k\|_{A_p} = \|m \cdot (f_j - f_k)\|_{A_p} \leq \|m\|_{\mathcal{M}_p}\|f_j - f_k\|_{A_p} \to 0.$$  

This can be easily verified as follows. Note that estimate (2) holds for every function $f \in A_p \cap L^2(\mathbb{R}^n)$ (this is obvious, since the Fourier transform and its inverse differ only in the sign of the variable and in the normalization factor). It is clear that the set $A_p \cap L^2(\mathbb{R}^n)$ is dense in $A_p(\mathbb{R}^n)$. Let $f \in A_p(\mathbb{R}^n)$ be an arbitrary function, and let $f_k \in A_p \cap L^2(\mathbb{R}^n)$, $k = 1, 2, \ldots$, be a sequence that converges to $f$ in $A_p(\mathbb{R}^n)$. We have

$$\|mf_j - mf_k\|_{A_p} = \|m \cdot (f_j - f_k)\|_{A_p} \leq \|m\|_{\mathcal{M}_p}\|f_j - f_k\|_{A_p} \to 0.$$  

It is obvious that the spaces $A_p(\mathbb{R}^n)$ are Banach spaces, so the sequence $mf_k$, $k = 1, 2, \ldots$, converges in $A_p(\mathbb{R}^n)$ to some function $g \in A_p(\mathbb{R}^n)$. Using the Hausdorff–Young inequality [1, Chap. V, Sec. 1], which, in our notation, has the form $\|f\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}^{1/p} \leq \|f\|_{L^q(\mathbb{R}^n)}^{1/q} = 1$, $1 \leq p \leq 2$, we see that the sequences $\{f_k\}$ and $\{mf_k\}$ converge in $L^q(\mathbb{R}^n)$ to $f$ and $g$, respectively. Thus, $mf = g$, and it remains to pass to the limit in the inequality $\|mf_k\|_{A_p} \leq \|m\|_{\mathcal{M}_p}\|f_k\|_{A_p}$. Let $D_1$ be a domain in $\mathbb{R}^m$, and let $D_2 = l(D_1)$ be its image under a nondegenerate affine map $l: \mathbb{R}^m \to \mathbb{R}^n$. It is easy to see that $1_{D_1} \in A_p(\mathbb{R}^m)$ if and only if $1_{D_2} \in A_p(\mathbb{R}^n)$. It suffices to observe that if $l(x) = Qx + b$, then, for each function $f \in L^1(\mathbb{R}^n)$, we have $|f \circ l(u)| = |\det Q|^{-1}|f((Q^{-1})^*u)|$, where $Q^{-1}$ is the inverse matrix of $Q$ and $(Q^{-1})^*$ is the transpose of $Q^{-1}$.

Let $E$ be an arbitrary set in $\mathbb{R}^m$, $m \geq 1$. Following [6], we say that a function $f$ defined on $E$ belongs to a space $A_p(\mathbb{R}^m, E)$ if there exists a function $F \in A_p(\mathbb{R}^m)$ such that its restriction $F|_E$ to the set $E$ coincides with $f$. We define the norm on $A_p(\mathbb{R}^m, E)$ by

$$\|f\|_{A_p(\mathbb{R}^m, E)} = \inf_{F|_E = f} \|F\|_{A_p(\mathbb{R}^m)}.$$  

Note that if $I$ is a parallelepiped in $\mathbb{R}^m$ and $f$ is a function on $I$, then, putting

$$\|f\|_{A_p(\mathbb{R}^m, I)} = \|f\|_{A_p(\mathbb{R}^m)}$$  

where $F$ is the function $f$ extended by zero to the complement $\mathbb{R}^m \setminus I$ (that is, $F = f$ on $I$ and $F = 0$ on $\mathbb{R}^m \setminus I$), we obtain a norm $\|f\|_{A_p(\mathbb{R}^m, I)}$ equivalent to the norm $\|f\|_{A_p(\mathbb{R}^m, I)}$ for $1 < p < 2$. This is so because, for $1 < p < \infty$, the characteristic function of a parallelepiped is an $L^p$-Fourier multiplier.

The proof of Theorem 2 is based on the following result, obtained by the author in [6, Theorem 1'']. Suppose that $1 \leq p < 2$. Let $V$ be a domain in $\mathbb{R}^m$, and let $\varphi \in C^1(\omega(V))$ be a real-valued function such that the gradient $\nabla \varphi$ of $\varphi$ is nondegenerate on $V$, i.e., the set $\nabla \varphi(V)$ is of positive measure. Then, for all those $\lambda \in \mathbb{R}$, $|\lambda| \geq 1$, for which $e^{i\lambda \varphi} \in A_p(\mathbb{R}^m, V)$, we have

$$\|e^{i\lambda \varphi}\|_{A_p(\mathbb{R}^m, V)} \geq c\left(|\lambda|^{1/p} \chi^{-1}\left(\frac{1}{|\lambda|}\right)^m\right),$$  

where $\chi^{-1}$ is the inverse function of $\chi(\delta) = \delta \omega(\delta)$ and $c = c(p, \varphi) > 0$ is independent of $\lambda$.

The simple Lemma 1 below (which we shall also use in Section 3) reduces the question of whether $1_D$ belongs to $A_p$ to the consideration of the behavior of the exponential functions $e^{i\lambda \varphi}$ in $A_p$.

For a vector $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and a number $a \in \mathbb{R}$, let $(x, a)$ denote the vector $(x_1, \ldots, x_m, a) \in \mathbb{R}^{m+1}$. 31
Let $I$ be an open parallelepiped in $\mathbb{R}^m$ with edges parallel to the coordinate axes, and let $\varphi$ be a continuous bounded function on $I$ such that $\varphi(t) > 0$ for all $t \in I$. Consider the following domain $G$ in $\mathbb{R}^{m+1}$:

$$G = \{(t, y) \in \mathbb{R}^m \times \mathbb{R} : t \in I, 0 < y < \varphi(t)\}.$$ 

This domain is called the special domain generated by the pair $(I, \varphi)$.

**Lemma 1.** Let $G \subseteq \mathbb{R}^{m+1}$ be the special domain generated by a pair $(I, \varphi)$. Suppose that $1 < p < 2$. Then $1_G \in A_p(\mathbb{R}^{m+1})$ if and only if $e^{i\lambda \varphi} \in A_p(\mathbb{R}^m, I)$ for almost all $\lambda \in \mathbb{R}$ and

$$\int_{\mathbb{R}} \frac{1}{|\lambda|^p} \|e^{i\lambda \varphi} - 1\|^p_{A_p(\mathbb{R}^m, I)} d\lambda < \infty.$$

**Proof.** For $\lambda \in \mathbb{R} \setminus \{0\}$, we define a function $F_\lambda$ on $\mathbb{R}^m$ by

$$F_\lambda(t) = \begin{cases} \frac{1}{-i\lambda} (e^{-i\lambda \varphi(t)} - 1) & \text{if } t \in I, \\ 0 & \text{if } t \in \mathbb{R}^m \setminus I. \end{cases}$$

Note that

$$\widehat{1_G}(u, \lambda) = \widehat{F_\lambda}(u), \quad (u, \lambda) \in \mathbb{R}^m \times \mathbb{R}, \quad \lambda \neq 0.$$

Indeed, direct calculation yields

$$\widehat{1_G}(u, \lambda) = \int_{t \in I, 0 < y < \varphi(t)} e^{-(u, t)} e^{-i\lambda y} dt dy = \int_I \left( \int_0^{\varphi(t)} e^{-i\lambda y} dy \right) e^{-(u, t)} dt$$

$$= \int_I \frac{1}{-i\lambda} (e^{-i\lambda \varphi(t)} - 1) e^{-(u, t)} dt = \widehat{F_\lambda}(u).$$

Thus, $1_G \in A_p(\mathbb{R}^{m+1})$ if and only if

$$\int_{\mathbb{R}} \|F_\lambda\|^p_{A_p(\mathbb{R}^m)} d\lambda < \infty.$$

It remains only to take into account the relation

$$\|F_\lambda\|_{A_p(\mathbb{R}^m)} = \frac{1}{|\lambda|} \|e^{i\lambda \varphi} - 1\|^p_{A_p(\mathbb{R}^m, I)},$$

where $\cdot \| \cdot \|_{A_p(\mathbb{R}^m, I)}$ is the equivalent norm on $A_p(\mathbb{R}^m, I)$ defined above (see (3)). The lemma is proved.

**Proof of Theorem 2.** Assume that, contrary to the assertion of the theorem, we have $1_D \in A_p(\mathbb{R}^n)$.

Each point $x$ of the boundary $\partial D$ can be surrounded by an open parallelepiped $\Pi_x \ni x$ so small that, after appropriate rotation and translation, the intersection $D \cap \Pi_x$ becomes a special domain. Consider a finite subcover of the cover $\{\Pi_x, x \in \partial D\}$ of $\partial D$. Note that $\nu_D(\partial D) = S^{m-1}$; hence, for at least one of the parallelepipeds $\Pi_x$, which we denote by $\Pi$, we have

$$\nu_D(\partial D \cap \Pi)_{|S^{m-1}} > 0. \quad (5)$$

Consider the domain $G = D \cap \Pi$. Since the characteristic function of any parallelepiped is an $L^p$-multiplier, we have $1_G = 1_{\Pi} \cdot 1_D \in A_p(\mathbb{R}^n)$. Replacing, if needed, the domain $G$ by its copy obtained by rotation and translation, we can assume that $G$ is a special domain. This domain is generated by a pair $(I, \varphi)$, where $I$ is a parallelepiped in $\mathbb{R}^m$, $m + 1 = n$, with edges parallel to the coordinate axes and $\varphi$ is a function in $C^{1,\omega}(I)$.

It is easy to see that condition (5) implies that the gradient of $\varphi$ is nondegenerate on $I$, that is, we have $|\nabla \varphi(I)| > 0$. Indeed (recall that if $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $a \in \mathbb{R}$, then $(x, a)$ denotes the vector $(x_1, \ldots, x_m, a) \in \mathbb{R}^{m+1}$), consider the map

$$\beta(t) = (t, \varphi(t)), \quad t \in I.$$
(β maps I onto the graph of ϕ). The normal map ν_D and the gradient ∇ϕ of ϕ are related by

$$\nu_D \circ \beta(t) = \frac{1}{\sqrt{|\nabla \varphi(t)|^2 + 1}}(-\nabla \varphi(t), 1), \quad t \in I.$$  

Thus, putting

$$\gamma(\xi) = \frac{1}{\sqrt{|\xi|^2 + 1}}(-\xi, 1), \quad \xi \in \mathbb{R}^m,$$

we obtain ν_D ◦ β = γ ◦ ∇ϕ. Therefore, for the set W = ∇ϕ(I), we have

$$\gamma(W) = \gamma(\nabla \varphi(I)) = \nu_D \circ \beta(I) = \nu_D(\partial D \cap \Pi),$$

and relation (5) implies |γ(W)|_{Sn−1} > 0. Since γ is a diffeomorphism of \( \mathbb{R}^m \) onto the upper hemisphere

$$S^m_+ = \{x = (x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} : |x| = 1, x_{m+1} > 0\}$$

(where \( m + 1 = n \)), we see that |W| > 0.

Thus, we see that the special domain G is generated by a pair (I, ϕ), where ϕ ∈ \( C^{1,\infty}(I) \) is a function with nondegenerate gradient; at the same time, we have \( I_G \in \mathcal{A}_p(\mathbb{R}^{m+1}) \).

By Lemma 1 we have

$$\int_{\mathbb{R}} \frac{1}{|\lambda|^p} \|e^{i\lambda \varphi} - 1\|^p \, d\lambda < \infty,$$

whence

$$\int_{\lambda \geq 1} \frac{1}{\lambda^p} \|e^{i\lambda \varphi} - 1\|^p \, d\lambda < \infty;$$

since \( 1 \in \mathcal{A}_p(\mathbb{R}^m, I) \), \( p > 1 \), we see that

$$\int_{1}^{\infty} \frac{1}{\lambda^p} \|e^{i\lambda \varphi}\|^p \, d\lambda < \infty.$$

Hence, putting \( V = I \) in estimate (4), we obtain

$$\int_{1}^{\infty} \lambda^{m-p} \left(\chi^{-1} \left(\frac{1}{\lambda}\right)\right)^m \, d\lambda < \infty,$$

that is (recall that \( m = n - 1 \)),

$$\int_{1}^{\infty} \lambda^{n-1-p} \left(\chi^{-1} \left(\frac{1}{\lambda}\right)\right)^{(n-1)p} \, d\lambda < \infty.$$

The following lemma of purely technical character completes the proof of the theorem.

**Lemma 2.** For \( n \geq 2 \) and \( 1 < p < 2 \), the following conditions are equivalent:

(i) \( \int_{1}^{\infty} \lambda^{n-1-p} \left(\chi^{-1} \left(\frac{1}{\lambda}\right)\right)^{(n-1)p} \, d\lambda < \infty; \)

(ii) \( \int_{0}^{1} \frac{\delta^{n(p-1)-1}}{(\omega(\delta)^{n-p})} \, d\delta < \infty. \)

To complete the proof of the theorem, it suffices to verify that (i) ⇒ (ii). The inverse implication for \( n = 2 \) will be used below in Section 3.

**Proof of Lemma 2.** For \( 0 < \varepsilon < 1 \), we put

$$I(\varepsilon) = \int_{1/\chi(1)}^{1/\chi(\varepsilon)} \lambda^{n-1-p} \left(\chi^{-1} \left(\frac{1}{\lambda}\right)\right)^{(n-1)p} \, d\lambda, \quad J(\varepsilon) = \int_{\varepsilon}^{1} \frac{\delta^{n(p-1)-1}}{(\omega(\delta)^{n-p})} \, d\delta.$$

We have

$$I(\varepsilon) = \frac{1}{n-p} \int_{1/\chi(1)}^{1/\chi(\varepsilon)} \left(\chi^{-1} \left(\frac{1}{\lambda}\right)\right)^{(n-1)p} \, d\lambda^{n-p}. \quad \text{33}$$
Making the change of variable $\lambda = 1/\chi(\delta)$ and integrating by parts, we obtain

$$I(\varepsilon) = \frac{1}{n-p} \left( \frac{\varepsilon^{n(p-1)}}{(\omega(\varepsilon))^{n-p}} - \frac{1}{(\omega(1))^{n-p}} \right) + \frac{(n-1)p}{n-p} J(\varepsilon). \quad (6)$$

Using this relation, we see that

$$I(\varepsilon) \geq \frac{-1}{(n-p)(\omega(1))^{n-p}} + \frac{(n-1)p}{n-p} J(\varepsilon);$$

therefore, (i) $\Rightarrow$ (ii).

Conversely, assume that condition (ii) holds. Then, since

$$\int_{\varepsilon/2}^{\varepsilon} \frac{\delta^{n(p-1)-1}}{(\omega(\delta))^{n-p}} d\delta \geq \frac{1}{(\omega(\varepsilon))^{n-p}} \int_{\varepsilon/2}^{\varepsilon} \delta^{n(p-1)-1} d\delta \geq c_{n,p} \frac{\varepsilon^{n(p-1)}}{(\omega(\varepsilon))^{n-p}},$$

we have

$$\frac{\varepsilon^{n(p-1)}}{(\omega(\varepsilon))^{n-p}} \to 0, \quad \varepsilon \to +0,$$

and using (6), we obtain condition (i). The lemma and, hence, the theorem are proved.

**Remark 2.** Theorem 2 (and Corollary 1) has local character. The theorem remains true if we assume that only a part of the boundary of $D$, i.e., the intersection $B \cap \partial D$, where $B$ is an open ball in $\mathbb{R}^n$, is $C^{1,\omega}$ and the normal map $\nu$ defined on $B \cap \partial D$ is nondegenerate, that is, $|\nu(B \cap \partial D)|_{S^{n-1}} > 0$. The condition that $D$ is bounded can be replaced by the weaker condition $|D| < \infty$. (The modification of the proof is obvious.)

For $n = 2$, the condition of nondegeneracy of the normal map on $B \cap \partial D$ means that $B \cap \partial D$ is not a straight line interval.

### 3. Domains in $\mathbb{R}^2$

In this section, for each class $C^{1,\omega}$ (under a certain simple condition imposed on $\omega$), we shall construct a bounded domain $D \subseteq \mathbb{R}^2$ with $C^{1,\omega}$ boundary such that the characteristic function $1_D$ belongs to $A_p$ for $p$ so close to $1$ as allowed by Theorem 2. In addition, the domain $D$ has the property that its boundary does not contain straight line intervals (thus, this domain is essentially different from polygons).

According to Theorem 2, if $D$ is a bounded domain in $\mathbb{R}^2$ with $\partial D \in C^{1,\omega}$ and

$$\int_0^1 \frac{\delta^{2p-3}}{\omega(\delta)^{2-p}} d\delta = \infty,$$

then $1_D \notin A_p(\mathbb{R}^2)$. In particular, this is the case when $\partial D \in C^{1,\alpha}$ and $p \leq 1 + \alpha/(2 + \alpha)$. The following theorem shows that this result is sharp.

**Theorem 3.** Suppose that $\omega(2\delta) < 2\omega(\delta)$ for all sufficiently small $\delta > 0$. There exists a bounded domain $D \subseteq \mathbb{R}^2$ with $C^{1,\omega}$ boundary such that $1_D \in A_p(\mathbb{R}^2)$ for all $p$, $1 < p < 2$, satisfying

$$\int_0^1 \frac{\delta^{2p-3}}{\omega(\delta)^{2-p}} d\delta < \infty. \quad (7)$$

In addition, the boundary of $D$ does not contain straight line intervals.

This theorem immediately implies the following corollaries.

**Corollary 3.** For each $\alpha$, $0 < \alpha < 1$, there exists a bounded domain $D \subseteq \mathbb{R}^2$ with $C^{1,\alpha}$ boundary such that $1_D \in A_p(\mathbb{R}^2)$ for all $p > 1 + \alpha/(2 + \alpha)$. The boundary of $D$ does not contain straight line intervals.

**Corollary 4.** There exists a bounded domain $D \subseteq \mathbb{R}^2$ with $C^1$ boundary such that $1_D \in \bigcap_{p>1} A_p(\mathbb{R}^2)$. The boundary of $D$ does not contain straight line intervals.

Note also that from Theorems 2 and 3 it follows that the existence of a domain $D \subseteq \mathbb{R}^2$ with $\partial D \in C^{1,\omega}$ and $1_D \in \bigcap_{p>1} A_p(\mathbb{R}^2)$ is equivalent to the condition that $\omega(\delta)$ tends to 0 slower than
any power of \( \delta \), i.e., to the condition that \( \lim_{\delta \to +0} \omega(\delta)/\delta^\varepsilon = \infty \) for all \( \varepsilon > 0 \). Theorem 2 implies the necessity of this condition, and Theorem 3 implies its sufficiency.\(^*\)

The author does not know whether similar results are true for domains in \( \mathbb{R}^n \) with \( n \geq 3 \).

**Proof of Theorem 3.** Let \( A_p(\mathbb{T}) \), \( 1 \leq p \leq \infty \), be the space of distributions \( f \) on the circle \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \) (where \( \mathbb{Z} \) is the set of integers) such that the sequence of Fourier coefficients \( \hat{f} = \{ \hat{f}(k) \mid k \in \mathbb{Z} \} \) belongs to \( l^p \). We put

\[
\| f \|_{A_p(\mathbb{T})} = \| \hat{f} \|_{l^p} = \left( \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p \right)^{1/p}. 
\]

(For \( 1 \leq p \leq 2 \), each distribution in \( A_p(\mathbb{T}) \) is a function in \( L^p(\mathbb{T}) \subseteq L^1(\mathbb{T}) \), \( 1/p + 1/q = 1 \).

For \( p > 1 \), we put

\[
\Theta_p(y) = \left( \int_{1}^{y} \left( \chi^{-1}(\tau) \right)^p d\tau \right)^{1/p}, \quad y > 1,
\]

where, as above, \( \chi^{-1} \) is the inverse function of \( \chi(\delta) = \delta \omega(\delta) \).

In Theorem 2 of [5], under the assumption that \( \omega(2\delta) < 2\omega(\delta) \) for all sufficiently small \( \delta > 0 \), we constructed a real-valued function \( \varphi \) on the circle \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \) such that \( \varphi \in C^{1,\omega}(\mathbb{T}) \) (that is, \( \varphi \) is a \( 2\pi \)-periodic function of class \( C^{1,\omega}(\mathbb{R}) \)) and, for all \( p, 1 < p < 2 \), we have

\[
\| e^{i\lambda \varphi} \|_{A_p(\mathbb{T})} \leq c_p \Theta_p(|\lambda|), \quad \lambda \in \mathbb{R}, \ |\lambda| \geq 2. \tag{8}
\]

In addition, the function \( \varphi \) is nowhere linear, that is, it is not linear on any interval\(\^{**}\).

It is clear that estimate (8) implies

\[
\| e^{i\lambda \varphi} - 1 \|_{A_p(\mathbb{T})} \leq c_p \Theta_p(|\lambda|), \quad \lambda \in \mathbb{R}, \ |\lambda| \geq 2. \tag{9}
\]

It is also clear that, for every continuously differentiable function \( f \) on \( \mathbb{T} \), we have \( f \in A_1(\mathbb{T}) \) and

\[
\| f \|_{A_1(\mathbb{T})} \leq c \| f \|_{C^1(\mathbb{T})},
\]

where

\[
\| f \|_{C^1(\mathbb{T})} = \max_{t \in \mathbb{T}} |f(t)| + \max_{t \in \mathbb{T}} |f'(t)|.
\]

Therefore,

\[
\| e^{i\lambda \varphi} - 1 \|_{A_p(\mathbb{T})} \leq \| e^{i\lambda \varphi} - 1 \|_{A_1(\mathbb{T})} \leq c \| e^{i\lambda \varphi} - 1 \|_{C^1(\mathbb{T})} \leq c_p |\lambda|, \quad \lambda \in \mathbb{R}. \tag{10}
\]

Consider the following set \( Q \) on the real line \( \mathbb{R} \):

\[
Q = \{ t \in (0, 2\pi) : \varphi'(t) > 0 \}.
\]

Since \( \varphi \neq \text{const} \) and \( \varphi(0) = \varphi(2\pi) \), it is clear that \( Q \neq \emptyset \) and \( Q \neq (0, 2\pi) \). Consider an interval \( (a, b) \) which is a connected component of the set \( Q \). The derivative \( \varphi' \) vanishes at least at one of its endpoints. We can assume that this is the right one, that is, \( \varphi'(b) = 0 \); otherwise, instead of \( \varphi(t) \) and the interval \( (a, b) \), we consider the function \( -\varphi(-t) \) and the interval \( (-b, -a) \). Choose a point \( c, a < c < b \). Replacing the function \( \varphi(t) \) by \( \varphi(t) - \varphi(c) \), we can assume that \( \varphi(c) = 0 \). We put \( I = (c, b) \).

Thus, we obtain a nowhere linear function \( \varphi \in C^{1,\omega}(\mathbb{T}) \) satisfying conditions (9) and (10) and an interval \( I = (c, b) \subseteq [0, 2\pi] \) such that \( \varphi(c) = 0 \), the function \( \varphi \) is strictly increasing on \( I \), and, in addition, \( \varphi'(c) > 0 \) and \( \varphi'(b) = 0 \).

\(\^{**}\)Theorem 2 of [5] contains also a similar result for \( p = 1 \).
Recall the well-known relation between the spaces $A_p(T)$ and $A_p(\mathbb{R})$ for $1 < p \leq 2$ [16, Sec. 44]. If $f$ is a $2\pi$-periodic function and $f^*$ is its restriction to $[0, 2\pi]$ extended by zero to $\mathbb{R}$, i.e., $f^* = f$ on $[0, 2\pi]$ and $f^* = 0$ on $\mathbb{R} \setminus [0, 2\pi]$, then $f \in A_p(T)$ if and only if $f^* \in A_p(\mathbb{R})$. The norms satisfy

$$c_1(p)\| f^* \|_{A_p(\mathbb{R})} \leq \| f \|_{A_p(T)} \leq c_2(p)\| f^* \|_{A_p(\mathbb{R})}.$$ 

Thus, for all $p$, $1 < p < 2$, from (9) and (10) we obtain

$$\| e^{i\lambda \varphi} - 1 \|_{A_p(\mathbb{R}, I)} \leq c_p \Theta_p(\| \lambda \|), \quad \lambda \in \mathbb{R}, \quad |\lambda| \geq 2,$$

and

$$\| e^{i\lambda \varphi} - 1 \|_{A_p(\mathbb{R}, I)} \leq c_p |\lambda|, \quad \lambda \in \mathbb{R},$$

respectively.

Consider the special domain $G \subseteq \mathbb{R}^2$ generated by the pair $(I, \varphi)$.

**Lemma 3.** If $p$, $1 < p < 2$, satisfies (7), then $1_G \in A_p(\mathbb{R}^2)$.

**Proof.** It is easy to verify that condition (7) implies

$$\int_1^{\infty} \frac{1}{\lambda^p}(\Theta_p(\lambda))^p \, d\lambda < \infty. \quad (13)$$

Indeed, for any $a > 1$, integrating by parts, we obtain

$$\int_1^a \frac{1}{\lambda^p}(\Theta_p(\lambda))^p \, d\lambda = \frac{1}{p + 1} \int_1^a (\Theta_p(\lambda))^p \, d\lambda \lambda^{-p+1}$$

$$= - \frac{1}{p + 1} \left( (\Theta_p(a))^p \lambda^{-p+1} - \int_1^a \lambda^{-p+1} \left( \left( \frac{1}{\lambda} \right)^p \right) \, d\lambda \right)$$

$$\leq \frac{1}{p - 1} \int_1^a \lambda^{-p+1} \left( \chi^{-1}(\frac{1}{\lambda}) \right)^p \, d\lambda,$$

and Lemma 2 with $n = 2$ implies (13).

Therefore (see (11) and (13)),

$$\int_{|\lambda| > 2} \frac{1}{|\lambda|^p} \| e^{i\lambda \varphi} - 1 \|_{A_p(\mathbb{R}, I)}^p \, d\lambda < \infty.$$

At the same time (see (12)),

$$\int_{|\lambda| < 2} \frac{1}{|\lambda|^p} \| e^{i\lambda \varphi} - 1 \|_{A_p(\mathbb{R}, I)}^p \, d\lambda < \infty.$$

Thus,

$$\int_{\mathbb{R}} \frac{1}{|\lambda|^p} \| e^{i\lambda \varphi} - 1 \|_{A_p(\mathbb{R}, I)}^p \, d\lambda < \infty.$$

It remains to use Lemma 1. The lemma is proved.

Let us complete the proof of Theorem 3. The domain $G \subseteq \mathbb{R}^2$, which we have constructed, is of the form

$$G = \{(t, y) : c < t < b, \quad 0 < y < \varphi(t)\},$$

where $\varphi \in C^1(\mathbb{R})$ is a nowhere linear function. Recall that, according to our construction, $\varphi(c) = 0$, and the function $\varphi$ strictly increases on the interval $(c, b)$. In addition, $\varphi'(c) > 0$ and $\varphi'(b) = 0$. By Lemma 3, for all $p$ satisfying (7), we have $1_G \in A_p(\mathbb{R}^2)$. Expanding (or shrinking) the domain $G$ in the vertical direction by an appropriate affine map, we can assume that $\varphi'(c) = 1$. Let $G^*$ be the domain symmetric to $G$ with respect to the line $t = b$, and let $W = G \cup G^* \cup \xi$, where $\xi$ is the interval with endpoints $(b, 0)$ and $(b, \varphi(b))$. Take a square $\Pi \subseteq \mathbb{R}^2$ with side length $2(b - c)$. We obtain the required domain $D$ by taking four rigid copies of the domain $W$ (that is, copies obtained by rotation and translation) and attaching them to the sides of the square $\Pi$ on its outer side. The theorem is proved.

**Remark 3.** Theorem 3 (and Corollaries 3 and 4) admit the following modification. The property that the boundary of the domain $D$ does not contain straight line intervals can be replaced by
the property that $D$ is convex. The author does not know if it is possible to achieve both of these properties simultaneously. This modification follows from the existence of a (real-valued) nonconstant function $\varphi \in C^{1,\omega}(\mathbb{T})$ satisfying (8) for which the interval $(0, 2\pi)$ is the union of three intervals such that the derivative $\varphi'$ is monotone on each of them. Such a function was constructed by the author in [5] (see the construction before the proof of Theorem 2 in [5]).

It is not clear, even without any assumptions on the smoothness of the boundary, whether there exists a strictly convex domain $D \subseteq \mathbb{R}^n$, $n \geq 2$, such that $1_D \in \bigcap_{p>1} A_p(\mathbb{R}^n)$ (we call a domain strictly convex if it is convex and its boundary does not contain straight line intervals).

References


Translated by V. V. Lebedev