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LOCALLY FLAT AND WILDLY EMBEDDED SEPARATRICES IN SIMPLEST MORSE-SMALE SYSTEMS

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ABSTRACT. Let $MS^{flow}(M^n,k)$ and $MS^{diff}(M^n,k)$ be Morse-Smale flows and diffeomorphisms respectively the non-wandering set of those consists of k fixed points on a closed n-manifold M^n ($n \geq 4$). We prove that the closure of any separatrix of $f^t \in MS^{flow}(M^n,3)$ is a locally flat $\frac{n}{2}$-sphere while there is $f^t \in MS^{flow}(M^n,4)$ the closure of separatrix of those is a wildly embedded codimension two sphere. For $n \geq 6$, one proves that the closure of any separatrix of $f \in MS^{diff}(M^n,3)$ is a locally flat $\frac{n}{2}$-sphere while there is $f \in MS^{diff}(M^4,3)$ such that the closure of any separatrix is a wildly embedded 2-sphere.

1. Introduction

In 1960, Steve Smale [28] introduced a class of dynamical systems called later Morse-Smale systems. It was proved that Morse-Smale systems are structurally stable and have zero entropy [23, 25, 27]. In this sense, Morse-Smale systems are simplest structurally stable ones. However, the dynamics is far from the complete classification beginning with 3-dimensional Morse-Smale systems (see the surveys [7, 21]). The reasons are heteroclinic intersections and the possibility for the closure of separatrices to be wildly embedded. This possibility was discovered by Pixton [22] who constructed the gradient-like Morse-Smale diffeomorphism $f : S^3 \to S^3$ with two sinks, a source, and saddle such that the closure of 2-dimensional separatrix of the saddle is a wildly embedded 2-sphere while the closure of the 1-dimensional separatrix forms a half of the wildly embedded Artin-Fox arc [5]. The similar examples was constructed in [6, 9], where the classification of gradient-like Morse-Smale diffeomorphisms was considered.

The effect of wildly embedded separatrices looks the most clear when the number of fixed points is minimal, and there are no heteroclinic intersections. It follows from [8, 22] that four is the minimal number of fixed points when the effect of wildly embedding holds for 3-dimensional Morse-Smale

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diffeomorphisms. One can easily prove that the closure of separatrix is locally flat for any 2-dimensional Morse-Smale systems (diffeomorphisms and flows) and 3-dimensional Morse-Smale flows. So, it is natural to consider the following aspects: 1) the possibility of wild embedding for Morse-Smale flows, and 2) to find the minimal number of fixed points for n-dimensional Morse-Smale systems, $n \geq 4$, when the effect of wild embedding holds.

Denote by $MS^{flow}(M^n, k)$ the set of Morse-Smale flows the non-wandering set of those consists of k fixed points on a closed n-manifold M^n. Note that a flow $f^t \in MS^{flow}(M^n, k)$ is a gradient one [28]. Similarly, denote by $MS^{diff}(M^n, k)$ the set of Morse-Smale diffeomorphisms the non-wandering set of those consists of k periodic points. First of all, we remark that $k \geq 2$ because of M^n is compact [28]. If $k = 2$ then the supporting manifold M^n is an n-sphere and the system is of north-south type [16, 26], see Fig. 1, (a). Since M^n is connected, for $k \geq 3$, a Morse-Smale system must have at least one saddle provide its non-wandering set consists of fixed points. Eells and Kuiper [14] proved that there are closed n-manifolds, $n \geq 4$, admitting Morse functions with exactly three critical points. As a consequence, there exist closed n-manifolds, $n \geq 4$, admitting gradient Morse-Smale systems (flows and diffeomorphisms) the non-wandering set of those consists of three fixed points. Thus, it is natural to exam the aspects above beginning with n-dimensional Morse-Smale systems, $n \geq 4$, the non-wandering set of those consists of three and four fixed points (a sink, source, and one or two saddles). The main results for Morse-Smale flows are the following theorems. We begin with flows with three fixed points.

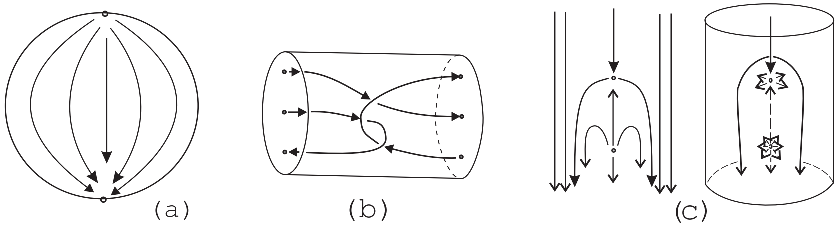

Fig. 1

Theorem 1. *Let $f^t \in MS^{flow}(M^n, 3)$, $n \geq 4$. Then the non-wandering set of f^t consists of a sink, say ω, source, say α, and saddle, say σ. In addition,*

- *M^n is a simply connected (hence, orientable) manifold, and $n \in \{4, 8, 16\}$;*
- *every separatrix of σ is $\frac{n}{2}$-dimensional;*

- *the closure of any separatrix of* σ *is a topologically embedded* $\frac{n}{2}$*-sphere that is a locally flat* $\frac{n}{2}$*-sphere.*

The last item of this theorem allows to us prove the following theorem concerning the topological equivalence.

Theorem 2. *Any flows* f^t, $g^t \in MS^{flow}(M^4, 3)$ *are topologically equivalent.*

For four fixed points, we prove the existence of wildly embedded separatrices.

Theorem 3. *Given any* $n \geq 4$*, there are a closed* n*-manifold* M^n *and gradient polar flow* $f^t \in MS^{flow}(M^n, 4)$ *such that* f^t *has no heteroclinic intersections, and the closure of some separatrix of* f^t *is a codimension two wildly embedded sphere.*

One can show that the manifold M^n in Theorem 3 is simply connected. However, $M^n \neq S^n$ [17, 18]. As to diffeomorphisms, our main result is the following theorem.

Theorem 4. *Let* $f \in MS^{diff}(M^n, 3)$*,* $n \geq 4$*. Then the non-wandering set of* f *consists of a sink, say* ω*, source, say* α*, and saddle, say* s_0*. In addition,*

- M^n *is an orientable manifold, and* n *is even;*
- *every separatrix of* s_0 *is* $\frac{n}{2}$*-dimensional;*
- *the closure of the unstable and stable separatrices* $Sep^u(s_0)$*,* $Sep^s(s_0)$ *are a topologically embedded* $\frac{n}{2}$*-spheres* $W^u(s_0) \cup \{\omega\} = S_\omega$*,* $W^s(s_0) \cup \{\alpha\} = S_\alpha$ *respectively.*

Moreover,

- *for* $n \geq 6$*, the spheres* S_ω*,* S_α *are locally flat;*
- *for* $n = 4$*, there is* $f \in MS^{diff}(M^4, 3)$ *such that the spheres* S_ω*,* S_α *are wildly embedded.*

The structure of the paper is the following. In Section 2, we give the main definitions and describe the special neighborhood of saddle fixed point. In Section 3, we prove Theorems 1, 2, and all items of Theorem 4 except the last one. In Section 4, we prove Theorem 3 and the last item of Theorem 4. Actually, this last proofs are the constructions of examples. For the flows, our construction is novel. For the diffeomorphisms, we follow [6, 22]. Therefore, for the reader convenient, we shortly give the main idea of the construction in [6, 22].

Let f^t_{NS} be the north-south type flow on the 3-sphere S^3, Fig. 1, (a). If S^3 thought of $\mathbb{R}^3$ completed by the infinity point, f^t_{NS} is defined by the system $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$, $\dot{x}_3 = x_3$. The origin $O = \alpha =$ north is a source, and the infinity point $\omega =$ south is a sink. Let $f_{NS} = f^1_{NS}$ be the shift-time $t = 1$ along the trajectories.

Take the Artin-Fox configuration consisting of three arcs, see Fig. 1, (b). One can assume that the Artin-Fox curve l_{AF} is the union of shifts f^m_{NS}, $m \in \mathbb{Z}$, so that l_{AF} connects ω and α, and l_{AF} is invariant under f_{NS}. Well-known that the Artin-Fox closed arc $l_{AF} \cup \{\alpha\} \cup \{\omega\}$ is wild at ω and α [5, 13]. Let T be a tubular neighborhood of l_{AF} such that T is invariant under f_{NS}. Actually, T is an infinite cylinder that can be thought of the support of Cherry type flow g^t with a saddle, say σ, of type $(2, 1)$ and an attracting node, Fig. 1, (c). One can assume that the shift-time-one $g^1 = g$ on ∂T coincides with the restriction $f_{NS}|_{\partial T}$. The Pixton-Bonatti-Grines diffeomorphism $f : S^3 \to S^3$ equals f_{NS} on $S^3 \setminus T$ and equals g on T. It is easy to see that f is a gradient-like Morse-Smale diffeomorphism such that the closure of unstable separatrix of σ is a topologically embedded 2-sphere that is wild at ω.

Developing this idea we consider a 4-sphere S^4 being the result after the rotation $\mathcal{R}$ of 3-sphere S^3 such that $\mathcal{R}(\omega) = \omega$, $\mathcal{R}(\alpha) = \alpha$. Instead of T, one takes $\mathcal{R}(T)$, and instead of Cherry type flow g^t we take the flow on the special neighborhood U_0 with a unique saddle of type $(2, 2)$, see details in Section 2.

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2. Main definitions and previous results

Basic definitions of dynamical systems see in [3, 27, 30]. A dynamical system (diffeomorphism or flow) is Morse-Smale if it is structurally stable and the non-wandering set consists of a finitely many periodic orbits (in particular, each periodic orbit is hyperbolic and, stable and unstable manifolds of periodic orbits intersect transversally). Many definitions for Morse-Smale diffeomorphisms and flows are similar. So, we shall give mainly the notation for diffeomorphisms giving the exact notation for flows if necessary.

Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism of n-manifold M^n. A periodic (in particular, fixed) point σ is called a *saddle* periodic point (in short, *saddle*) if $1 \le \dim W^u(\sigma) \le n - 1$, $1 \le \dim W^s(\sigma) \le n - 1$ where $W^u(\sigma)$ and $W^s(\sigma)$ are unstable and stable manifolds of σ respectively. A component of $W^u(\sigma) \setminus \sigma$ denoted by $Sep^u(\sigma)$ is called an *unstable separatrix* of σ. If $\dim W^u(\sigma) \ge 2$, then $Sep^u(\sigma)$ is unique. The similar notation holds for a stable separatrix. Following [1], one says that the saddle σ *is of type*

(μ, ν), if $\mu = \dim W^u(\sigma)$, $\nu = \dim W^s(\sigma)$. The number μ (ν) is called an *unstable (stable) Morse index.*

Special neighborhood. Let $\mathbb{R}^n$ be Euclidean space endowed with coordinates $(x_1, \dots, x_n)$, and a vector field $\vec{V}_s$ defined by the system

$$\dot{x}_1 = -x_1, \; \dots, \; \dot{x}_k = -x_k, \quad \dot{x}_{k+1} = x_{k+1}, \; \dots, \; \dot{x}_n = x_n. \tag{1}$$

We assume that $k \geq 2$, $n - k \geq 2$. The origin $O = (0, \dots, 0)$ is a saddle of $\vec{V}_s$ whose k-dimensional stable separatrix $W^s(O)$ and $(n-k)$-dimensional unstable separatrix $W^u(O)$ are the following

$$W^s(O) = \{(x_1, \dots, x_n) \,|\, x_{k+1} = 0, \dots, x_n = 0\} = \mathbb{R}^k \subset \mathbb{R}^n,$$
$$W^u(O) = \{(x_1, \dots, x_n) \,|\, x_1 = 0, \dots, x_k = 0\} \overset{\text{def}}{=} \mathbb{R}^n_{k+1} \subset \mathbb{R}^n.$$

Lemma 5. *The function* $F(x_1, \dots, x_n) = \sum_{i=1}^k x_i^2 \sum_{j=k+1}^n x_j^2$ *is the first integral for the system* (1).

Proof. Taking in mind (1), one gets

$$\begin{aligned}\frac{dF}{dt} &= \sum_{\nu=1}^{n} \frac{\partial F}{\partial x_\nu} \dot{x}_\nu = \sum_{i=1}^{k} (2x_i)\dot{x}_i \sum_{j=k+1}^{n} x_j^2 + \sum_{j=k+1}^{n} (2x_j)\dot{x}_j \sum_{i=1}^{k} x_i^2 \\ &= -2 \sum_{i=1}^{k} x_i^2 \sum_{j=k+1}^{n} x_j^2 + 2 \sum_{j=k+1}^{n} x_j^2 \sum_{i=1}^{k} x_i^2 \\ &= 2F(x_1, ., x_n) - 2F(x_1, ., x_n) \equiv 0.\end{aligned}$$

□

By Lemma 5, $F = 1$ defines an $(n-1)$-manifold, denoted H^{n-1}, that divides $\mathbb{R}^n$ into the two open sets

$$\{\vec{x} = (x_1, \dots, x_n) \,|\, F(\vec{x}) < 1\} \overset{\text{def}}{=} U_0, \quad \{\vec{x} = (x_1, \dots, x_n) \,|\, F(\vec{x}) > 1\} \overset{\text{def}}{=} U_\infty.$$

Clearly, U_0 is an invariant neighborhood of O, called *special.*

Fix $k \geq 2$ and denote by T_r^{n-2} the set of points whose coordinates satisfy the equations

$$x_1^2 + \dots + x_k^2 = r^2, \quad r^2(x_{k+1}^2 + \dots + x_n^2) = 1.$$

Then $T_r^{n-2} \subset H^{n-1}$ and T_r^{n-2} is naturally homeomorphic to the product of the spheres $S_{1,k}^{k-1}(r) \times S_{k+1,n}^{n-k-1}\left(\frac{1}{r}\right)$ where

$$S_{1,k}^{k-1}(r) = \left\{ (x_1, \dots, x_k, 0, \dots, 0) \,|\, \sum_{i=1}^{k} x_i^2 = r^2 \right\} \subset \mathbb{R}^k,$$
$$S_{k+1,n}^{n-k-1}\left(\frac{1}{r}\right) = \left\{ (0, \dots, 0, x_{k+1}, \dots, x_n) \,|\, \sum_{j=k+1}^{n} x_j^2 = \frac{1}{r^2} \right\} \subset \mathbb{R}^n_{k+1}.$$

The sphere $S^{k-1}_{1,k}(r)$ bounds the disk $D^k_{1,k} = \{(x_1, \dots, x_k, 0, \dots, 0) \mid \sum_{i=1}^{k} x_i^2 \leq r^2\} \subset \mathbb{R}^k$. For $r = 1$, denote $S^{k-1}_{1,k}(1)$ by $S^{k-1}_{1,k}$. Similarly, $S^{n-k-1}_{k+1,n}(1) = S^{n-k-1}_{k+1,n}$. One can check that every trajectory of $\vec{V}_s$ belonging to H^{n-1} intersects T^{n-2}_r at a unique point. Therefore, H^{n-1} is homeomorphic to $T^{n-2}_r \times \mathbb{R}$.

Let $H^{n-1}_c(0 \leq \tau \leq 1)$ be the union of trajectory arcs of $\vec{V}_s$ that start at $S^{k-1}_{1,k} \times S^{n-k-1}_{k+1,n}$ and finish at $S^{k-1}_{1,k}\left(\frac{1}{\sqrt{e}}\right) \times S^{n-k-1}_{k+1,n}(\sqrt{e})$. In other words,

$$H^{n-1}(0 \leq \tau \leq 1) = \bigcup_{0 \leq \tau \leq 1} f_\tau \left(S^{k-1}_{1,k} \times S^{n-k-1}_{k+1,n}\right).$$

Certainly, $H^{n-1}(0 \leq \tau \leq 1) \subset H^{n-1}$.

Flatness and wildness. For $1 \leq m \leq n$, we presume Euclidean space $\mathbb{R}^m$ to be included naturally in $\mathbb{R}^n$ as the subset whose final $(n - m)$ coordinates each equals 0. Let $e : M^m \to N^n$ be an embedding of closed m-manifold M^m in the interior of n-manifold N^n. One says that $e(M^m)$ is *locally flat at* $e(x)$, $x \in M^m$, if there exists a neighborhood $U(e(x)) = U$ and a homeomorphism $h : U \to \mathbb{R}^n$ such that $h(U \cap e(M^m)) = \mathbb{R}^m \subset \mathbb{R}^n$. Otherwise, $e(M^m)$ is *wild at* $e(x)$ [13]. The similar notation for a compact M^m, in particular $M^m = [0; 1]$).

For the reference, we formulate the following lemma proved in [17] (see also [16, 18]).

Lemma 6. *Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism, and $Sep^\tau(\sigma)$ a separatrix of dimension $1 \leq d \leq n-1$ of a saddle σ. Suppose that $Sep^\tau(\sigma)$ has no intersections with other separatrices. Then $Sep^\tau(\sigma)$ belongs to unstable (if $\tau = s$) or stable (if $\tau = u$) manifolds of some node periodic point, say N, and the topological closure of $Sep^\tau(\sigma)$ is a topologically embedded d-sphere that equals $W^\tau(\sigma) \cup \{N\}$.*

Note that a separatrix $Sep^\tau(\sigma)$ is a smooth manifold. Hence, $Sep^\tau(\sigma)$ is locally flat at every point [13]. However a-priori, *clos* $Sep^\tau(\sigma) = W^\tau(\sigma) \cup \{N\}$ could be wild at the unique point N.

3. Local flatness

Here we prove Theorems 1, 2, and all items of Theorem 4 except the last one.

Morse-Smale flows. The crucial statement for the proof of Theorem 1 is the following lemma.

Lemma 7. *Let M^4_* be a compact 4-manifold the boundary of whose consists of two 3-spheres S^3_1 and S^3_2, $\partial M^4_* = S^3_1 \cup S^3_2$. Suppose that there is a vector field $\vec{V}$ on M^4_* such that*

1. *$\vec{V}$ has a unique fixed point s_* which is a hyperbolic saddle of type* $(2,2)$*;*
2. *$\vec{V}$ is transversal to ∂M^4_*, to be precise $\vec{V}|_{S^3_1}$ is inside and $\vec{V}|_{S^3_2}$ is outside of M^4_*;*
3. *Every trajectory of $\vec{V}$, except the trajectories belonging to the separatrices $W^s(s_*)$, $W^u(s_*)$ of the saddle s_*, intersects the both spheres S^3_1, S^3_2;*
4. *The stable separatrix $W^s(s_*)$ and unstable separatrix $W^u(s_*)$ intersects the spheres S^3_1, S^3_2 along the closed curves $W^s(s_*) \cap S^3_1 = C_1$, $W^u(s_*) \cap S^3_2 = C_2$ respectively.*

Then each of the curves C_1, C_2 is unknotted in S^3_1, S^3_2 respectively.

Proof. The curves C_1, C_2 bound in $W^s(s_*)$, $W^u(s_*)$ the closed disks D_1, D_2 respectively. Since S^3_1, S^3_2 are transversal to $\vec{V}$, s is inside of each D_1, D_2, and $s_* = D_1 \cap D_2$.

Suppose the contradiction, and assume that C_1 is knotted in S^3_1 (the case when C_2 is knotted in S^3_2 is similar). Due to the extended version of Grobman-Hartman theorem, there is a neighborhood U of $D_1 \cup D_2$ such that $\vec{V}|_U$ is locally equivalent to the vector field defined by the linear part of $\vec{V}$ at s_*. By Proposition 2.15 [24], $\vec{V}|_U$ is equivalent to $\vec{V}_s$ when $n = k = 2$.

By conditions, the exterior of C_1 in S^3_1 is homeomorphic to the exterior of C_2 in S^3_2. Hence, C_2 is knotted in S^3_2, [15]. Moreover, S^3_2 can be considered as a result of knot surgery of S^3_1 along the knot C_1. Because of this surgery is not trivial, S^3_2 is not homeomorphic to a 3-sphere [15,20]. The contradiction follows the statement. □

Proof of Theorem 1. Note that any Morse-Smale flow without periodic trajectories has at least one source and sink [28]. In addition, any such flow is gradient in some Riemannian metric [28, 29]. It follows from [14] that $n \in \{4, 8, 16\}$ and the dimension of each separatrix is $\frac{n}{2}$ provided $f^t \in MS^{flow}(M^n, 3)$, $n \geq 4$.

It remains to prove that the both $\operatorname{clos} W^u(\sigma) \overset{\text{def}}{=} S_\omega$ and $\operatorname{clos} W^s(\sigma) \overset{\text{def}}{=} S_\alpha$ are locally flat. If $n = 8$ or $n = 16$, each S_ω and S_α is of codimension ≥ 3. By [31], S_ω and S_α are locally flat. Let us consider the case $n = 4$. There are neighborhoods U_α, U_ω of α and ω respectively homeomorphic to a 4-ball such that $\partial U_\alpha \cap W^s(\sigma)$ (resp., $\partial U_\omega \cap W^u(\sigma)$) is a simple closed curve, say C_α (resp., C_ω). By Lemma 7, C_α (resp., C_ω) is unknotted in ∂U_α (resp., ∂U_ω). This implies that S_ω and S_α are locally flat. □

Proof of Theorem 2. Let ω_f, α_f, σ_f be the sink, source, and saddle of f^t respectively. There are neighborhoods $U(\omega_f)$, $U(\alpha_f)$ of α_f, σ_f respectively such that the boundaries $\partial U(\omega_f)$, $\partial U(\alpha_f)$ are transverse to f^t, and $\sigma_f \notin U(\omega_f) \cup U(\alpha_f)$. Without loss of generality, one can assume that the both $U(\omega_f)$ and $U(\alpha_f)$ homeomorphic to a 4-ball. The similar notation holds for g^t.

By Proposition 2.15 [24], there are neighborhoods $V(\sigma_f)$, $V(\sigma_g)$ of σ_f, σ_g respectively such that each flow $f^t|_{V(\sigma_f)}$, $g^t|_{V(\sigma_g)}$ is equivalent to $f^t_s|_{U_0}$, where U_0 is the special neighborhood. In particular, each intersection $V(\sigma_f) \cap \partial U(\alpha_f) = T_f$, $V(\sigma_g) \cap \partial U(\alpha_g) = T_g$ is a solid torus. We see that there is a homeomorphism $h : V(\sigma_f) \to V(\sigma_g)$ taking the trajectories of $f^t|_{V(\sigma_f)}$ to the trajectories of $g^t|_{V(\sigma_g)}$. Since T_f and T_g are transversal to the flows f^t and f^g respectively, h induces the homeomorphism $T_f \to T_g$ denoted again by h. Obviously, h takes $Sep^u(\sigma_f)$ to $Sep^u(\sigma_g)$. Therefore, h takes $Sep^u(\sigma_f) \cap T_f$ to $Sep^u(\sigma_g) \cap T_g$. According to the flow structure in the special neighborhood U_0, the both $Sep^u(\sigma_f) \cap T_f$ and $Sep^u(\sigma_g) \cap T_g$ are axes of solid toruses T_f and T_g respectively.

By Lemma 7, the curves $Sep^u(\sigma_f) \cap T_f$ and $Sep^u(\sigma_g) \cap T_g$ are unknotted in the 3-spheres $\partial U(\alpha_f)$ and $\partial U(\alpha_g)$ respectively. Hence, the complements to T_f and T_g are solid toruses, and h can be extended to a homeomorphism $\partial U(\alpha_f) \to \partial U(\alpha_g)$. It follows that there is a homeomorphism $h_* : M^4 \setminus (\alpha_f \cup \omega_f) \to M^4 \setminus (\alpha_g \cup \omega_g)$ taking the trajectories to the trajectories. Then h_* is easily extended to M^4 to get a homeomorphism taking the trajectories of f^t to the trajectories of g^t. $\Box$

Morse-Smale diffeomorphisms. Here, $f \in MS^{diff}(M^n, 3)$, $n \geq 4$. By a connectedness of M^n, the non-wandering set of f consists of a sink, say ω, source, say α, and saddle, say σ. Let us show that $2 \leq d = \dim Sep^\tau(\sigma) \leq n-2$. Suppose the contradiction. By Lemma 6, $W^s(\sigma) \cup \{\alpha\} \stackrel{\text{def}}{=} S^1_\alpha$, $W^u(\sigma) \cup \{\omega\} \stackrel{\text{def}}{=} S^{n-1}_\omega$ are topologically embedded circle and $(n-1)$-sphere respectively. Since $n \geq 4$, there is a neighborhood U_ω of S^{n-1}_ω homeomorphic to $S^{n-1}_\omega \times (-1; +1)$ [10,13]. Without loss of generality, one can assume that $f(U_\omega) \subset U_\omega$. The sphere S^{n-1}_ω does not divide M^n because S^{n-1}_ω intersects S^1_α at a unique point σ. As a consequence, $M^n_1 = M^n \setminus U_\omega$ is a connected manifold with two boundary component each homeomorphic to S^{n-1}_ω. Gluing n-balls to this components, one gets a closed manifold M^n_2. Since $f(U_\omega) \subset U_\omega$, one can extend f to M^n_2 such that f will have a source and two sinks. This is impossible.

By [16], the absence of one-dimensional separatrices implies that a Morse-Smale diffeomorphism has unique source and unique sink. It follows that M^n is orientable.

Let M_j be the number of periodic points $p \in Per\ (f)$ those stable Morse index equals $j = \dim W^s(p)$, and $\beta_i = rank\ H_i(M^n, \mathbb{Z})$ the Betti numbers. According to [28],

$$M_0 \geq \beta_0, \quad M_1 - M_0 \geq \beta_1 - \beta_0, \quad M_2 - M_1 + M_0 \geq \beta_2 - \beta_1 + \beta_0, \cdots \quad (2)$$

$$\sum_{i=0}^{n} (-1)^i M_i = \sum_{i=0}^{n} (-1)^i \beta_i. \quad (3)$$

Suppose σ is of type $(n-k, k)$. Then $M_0 = M_n = M_k = 1$. For f^{-1}, one holds $M_0 = M_n = M_{n-k} = 1$. For $j \neq 0$, n, k, $n-k$, one holds $M_j = 0$. Since the left parts of (3) for f and f^{-1} are equal, $(-1)^k = (-1)^{n-k}$. Hence, $n = 2m$ is even, where $m \geq 2$.

We show above that $k \neq 1$ and $k \neq n-1$. As a consequence, $M_1 = M_{n-1} = 0$. Let us show that $k = m$. Suppose the contradiction. Assume for definiteness that $k > m$. It follows from (2) that $\beta_1 = \ldots = \beta_{n-k-1} = 0$ because of $M_1 = \ldots = M_{n-k-1} = 0$. The Poincare duality implies that $\beta_1 = \ldots = \beta_{k-1} = 0$. Hence, $\beta_i = 0$ for all $i = 1, \ldots, n-1$. Then (3) becomes $1 + (-1)^k + (-1)^n = 1 + (-1)^n$. This is impossible.

It remains to prove that $S^k_\omega = W^u(s_0) \cup \{\omega\}$, $S^k_\alpha = W^s(s_0) \cup \{\alpha\}$ are locally flat. It follows from [12] (see [11,31]) that k-manifold has no isolated wild points provided $n \geq 5$, $k \neq n-2$. As a consequence, S^k_ω, S^k_α are locally flat k-spheres. This completes the proof of Theorem 4 except the last item.

4. Wild embedding

Here, we prove Theorem 3 and the last item of Theorem 4.

Examples of Morse-Smale flows. First, we introduce the special Morse-Smale flows $MS^{flow}_{k,n-k}(M^n, 4) \subset MS^{flow}(M^n, k)$, where $k \geq 2$, $n-k \geq 2$. In $\mathbb{R}^n$, consider the linear vector field $\vec{V}_n$ defined by the system

$$\dot{x}_1 = x_1, \quad \dot{x}_2 = x_2, \quad \dot{x}_{n-1} = x_{n-1}, \quad \dot{x}_n = x_n. \tag{4}$$

Clearly, $O = (0, \ldots, 0)$ is a repelling node, and $(n-1)$-sphere $S^{n-1}_j = \{\vec{x} = (x_1, \ldots, x_n) \,|\, x_1^2 + \cdots + x_n^2 = j^2\}$ is transversal to $\vec{V}_n$ for any $j \in \mathbb{N}$. Let S^{k-1}_1 be a smoothly embedded in S^{n-1}_1 $(k-1)$-sphere. Denote by $T(S^{k-1}_1) \subset S^{n-1}_1$ a closed tubular neighborhood of S^{k-1}_1 diffeomorphic to $S^{k-1}_1 \times D^{n-k}$. Let Q^n be the union of rays starting at $O = (0, \ldots, 0)$ through $T(S^{k-1}_1)$. Actually, each ray is the node O and a trajectory through $T(S^{k-1}_1)$. Since $\partial T(S^{k-1}_1)$ is diffeomorphic to $S^{k-1} \times S^{n-k-1}$, the boundary of the set $R \overset{\text{def}}{=} \mathbb{R}^n \setminus (O \cup \ int\ Q^n)$ is diffeomorphic to $S^{k-1} \times S^{n-k-1} \times \mathbb{R}$ where the last factor $\mathbb{R}$ corresponds to the time parameter of (4).

Recall that $\partial U_0 = S^{k-1} \times S^{n-k-1} \times \mathbb{R}$ where the last factor $\mathbb{R}$ corresponds to the time parameter of (1). Let $\eta : \partial U_0 \to \partial R$ be the natural identification. Then $(\text{clos } U_0) \bigcup_\eta R$ is a manifold. Because of η is a homotopy identity on the factor $S^{k-1} \times S^{n-k-1}$, one can extend the structure of smooth manifold to $O \cup (\text{clos } U_0) \bigcup_\eta R \overset{\text{def}}{=} R_n$ such that the set $\left(S^{n-1}_1 \setminus T(S^{k-1}_1)\right) \bigcup_\eta (S^{k-1}_{1,k} \times D^{n-k}_{k+1,n})$ is homeomorphic to S^{n-1}_1 that bounds the neighborhood of O in R_n homeomorphic to $\mathbb{R}^n$.

Let A be a closed annulus bounded by S^{n-1}_1, S^{n-1}_2 in $\mathbb{R}^n$, and $\mathbb{B}^n_2 \subset \mathbb{R}^n$ the closed n-ball bounded by S^{n-1}_2. By construction, η glue $H^{n-1}(0 \leq \tau \leq$

1) with $\partial(A \setminus Q^n)$. Therefore, η glue $\partial(S_2^{n-1} \setminus Q^n)$ with

$$\partial\left(D^k_{1,k}\left(\frac{r}{\sqrt{e}}\right) \times S^{n-k-1}_{k+1,n}\left(\frac{\sqrt{e}}{r}\right)\right) = S^{k-1}_{1,k}\left(\frac{r}{\sqrt{e}}\right) \times S^{n-k-1}_{k+1,n}\left(\frac{\sqrt{e}}{r}\right).$$

Put by definition,

$$D^n(\tau \leq 0) = \cup_{0\leq\tau\leq 1} f_\tau\left(D^k_{1,k}\left(\frac{r}{\sqrt{e}}\right) \times S^{n-k-1}_{k+1,n}\left(\frac{\sqrt{e}}{r}\right)\right),$$
$$B_n = D^n(\tau \leq 0) \cup_\eta \partial\left(\mathbb{B}_2^n \setminus Q^n\right).$$

The set B_n is a part of R_n with the piecewise smooth boundary

$$\partial B_n = (S_2^{n-1} \setminus Q^n) \cup_\eta \left(D^k_{1,k}\left(\frac{r}{\sqrt{e}}\right) \times S^{n-k-1}_{k+1,n}\left(\frac{\sqrt{e}}{r}\right)\right).$$

The vector fields $\vec{V}_s$, $\vec{V}_n$ define the vector field $\vec{v}$ on $\int B_n$. Smoothing the boundary of B_n and $\vec{v}$ to get a smooth vector field (denoted by $\vec{v}$ again) that is transversal to ∂B_n. By construction, $\vec{v}$ has the repelling node O and the saddle, say s_0, of the type $(n-k,k)$. Note that $S^{k-1}_{1,k} = W^s(s_0) \cap S_1^{n-1} = S^{k-1}_{1,k}$, $S^{n-k-1}_{k+1,n} = W^u(s_0) \cap \partial B_n = S^{n-k-1}_{k+1,n}$. Take the copy B'_n of B_n with the vector field $-\vec{v}$. Clearly, $-\vec{v}$ has an attracting node, say O', and saddle, say s'_0, of the type $(k, n-k)$. The intersection of $W^s(s'_0)$ with $\partial B'_n$ is a sphere $S^{n-k-1,*}_{k+1,n}$. Without loss of generality, one can assume that $S^{n-k-1,*}_{k+1,n} \cap S^{n-k-1,*}_{k+1,n} = \emptyset$ because of $k \geq 2$.

Let $B_n \cup_\psi B'_n \stackrel{\text{def}}{=} M^n$ be the manifold obtained by the identification ψ of the boundaries of B_n, B'_n [19]. The fields $\vec{v}$, $-\vec{v}$ defines on M^n the Morse-Smale vector field $\vec{V}$ that induces the Morse-Smale flow denoted by $f^t_{k,n-k}(S^{k-1}_{1,k},)$. Obviously, $f^t_{k,n-k}(S^{k-1}_{1,k},) \in MS^t_{k,n-k}(M^n, 4)$. For $k = n-2$ and $n \geq 4$, take $S_1^{k-1} = S_1^{n-3}$ to be smoothly embedded and knotted codimension two sphere. Well-known that such spheres exist [13]. According to [4], [2]), and [12], the spheres $W^s(s_0) \cup O$, $W^u(s'_0) \cup O'$ are wild at O and O' respectively. This completes the proof of Theorem 3.

Examples of Morse-Smale diffeomorphisms. Here, we keep the notation of section 2 for $n = 4$, $k = 2$. Given a 2-torus T^2 that is the boundary of solid torus $P^3 = S^1 \times D^2$, $T^2 = \partial P^3 = S^1 \times \partial D^2$, any curve $\{\cdot\} \times \partial D^2$ is called a *meridian*, and $S^1 \times \{\cdot\}$ is a *parallel*. Recall that a 3-sphere S^3 can be obtained after a gluing of two copy of solid torus P^3, $S^3 = P^3 \cup_\nu P^3$, where the glue mapping $\nu : T^2 \to T^2$ takes a meridian to parallel and vise versa. This representation of S^3 is a standard Heegaard splitting of genus 1.

Given any $t \in \mathbb{R}$, we introduce 2-torus

$$\mathbb{T}^2_t = \{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 = \exp(-2t),\ x_3^2 + x_4^2 = \exp 2t\} \subset H^3$$

that is the boundary of the following solid toruses

$$P^3_{12,t} = \left\{(x_1, x_2, x_3, x_4) \middle|\, x_1^2 + x_2^2 = \exp(-2t),\ x_3^2 + x_4^2 \leq \exp 2t\right\},$$

$$P^3_{34,t} = \left\{(x_1, x_2, x_3, x_4) |\, x_1^2 + x_2^2 \le \exp(-2t),\ x_3^2 + x_4^2 = \exp 2t\right\},$$

that form a standard Heegaard splitting $P^3_{12,t} \cup_{id} P^3_{34,t}$ of genus 1. The 3-sphere $P^3_{12,t} \cup_{id} P^3_{34,t}$ bounds the 4-ball, say B^4_0. Moreover, P^3_{12,t_0} and P^3_{34,t_0} divide U_0 into three domains B^4_0, $U_{12}(t \le t_0)$, $U_{34}(t \ge t_0)$, Fig. 2, where

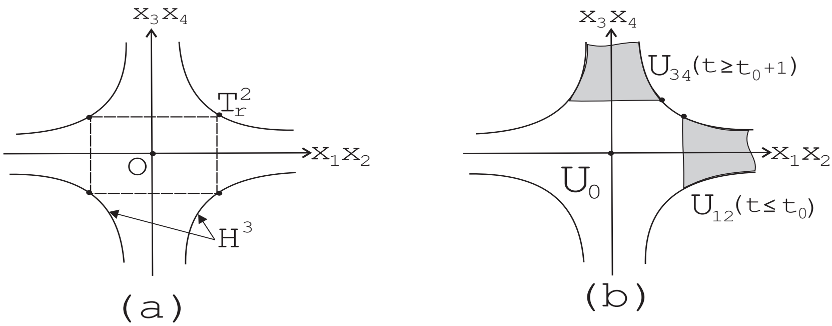

Fig. 2

$$U^4_{12}(t \le t_0) = \Big\{(x_1, x_2, x_3, x_4) |\, (x_1^2 + x_2^2 > \exp(-2t_0), x_3^2 + x_4^2 < \exp 2t_0, (x_1^2 + x_2^2)(x_3^2 + x_4^2) < 1\Big\},$$

$$U^4_{34}(t \ge t_0) = \Big\{(x_1, x_2, x_3, x_4) |\, (x_1^2 + x_2^2 < \exp(-2t_0), x_3^2 + x_4^2 > \exp 2t_0, (x_1^2 + x_2^2)(x_3^2 + x_4^2) < 1\Big\}.$$

We see that a 2-torus $\mathbb{T}^2_{t_0}$ divides H^3 into two parts $\mathbb{T}^2_{t \le t_0} = \cup_{t \le t_0} \mathbb{T}^2_t$, $\mathbb{T}^2_{t \ge t_0} = \cup_{t \ge t_0} \mathbb{T}^2_t$ such that $\partial U_{12}(t \le t_0) = \mathbb{T}^2_{t \le t_0}$ and $\partial U_{34}(t \ge t_0) = \mathbb{T}^2_{t \ge t_0}$.

Let us introduce the coordinates (t, u, v) on $H^3 = \partial U_0$ as follows

$$\begin{aligned} x_1 &= e^{-t} \cos 2\pi u, \quad x_2 = e^{-t} \sin 2\pi u, \\ x_3 &= e^{t} \cos 2\pi v, \qquad x_4 = e^{t} \sin 2\pi v, \end{aligned} \tag{5}$$

where u, $v \in [0; 1)$ are cyclic coordinates on meridians or parallels on $\mathbb{T}^2_t$. Later on, (t, u, v) becomes (t_2, u_2, v_2).

Take the copy of $\mathbb{R}^4$ endowed with the coordinates (x_1, x_2, x_3, x_4), and the flow f^t_{NS} that is defined by (4) for $n = 4$. The diffeomorphism

$$f_{NS} = f^1_{NS} : (x_1, x_2, x_3, x_4) \to (ex_1, ex_2, ex_3, ex_4)$$

is a shift-time-one along the trajectories of f^t_{NS}. Clearly, the family of spheres

$$S^3_m = \{(x_1, \dots, x_4) \,:\, x_2^2 + x_2^2 + x_3^2 + x_4^2 = e^{2m}\}, \quad S^2_m = S^4_m \cap \{x_4 = 0\}$$

is invariant under f_{NS}.

Now we construct the special Artin-Fox curve as follows. On S_0^2, one takes the points $Y_0^0\left(\frac{1}{2};0;\frac{\sqrt{3}}{2};0\right)$, $Y_1^0(0;0;1;0)$, $Y_2^0\left(\frac{\sqrt{3}}{2};0;\frac{1}{2};0\right)$. Between the spheres S_0^3 and $f_{NS}(S_0^3) = S_1^3$, one takes arcs d_1, d_2, d_3 forming Artin-Fox configuration such that d_1 connects Y_0^0, Y_1^0, and d_2 connects $f_{NS}(Y_1^0)$, $f_{NS}(Y_2^0)$, and d_3 connects Y_2^0, $f_{NS}(Y_0^0)$, Fig 3, (a). One can assume that the union

$$l^\circ \stackrel{\text{def}}{=} \cup_{k\in\mathbb{Z}} f_{NS}^k(d_1 \cup d_2 \cup d_3)$$

is a simple arc connecting the origin $O = N$ and the infinity point S such that $l = \{S, N\} \cup l^\circ$ is Artin-Fox curve [5]. Here, we consider the 3-sphere

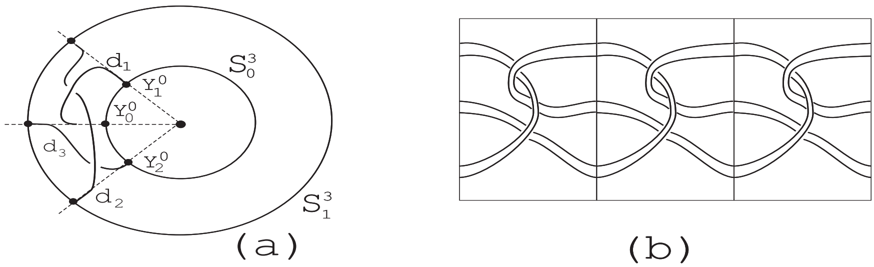

Fig. 3

S^3 the both as $\mathbb{R}^3$ completed by the infinity point S, and as the natural part of the 4-sphere S^4. Without loss of generality, we can suppose that l intersects transversally all spheres S_m^3, $m \in \mathbb{Z}$.

Let $\mathcal{R}$ be the rotation of the half-space $\mathbb{R}_+^3$ about 2-plane $x_4 = 0 = x_3$ that is defined as

$$\begin{aligned} \overline{x}_1 &= x_1, & \overline{x}_2 &= x_2, \\ \overline{x}_3 &= x_3\cos 2\pi v - x_4 \sin 2\pi v, & \overline{x}_4 &= x_3 \sin 2\pi v + x_4 \cos 2\pi v, \end{aligned} \tag{6}$$

where $v \in [0,1]$. Since S and N are fixed under $\mathcal{R}$ and $l^\circ = l \setminus (\{S, N\}) \subset \mathbb{R}_+^3$, $\mathcal{R}(l) = l_{\mathcal{R}}$ is a topologically embedded 2-sphere. It follows from [2,12] that $l_{\mathcal{R}}$ is wild at S and N.

One can introduce the smooth injective parametrization $\theta : \mathbb{R} \to l^\circ$ such that l intersects every S_m^3, $m \in \mathbb{Z}$, at three points $l^\circ(m)$, $l^\circ\left(m + \frac{1}{3}\right)$, $l^\circ\left(m + \frac{2}{3}\right)$ with parameters $t = m$, $m + \frac{1}{3}$, $m + \frac{2}{3}$ respectively. Since l is invariant under f_{NS}, there is a tubular neighborhood $T(l^\circ)$ of l° such that $T(l^\circ)$ is invariant under f_{NS}, and $T(l^\circ)$ is diffeomorphic to $\mathbb{R} \times D^2$, and

$T(l^\circ)$ intersects every S^3_m, $m \in \mathbb{Z}$, at three disks

$$D_{m,0} = \{m\}\times D^2, \quad D_{m+\dfrac{1}{3},1} = \left\{m|\frac{1}{3}\right\}\times D^2, \quad D_{m+\frac{2}{3},2} = \left\{m+\frac{2}{3}\right\}\times D^2,$$

where $Y_i^0 \in D_{\frac{i}{3},i}$, $i = 1,2,3$.

Clearly, $\mathcal{R}\left(\mathbb{R}\times D^2\right)_{AF}$ is a neighborhood of $\mathcal{R}(l^\circ) = l^\circ_{\mathcal{R}}$ which is homeomorphic to $\mathbb{R}\times D^2\times S^1$ the boundary of those is $\mathbb{R}\times S^1\times S^1$. Therefore, this boundary is endowed with the coordinates (t,u,v) defined by (6). Here, we denote (t,u,v) by (t_1,u_1,v_1).

Put by definition, $\mathcal{I} = \int\left(\mathbb{R}\times D^2\right)_{AF}\cup\{N,S\}$, $M_1 = S^4\setminus\mathcal{I}$, and $M_2 =$ clos U_0. We see that ∂M_1 is homeomorphic to $\mathbb{R}\times S^1\times S^1 \simeq \partial M_2 = H^3$. Recall that H^3 endowed with the coordinates (t_2,u_2,v_2). The mapping $\Xi : \partial M_2 \to \partial M_1$ is defined as follows:

$$t_1 = t_2, \quad u_1 = u_2 - v_2, \quad v_1 = v_2. \tag{7}$$

According to [19], the set $M^4_* = M_1\cup_\Xi M_2$ is a noncompact manifold. Clearly, the set $M_1' = S^4\setminus\int\left(\mathbb{R}\times D^2\right)_{AF}$ is compact, and $M_1 = M_1'\setminus\{N,S\}$.

The toruses $\mathbb{T}^2_0$, $\mathbb{T}^2_1$ divide H^3 into three sets $\mathbb{T}^2_{t\ge1}$, $\mathbb{T}^2_{0\le t\le1}$, $\mathbb{T}^2_{t\le0}$ where $\mathbb{T}^2_{0\le t\le1}$ is compact while the others are non-compact. Denote by $\Xi_{t\ge1}$, $\Xi_{0\le t\le1}$, $\Xi_{t\le0}$ the restriction of Ξ on $\mathbb{T}^2_{t\ge1}$, $\mathbb{T}^2_{0\le t\le1}$, $\mathbb{T}^2_{t\le0}$ respectively. Similarly, the circles $\left(\{0\}\times\partial D^2\right)_{AF}$, $\left(\{1\}\times\partial D^2\right)_{AF}$ divide the boundary of $\left(\mathbb{R}\times D^2\right)_{AF}$ into three cylinders $C_{0\le t\le1} = \left([0;1]\times S^1\right)_{AF}$, $C_{t\ge1} = \left([1;+\infty)\times S^1\right)_{AF}$, $C_{t\le0} = \left((-\infty;0]\times S^1\right)_{AF}$ where $C_{0\le t\le1}$ is compact while the others are not. Denote $\mathcal{R}(C_{0\le t\le1})$, $\mathcal{R}(C_{t\ge1})$, $\mathcal{R}(C_{t\le0})$ by

$$C_{0\le t\le1,\mathcal{R}} \simeq [0;1]\times S^1\times S^1, \quad C_{t\ge1,\mathcal{R}} \simeq [1;+\infty)\times S^1\times S^1,$$
$$C_{t\le0,\mathcal{R}} \simeq (-\infty;0]\times S^1\times S^1$$

respectively.

Clearly, $\mathbb{R}(S^2_m) = S^3_m \subset \mathbb{R}^4\setminus\{N,S\}$. Let $K(\ge m) = \{(x_1,x_2,x_3,x_4) : x_2^2+x_2^2+x_3^2+x_4^2 \ge e^{2m}\}$ be the exterior of S^3_m, and $K(\le m)$ the interior of S^3_m with the hole N. Denote by $K(m_1,m_2)$ the closed annulus between the spheres $S^3_{m_1}$, $S^3_{m_2}$. Because of (7), $M^4_* = M_1\cup_\Xi M_2$ is the union of the following sets:

1) $U_{12}(t\le0)\cup_{\Xi_{t\le0}}[K(\le0)\setminus\mathcal{I}] \overset{\text{def}}{=} B_N$,

2) $U_{34}(t\ge1)\cup_{\Xi_{t\ge1}}[K(r\ge1)\setminus\mathcal{I}] \overset{\text{def}}{=} B_S$

3) $B^4(0\le t\le1)\cup_{\Xi_{0\le t\le1}}[K(-1,1)\setminus\mathcal{I}] \overset{\text{def}}{=} B_*$.

It follows from (7) that the set

$$S_{*,-m}\cup_{\Xi|_{t\le0}}\left(P^3_{12,-m}\cup P^3_{12,-m+\frac{1}{3}}\cup P^3_{12,-m+\frac{2}{3}}\right) \overset{\text{def}}{=} S^3_{m,*}$$

is a 3-sphere, since the lens $L(1,-1)$, $L(1,1)$ are the 3-sphere $S^3 = L(1,0)$. Note that if l deforms outside of some compact part to being rays, l becomes a locally flat arc. It follows that the set $K_N(-m,-m-1)$ between $S^3_{m,*}$, $S^3_{m+1,*}$ can be embedded in $\mathbb{R}^4$. This implies that $K_N(-m,-m-1)$ homeomorphic to the annulus $S^3\times[0;1]$, and hence B_N can be completed by a point to be a smooth manifold. Similarly, B_S. We see that $M^4=M_1'\cup_\Xi M_2$ admits a structure of closed smooth 4-manifold.

The shift-time-one diffeomorphisms $f_{NS}:M_1\to M_1$, $f^1_s:M_2\to M_2$ induce the diffeomorphism $f:M^4\to M^4$ with two nodes, say α and ω, and a saddle. By construction, the spheres S_ω, S_α are wildly embedded. This completes the proof of Theorem 4.

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