LOG TERMINAL SINGULARITIES, PLATONIC TUPLES AND ITERATION OF COX RINGS

IVAN ARZHANTSEV, LUKAS BRAUN, JÜRGEN HAUSEN AND MILENA WROBEL

Abstract. Looking at the well understood case of log terminal surface singularities, one observes that each of them is the quotient of a factorial one by a finite solvable group. The derived series of this group reflects an iteration of Cox rings of surface singularities. We extend this picture to log terminal singularities in any dimension coming with a torus action of complexity one. In this setting, the previously finite groups become solvable torus extensions. As explicit examples, we investigate compound du Val threefold singularities. We give a complete classification and exhibit all the possible chains of iterated Cox rings.

1. Introduction

We begin with a brief discussion of the well known surface case [1, 6, 11]. The two-dimensional log terminal singularities are exactly the quotient singularities $\mathbb{C}^2/G$, where G is a finite subgroup of the general linear group $\mathrm{GL}(2)$. The particular case that G is a subgroup of $\mathrm{SL}(2)$ leads to the du Val singularities A_n, D_n, E_6, E_7 and E_8, named according to their resolution graphs. They are precisely the rational double points, and are also characterized by being the canonical surface singularities. The du Val singularities fill the middle row of the following commutative diagram involving *all* two-dimensional log terminal singularities:

$$
\begin{tikzcd}
& \mathbb{C}^2 \arrow[dl] \arrow[dr, "\mathrm{CR}"] \arrow[drrr, bend left] \arrow[drrrrr, bend left] \arrow[drrrrrr, bend left] & & & & & & \\
E_8 \arrow[d] & & A_n \arrow[rr, "\mathrm{CR}"] \arrow[d] \arrow[dr, "n \text{ odd}"] & & D_{n+3} \arrow[rr, "\mathrm{CR}", "n=1"'] \arrow[d] \arrow[dr, "n=1"] & & E_6 \arrow[r, "\mathrm{CR}"] \arrow[d] & E_7 \arrow[d] \\
E_8^{\imath} & & A_{n,k}^{\imath} & D_{(n+3)/2}^{2,\imath} & D_{n+3}^{\imath} & E_6^{3,\imath} & E_6^{\imath} & E_7^{\imath}
\end{tikzcd}
$$

Here, all arrows indicate quotients by finite groups. The label "CR" tells us that this quotient represents a Cox ring; recall that the Cox rings of (the resolutions of) the du Val singularities $\mathbb{C}^2/G$ have been computed in [10, 13], see also the example given below. So, E_6 is the spectrum of the Cox ring of E_7 etc.. In fact, the chain of Cox rings reflects the derived series of the binary octahedral group $\widetilde{S}_4 \subseteq \mathrm{SL}(2)$, producing the E_7 singularity:

$$\widetilde{S}_4 \supseteq \widetilde{A}_4 \supseteq \widetilde{D}_4 \supseteq \{\pm I_2\} \supseteq \{I_2\},$$

where $\widetilde{A}_4$ is the binary tetrahedral group, $\widetilde{D}_4$ the binary dihedral group, and I_2 stands for the 2×2 unit matrix. The respective CR labelled arrows stand for quotients by the factors of this derived series. The arrows passing from the middle to the lower row indicate index-one covers: the upper surface is Gorenstein, one divides

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by a cyclic group of order $\imath$ and the lower surface is of Gorenstein index $\imath$. Finally, the superscripts 2 in $D^{2,\imath}_{(n+3)/2}$ and 3 in $E^{3,\imath}_6$ denote the "canonical multiplicity" of the singularity, generalizing the "exponent" discussed in [9, 12]; see 4.2. For a discussion of the surface case based on the methods provided in this article, see Example 4.8.

Another feature of the log terminal surface singularities is that, as quotients $\mathbb{C}^2/G$ by a finite subgroup $G \subseteq \mathrm{GL}(2)$, they all come with a non-trivial $\mathbb{C}^*$-action, induced by scalar multiplication on $\mathbb{C}^2$. The higher dimensional analogue of $\mathbb{C}^*$-surfaces are T-varieties X of complexity one, that means varieties X with an effective action of an algebraic torus T which is of dimension one less than X. The notion of log terminality is defined in general via discrepancies in the ramification formula; see Section 3 for a brief reminder. In higher dimensions, log terminal singularities form a larger class than the quotient singularities $\mathbb{C}^n/G$ with G a finite subgroup of $\mathrm{GL}(n)$. Our aim is, however, to extend the picture drawn at the beginning for the surface case to log terminal singularities with a torus action of complexity one in any dimension.

We use the Cox ring based approach developed in [17, 18, 19]. Recall that the Cox ring of a normal variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ and only constant globally invertible functions is

$$\mathcal{R}(X) \ := \ \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),$$

where we refer to [2] for the necessary background. If X comes with a torus action of complexity one, then the Cox ring $\mathcal{R}(X)$ admits an explicit description in terms of generators and very specific trinomial relations. Vice versa, one can abstractly write down all rings that arise as the Cox ring of some T-variety X of complexity one. Let us briefly summarize the procedure; see Section 2 and [17, 19] for the details.

Construction 1. Fix integers $m \ge 0$, $\iota \in \{0,1\}$ and $r, n > 0$ and a partition $n = n_\iota + \dots + n_r$. For every $i = \iota, \dots, r$ let $l_i := (l_{i1}, \dots, l_{in_i}) \in \mathbb{Z}^{n_i}_{>0}$ with $l_{i1} \ge \ldots \ge l_{in_i}$ and $l_{\iota 1} \ge \ldots \ge l_{r1}$ and define a monomial

$$T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Denote the polynomial ring $\mathbb{C}[T_{ij}, S_k;\ i = \iota, \dots, r,\ j = 1, \dots, n_i,\ k = 1, \dots, m]$ for short by $\mathbb{C}[T_{ij}, S_k]$. We distinguish two types of rings:

Type 1. Take $\iota = 1$ and pairwise different scalars $\theta_1 = 1, \theta_2, \dots, \theta_{r-1} \in \mathbb{C}^*$ and define for each $i = 1, \dots, r-1$ a trinomial

$$g_i \ := \ T_i^{l_i} - T_{i+1}^{l_{i+1}} - \theta_i.$$

Then we obtain a factor ring

$$R \ = \ \mathbb{C}[T_{ij}, S_k] / \langle g_1, \dots, g_{r-1} \rangle.$$

Type 2. Take $\iota = 0$ and pairwise different scalars $\theta_0 = 1, \theta_1, \dots, \theta_{r-2} \in \mathbb{C}^*$ and define for each $i = 0, \dots, r-2$ a trinomial

$$g_i \ := \ \theta_i T_i^{l_i} + T_{i+1}^{l_{i+1}} + T_{i+2}^{l_{i+2}}.$$

Then we obtain a factor ring

$$R \ = \ \mathbb{C}[T_{ij}, S_k] / \langle g_0, \dots, g_{r-2} \rangle.$$

As we explain later, the rings R come with a natural grading by a finitely generated abelian group K_0 and suitable downgradings $K_0 \to K$ give us Cox rings of rational, normal, varieties X with $\mathrm{Cl}(X) = K$ that come with a torus action of complexity one. More geometrically, X arises as a quotient of an open set $\widehat{X} \subseteq \overline{X}$ of the total coordinate space $\overline{X} = \operatorname{Spec} R$ by the quasitorus H having K as its character group. Conversely, basically every rational, normal variety X with a torus action of complexity one can be presented this way.

Geometrically speaking, Type 1 leads to the T-varieties of complexity one that admit non-constant global invariant functions and Type 2 to those having only constant global invariant functions. The varieties of Type 1 turn out to be locally isomorphic to toric varieties. In particular, they are all log terminal and the study of their singularities is essentially toric geometry, see Corollary 3.5 for a precise formulation. We therefore mainly concentrate on Type 2. There, the true non-toric phenomena occur, as for instance the singularities D_n, E_6, E_7 and E_8 in the surface case.

Characterizing log terminality for a T-variety of complexity one of Type 2 involves platonic triples, that means, triples of the form

$$(5,3,2), \quad (4,3,2), \quad (3,3,2), \quad (x,2,2), \quad (x,y,1),$$

where $x \geq y \in \mathbb{Z}_{\geq 1}$. We say that positive integers $a_0, \ldots, a_r$ form a *platonic tuple* if, after reordering decreasingly, the first three numbers are a platonic triple and all others equal one. Moreover, in the setting of Construction 1, we say that a ring R of Type 2 is *platonic* if every $(l_{0j_0}, \ldots, l_{rj_r})$ is a platonic tuple.

Example 1. The platonic rings of Type 2 in dimension two are the polynomial ring $\mathbb{C}[T_1, T_2]$ and the factor rings $\mathbb{C}[T_1, T_2, T_3]/\langle f \rangle$, where f is one of

$$T_1^y + T_2^2 + T_3^2,\ y \in \mathbb{Z}_{>1}, \quad T_1^3 + T_2^3 + T_3^2, \quad T_1^4 + T_2^3 + T_3^2, \quad T_1^5 + T_2^3 + T_3^2.$$

Endowed with a suitable grading, $\mathbb{C}[T_1, T_2]$ is the Cox ring of A_n, and the other rings, according to the above order of listing, of D_{y-2}, E_6, E_7 and E_8.

Our first result says that a rational, normal variety X with a torus action of complexity one of Type 2 has at most log terminal singularities if and only if there occur enough platonic tuples $(l_{0j_0}, \ldots, l_{rn_r})$ in the Cox ring R; see Theorem 3.13 for the precise meaning of "enough". In the affine case, the result specializes to the following; compare also [14, Ex. 2.20] for an earlier result in a particular case and [22, Cor. 5.8] for a related characterization.

Theorem 1. *An affine, normal, $\mathbb{Q}$-Gorenstein, rational variety X with torus action of complexity one of Type 2 has at most log terminal singularities if and only if its Cox ring R is a platonic ring.*

Set for the moment $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i})$. Then, by [19], a ring R of Type 1 is factorial if and only if $\mathfrak{l}_i = 1$ holds for all $i = 1, \ldots, r$. Moreover, a ring R of Type 2 is factorial if and only if the $\mathfrak{l}_i$ are pairwise coprime for $i = 0, \ldots, r$, see [17, Thm. 1.1].

Example 2. In dimension two, the factorial platonic rings R of Type 2 are the polynomial ring $\mathbb{C}[T_1, T_2]$ and the ring $\mathbb{C}[T_1, T_2, T_3]/\langle T_1^5 + T_2^3 + T_3^2 \rangle$.

To extend the iteration of Cox rings $\mathbb{C}^2 \to A_1 \to D_4 \to E_6 \to E_7$ observed in the surface case to higher dimensions, we have to allow instead of only finite abelian groups also non-finite abelian groups in the respective quotients.

Theorem 2. *Let X_1 be a rational, normal, affine variety with a torus action of complexity one of Type 2 and at most log terminal singularities. Then there is a unique chain of quotients*

$$X_p \xrightarrow{/\!\!/ H_{p-1}} X_{p-1} \xrightarrow{/\!\!/ H_{p-2}} \quad \ldots \quad \xrightarrow{/\!\!/ H_3} X_3 \xrightarrow{/\!\!/ H_2} X_2 \xrightarrow{/\!\!/ H_1} X_1 \ ,$$

where $X_i = \mathrm{Spec}(R_i)$ holds with a platonic ring R_i for $i \geq 2$, the ring R_p is factorial and each $X_i \to X_{i-1}$ is the total coordinate space.

Note that iteration of Cox rings requires in each step finite generation of the divisor class group $\mathrm{Cl}(\overline{X})$ of the total coordinate space of X. The latter merely means that the curve Y with function field $\mathbb{C}(\overline{X})^{H_0^0}$ is of genus zero, where $H_0^0 \subseteq H_0$ is the unit component of the quasitorus H_0 with character group $\mathrm{Cl}(X)$. In Theorem 5.3, we establish a formula for the genus of Y in terms of the entries l_{ij} of the defining matrix P of $R = \mathcal{R}(X)$, generalizing the case of $\mathbb{C}^*$-surfaces settled in [25, Prop. 3, p. 64]. This allows us to conclude that for log terminal affine X, the total coordinate space is always rational. Together with the fact that the total coordinate space of a log terminal affine X is canonical, see Proposition 5.1, we obtain that Cox ring iteration is possible in the log terminal case; see Remark 5.12 for a discussion of a non log terminal example with rational Cox ring. The final step in proving Theorem 2 is to show that the Cox ring iteration even stops after finitely many steps. For this, we compute explicitly in Proposition 6.6 the equations of the iterated Cox ring.

The next result shows that, in a large sense, the log terminal singularities with torus action of complexity one still can be regarded as quotient singularities: the affine plane $\mathbb{C}^2$ and the finite group $G \subseteq \mathrm{GL}(2)$ of the surface case have to be replaced with a factorial affine T-variety of complexity one and a solvable reductive group.

Theorem 3. *Let X be a rational, normal, affine variety of Type 2 with a torus action of complexity one and at most log terminal singularities.*

(i) *X is a quotient $X = X' /\!\!/ G$ of a factorial affine variety $X' := \mathrm{Spec}(R')$ by a solvable reductive group G, where R' is a factorial platonic ring.*
(ii) *The presentation of Theorem 2 is regained by $H_i := G^{(i-1)}/G^{(i)}$ and $X_i := X'/G^{(i-1)}$, where $G^{(i)}$ is the i-th derived subgroup of G.*

Example 3. Every log terminal affine $\mathbb{C}^*$-surface is a quotient of $\mathbb{C}^2$ or the E_8-singular surface $V(T_1^5 + T_2^3 + T_3^2) \subseteq \mathbb{C}^3$ by a finite solvable group.

A natural three-dimensional generalization of du Val singularities are the *compound du Val singularities*, introduced in [26]: these are normal, canonical Gorenstein threefold singularities $x \in X$ such that a general hypersurface section through x has a du Val (surface) singularity at x. The isolated compound du Val singularities are precisely the terminal Gorenstein singularities. If a threefold X admits at most compound du Val singularities, then, for a given singular point $x \in X$, we have possible one-dimensional irreducible components $C_1, \ldots, C_r$ of the singular locus that contain x. The compound du Val singularity type (cDV-type) of x is denoted by $S(x_1), \ldots, S(x_r) \to cS(x)$, where $S(x_i)$ stands for the type of the du Val surface singularity obtained by a general hypersurface section through a general point of $x_i \in C_i$ and $S(x)$ for that through x; the c just indicates compound du Val. The following result goes one step beyond the known [8] case of toric compound du Val singularities.

Theorem 4. *The following table provides the equations for the affine threefolds with at most compound du Val singularities which are toric (nos. 1 – 3) or non-toric with a torus action of complexity one of Type 2 (nos. 4 – 18).*

No.	*cDV-type*	*equation in* $\mathbb{C}^4$
1	$A_l \times \mathbb{C}$	$T_1T_2 + T_3^{l+1}$
2	$A_{l_1-1}, A_{l_2-1} \to cA_{l_1+l_2-1}$	$T_1T_2 + T_3^{l_1}T_4^{l_2}$
3	$A_1, A_1, A_1 \to cD_4$	$T_1^2 + T_2T_3T_4$
4	$D_{l+3} \times \mathbb{C}$	$T_1^2 + T_2^2T_3 + T_3^{l+2}$
5	$A_1, A_{l-1} \to cD_{l+4}$	$T_1^2 + T_2^2T_3 + T_3T_4^{l+2}$
6	$E_6 \times \mathbb{C}$	$T_1^2 + T_2^3 + T_3^4$
7	$E_7 \times \mathbb{C}$	$T_1^2 + T_2^3 + T_2T_3^3$
8	$E_8 \times \mathbb{C}$	$T_1^2 + T_2^3 + T_3^5$
9a	$A_{l-1} \to cA_L$	$T_1T_2 + \left(T_3^{L_1+1} + T_4^{L_2+1}\right)^l$, $L = \min(L_1+1, L_2+1)l - 1$
9b	$A_{l_j-1} \to cA_L$	$T_1T_2 + \prod_{j=1}^{r-1}\left(jT_3^{L_1+1} + (2j-1)T_4^{L_2+1}\right)^{l_j}$, $L = \min(L_i+1)\sum l_j - 1$
9c	$A_{L_3-1}, A_{l_j-1} \to cA_L$	$T_1T_2 + T_3^{L_3}\prod_{j=1}^{r-1}\left(jT_3^{L_1} + (2j-1)T_4^{L_2+1}\right)^{l_j}$, $L = \min(L_3 + L_1\sum l_j - 1, L_2\sum l_j - 1)$
9d	$A_{L_3-1}, A_{L_4-1}, A_{l_j-1} \to cA_L$	$T_1T_2 + T_3^{L_3}T_4^{L_4}\prod_{j=1}^{r-1}\left(jT_3^{L_1} + (2j-1)T_4^{L_2}\right)^{l_j}$, $L = \min_{k=3,4}(L_k + l_{k-2}\sum l_j - 1)$
10	$A_{l+1} \to cD_{l+3}$	$T_1^2 + T_2^2T_3 + T_4^{l+2}$
11	$A_{2l+1} \to cD_{2l+2}$	$T_1^2 + T_2^2T_3 + T_2T_4^{l+1}$
12	$A_{l_2-1}, D_{l_1+2} \to cD_{l_1+l_2+2}$	$T_1^2 + T_2^2T_3 + T_3^{l_1+1}T_4^{l_2}$
13	$A_1, A_1 \to cD_{l+3}$	$T_1^2 + T_2T_3T_4 + T_4^{l+2}$
14	$A_1, A_1, A_2 \to cE_6$	$T_1^2 + T_2^3 + T_3^2T_4^2$
15	$D_4 \to cE_6, cE_7$	$T_1^2 + T_2^3 + T_3^3T_4$
16	$A_1, D_4 \to cE_7$	$T_1^2 + T_2^3 + T_2T_3T_4^2$
17	$A_2, D_4 \to cE_8$	$T_1^2 + T_2^3 + T_3^2T_4^3$
18	$E_6 \to cE_8$	$T_1^2 + T_2^3 + T_3T_4^4$

Here, parameters are integers greater than zero with the exponents containing L_1, L_2 *in nos. 9a to 9d being coprime,* A_0 *means that there is no singularity and* $D_l \cong A_l$ *for* $l \leq 3$.

The defining data as toric or T-varieties of complexity one for the varieties listed in Theorem 4 are provided in Section 7. Finally, we study the possible Cox ring iterations of the compound du Val singularities.

Theorem 5. *For the singularities from Theorem 4, one has the following Cox ring iterations; the respective total coordinate spaces are indicated by the downward arrows:*

$$\begin{tikzcd}
 & \mathbb{C}^3 \arrow[dl] \arrow[d] & \mathbb{C}^4 \arrow[d] & (10\text{-}o) \arrow[d] & X_1 \arrow[d] & X_2 \arrow[d] & X_3 \arrow[d] & X_4 \arrow[d] & X_5 \arrow[d] & (9a_{l=1}) \arrow[d] \\
(3) & (1) \arrow[dl] \arrow[d] & (2) \arrow[d] & (11\text{-}o) & (10\text{-}e) \arrow[dl] \arrow[d] & (13\text{-}e) & (9b) & (9c) \arrow[d] & (9d) \arrow[d] & (9a_{l\ge 2}) \\
(5) & (4) \arrow[d] & (12) & (11\text{-}e) & (16) & & & (13\text{-}o) & (14) & \\
 & (6) \arrow[d] & & & & & & & & \\
 & (7) & & & & & & & &
\end{tikzcd}$$

Here 10-e (10-*o*) *denotes the singularity* 10 *with even (odd) parameter; similarly in the other cases. Moreover with the respective parameters from Theorem 4:*

$$X_1 \;=\; V(T_1^2T_2 + T_3^2T_4 + T_5^{\frac{l+2}{2}}), \qquad X_2 \;=\; V(T_1T_2 + T_3T_4 + T_5^{l-1}),$$

$$X_5 \;=\; V(T_1^{L_1}T_2^{L_1} + T_3^{L_2}T_4^{L_2} + T_5T_6,\; T_3^{L_2}T_4^{L_2} + 2T_5T_6 + T_7T_8,\; \ldots).$$

To obtain X_4, set $T_4 = 1$ and for X_3 in addition $T_2 = 1$ in the equations of X_5. The singularities 8, 15, 17 and 18 are factorial.

The varieties $X_1, \ldots, X_5$ in Theorem 5 are of dimension four or higher. They enjoy a generalized compound du Val property in the sense that the hyperplane section $X_i \cap V(T_4 - T_3)$ has at most canonical singularities. For instance, for X_2, the hyperplane section gives a compound du Val singularity of Type 9a. The composition $\mathbb{C}^3 \to (1) \to (5)$ is a quotient by the dihedral group D_{2l+4}, which is not a subgroup of SL(2).

Contents

2. Rational varieties with torus action of complexity one

We recall the basic concepts and facts on normal rational *T-varieties X of complexity one*, i.e., the variety X is endowed with an effective action $T \times X \to X$ of an algebraic torus T such that $\dim(T) = \dim(X) - 1$ holds. We work over the field $\mathbb{C}$ of complex numbers. For the proofs and full details, we refer to [2, 17, 18, 19].

The approach follows the general philosophy behind [2, Chap. 3]: one starts with a Cox ring $R = \mathcal{R}(X)$ and then obtains X as a quotient $X = \widehat{X} /\!\!/ H$ of an open subset $\widehat{X} \subseteq \overline{X}$ of the *total coordinate space* $\overline{X} = \operatorname{Spec} R$ by the action of the *characteristic quasitorus* $H = \operatorname{Spec} \mathbb{C}[K]$, where $K \cong \operatorname{Cl}(X)$ is the divisor class group of X. The quotient map $\widehat{X} \to X$ is called the *characteristic space* over X. In our concrete case of T-varieties of complexity one, the total coordinate space $\overline{X}$ will be acted on by a larger quasitorus $H_0 = \operatorname{Spec} \mathbb{C}[K_0]$ containing the characteristic quasitorus H as a closed subgroup and the torus action on $X = \widehat{X} /\!\!/ H$ will be the induced action of $T = H_0/H$.

Our first step provides K_0-graded rings R, which after suitable downgrading become prospective Cox rings of our T-varieties. The construction depends on continuous data A and discrete data P_0 introduced below. There are two types of input data (A, P_0): for Type 1, we will have the affine line as a generic quotient of the action of H_0 on $\overline{X}$ and Type 2 will lead to the projective line.

Construction 2.1. Fix integers $r, n > 0$, $m \ge 0$ and a partition $n = n_\iota + \ldots + n_r$ starting at $\iota \in \{0, 1\}$. For each $\iota \le i \le r$, fix a tuple $l_i \in \mathbb{Z}^{n_i}_{>0}$ and define a monomial

$$T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{C}[T_{ij}, S_k; \ \iota \le i \le r, \ 1 \le j \le n_i, \ 1 \le k \le m].$$

We will also write $\mathbb{C}[T_{ij}, S_k]$ for the above polynomial ring. We distinguish two settings for the input data A and P_0 of the graded $\mathbb{C}$-algebra $R(A, P_0)$.

Type 1. Take $\iota = 1$. Let $A := (a_1, \ldots, a_r)$ be a list of pairwise different elements of $\mathbb{C}$. Set $I := \{1, \ldots, r-1\}$ and define for every $i \in I$ a polynomial

$$g_i \ := \ T_i^{l_i} - T_{i+1}^{l_{i+1}} - (a_{i+1} - a_i) \ \in \ \mathbb{C}[T_{ij}, S_k].$$

We build up an $r \times (n+m)$ matrix from the exponent vectors $l_1, \ldots, l_r$ of these polynomials:

$$P_0 \ := \ \begin{bmatrix} l_1 & & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & & l_r & 0 & \ldots & 0 \end{bmatrix}.$$

Type 2. Take $\iota = 0$. Let $A := (a_0, \ldots, a_r)$ be a $2 \times (r+1)$-matrix with pairwise linearly independent columns $a_i \in \mathbb{C}^2$. Set $I := \{0, \ldots, r-2\}$ and for every $i \in I$ define

$$g_i \ := \ \det \begin{bmatrix} T_i^{l_i} & T_{i+1}^{l_{i+1}} & T_{i+2}^{l_{i+2}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix} \ \in \ \mathbb{C}[T_{ij}, S_k].$$

We build up an $r \times (n+m)$ matrix from the exponent vectors $l_0, \ldots, l_r$ of these polynomials:

$$P_0 \ := \ \begin{bmatrix} -l_0 & l_1 & & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -l_0 & 0 & & l_r & 0 & \ldots & 0 \end{bmatrix}.$$

We now define the ring $R(A, P_0)$ simultaneously for both types in terms of the data A and P_0. Denote by P_0^* the transpose of P_0 and consider the projection

$$Q \colon \mathbb{Z}^{n+m} \ \to \ K_0 \ := \ \mathbb{Z}^{n+m} / \operatorname{im}(P_0^*).$$

Denote by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables T_{ij}, S_k. Define a K_0-grading on $\mathbb{C}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ Q(e_{ij}) \ \in \ K_0, \qquad \deg(S_k) \ := \ Q(e_k) \ \in \ K_0.$$

This is the coarsest possible grading of $\mathbb{C}[T_{ij}, S_k]$ leaving the variables and the g_i homogeneous. In particular, we have a K_0-graded factor algebra

$$R(A, P_0) \ := \ \mathbb{C}[T_{ij}, S_k] / \langle g_i; \ i \in I \rangle.$$

The $\mathbb{C}$-algebra $R(A, P_0)$ just constructed is an integral normal complete intersection of dimension $n + m + 1 - r$ admitting only constant invertible homogeneous elements. Moreover, $R(A, P_0)$ is K_0-factorial in the sense that every non-zero homogeneous non-unit is a product of K_0-primes. The latter merely means that on $\overline{X} = \operatorname{Spec} R(A, P_0)$ every H_0-invariant divisor is the divisor of an H_0-homogeneous rational function. Moreover, every affine variety with a quasitorus action of complexity one having this property and admitting only constant invertible homogeneous functions arises from Construction 2.1, see [2, Sec. 4.4.2].

In the second construction step, we introduce the downgradings $K_0 \to K$ that will turn $R(A, P_0)$ into a Cox ring. More geometrically speaking, we figure out the possible characteristic quasitori $H \subseteq H_0$. This is achieved by suitably enhancing the matrix P_0.

Construction 2.2. Let integers r, $n = n_\iota + \ldots + n_r$, m and data A and P_0 of Type 1 or Type 2 as in Construction 2.1. Fix $1 \le s \le n + m - r$, choose an integral $s \times (n + m)$ matrix d and build the $(r + s) \times (n + m)$ stack matrix

$$P \ := \ \left[\begin{array}{c} P_0 \\ d \end{array} \right].$$

We require the columns of P to be pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a vector space. Let P^* denote the transpose of P and consider the projection

$$Q\colon \mathbb{Z}^{n+m} \ \to \ K \ := \ \mathbb{Z}^{n+m}/\mathrm{im}(P^*).$$

Denoting as before by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables T_{ij} and S_k, we obtain a K-grading on $\mathbb{C}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ Q(e_{ij}) \ \in \ K, \qquad\qquad \deg(S_k) \ := \ Q(e_k) \ \in \ K.$$

This K-grading coarsens the K_0-grading of $\mathbb{C}[T_{ij}, S_k]$ given in Construction 2.1. In particular, we have the K-graded factor algebra

$$R(A, P) \ := \ \mathbb{C}[T_{ij}, S_k]/\langle g_i; \ i \in I \rangle.$$

So, as algebras $R(A, P_0)$ and $R(A, P)$ coincide, but the latter comes with the coarser K-grading. Again, $R(A, P)$ is K-factorial, i.e., for the action of $H = \operatorname{Spec} \mathbb{C}[K]$ on $\overline{X} = \operatorname{Spec} R(A, P)$, every H-invariant divisor is the divisor of an H-homogeneous function.

Remark 2.3. Consider the defining matrix P of a K-graded ring $R(A, P)$ as in Construction 2.2. Write $v_{ij} = P(e_{ij})$ and $v_k = P(e_k)$ for the columns of P. The i-th column block of P is $(v_{i1}, \ldots, v_{in_i})$ and by the data of this block we mean l_i and the $s \times n_i$ block d_i of d. We introduce *admissible operations* on P:

(i) swap two columns inside a block $v_{i1}, \ldots, v_{in_i}$,
(ii) exchange the data l_{i_1}, d_{i_1} and l_{i_2}, d_{i_2} of two column blocks,
(iii) add multiples of the upper r rows to one of the last s rows,
(iv) any elementary row operation among the last s rows,
(v) swapping among the last m columns.

The operations of type (iii) and (iv) do not change the associated ring $R(A, P)$, whereas the types (i), (ii), (v) correspond to certain renumberings of the variables of $R(A, P)$ keeping the (graded) isomorphy type.

Remark 2.4. If $R(A, P)$ is not a polynomial ring, then we can always assume that P is *irredundant* in the sense that $l_{i1} + \ldots + l_{in_i} > 1$ holds for $i = 0, \ldots, r$. Indeed, if P is redundant, then we have $n_i = 1$ and $l_{i1} = 1$ for some i. After an admissible operation of type (ii), we may assume $i = r$. Now, erasing v_{r1} and the r-th row of P and the last column from A produces new data defining a ring $R(A, P)$ isomorphic to the previous one. Iterating this procedure leads to an $R(A, P)$ isomorphic to the initial one but with irredundant P.

Remark 2.5. Construction 2.2 allows more flexibility than the simpler version presented in the introduction. However, given any $R(A, P)$ as in Construction 2.2, we can achieve $l_{i1} \ge \ldots \ge l_{in_i}$ for all i and $l_{\iota 1} \ge \ldots \ge l_{r1}$ by means of admissible operations of type (i) and (ii). Moreover, via suitable scalings of the variables T_{ij}, we can turn the coefficients of the relations g_i into those presented in the introduction.

The algebras $R(A,P)$ will be our prospective Cox rings. The remaining task is to determine the open H-invariant sets $\widehat{X} \subseteq \overline{X} = \operatorname{Spec} R(A,P)$ that give rise to suitable quotients $X = \widehat{X} /\!\!/ H$. This is done via geometric invariant theory: the respective open sets $\widehat{X} \subseteq \overline{X}$ are in correspondence with "bunches of cones", certain collections Φ of convex polyhedral cones in $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$; we refer to [2, Sec. 3.2.1] for a detailed introduction.

Construction 2.6. Let $R(A,P)$ be a K-graded ring as provided by Construction 2.2 and $\mathfrak{F} = (T_{ij}, S_k)$ the canonical system of generators. Consider

$$H \ := \ \operatorname{Spec} \mathbb{C}[K], \qquad\qquad \overline{X}(A,P) \ := \ \operatorname{Spec} R(A,P).$$

Then H is a quasitorus and the K-grading of $R(A,P)$ defines an action of H on $\overline{X}$. Any true $\mathfrak{F}$-bunch Φ defines an H-invariant open set and a good quotient

$$\widehat{X}(A,P,\Phi) \ \subseteq \ \overline{X}(A,P), \qquad\qquad X(A,P,\Phi) \ := \ \widehat{X}(A,P,\Phi) /\!\!/ H.$$

The action of $H_0 = \operatorname{Spec} \mathbb{C}[K_0]$ leaves $\widehat{X}(A,P,\Phi)$ invariant and induces an action of the torus $T = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^s]$ on $X(A,P,\Phi)$.

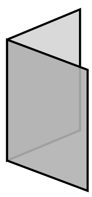
Recall from [2, Thm. 3.4.3.7] that the resulting variety $X = X(A,P,\Phi)$ is rational, normal, admits only constant invertible functions and is of dimension $n+m+1-r-\dim(K_{\mathbb{Q}}) = s+1$. Moreover, the divisor class group of X is isomorphic to K and the Cox ring to $R(A,P)$. Conversely, the basic result of the theory says that if X is a rational, normal variety with a torus action of complexity one having only constant globally invertible functions and satisfies a certain maximality property with respect to embeddability into toric varieties, then X is equivariantly isomorphic to some $X(A,P,\Phi)$, see [19, Thm. 1.8].

Toric embeddability is important in our subsequent considerations. More specifically, there is even a canonical embedding $X \to Z$ into a toric variety such that X inherits many geometric properties from Z. The construction makes use of the tropical variety of X.

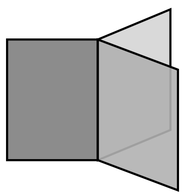
Construction 2.7. Let $X = X(A,P,\Phi)$ be obtained from Construction 2.6. The *tropical variety* of X is the fan $\operatorname{trop}(X)$ in $\mathbb{Q}^{r+s}$ consisting of the cones

$$\lambda_i \ := \ \operatorname{cone}(v_{i1}) + \operatorname{lin}(e_{r+1}, \ldots, e_{r+s}) \text{ for } i \ = \ \iota, \ldots, r, \qquad \lambda \ := \ \lambda_\iota \cap \ldots \cap \lambda_r,$$

where $v_{ij} \in \mathbb{Z}^{r+s}$ denote the first n columns of P and $e_k \in \mathbb{Z}^{r+s}$ the k-th canonical basis vector; we call λ_i a *leaf* and λ the *lineality part* of $\operatorname{trop}(X)$.



Type 1



Type 2

Construction 2.8. Let $X = X(A,P,\Phi)$ be obtained from Construction 2.6. For a face $\delta_0 \preceq \delta$ of the orthant $\delta \subseteq \mathbb{Q}^{n+m}$, let $\delta_0^* \preceq \delta$ denote the complementary face and call δ_0 *relevant* if

- the relative interior of $P(\delta_0)$ intersects $\operatorname{trop}(X)$,
- the image $Q(\delta_0^*)$ comprises a cone of Φ,

where $Q\colon \mathbb{Z}^{n+m} \to K = \mathbb{Z}^{n+m}/P^*(\mathbb{Z}^{r+s})$ is the projection. Then we obtain fans $\widehat{\Sigma}$ in $\mathbb{Z}^{n+m}$ and Σ in $\mathbb{Z}^{r+s}$ of pointed cones by setting

$$\widehat{\Sigma} \ := \ \{\delta_1 \preceq \delta_0;\ \delta_0 \preceq \delta \text{ relevant}\}, \qquad \Sigma \ := \ \{\sigma \preceq P(\delta_0);\ \delta_0 \preceq \delta \text{ relevant}\}.$$

The toric varieties $\widehat{Z}$ and Z associated with $\widehat{\Sigma}$ and Σ, respectively, and $\overline{Z} = \mathbb{C}^{n+m}$ fit into a commutative diagramm of characteristic spaces and total coordinate spaces

$$\begin{tikzcd}
\overline{X}(A,P) \arrow[r, phantom, "\subseteq"] & \overline{Z} \\
\widehat{X}(A,P,\Phi) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, phantom, "\subseteq"] \arrow[d, "/\!\!/ H"'] & \widehat{Z} \arrow[u, phantom, sloped, "\subseteq"] \arrow[d, "/\!\!/ H"] \\
X(A,P,\Phi) \arrow[r, phantom, "\subseteq"] & Z
\end{tikzcd}$$

The horizontal inclusions are T-equivariant closed embeddings, where T acts on Z as the subtorus of the $(r+s)$-torus corresponding to $0 \times \mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$. Moreover, $X(A,P,\Phi)$ intersects every closed toric orbit of Z.

We call Z from Construction 2.8 the *minimal toric ambient variety* of $X = X(A,P,\Phi)$. Observe that the rays of the fan Σ of Z have precisely the columns of the matrix P as its primitive generators. In particular, every ray of Σ lies on the tropical variety $\mathrm{trop}(X)$. The minimal toric ambient variety is crucial for the resolution of singularities. The following recipe for resolving singularities directly generalizes [2, Thm. 3.4.4.9]; a related approach using polyhedral divisors is presented in [22].

Construction 2.9. Let $X = X(A,P,\Phi)$ be obtained from Construction 2.6 and consider the canonical toric embedding $X \subseteq Z$ and the defining fan Σ of Z.

- Let $\Sigma' = \Sigma \sqcap \mathrm{trop}(X)$ be the coarsest common refinement.
- Let Σ'' be any regular subdivision of the fan Σ'.

Then $\Sigma'' \to \Sigma$ defines a proper toric morphism $Z'' \to Z$ and with the proper transform $X'' \subseteq Z''$ of $X \subseteq Z$, the morphism $X'' \to X$ is a resolution of singularities.

Remark 2.10. In the setting of Construction 2.9, the variety X'' has again a torus action of complexity one and thus is of the form $X'' = X(A'', P'', \Phi'')$. We have $A'' = A$ and P'' is obtained from P by inserting the primitive generators of Σ'' as new columns. Moreover, Φ'' is the Gale dual of Σ'', that means that with the corresponding projection Q'' and orthant δ'' we have

$$\Phi'' \ = \ \{Q''(\delta_0^*);\ \delta_0 \preceq \delta'';\ P''(\delta_0) \in \Sigma''\}.$$

Proposition 2.11. *Consider a variety $X = X(A,P,\Phi)$ of Type 2 as provided by Construction 2.6. Then the following statements are equivalent.*

(i) *One has $\widehat{X} = \overline{X}$*
(ii) *The variety X is affine.*
(iii) *The minimal toric ambient variety Z of X is affine.*
(iv) *One has $\widehat{Z} = \overline{Z} = \mathbb{C}^{n+m}$.*

If one of these statements holds, then the columns of P generate the extremal rays of a full-dimensional cone $\sigma \subseteq \mathbb{Q}^{r+s}$ and we have $Z = \mathrm{Spec}\,\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^{r+s}]$.

Proof. Only for the implication "(ii)⇒(iii)" there is something to show. As X is of Type 2, we have $0 \in \overline{X} \subseteq \overline{Z} = \mathbb{C}^{n+m}$. Since X is affine, we have $\overline{X} = \widehat{X}$ and thus $0 \in \widehat{Z}$. We conclude $\widehat{Z} = \overline{Z}$ and thus $Z = \overline{Z} /\!\!/ H$ is affine. $\square$

The characterization 2.11 (i) allows us to omit the bunch of cones Φ in the affine case: we may just speak of the affine variety $X = X(A,P) := \overline{X} /\!\!/ H$.

Corollary 2.12. *Let $X = X(A, P)$ be affine of Type 2. Then the following statements are equivalent.*

(i) *The variety X is $\mathbb{Q}$-factorial.*
(ii) *The variety Z is $\mathbb{Q}$-factorial.*
(iii) *The columns of P are linearly independent.*

Proof. The equivalence of (i) and (ii) is [2, Cor. 3.3.1.7], The equivalence of (ii) and (iii) is [7, Thm. 3.1.19 (b)]. □

Corollary 2.13. *Let $X = X(A, P)$ be affine of Type 2. Then the Picard group of X is trivial.*

Proof. Proposition 2.11 says that the minimal toric ambient variety Z is affine. Thus, Z has trivial Picard group; see [7, Prop. 4.2.2]. According to [2, Cor. 3.3.1.12], the Picard group of X equals that of Z. □

3. The anticanonical complex and singularities

First recall the basic singularity types arising in the minimal model programme. Let X be a $\mathbb{Q}$-Gorenstein variety, i.e., some non-zero multiple of a canonical divisor D_X on X is an integral Cartier divisor. Then, for any resolution of singularities $\varphi\colon X' \to X$, one has the ramification formula

$$D_{X'} - \varphi^*(D_X) \;=\; \sum a_i E_i,$$

where the E_i are the prime components of the exceptional divisors and the coefficients $a_i \in \mathbb{Q}$ are the discrepancies of the resolution. The variety X is said to have at most *log terminal (canonical, terminal)* singularities, if for every resolution of singularities the discrepancies a_i satisfy $a_i > -1$ ($a_i \geq 0$, $a_i > 0$).

In [5], the “anticanonical complex” has been introduced for Fano varieties $X(A, P, \Phi)$ and served as a tool to study singularities of the above type. The purpose of this section is to extend this approach and to generalize results from [5] to the non-complete and non-$\mathbb{Q}$-factorial cases. As an application, we characterize log terminality in Theorem 3.13 via platonic triples occuring in the Cox ring. For the affine case, the result specializes to Theorem 1.

Now, let $X = X(A, P, \Phi)$ be a rational T-variety of complexity one arising from Construction 2.6. Consider the embedding $X \subseteq Z$ into the minimal toric ambient variety. Then X and Z share the same divisor class group

$$K \;=\; \mathrm{Cl}(X) \;=\; \mathrm{Cl}(Z)$$

and the same degree map $Q\colon \mathbb{Z}^{n+m} \to K$ for their Cox rings. Let $e_Z \in \mathbb{Z}^{n+m}$ denote the sum over the canonical basis vectors e_{ij} and e_k of $\mathbb{Z}^{n+m}$. Then, with the defining relations $g_\iota, \ldots, g_{r-2}$ of the Cox ring $R(A, P)$, the canonical divisor classes of Z and X are given as

$$\mathcal{K}_Z \;=\; -Q(e_Z) \;\in\; K, \qquad\qquad \mathcal{K}_X \;=\; \sum_{i=\iota}^{r-2+\iota} \deg(g_i) + \mathcal{K}_Z \;\in\; K.$$

Observe that if X is of Type 1, then its canonical divisor class equals that of the minimal toric ambient variety Z. Define a (rational) polyhedron

$$B(-\mathcal{K}_X) \;:=\; Q^{-1}(-\mathcal{K}_X) \cap \mathbb{Q}^{n+m}_{\geq 0} \;\subseteq\; \mathbb{Q}^{n+m}$$

and let $B := B(g_\iota) + \ldots + B(g_{r-2+\iota}) \subseteq \mathbb{Q}^{n+m}$ denote the Minkowski sum of the Newton polytopes $B(g_i)$ of the relations $g_\iota, \ldots, g_{r-2+\iota}$ of $R(A, P)$.

Definition 3.1. Let $X = X(A, P, \Phi)$ such that $-\mathcal{K}_X$ is ample and denote by Σ the fan of the minimal toric ambient variety Z of X.

(i) The *anticanonical polyhedron* of X is the dual polyhedron $A_X \subseteq \mathbb{Q}^{r+s}$ of the polyhedron

$$B_X \ := \ (P^*)^{-1}(B(-\mathcal{K}_X) + B - e_\Sigma) \ \subseteq \ \mathbb{Q}^{r+s}.$$

(ii) The *anticanonical complex* of X is the coarsest common refinement of polyhedral complexes

$$A_X^c \ := \ \mathrm{faces}(A_X) \sqcap \Sigma \sqcap \mathrm{trop}(X).$$

(iii) The *relative interior* of A_X^c is the interior of its support with respect to the intersection $\mathrm{Supp}(\Sigma) \cap \mathrm{trop}(X)$.

(iv) The *relative boundary* ∂A_X^c is the complement of the relative interior of A_X^c in A_X^c.

A first statement expresses the discrepancies of a given resolution of singularities via the anticanonical complex; the proof is a straightforward generalization of the one given in [5] for the Fano case and will be made available elsewhere.

Proposition 3.2. *Let $X = X(A, P, \Phi)$ such that $-\mathcal{K}_X$ is ample and $X'' \to X$ a resolution of singularities as in Construction 2.9. For any ray $\varrho \in \Sigma''$, let v_ϱ be its primitive generator, v'_ϱ its leaving point of A_X^c provided $\varrho \not\subseteq A_X^c$ and D_ϱ the corresponding prime divisor on X''. Then the discrepancy a_ϱ along D_ϱ satisfies*

$$a_\varrho \ = \ -1 + \frac{||v_\varrho||}{||v'_\varrho||} \quad \text{if } \varrho \not\subseteq A_X^c, \qquad\qquad a_\varrho \leq -1 \quad \text{if } \varrho \subseteq A_X^c.$$

The next result characterizes the existence of at most log terminal (canonical, terminal) singularities in terms of the anticanonical complex; again, this generalizes a result from [5] and the proof will be made available elswhere.

Theorem 3.3. *Let $X = X(A, P, \Phi)$ be such that $-\mathcal{K}_X$ is ample. Then the following statements hold.*

(i) *A_X^c contains the origin in its relative interior and all primitive generators of the fan Σ are vertices of A_X^c.*

(ii) *X has at most log terminal singularities if and only if the anticanonical complex A_X^c is bounded.*

(iii) *X has at most canonical singularities if and only if 0 is the only lattice point in the relative interior of A_X^c.*

(iv) *X has at most terminal singularities if and only if 0 and the primitive generators v_ϱ for $\varrho \in \Sigma^{(1)}$ are the only lattice points of A_X^c.*

We describe the structure of the anticanonical complex in more detail, which generalizes in particular statements on the $\mathbb{Q}$-factorial Fano case obtained in [5]. For Type 1, the situation turns out to be simple, whereas Type 2 is more involved.

Proposition 3.4. *Let $X = X(A, P, \Phi)$ be of Type 1 such that $-\mathcal{K}_X$ is ample. Let Σ be the fan of the minimal toric ambient variety of X and denote by $\lambda_0, \ldots, \lambda_r$ the leaves of* $\mathrm{trop}(X)$.

(i) *Every cone $\sigma \in \Sigma$ is contained in a leaf $\lambda_i \subseteq \mathrm{trop}(X)$. In particular, $\Sigma \sqcap \mathrm{trop}(X)$ equals Σ.*

(ii) *The boundary of A_X^c is the union of all faces of A_X that are contained in* $\mathrm{Supp}(\Sigma)$.

(iii) *The non-zero vertices of A_X^c are the primitive generators of Σ, i.e. the columns of P.*

Corollary 3.5. *Let $X = X(A, P, \Phi)$ be a T-variety of Type 1. Then X has at most log-terminal singularities. Moreover, it has at most canonical (terminal) singularities if and only if its minimal toric ambient variety Z does so.*

Construction 3.6. Let $X = X(A, P, \Phi)$ be of Type 2 and Σ the fan of the minimal toric ambient variety of Z. Write $v_{ij} := P(e_{ij})$ and $v_k := P(e_k)$ for the columns of P. Consider a pointed cone of the form

$$\tau \;=\; \operatorname{cone}(v_{0j_0}, \ldots, v_{rj_r}) \;\subseteq\; \mathbb{Q}^{r+s},$$

that means that τ contains exactly one v_{ij} for every $i = 0, \ldots, r$. We call such τ a *P-elementary cone* and associate the following numbers with τ:

$$\ell_{\tau,i} \;:=\; \frac{l_{0j_0} \cdots l_{rj_r}}{l_{ij_i}} \text{ for } i = 0, \ldots, r, \qquad \ell_\tau \;:=\; (1-r) l_{0j_0} \cdots l_{rj_r} + \sum_{i=0}^{r} l_{\tau,i}.$$

Moreover, we set

$$v_\tau \;:=\; \ell_{\tau,0} v_{0j_0} + \ldots + \ell_{\tau,r} v_{rj_r} \;\in\; \mathbb{Z}^{r+s}, \qquad \varrho_\tau \;:=\; \mathbb{Q}_{\geq 0} \cdot v_\tau \;\in\; \mathbb{Q}^{r+s}.$$

We denote by $\mathrm{T}(A, P, \Phi)$ the set of all P-elementary cones $\tau \in \Sigma$. For a given $\sigma \in \Sigma$, we denote by $\mathrm{T}(\sigma)$ the set of all P-elementary faces of σ.

Remark 3.7. Let $X = X(A, P, \Phi)$ be of Type 2. Let Σ be the fan of the minimal toric ambient variety of X and $\lambda_0, \ldots, \lambda_r \subseteq \operatorname{trop}(X)$ the leaves of the tropical variety of X. As in [5, Def. 4.1], we say that

(i) a cone $\sigma \in \Sigma$ is a *leaf cone* if $\sigma \subseteq \lambda_i$ holds for some $i = 0, \ldots, r$,
(ii) a cone $\sigma \in \Sigma$ is called *big* if $\sigma \cap \lambda_i^\circ \neq \emptyset$ holds for all $i = 0, \ldots, r$.

Observe that a given cone $\sigma \in \Sigma$ is big if and only if σ contains some P-elementary cone as a subset.

Proposition 3.8. *Let $X = X(A, P, \Phi)$ be of Type 2 such that $-\mathcal{K}_X$ is ample. Let Σ be the fan of the minimal toric ambient variety of X, denote by $\lambda_0, \ldots, \lambda_r$ the leaves of* $\operatorname{trop}(X)$ *and by $\lambda = \lambda_0 \cap \ldots \cap \lambda_r$ its lineality part.*

(i) *The fan $\Sigma \sqcap \operatorname{trop}(X)$ consists of the cones $\sigma \cap \lambda$ and $\sigma \cap \lambda_i$, where $\sigma \in \Sigma$ and $i = 0, \ldots, r$. Here, one always has $\sigma \cap \lambda \preceq \sigma \cap \lambda_i$.*
(ii) *The fan $\Sigma \sqcap \operatorname{trop}(X)$ is a subfan of the normal fan of the polyhedron B_X. In particular, for every cone $\sigma \cap \lambda_i$, there is a vertex $u_{\sigma,i} \in B_X$ with*

$$\partial A_X^{\mathrm{c}} \cap \sigma \cap \lambda_i \;=\; \{v \in \sigma \cap \lambda_i;\ \langle u_{\sigma,i}, v\rangle = -1\}.$$

(iii) *If a P-elementary cone τ is contained in some $\sigma \in \Sigma$, then τ is simplicial, $v_\tau \in \tau^\circ$ holds, ϱ_τ is a ray, $\varrho_\tau = \tau \cap \lambda$ holds as well as $\mathbb{Q}\varrho_\tau = \mathbb{Q}\tau \cap \lambda$.*
(iv) *Let $\sigma \in \Sigma$ be any cone. Then, for every $i = 0, \ldots, r$, the set of extremal rays of $\sigma \cap \lambda_i \in \Sigma \sqcap \operatorname{trop}(X)$ is given by*

$$(\sigma \cap \lambda_i)^{(1)} \;=\; \{\varrho(\sigma_0);\ \sigma_0 \in \mathrm{T}(\sigma)\} \cup \{\varrho \in \sigma^{(1)};\ \varrho \subseteq \lambda_i\}.$$

(v) *The set of rays of $\Sigma \sqcap \operatorname{trop}(X)$ consists of the rays of Σ and the rays $\varrho(\sigma_0)$, where $\sigma_0 \in \mathrm{T}(A, P, \Phi)$.*
(vi) *If a P-elementary cone τ is contained in some $\sigma \in \Sigma$, then the minimum value among all $\langle u, v_\tau\rangle$, where $u \in B_X$, equals $-\ell_\tau$.*
(vii) *Let the P-elementary cone τ be contained in $\sigma \in \Sigma$. Then $\varrho_\tau \not\subseteq A_X^{\mathrm{c}}$ holds if and only if $\ell_\tau > 0$ holds; in this case, ϱ_τ leaves A_X^{c} at $v'_\tau = \ell_\tau^{-1} v_\tau$.*
(viii) *The vertices of A_X^{c} are the primitive generators of Σ, i.e. the columns of P, and the points $v(\sigma_0)' = \ell_{\sigma_0}^{-1} v(\sigma_0)$, where $\sigma_0 \in \mathrm{T}(A, P, \Phi)$ and $\ell_{\sigma_0} > 0$.*

Proof. Assertion (i) holds more generally. Indeed, the coarsest common refinement $\Sigma_1 \sqcap \Sigma_2$ of any two quasifans Σ_i in a common vector space consists of the intersections $\sigma_1 \cap \sigma_2$, where $\sigma_i \in \Sigma_i$. Moreover, the faces of a given cone $\sigma_1 \cap \sigma_2$ of $\Sigma_1 \sqcap \Sigma_2$ are precisely the cones $\sigma'_1 \cap \sigma'_2$, where $\sigma'_i \preceq \sigma_i$.

We show (ii). Let Σ' be the complete fan in $\mathbb{Q}^{r+s}$ defined by the class $-\mathcal{K}_X \in K$. Since $-\mathcal{K}_X$ is ample, the fan Σ is a subfan of Σ'. The preimage $P^{-1}(\Sigma')$ consists the cones $P^{-1}(\sigma')$, where $\sigma' \in \Sigma'$, and is the normal fan of $B(-\mathcal{K}_X) \subseteq \mathbb{Q}^{n+m}$.

Moreover, $P^{-1}(\mathrm{trop}(X))$ turns out to be a subfan of the normal fan of $B \subseteq \mathbb{Q}^{n+m}$. It follows that $P^{-1}(\Sigma') \sqcap P^{-1}(\mathrm{trop}(X))$ is a subfan of the normal fan of $B(-\mathcal{K}_X)+B$. Projecting the involved fans via P to $\mathbb{Q}^{r+s}$ gives the assertion.

To obtain (iii), consider first any P-elememtary $\tau = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$. Then $v_{0j_0}, \ldots, v_{rj_r}$ is linearly dependent if and only if $v_\tau = 0$ holds. The latter is equivalent to 0 being an inner point of τ. Thus, if τ is contained in some $\sigma \in \Sigma$, then τ is pointed an thus must be simplicial. The remaining part is then obvious; recall that the lineality part of $\mathrm{trop}(X)$ equals the vector subspace $0 \times \mathbb{Q}^s \subseteq \mathbb{Q}^{r+s}$.

We turn to (iv). First, we claim that if $\sigma_0 \in \Sigma$ is big and $\varrho_\tau = \varrho_{\tau'}$ holds for any two P-elementary cones $\tau, \tau' \subseteq \sigma$, then σ_0 is P-elementary. Assume that σ_0 is not P-elementary. Then we find some $1 \le t \le r$ and cones

$$\tau \;=\; \mathrm{cone}(v_{0j_0}, \ldots, v_{tj_{t-1}}, v_{tj_t}, v_{tj_{t+1}}, \ldots, v_{rj_r}) \;\subseteq\; \sigma_0,$$

$$\tau' \;=\; \mathrm{cone}(v_{0j_0}, \ldots, v_{tj_{t-1}}, v_{tj'_t}, v_{tj_{t+1}}, \ldots, v_{rj_r}) \;\subseteq\; \sigma_0$$

with $j_t \ne j'_t$ and thus $\tau \ne \tau'$. Here, we may assume that $c_\tau^{-1} l_{tj_t} \ge c_{\tau'}^{-1} l_{tj'_t}$ holds with the greatest common divisors c_τ and $c_{\tau'}$ of the entries of v_τ and $v_{\tau'}$ respectively. Then even $c_\tau^{-1}\ell_{\tau,i} \ge c_{\tau'}^{-1}\ell_{\tau',i}$ must hold for all $1 \le i \le r$. Since, the rays ϱ_τ and $\varrho_{\tau'}$ coincide, also their primitive generators $c_{\tau'}^{-1} v_{\tau'}$ and $c_\tau^{-1} v_\tau$ coincide. By the definition of v_τ and $v_{\tau'}$, this implies

$$c_{\tau'}^{-1}\ell_{\tau',t} v_{tj'_t} \;=\; c_\tau^{-1}\ell_{\tau,k} v_{tj_t} + \sum_{i\ne t} (c_\tau^{-1}\ell_{\tau,i} - c_{\tau'}^{-1}\ell_{\tau',i}) v_{ij_i}.$$

We conclude $v_{tj'_t} \in \tau$. Since $v_{tj'_t}$ is an extremal ray of σ_0 and $\tau' \subseteq \sigma_0$ holds, $v_{tj'_t}$ generates an extremal ray of τ. This contradicts to the choice of j'_t and the claim is verified.

Now, consider the equation of (iv). To verify "$\subseteq$", let ϱ be an extremal ray of $\sigma \cap \lambda_i$. We have to show that $\varrho = \varrho(\sigma_0)$ holds for some $\sigma_0 \in \mathrm{T}(\sigma)$ or that ϱ is a ray of σ with $\varrho \subseteq \lambda_i$. According to (ii), there is a face $\sigma_\varrho \preceq \sigma$ such that $\varrho = \sigma_\varrho \cap \lambda$ or $\varrho = \sigma_\varrho \cap \lambda_i$ holds. We choose σ_ϱ minimal with respect to this property, that means that we have $\varrho^\circ \subseteq \sigma_\varrho^\circ$. We distinguish the following cases.

Case 1. We have $\varrho = \sigma_\varrho \cap \lambda$. If $\sigma_\varrho \subseteq \lambda$ holds, then we obtain $\varrho = \sigma_\varrho$ and thus $\varrho \subseteq \lambda_i$ is an extremal ray of σ. So, assume that σ_ϱ is not contained in λ. Then, because of $\sigma_\varrho^\circ \cap \lambda \ne \emptyset$, there is a P-elementary cone $\tau \subseteq \sigma_\varrho$. Using (i), we obtain

$$\varrho_\tau \;=\; \tau \cap \lambda \;\subseteq\; \sigma_\varrho \cap \lambda \;=\; \varrho$$

and thus $\varrho = \varrho_\tau$. As this does not depend on the particular choice of the P-elementary cone $\tau \subseteq \sigma_\varrho$, the above claim yields $\sigma_0 := \sigma_\varrho \in \mathrm{T}(\sigma)$ and $\varrho = \varrho(\sigma_0)$.

Case 2. We don't have $\varrho = \sigma_\varrho \cap \lambda$. Then $\varrho = \sigma_\varrho \cap \lambda_i$ and $\varrho^\circ \subseteq \lambda_i^\circ$ hold. If $\sigma_\varrho \subseteq \lambda_i$ holds, then we obtain $\varrho = \sigma_\varrho$ and thus $\varrho \subseteq \lambda_i$ is an extremal ray of σ. So, assume that σ_ϱ is not contained in λ_i. Then $\sigma_\varrho \cap \lambda_j^\circ$ is non-empty for all $j = 0, \ldots, r$. Thus, there is a P-elementary cone $\tau \subseteq \sigma_\varrho$. Using (i), we obtain

$$\varrho_\tau \;=\; \tau \cap \lambda \;\subseteq\; \sigma_\varrho \cap \lambda \;=\; \varrho$$

and thus $\varrho = \varrho_\tau$. As this does not depend on the particular choice of the P-elementary cone $\tau \subseteq \sigma_\varrho$, the above claim yields $\sigma_0 := \sigma_\varrho \in \mathrm{T}(\sigma)$ and $\varrho = \varrho(\sigma_0)$.

We verify the inclusion "$\supseteq$". Consider a face $\sigma_0 \in \mathrm{T}(\sigma)$. As seen just before, the extremal rays of $\sigma_0 \cap \lambda_i$ are $\varrho(\sigma_0)$ and the rays of σ_0 that lie in λ_i. Since $\sigma_0 \cap \lambda_i$ is a face of $\sigma \cap \lambda_i$, the ray $\varrho(\sigma_0)$ is an extremal ray of $\sigma \cap \lambda_i$. Finally, consider an extremal ray $\varrho \preceq \sigma$ with $\varrho \subseteq \lambda_i$. Then $\varrho = \varrho \cap \lambda_i$ is a face of $\sigma \cap \lambda_i$.

The proof of Assertion (iv) is complete now. Assertion (v) is a direct consequence of (iv).

We turn to Assertions (vi), (vii) and (viii). Let $\widehat{\tau} \preceq \widehat{\sigma} \preceq \mathbb{Q}^{n+m}$ be the faces with $P(\widehat{\tau}) = \tau$ and $P(\widehat{\sigma}) = \sigma$. Moreover, let $e_\tau \in \widehat{\tau}$ be the (unique) point with $P(e_\tau) = v_\tau$. The minimum value $\langle u, v_\tau \rangle$ is attained at some vertex $u \in B_X$. For this u, we find vertices $e_\sigma \in B(-\mathcal{K}_X)$ and $e_B \in B$ with

$$u \ = \ (P^*)^{-1}(e_\sigma + e_B - e_Z).$$

Here, e_σ is any vertex of $B(-\mathcal{K}_X)$ such that $\widehat{\sigma}$ is contained in the cone of the normal fan of $B(-\mathcal{K}_X)$ associated with e_σ; such e_σ exists due to ampleness of $-\mathcal{K}_X$ and e_σ vanishes along $\widehat{\sigma}$. Together we have

$$e_\tau \ = \ \sum_{i=0}^{r} l_{ij_i} e_{ij_i}, \qquad\qquad \langle u, v_\tau \rangle \ = \ \langle e_\sigma + e_B - e_Z, e_\tau \rangle.$$

As mentioned, $\langle e_\sigma, e_\tau \rangle = 0$ holds. Moreover, $\langle e, e_\tau \rangle = (r-1) l_{0j_0} \cdots l_{rj_r}$ holds for every $e \in B$. We conclude $\langle u, v_\tau \rangle = -\ell_\tau$ and Assertion (vi). Moreover, Assertions (vii) and (viii) are direct consequences of (vi) and (ii). $\square$

Example 3.9. Consider the E_6-singular affine surface $X = V(z_1^4 + z_2^3 + z_3^2) \subseteq \mathbb{C}^3$. It inherits a $\mathbb{C}^*$-action from the action

$$t \cdot (z_1, z_2, z_3) \ = \ (t^3 z_1, t^4 z_2, t^6 z_3)$$

on $\mathbb{C}^3$. The divisor class group and the Cox ring of the surface X are explicitly given by

$$\mathrm{Cl}(X) \ = \ \mathbb{Z}/3\mathbb{Z}, \qquad\qquad \mathcal{R}(X) \ = \ \mathbb{C}[T_1, T_2, T_3]/\langle T_1^3 + T_2^3 + T_3^2 \rangle,$$

where the $\mathrm{Cl}(X)$-degrees of T_1, T_2, and T_3 are $\bar{1}$, $\bar{2}$ and $\bar{0}$. The minimal toric ambient variety is affine and corresponds to the cone

$$\sigma \ = \ \mathrm{cone}((-3,-3,-2), (3,0,1), (0,2,1)).$$

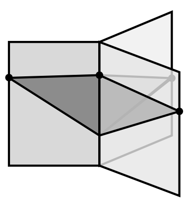
Denoting by $e_i \in \mathbb{Q}^3$ the i-th canonical basis vector, the tropical variety $\mathrm{trop}(X)$ in $\mathbb{Q}^3$ is given as

$$\mathrm{trop}(X) \ = \ \mathrm{cone}(e_1, \pm e_3) \cup \mathrm{cone}(e_2, \pm e_3) \cup \mathrm{cone}(-e_1 - e_2, \pm e_3).$$

The anticanonical polyhedron $A_X \subseteq \mathbb{Q}^3$ is non bounded with recession cone generated by $(-1,-1,-1)$, $(1,0,0)$, $(0,1,0)$. The vertices of A_X are

$$(-3,-3,-2), \ (3,0,1), \ (0,2,1), \ (0,0,1).$$

The anticanonical complex $A_X^c = A_X \sqcap \Sigma \sqcap \mathrm{trop}(X)$ lives inside $\mathrm{trop}(X)$ and looks as follows.



Corollary 3.10. *Let $X = X(A, P, \Phi)$ be of Type 2 such that $-\mathcal{K}_X$ is ample. Let τ be a P-elementary cone contained in some $\sigma \in \Sigma$. Assume $\varrho_\tau \not\subseteq A_X^c$ and denote by c_τ the greatest common divisor of the entries of v_τ. Then, for any resolution of singularities $\varphi\colon X'' \to X$ provided by 2.9, the discrepancy along the prime divisor of X'' corresponding to ϱ_τ equals $c_\tau^{-1}\ell_\tau - 1$.*

Corollary 3.11. *Let $X = X(A, P, \Phi)$ be of Type 2 such that $-\mathcal{K}_X$ is ample and let $\tau = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$ be contained in some $\sigma \in \Sigma$.*

(i) *If X has at most log terminal singularities, then $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1} > r - 1$ holds.*

(ii) *If X has at most canonical singularities, then $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1} \geq r - 1 + c_\tau l_{0j_0}^{-1} \cdots l_{rj_r}^{-1}$ holds.*
(iii) *If X has at most terminal singularities, then $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1} > r - 1 + c_\tau l_{0j_0}^{-1} \cdots l_{rj_r}^{-1}$ holds.*

Remark 3.12. Let $a_0, \ldots, a_r$ be positive integers. Then $a_0^{-1} + \ldots + a_r^{-1} > r - 1$ holds if and only if $(a_0, \ldots, a_r)$ is a platonic tuple.

Theorem 3.13. *Let $X = X(A, P, \Phi)$ be of Type 2 such that $-\mathcal{K}_X$ is ample and let Σ be the fan of the minimal toric ambient variety of X. Then the following statements are equivalent.*
(i) *The variety X has at most log terminal singularities.*
(ii) *For every P-elementary $\tau = \operatorname{cone}(v_{0j_0}, \ldots, v_{rj_r})$ contained in a cone of Σ, the exponents $l_{0j_0}, \ldots, l_{rj_r}$ form a platonic tuple.*

Proof. Assume that $X = X(A, P, \Phi)$ is log terminal. Then Corollary 3.11 (i) tells us that for every P-elementary $\tau = \operatorname{cone}(v_{0j_0}, \ldots, v_{rj_r})$ contained in a cone of Σ, the corresponding exponents $l_{0j_0}, \ldots, l_{rj_r}$ form a platonic tuple.

Now assume that (ii) holds. Then every $(l_{0j_0}, \ldots, l_{rj_r})$ is a platonic tuple. Consequently, we have $\ell_\tau > 0$ for every P-elementary cone τ. Proposition 3.8 shows that A_X^c is bounded for $X = X(A, P, \Phi)$. Theorem 3.3 (ii) tells us that X is log terminal. □

Remark 3.14. Let $X = X(A, P, \Phi)$ be affine of Type 2 such that $\mathcal{K}_X$ is $\mathbb{Q}$-Cartier. Then $-\mathcal{K}_X$ is ample. The fan Σ of the minimal toric ambient variety Z of X consists of all the faces of the cone σ generated by the columns of P. In particular, every P-elementary cone is contained in σ. Thus, Theorem 1 follows from Theorem 3.13. Moreover, the rays $\varrho(\sigma_0)$, where $\sigma_0 \in \mathrm{T}(A, P, \Phi)$, are precisely the extremal rays of the intersection of σ and the lineality part of $\operatorname{trop}(X)$.

4. Gorenstein index and canonical multiplicity

If a normal variety X is $\mathbb{Q}$-Gorenstein, then, by definition, some multiple of its canonical class $\mathcal{K}_X$ is Cartier. The *Gorenstein index* of X is the smallest positive integer $\imath_X$ such that $\imath_X \mathcal{K}_X$ is Cartier. We attach another invariant to the canonical divisor of X.

Remark 4.1. Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2. We consider canonical divisors D_X on X that are of the following form:

$$(4.1.1) \qquad - \sum_{i,j} D_{ij} - \sum_k E_k + \sum_{\alpha=1}^{r-1} \sum_{j=0}^{n_{i_\alpha}} l_{i_\alpha j} D_{i_\alpha j}, \qquad 0 \leq i_\alpha \leq r.$$

Corollary 2.13 says that $\imath_X D_X$ is the divisor of a T-homogeneous rational function. Any two $\imath_X D_X$ with D_X of shape (4.1.1) differ by the divisor of a T-invariant rational function, and thus, all the functions with divsors $\imath_X D_X$, where D_X as in (4.1.1), are homogeneous with respect to the same weight $\eta_X \in \mathbb{X}(T)$.

Definition 4.2. Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2. We call $\eta_X \in \mathbb{X}(T)$ of Remark 4.1 the *canonical weight* of X. The *canonical multiplicity* of X is the minimal non-negative integer ζ_X such that $\eta_X = \zeta_X \cdot \eta'_X$ holds with a primitive element $\eta'_X \in \mathbb{X}(T)$.

Proposition 4.3. *Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2 with at most log terminal singularities. Then $\zeta_X > 0$ holds. Moreover, for any positive integer $\imath$, the following statements are equivalent.*
(i) *The variety X is of Gorenstein index $\imath$.*

(ii) *There exist integers* $\mu_1, \ldots, \mu_r$ *with* $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath) = 1$ *such that with* $\mu_0 := \imath(r-1) - \mu_1 - \ldots - \mu_r$ *we obtain integral vectors*
$$\nu_i := (\nu_{i1}, \ldots, \nu_{in_i}) \text{ with } \nu_{ij} \ := \ \frac{\imath - \mu_i l_{ij}}{\zeta_X},$$
$$\nu' := (\nu'_1, \ldots, \nu'_m) \text{ with } \nu'_k \ := \ \frac{\imath}{\zeta_X}$$
and by suitable elementary row operations on the (d, d')*-block, the matrix* P *gains* $(\nu_0, \ldots, \nu_r, \nu')$ *as its last row, i.e., turns into the shape*
$$\tilde{P} \ = \ \begin{pmatrix} -l_0 & l_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -l_0 & 0 & \ldots & l_r & 0 \\ * & * & \ldots & * & * \\ \nu_0 & \nu_1 & \ldots & \nu_r & \nu' \end{pmatrix}.$$

Proof. We work with an anticanonical divisor D_X on X such that $-D_X$ is of the form (4.1.1):
$$D_X \ := \ \sum_{i,j} D_{ij} + \sum_k E_k - (r-1)\sum_{j=1}^{n_0} l_{0j} D_{0j}.$$
According to Corollary 2.13, the Picard group of X is trivial. Thus, $\imath_X D_X$ is the divisor of some toric character χ^u, where
$$u \ = \ (\mu_1, \ldots, \mu_r, \eta_1, \ldots, \eta_s) \ \in \ \mathbb{Z}^{r+s}.$$
Note that $-(\eta_1, \ldots, \eta_s) \in \mathbb{Z}^s = \mathbb{X}(T)$ is the canonical weight η_X of X. Moreover, the divisor $\imath_X D_X = \operatorname{div}(\chi^u)$ corresponds to the vector $P^* \cdot u \in \mathbb{Z}^{m+n}$ under the identification of toric divisors with lattice points via $D_{ij} \mapsto e_{ij}$ and $E_k \mapsto e_k$.

We claim that η_X is non-trivial. Otherwise, $\eta_1 = \ldots = \eta_s = 0$ holds. As noted, the ij-th and k-th components of the vector $P^* \cdot u$ are the multiplicities of D_{ij} and D_k in $\imath_X D_X$, respectively. More explicitly, this leads to the conditions
$$m \ = \ 0, \qquad \imath_X((r-1)l_{0j} - 1) \ = \ (\mu_1 + \ldots + \mu_r) l_{0j}, \qquad \imath_X \ = \ \mu_i l_{ij}$$
for all i and j. Plugging the third into the second one, we obtain that $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1}$ equals $r-1$ for any choice of $1 \le j_i \le n_i$. According to Corollary 3.11 (i), this contradicts to log terminality of X. Knowing that η_X is non-zero, we obtain that ζ_X is non-zero.

Now, assume that (i) holds, i.e., we have $\imath = \imath_X$. Let $u \in \mathbb{Z}^{r+s}$ as above. Then we have $\zeta_X = \gcd(\eta_1, \ldots, \eta_s)$ and $\operatorname{div}(\chi^u) = \imath D_X$ implies $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath) = 1$. Next, choose a unimodular $s \times s$ matrix $\mathcal{B}$ with $\mathcal{B}^{-1} \cdot (\eta_1, \ldots, \eta_s) = (0, \ldots, 0, \zeta_X)$. Consider $\tilde{P} := \operatorname{diag}(E_r, \mathcal{B}^*) \cdot P$ and
$$\tilde{u} \ = \ (\mu_1, \ldots, \mu_r, 0, \ldots, 0, \zeta_X) \ \in \ \mathbb{Z}^{r+s}.$$
Observe that we have $P^* \cdot u = \tilde{P}^* \cdot \tilde{u}$. Comparing the entries of $\tilde{P}^* \cdot \tilde{u}$ with the multiplicities of the prime divisors D_{ij} and D_k in $\imath D_X$ shows that the last row of $\tilde{P}$ is as claimed.

Conversely, if (ii) holds, consider $u := (\mu_1, \ldots, \mu_r, 0, \ldots, 0, \zeta_X)$. Then we obtain $\imath D_X = \operatorname{div}(\chi^u)$. Using $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath) = 1$, we conclude that $\imath$ is the Gorenstein index of X. □

Remark 4.4. Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2 and D_X a canonical divisor on X as in (4.1.1). Then $\imath_X D_X$ is the divisor of some toric character χ^u, where
$$u \ = \ (\mu_1, \ldots, \mu_r, \eta_1, \ldots, \eta_s) \ \in \ \mathbb{Z}^{r+s}.$$

In this situation, we have $\eta_X = (\eta_1, \ldots, \eta_s) \in \mathbb{X}(T)$ for the canonical weight of X and the canonical multiplicity of X is given by $\zeta_X = \gcd(\eta_1, \ldots, \eta_s)$. If P is in the shape of Proposition 4.3, then $\eta_X = (0, \ldots, 0, \zeta_X)$ holds and $-\mu_1, \ldots, -\mu_r$ satisfy the conditions of 4.3 (ii).

Remark 4.5. The defining matrix P of a given $\mathbb{Q}$-Gorenstein, affine T-variety $X = X(A, P)$ is in the shape of Proposition 4.3 if and only if for every $i = 0, \ldots, r$, the numbers $\mu_i := (\imath_X - \zeta_X \nu_{i1}) l_{i1}^{-1}$ satisfy

(i) $\zeta_X \nu_{ij} + \mu_i l_{ij} = \imath_X$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n_i$,
(ii) $\zeta_X \nu_{0j} + \mu_0 l_{0j} = \imath_X$, for $\mu_0 := \imath_X(r-1) - \mu_1 - \ldots - \mu_r$ and $j = 1, \ldots, n_0$,
(iii) $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath_X) = 1$,
(iv) $\zeta_X \nu'_k = \imath_X$ for $k = 1, \ldots, m$.

Corollary 4.6. *Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2 with at most log terminal singularities. Then, for every $\imath \in \mathbb{Z}_{\geq 1}$, the following statements are equivalent.*

(i) *The variety X is of Gorenstein index $\imath$ and of canonical multiplicity one.*
(ii) *One can choose the defining matrix P to be of the shape*

$$\begin{pmatrix} -l_0 & l_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -l_0 & 0 & \ldots & l_r & 0 \\ * & * & \ldots & * & * \\ \boldsymbol{\imath} - \imath(r-1)l_0 & \boldsymbol{\imath} & \ldots & \boldsymbol{\imath} & \boldsymbol{\imath} \end{pmatrix},$$

where $\boldsymbol{\imath}$ stands for a vector $(\imath, \ldots, \imath)$ of suitable length.

Proof. If (i) holds, then we may assume P to be as $\tilde{P}$ in Proposition 4.3. Adding the μ_i-fold of the i-th row of to the last row brings P into the desired form. If (ii) holds, take $u = (0, \ldots, 0, -\imath) \in \mathbb{Z}^{r+s}$. Then $P^* \cdot u \in \mathbb{Z}^{n+m}$ defines a divisor $\imath D_X$ with D_X a canonical divisor of shape (4.1.1) and we see $\zeta_X = 1$. $\square$

Proposition 4.7. *Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein affine T-variety of Type 2 with at most log terminal singularities and canonical multiplicity $\zeta_X > 1$. Then we can choose P of shape 4.3 (ii) such that $l_{ij} = 1$ and $\nu_{ij} = 0$ holds for $i = 3, \ldots, r$ and $j = 1, \ldots, n_i$ and, moreover, P satisfies one of the following cases:*

Case	(l_{01}, l_{11}, l_{21})	(ν_0, ν_1, ν_2)	ζ_X	$\imath_X$
(i)	$(4,3,2)$	$\frac{1}{2}(\boldsymbol{\imath_X} + l_0,\, \boldsymbol{\imath_X} - \imath_X l_1,\, \boldsymbol{\imath_X} - l_2)$	2	$0 \mod 2$
(ii)	$(3,3,2)$	$\frac{1}{3}(\boldsymbol{\imath_X} - l_0,\, \boldsymbol{\imath_X} + l_1,\, \boldsymbol{\imath_X} - \imath_X l_2)$	3	$0 \mod 3$
(iii)	$(2k+1,2,2)$	$\frac{1}{4}(\boldsymbol{\imath_X} - \imath_X l_0,\, \boldsymbol{\imath_X} - l_1,\, \boldsymbol{\imath_X} + l_2)$	4	$2 \mod 4$
(iv)	$(2k,2,2)$	$\frac{1}{2}(\boldsymbol{\imath_X} - l_0,\, \boldsymbol{\imath_X} + l_1,\, \boldsymbol{\imath_X} - \imath_X l_2)$	2	$0 \mod 2$
(v)	$(k,2,2)$	$\frac{1}{2}(\boldsymbol{\imath_X} - \imath_X l_0,\, \boldsymbol{\imath_X} - l_1,\, \boldsymbol{\imath_X} + l_2)$	2	$0 \mod 2$
(vi)	$(k_0,k_1,1)$	$(\nu_0,\, \nu_1,\, \zeta_X^{-1}(\boldsymbol{\imath_X} - \imath_X l_2))$		

where $\boldsymbol{\imath_X}$ stands for a vector $(\imath_X, \ldots, \imath_X)$ of suitable length, and in Case (vi), all the numbers $(\imath_X - \nu_{0j_0}\zeta_X)/l_{0j_0}$ and $(\nu_{1j_1}\zeta_X - \imath_X)/l_{1j_1}$ are integral and coincide.

Proof. Since $X = X(A, P)$ has at most log terminal singularities, Theorem 1 guarantees that the Cox ring $\mathcal{R}(X) = R(A, P)$ is platonic. Thus, suitably exchanging data column blocks, we achieve $l_{ij} = 1$ for all $i \geq 3$. Next, we bring P in to the form of Proposition 4.3 (ii). Finally, subtracting the ν_{ij}-fold of the i-th row from the last one, we achieve $\nu_{ij} = 0$ for $i = 3, \ldots, r$.

Observe that our new matrix P still satisfies the conditions of Remark 4.5. For the integers μ_i defined there, we have

$$(4.7.1) \qquad \mu_0 + \mu_1 + \mu_2 \ = \ \mu_3 \ = \ \ldots \ = \ \mu_r \ = \ \imath_X.$$

Moreover, for $i = 0, 1, 2$ set $\ell_i := l_{01}l_{11}l_{21}/l_{i1}$. Then, because of $\imath_X + \mu_i l_{ij} = \nu_{ij}\zeta_X$, we obtain

$$(4.7.2) \qquad \gcd(\ell_0, \ell_1, \ell_2)^{-1} \sum_{i=0}^{2} \ell_i(\imath_X - \mu_i l_{ij}) \ = \ \alpha\zeta_X \quad \text{for some } \alpha \in \mathbb{Z}.$$

Finally, Remark 4.5 ensures

$$(4.7.3) \qquad 1 \ = \ \gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath_X) \ = \ \gcd(\mu_1, \mu_2, \zeta_X, \imath_X).$$

We will now apply these conditions to establish the table of the assertion. Since (l_{01}, l_{11}, l_{21}) is a platonic triple, we have to discuss the following cases.

Case 1: (l_{01}, l_{11}, l_{21}) equals $(5, 3, 2)$. Our task is to rule out this case. Using (4.7.1) and (4.7.2), we see that ζ_X divides

$$\imath_X \ = \ 31\imath_X - 30(\mu_0 + \mu_1 + \mu_2) \ = \ 6(\imath_X - 5\mu_0) + 10(\imath_X - 3\mu_1) + 15(\imath_X - 2\mu_2).$$

Consequently, (4.7.3) becomes $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and from $\imath_X - \mu_i l_{ij} = \nu_{ij}\zeta_X$ we infer that ζ_X divides $5\mu_0$, $3\mu_1$ and $2\mu_2$. This leaves us with the three possibilities $\zeta_X = 2, 3, 6$.

If $\zeta_X = 2$ holds, then ζ_X divides μ_0 and μ_1 but not μ_2; if $\zeta_X = 3$ holds, then ζ_X divides μ_0 and μ_2 but not μ_1. Both contradicts to the fact that ζ_X divides $\imath_X = \mu_0 + \mu_1 + \mu_2$. Thus, only $\zeta_X = 6$ is left. In that case, ζ_X must divide μ_0. Since ζ_X divides $\imath_X = \mu_0 + \mu_1 + \mu_2$, we see that ζ_X divides $\mu_1 + \mu_2$. Moreover, $\zeta_X \mid 3\mu_1$ gives $\mu_1 = 2\mu_1'$ and $\zeta_X \mid 2\mu_2$ gives $\mu_2 = 3\mu_2'$ with integers μ_1', μ_2'. Now, as $\zeta_X = 6$ divides $2\mu_1' + 3\mu_2'$, we obtain that μ_2' and hence μ_2 are even. This contradicts $\gcd(\mu_1, \mu_2, \zeta_X) = 1$.

Case 2: (l_{01}, l_{11}, l_{21}) equals $(4, 3, 2)$. Similarly as in the preceding case, we apply (4.7.1) and (4.7.2) to see that ζ_X divides

$$\imath_X \ = \ 13\imath_X - 12(\mu_0 + \mu_1 + \mu_2) \ = \ \frac{1}{2}\big(6(\imath_X - 4\mu_0) + 8(\imath_X - 3\mu_1) + 12(\imath_X - 2\mu_2)\big).$$

As before, we conclude $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and obtain that ζ_X divides $4\mu_0$, $3\mu_1$ and $2\mu_2$. This reduces to $\zeta_X = 2, 3, 6$.

If $\zeta_X = 3$ holds, then ζ_X divides μ_0 and μ_2 but not μ_1, contradicting the fact that ζ_X divides $\imath_X = \mu_0 + \mu_1 + \mu_2$. If $\zeta_X = 6$ holds, then we obtain $\mu_0 = 3\mu_0'$, $\mu_1 = 2\mu_1'$ and $\mu_2 = 3\mu_2'$ with suitable integers μ_i'. Since ζ_X divides $\imath_X = \mu_0 + \mu_1 + \mu_2$, we obtain that μ_2 is divisible by 3, contradicting $\gcd(\mu_1, \mu_2, \zeta_X) = 1$.

Thus, the only possibility left is $\zeta_X = 2$. We show that this leads to Case (i) of the assertion. Observe that μ_1 is even, μ_2 is odd because of $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and μ_2 is odd because $\imath_X = \mu_0 + \mu_1 + \mu_2$ is even. Recall that the vectors ν_i in the last row of P are given as

$$\nu_i \ = \ \frac{1}{\zeta_X}(\boldsymbol{\imath_X} - \mu_i l_i) \ = \ \frac{1}{2}\boldsymbol{\imath_X} - \frac{\mu_i}{2} l_i.$$

Thus, adding the $(-\mu_0 - \mu_2)/2$-fold of the first row and the $(\mu_2 - 1)/2$-fold of the second row to the last row brings P into the shape of Case (i).

Case 3: (l_{01}, l_{11}, l_{21}) equals $(3,3,2)$. As in the two preceding cases, we infer from (4.7.1) and (4.7.2) that ζ_X divides

$$\imath_X \;=\; 7\imath_X - 6(\mu_0+\mu_1+\mu_2) \;=\; \frac{1}{3}\big(6(\imath_X - 3\mu_0) + 6(\imath_X - 3\mu_1) + 9(\imath_X - 2\mu_2)\big).$$

Since $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and ζ_X divides $3\mu_0, 3\mu_1, 2\mu_2$, we are left with $\zeta_X = 2,3,6$. If $\zeta_X = 2$ or $\zeta_X = 6$ holds, then μ_0, μ_1 and $\imath_X = \mu_0+\mu_1+\mu_2$ must be even. Thus also μ_2 must be even, contradicting $\gcd(\mu_1, \mu_2, \zeta_X) = 1$.

Let $\zeta_X = 3$. We show that this leads to Case (ii) of the assertion. First, 3 divides μ_2 and $\imath_X = \mu_0+\mu_1+\mu_2$, hence also $\mu_0+\mu_1$. Moreover, 3 divides neither μ_0 nor μ_1 because of $\gcd(\mu_1, \mu_2, \zeta_X) = 1$. Interchanging, if necessary, the data of the column blocks no. 0 and 1, we achieve that 3 divides $\mu_0 - 1$ and $\mu_1 + 1$. So, at the moment, the ν_i in the last row of P are of the form

$$\nu_i \;=\; \frac{1}{\zeta_X}(\boldsymbol{\imath_X} - \mu_i l_i) \;=\; \frac{1}{3}\boldsymbol{\imath_X} - \frac{\mu_i}{3} l_i.$$

Adding the $(\mu_1+1)/3$-fold of the first and the $(-\mu_0-\mu_1)/3$-fold of the second to the last row of P, we arrive at Case (ii).

Case 4: (l_{01}, l_{11}, l_{21}) equals $(k,2,2)$ with $k \ge 3$ odd. Then (4.7.1) and (4.7.2) show that ζ_X divides

$$2\imath_X \;=\; (2+2k)\imath_X - 2k(\mu_0+\mu_1+\mu_2) \;=\; \frac{1}{2}(4(\imath_X - k\mu_0) + 2k(\imath_X - 2\mu_1) + 2k(\imath_X - 2\mu_2)).$$

Case 4.1: ζ_X doesn't divide $\imath_X$. Then we have $2\imath_X = \alpha\zeta_X$ with $\alpha \in \mathbb{Z}$ odd. Thus, ζ_X is even and $2\mu_i = \imath_X - \nu_{ij}\zeta_X$ implies that $4\mu_i$ is an odd multiple of ζ_X for $i = 1,2$. In particular, 4 divides ζ_X. Moreover, (4.7.3) implies $\gcd(\mu_1, \mu_2, \zeta_X/2) = 1$ and we obtain $\zeta_X = 4$. That means $\imath_X \equiv 2 \mod 4$. Since $\zeta_X = 4$ divides $\imath_X - k\mu_0$ and k is odd, we conclude $\mu_0 \equiv 2 \mod 4$. Then $\mu_0+\mu_1+\mu_2 = \imath_X \equiv 2 \mod 4$ implies that 4 divides $\mu_1+\mu_2$. Interchanging, if necessary, the data of the column blocks no. 1 and 2, we can assume $\mu_1 \equiv -\mu_2 \equiv 1 \mod 4$. Then, adding the $(\mu_1-1)/4$-fold of the first and the $(\mu_2+1)/4$-fold of the second to the last row of P, we arrive at Case (iii) of the assertion.

Case 4.2: ζ_X divides $\imath_X$. Then (4.7.3) becomes $\gcd(\mu_1, \mu_2, \zeta_X) = 1$. Since ζ_X divides $2\mu_1$ and $2\mu_2$, we see that $\zeta = 2$ holds and μ_1, μ_2 are odd. Adding the $(\mu_1-1)/2$-fold of the first and the $(\mu_2+1)/2$-fold of the second to the last row of P leads to Case (v) of the assertion.

Case 5: (l_{01}, l_{11}, l_{21}) equals $(k,2,2)$ with $k \ge 2$ even. Then (4.7.1) and (4.7.2) show that ζ_X divides

$$\imath_X \;=\; (k+1)\imath_X - k(\mu_0+\mu_1+\mu_2) \;=\; \frac{1}{4}(4(\imath_X - k\mu_0) + 2k(\imath_X - 2\mu_1) + 2k(\imath_X - 2\mu_2)).$$

As earlier, we conclude that $\zeta_X | 2\mu_i$ for $i = 1,2$ and $\zeta_X = 2$. Since $\gcd(\mu_1, \mu_2, 2) = 1$ holds and $\mu_0+\mu_1+\mu_2 = \imath_X$ is even, two of the μ_i are be odd and one is even. If μ_1 and μ_2 are odd, then adding the $(\mu_1-1)/2$-fold of the first and the $(\mu_2+1)/2$-fold of the second to the last row of P leads to Case (v). Now, let μ_0 be odd. Interchanging, if necessary, the data of the column blocks no. 1 and 2, we achieve that μ_1 is odd. Then we add the $(\mu_1+1)/2$-fold of the first and the $(-\mu_0-\mu_1)/2$-fold of the second to the last row of P and arrive at Case (iv) of the assertion.

Case 6. (l_{01}, l_{11}, l_{21}) equals $(k_0, k_1, 1)$, where $k_0, k_1 \in \mathbb{Z}_{>0}$. We subtract the ν_{21}-fold of the second row of P from the last one. Since $\nu_{21} = (\imath_X - \mu_2)/\zeta_X$ holds, we obtain $\nu_2 = \zeta_X^{-1}(\boldsymbol{\imath_X} - \imath_X l_2)$. Moreover, (4.7.1) becomes $\mu_0 + \mu_1 = 0$. We arrive at Case (vi) of the assertion by observing

$$(\imath_X - \nu_{0j_0}\zeta_X)/l_{0j_0} \;=\; \mu_0 \;=\; -\mu_1 \;=\; (\nu_{1j_1}\zeta_X - \imath_X)/l_{1j_1}.$$

□

Example 4.8. We discuss the rational affine $\mathbb{C}^*$-surfaces X with at most log terminal singularities. First, the affine toric surfaces $X = \mathbb{C}^2/C_k$ show up here, where C_k is the cyclic group of order k acting diagonally. In terms of toric geometry, these surfaces are given as

$$X \;=\; \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2], \qquad \sigma = \operatorname{cone}((k,\imath),\, (\imath, k+m)),$$

where $k, m \in \mathbb{Z}_{>0}$ with $\gcd(k,\imath) = \gcd(k+m,\imath) = 1$ and $\imath$ is the Gorenstein index of X; see [7, Chap. 10] for more background. Now consider a non-toric $\mathbb{C}^*$-surface $X = X(A,P)$ of Type 2. As a quotient of $\mathbb{C}^2$ by a finite group, X has finite divisor class group and thus P is a 3×3 matrix of the shape

$$P \;=\; \begin{bmatrix} -l_{01} & l_{11} & 0 \\ -l_{01} & 0 & l_{21} \\ d_{01} & d_{11} & d_{21} \end{bmatrix}.$$

Theorem 1 says that (l_{01}, l_{11}, l_{21}) is a platonic triple. Moreover, Corollary 4.6 and Proposition 4.7 provide us with constraints on the d_{i1}. Having in mind that P is of rank three with primitive columns, one directly arrives at the following possibilities, where $\zeta = \zeta_X$ is the canonical multiplicity and $\imath = \imath_X$ the Gorenstein index:

Type	P	ζ	$\imath$
$D_n^{1,\imath}$	$\begin{bmatrix} -n+2 & 2 & 0 \\ -n+2 & 0 & 2 \\ -n\imath+3\imath & \imath & \imath \end{bmatrix}$	1	$\gcd(\imath, 2n) = 1$
$D_{2n+1}^{2,\imath}$	$\begin{bmatrix} -2n+1 & 2 & 0 \\ -2n+1 & 0 & 2 \\ (1-n)\imath & \imath/2+1 & \imath/2-1 \end{bmatrix}$	2	$\gcd(\imath, 8n-4) = 4$
$E_6^{1,\imath}$	$\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ -2\imath & \imath & \imath \end{bmatrix}$	1	$\gcd(\imath, 6) = 1$
$E_6^{3,\imath}$	$\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ \imath/3-1 & \imath/3+1 & -\imath/3 \end{bmatrix}$	3	$\gcd(\imath, 18) = 9$
$E_7^{1,\imath}$	$\begin{bmatrix} -4 & 3 & 0 \\ -4 & 0 & 2 \\ -3\imath & \imath & \imath \end{bmatrix}$	1	$\gcd(\imath, 6) = 1$
$E_8^{1,\imath}$	$\begin{bmatrix} -5 & 3 & 0 \\ -5 & 0 & 2 \\ -4\imath & \imath & \imath \end{bmatrix}$	1	$\gcd(\imath, 30) = 1$

For geometric details on these surfaces, we refer to the work of Brieskorn [6], and, in the context of the McKay Correspondence, Wunram [30] and Wemyss [29].

5. Geometry of the total coordinate space

We take a closer look at the geometry of the total coordinate space $\overline{X}$ of a T-variety X of complexity one. The first result says in particular that $\overline{X}$ is Gorenstein and canonical provided that X is log terminal and affine.

Proposition 5.1. *Let $R(A, P_0)$ be a platonic ring of Type 2. Then the affine variety $\overline{X} = \operatorname{Spec} R(A, P_0)$ is Gorenstein and has at most canonical singularities.*

Proof. Adding suitable rows, we complement the matrix P_0 to a square matrix P of full rank with last row $(\mathbf{1}-(r-1)l_0, \mathbf{1}, \dots, \mathbf{1})$, where $\mathbf{1}$ indicates vectors of length n_i with all entries equal to one; this is possible, because the last row is not in the row space of P_0. Then $X = X(A,P)$ is a $\mathbb{Q}$-factorial affine T-variety. Theorem 1 tells us that X has at most log terminal singularities and Corollary 4.6 ensures that X is Gorenstein. Thus, X has at most canonical singularities. Since $\overline{X} \to X$ is finite with ramification locus of codimension at least two, we can use [20, Thm. 6.2.9] to see that $\overline{X}$ is Gorenstein with at most canonical singularities. $\square$

Now we investigate the generic quotient Y of $\overline{X}$ by the action of the unit component $H_0^0 \subseteq H_0$, in other words, the smooth projective curve Y with function field $\mathbb{C}(Y) = \mathbb{C}(\overline{X})^{H_0^0}$.

Definition 5.2. Consider the defining matrix P_0 of a ring $R(A, P_0)$ of Type 2 and the vectors $l_i = (l_{i1}, \ldots, l_{in_i})$ occuring in the rows of P_0. Set

$$\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i}), \qquad \mathfrak{l} := \gcd(\mathfrak{l}_0, \ldots, \mathfrak{l}_r), \qquad \mathfrak{l}_{ij} := \gcd(\mathfrak{l}^{-1}\mathfrak{l}_i, \mathfrak{l}^{-1}\mathfrak{l}_j),$$

$$\bar{\mathfrak{l}} := \operatorname{lcm}(\mathfrak{l}_0, \ldots, \mathfrak{l}_r), \qquad b_i := \mathfrak{l}_i^{-1}\bar{\mathfrak{l}}, \qquad b(i) := \gcd(b_j;\ j \neq i).$$

Theorem 5.3. *Let $R(A, P_0)$ be of Type 2 and consider the action of the unit component $H_0^0 \subseteq H_0$ of the quasitorus $H_0 = \operatorname{Spec} \mathbb{C}[K_0]$ on $\overline{X} = \operatorname{Spec} R(A, P_0)$. Then the smooth projective curve Y with function field $\mathbb{C}(Y) = \mathbb{C}(\overline{X})^{H_0^0}$ is of genus*

$$g(Y) \ = \ \frac{\mathfrak{l}_0 \cdots \mathfrak{l}_r}{2\bar{\mathfrak{l}}} \left((r-1) - \sum_{i=0}^{r} \frac{b(i)}{\mathfrak{l}_i} \right) + 1.$$

Lemma 5.4. *Let $R(A, P_0)$ be of Type 2, consider the degree $u := \deg(g_0) \in K_0$ of the defining relations and the subgroup*

$$K_0(u) \ := \ \{w \in K_0;\ \alpha w \in \mathbb{Z}u \text{ for some } \alpha \in \mathbb{Z}_{>0}\} \ \subseteq \ K_0.$$

Then the Veronese subalgebra $R(A, P_0)(u)$ of $R(A, P_0)$ associated with $K_0(u)$ of K_0 is generated by the monomials $T_0^{l_0/\mathfrak{l}_0}, \ldots, T_r^{l_r/\mathfrak{l}_r}$.

Proof. First, observe that every element of $R(A, P_0)(u)$ is a polynomial in the variables T_{ij}. Now consider a monomial T^l in the T_{ij} of degree $w \in K_0(u)$, where $l \in \mathbb{Z}^{n+m}$. Then $\alpha w \in \beta_0 u$ holds for some $\alpha \in \mathbb{Z}_{>0}$ and $\beta_0 \in \mathbb{Z}$. Moreover, there are $\beta_1, \ldots, \beta_r \in \mathbb{Z}$ with

$$\alpha l = \beta_0 l_0' + \beta_1(l_0' - l_1') + \ldots + \beta_r(l_0' - l_r'), \text{ where } l_i' := l_{i1}e_{i1} + \ldots + l_{in_i}e_{in_i},$$

reflecting the fact that $\alpha l - \beta_0 l_0'$ lies in the row space of P_0. Consequently, we obtain $l = \beta_0' l_0' + \ldots + \beta_r' l_r'$ for suitable $\beta_i' \in \mathbb{Q}$. Since l has only non-negative integer entries, we conclude that every β_i' is a non-negative integral multiple of $\mathfrak{l}_i^{-1}$. Thus, T^l is a monomial in the $T_i^{l_i/\mathfrak{l}_i}$. The assertion follows. $\square$

Proof of Theorem 5.3. The curve Y occurs as a GIT-quotient: $Y = \overline{X}^{ss}(u^0)/H_0^0$, where $u^0 \in \mathbb{X}(H_0^0)$ represents the character induced by $u = \deg(g_0) \in K_0 = \mathbb{X}(H_0)$. In other words, we have $Y = \operatorname{Proj} R(A, P_0)(u^0)$ with the Veronese subalgebra defined by u^0. We may replace u^0 with

$$w^0 \ := \ \frac{1}{\bar{\mathfrak{l}}} u^0 \ \in \ \mathbb{X}(H_0^0).$$

Then $R(A, P_0)(u^0)$ is replaced with $R(A, P_0)(w^0)$ which in turn equals the Veronese subalgebra treated in Lemma 5.4. Moreover, the generators $T_i^{l_i/\mathfrak{l}_i} \in R(A, P_0)(w^0)$ are of degree $b_i w^0 \in \mathbb{X}(H_0^0)$, respectively. We obtain a closed embedding into a weighted projective space

$$Y \ = \ V(h_0, \ldots, h_{r-2}) \ \subseteq \ \mathbb{P}(b_0, \ldots, b_r), \qquad h_i \ := \ \det \begin{bmatrix} T_i^{\mathfrak{l}_i} & T_{i+1}^{\mathfrak{l}_{i+1}} & T_{i+2}^{\mathfrak{l}_{i+2}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix},$$

where the h_i generate the ideal of relations among the generators of the Veronese subalgebra $R(A, P_0)(w^0)$. The idea is now to construct a ramified covering $Y' \to Y$

with a suitable curve Y' and then to compute the genus of Y via the Hurwitz formula. Consider

$$Y' \;=\; V(h'_0, \dots, h'_{r-2}) \;\subseteq\; \mathbb{P}_r, \qquad h'_i \;:=\; \det \begin{bmatrix} T_i^{\bar{\mathfrak{l}}} & T_{i+1}^{\bar{\mathfrak{l}}} & T_{i+2}^{\bar{\mathfrak{l}}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix}.$$

The $Y' \subseteq \mathbb{P}_r$ is a smooth complete intersection curve. Computing the genus of Y' according to [15], we obtain

$$g(Y') \;=\; \frac{1}{2}((r-1)\bar{\mathfrak{l}}^r - (r+1)\bar{\mathfrak{l}}^{r-1}) + 1.$$

The morphism $\mathbb{P}_r \to \mathbb{P}(b_0, \dots, b_r)$ sending $[z_0, \dots, z_r]$ to $[z_0^{b_0}, \dots, z_r^{b_r}]$ restricts to a morphism $Y' \to Y$ of degree $b_0 \cdots b_r$. The intersection $Y \cap U_i$ with the i-th coordinate hyperplane $U_i \subseteq \mathbb{P}_r$ contains precisely $\bar{\mathfrak{l}}^{r-1}$ points and each of these points has ramification order $b_i \cdot b(i) - 1$. Outside the U_i, the morphism $Y' \to Y$ is unramified. The Hurwitz formula then gives $g(Y)$. $\square$

We now use Theorem 5.3 to characterize rationality of $\overline{X} = \operatorname{Spec} R(A, P_0)$. For the special case of Pham-Brieskorn surfaces, the following statement has been obtained in [4].

Proposition 5.5. *Let $R(A, P_0)$ be of Type 2 with $r = 2$, that means that $\overline{X} = \operatorname{Spec} R(A, P_0)$ is given as*

$$\overline{X} \;\cong\; V(T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + T_{11}^{l_{11}} \cdots T_{1n_1}^{l_{1n_1}} + T_{21}^{l_{21}} \cdots T_{2n_2}^{l_{2n_2}}) \;\subseteq\; \mathbb{C}^n.$$

Then the hypersurface $\overline{X}$ is rational if and only if one of the following conditions holds:

(i) *there are pairwise coprime positive integers c_0, c_1, c_2 and a positive integer s such that, after suitable renumbering, one has*

$$\gcd(c_2, s) = 1, \quad \mathfrak{l}_0 = sc_0, \quad \mathfrak{l}_1 = sc_1, \quad \mathfrak{l}_2 = c_2;$$

(ii) *there are pairwise coprime positive integers c_0, c_1, c_2 such that*

$$\mathfrak{l}_0 = 2c_0, \quad \mathfrak{l}_1 = 2c_1, \quad \mathfrak{l}_2 = 2c_2.$$

Lemma 5.6. *For $i = 0, 1, 2$, let $l_i = (l_{i1}, \dots, l_{in_i})$ be tuples of positive integers. Define $\mathfrak{l}$, $\mathfrak{l}_i$ and $\mathfrak{l}_{ij}$ as in Definition 5.2 for $r = 2$. Then the following statements are equivalent.*

(i) *We have $\mathfrak{l}(\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12} - (\mathfrak{l}_{01} + \mathfrak{l}_{02} + \mathfrak{l}_{12})) = -2$.*

(ii) *One of the following two conditions holds:*

(a) *there are pairwise coprime positive integers c_0, c_1, c_2 and a positive integer s such that, after suitable renumbering, one has*

$$\gcd(c_2, s) = 1, \quad \mathfrak{l}_0 = sc_0, \quad \mathfrak{l}_1 = sc_1, \quad \mathfrak{l}_2 = c_2;$$

(b) *there are pairwise coprime positive integers c_0, c_1, c_2 such that*

$$\mathfrak{l}_0 = 2c_0, \quad \mathfrak{l}_1 = 2c_1, \quad \mathfrak{l}_2 = 2c_2.$$

Proof. If (ii) holds, then a simple computation shows that (i) is valid. Now, assume that (i) holds. Then the following cases have to be considered.

Case 1. We have $\mathfrak{l} = 1$. Then $\mathfrak{l}_{01}(\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) = \mathfrak{l}_{02} + \mathfrak{l}_{12} - 2$ holds. From this we deduce

$$\begin{aligned} \mathfrak{l}_{01}(\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) \;&=\; (\mathfrak{l}_{01} - 1)(\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) + (\mathfrak{l}_{02} - 1)(\mathfrak{l}_{12} - 1) + \mathfrak{l}_{02} + \mathfrak{l}_{12} - 2 \\ &\geq\; \mathfrak{l}_{02} + \mathfrak{l}_{12} - 2, \end{aligned}$$

where equality holds if and only if at least two of $\mathfrak{l}_{01}, \mathfrak{l}_{02}, \mathfrak{l}_{12}$ equal one. So, we arrive at Condition (a).

Case 2. We have $\mathfrak{l} = 2$. Then we have $\mathfrak{l}_{01}(2\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) + 1 = \mathfrak{l}_{02} + \mathfrak{l}_{12}$. In this situation, we conclude

$$\begin{aligned} \mathfrak{l}_{01}(2\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) + 1 &= (\mathfrak{l}_{01} - 1)(2\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) + \mathfrak{l}_{02}\mathfrak{l}_{12} \\ &\quad +(\mathfrak{l}_{02} - 1)(\mathfrak{l}_{12} - 1) + \mathfrak{l}_{02} + \mathfrak{l}_{12} - 1 \\ &\geq \mathfrak{l}_{02} + \mathfrak{l}_{12}, \end{aligned}$$

where equality holds if and only if we have $\mathfrak{l}_{01} = \mathfrak{l}_{02} = \mathfrak{l}_{12} = 1$. Thus, we arrive at Condition (b). ☐

Proof of Proposition 5.5. First, observe that $\overline{X}$ is rational if and only if Y is rational or, in other words, of genus zero. For $r = 2$, Theorem 5.3 gives

$$g(Y) = \frac{\mathfrak{l}}{2}(\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12} - \mathfrak{l}_{01} - \mathfrak{l}_{02} - \mathfrak{l}_{12}) + 1.$$

Thus, according to Lemma 5.6, condition $g(Y) = 0$ holds if and only if (i) or (ii) of the proposition holds. ☐

Remark 5.7. If the defining polynomial in Proposition 5.5 is classically homogeneous, then it defines a projective hypersurface $X' \subseteq \mathbb{P}^{n-1}$ and the following statements are equivalent.

(i) X' is rational.
(ii) $\mathrm{Cl}(X')$ is finitely generated.
(iii) Condition 5.5 (i) or (ii) holds.

Corollary 5.8. *Let $R(A, P_0)$ be of Type 2. Then $\overline{X} = \operatorname{Spec} R(A, P_0)$ is rational if and only if one of the following conditions holds:*

(i) *We have $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $0 \leq i < j \leq r$, in other words, $R(A, P_0)$ is factorial.*
(ii) *There are $0 \leq i < j \leq r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) > 1$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j\}$.*
(iii) *There are $0 \leq i < j < k \leq r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = \gcd(\mathfrak{l}_i, \mathfrak{l}_k) = \gcd(\mathfrak{l}_j, \mathfrak{l}_k) = 2$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j, k\}$.*

Lemma 5.9. *Let A, P_0 be defining data of Type 2, enhance A to A' by attaching a further column and P_0 to P_0' by attaching l_{r+1} to $l_0, \ldots, l_r$. If $\gcd(\mathfrak{l}_i, \mathfrak{l}_{r+1}) = 1$ holds for $i = 0, \ldots, r$, then we have $g(Y) = g(Y')$ for the curves associated with $R(A, P)$ and $R(A', P_0')$ respectively.*

Proof. Denote the numbers arising from P' in the sense of Definition 5.2 by $\mathfrak{l}_i'$, $\mathfrak{l}'$ etc. Then we have

$$\bar{\mathfrak{l}}' = \bar{\mathfrak{l}}\mathfrak{l}_{r+1}, \quad b'(i) = \gcd(\bar{\mathfrak{l}}, \bar{\mathfrak{l}}'/\bar{\mathfrak{l}}_j;\ j \neq i) = b(i), \quad i = 0, \ldots, r,$$

$$b(r+1) = \gcd(\bar{\mathfrak{l}}'/\bar{\mathfrak{l}}_0, \ldots, \bar{\mathfrak{l}}'/\bar{\mathfrak{l}}_r) = \mathfrak{l}_{r+1}.$$

Plugging these identities into the genus formula of Theorem 5.3, we directly obtain $g(Y') = g(Y)$. ☐

Lemma 5.10. *Let $R(A, P_0)$ be of Type 2 and assume that the curve Y associated with $R(A, P)$ is of genus zero. Then there are $0 \leq i \leq j \leq k \leq r$ with $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j, k\}$.*

Proof. According to Theorem 5.3, the fact that the curve Y associated with $R(A, P)$ is of genus zero implies

$$\sum_{i=0}^{r} \frac{b(i)}{\mathfrak{l}_i} = (r-1) + \frac{2\bar{\mathfrak{l}}}{\mathfrak{l}_0 \cdots \mathfrak{l}_r} > r - 1.$$

As $b(i)$ divides $\mathfrak{l}_i$, we see that $b(i) \neq \mathfrak{l}_i$ can happen at most three times. Moreover, $b(i) = \mathfrak{l}_i$ is equivalent to $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $j \neq i$. $\square$

Proof of Corollary 5.8. We may assume that the indices i, j and k of Lemma 5.10 are $0, 1$ and 2. Then Lemma 5.9 says that $\overline{X}$ is rational if and only if the trinomial hypersurface defined by the exponent vectors l_0, l_1, l_2 is rational. Thus, Proposition 5.5 gives the assertion. $\square$

Corollary 5.11. *Let $R(A, P_0)$ be a platonic ring of Type 2. Then $\overline{X} = \operatorname{Spec} R(A, P_0)$ is rational.*

Remark 5.12. It may happen that for a rational T-variety X of complexity one, the total coordinate space $\overline{X}$ is rational, but the total coordinate space of $\overline{X}$ not any more. For instance consider

$$X_3 \ := \ V(T_1^4 + T_2^4 + T_3^4) \ \subseteq \ \mathbb{C}^3.$$

Then, according to Proposition 5.5, the surface X_3 is not rational. Moreover, X_3 is the total coordinate space of an affine rational $\mathbb{C}^*$-surface X_2 with defining matrix

$$P_2 \ = \ \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \\ -3 & 1 & 1 \end{bmatrix}.$$

The divisor class group of X_2 is $\mathrm{Cl}(X_2) = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and the $\mathrm{Cl}(X_2)$-grading of the Cox ring $\mathcal{R}(X_2) = \mathbb{C}[T_1, T_2, T_3]/\langle T_1^4 + T_2^4 + T_3^4 \rangle$ is given by

$$\deg(T_1) \ = \ (\overline{1}, \overline{1}), \quad \deg(T_2) \ = \ (\overline{1}, \overline{2}), \quad \deg(T_3) \ = \ (\overline{2}, \overline{1}).$$

For an equation for X_2, compute the degree zero subalgebra of $\mathcal{R}(X_2)$: it has three generators S_1, S_2, S_3 and $S_1^3 + S_2^3 + S_3^4$ as defining relation. Thus,

$$X_2 \ \cong \ V(S_1^3 + S_2^3 + S_3^4) \ \subseteq \ \mathbb{C}^3.$$

To obtain a rational affine $\mathbb{C}^*$-surface having X_2 as its total coordinate space, we take X_1, defined by

$$P_1 \ := \ \begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 4 \\ -2 & 1 & 1 \end{bmatrix}.$$

The divisor class group of X_1 is $\mathrm{Cl}(X_1) = \mathbb{Z}/3\mathbb{Z}$ and the $\mathrm{Cl}(X_1)$-grading of the Cox ring $\mathcal{R}(X_1) = \mathbb{C}[S_1, S_2, S_3]/\langle S_1^3 + S_2^3 + S_3^4 \rangle$ is given by

$$\deg(T_1) \ = \ \overline{1}, \quad \deg(T_2) \ = \ \overline{2}, \quad \deg(T_3) \ = \ \overline{0}.$$

We have constructed a chain of total coordinate spaces $X_3 \to X_2 \to X_1$, where X_1 is a rational affine $\mathbb{C}^*$-surface, X_2 is rational and X_3 not.

Finally, we determine the factor group of the maximal quasitorus by its unit component acting on a given trinomial hypersurface; the proof is a direct consequence of the subsequent lemma.

Proposition 5.13. *Let $R(A, P)$ be any ring of Type 2, where $r = 2$. Then, for the quasitorus H_0 acting on the corresponding trinomial hypersurface*

$$\overline{X} \ \cong \ V(T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + T_{11}^{l_{11}} \cdots T_{1n_1}^{l_{1n_1}} + T_{21}^{l_{21}} \cdots T_{2n_2}^{l_{2n_2}}) \subseteq \mathbb{C}^n,$$

the factor group H_0/H_0^0 by the unit component $H_0^0 \subseteq H_0$ is isomorphic to the product of cyclic groups $C(\mathfrak{l}) \times C(\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12})$.

Lemma 5.14. *Consider a matrix P_0 with $m = 0$ and $r = 2$ as in Type 2 of Construction 2.2:*

$$P_0 \;=\; \begin{bmatrix} -l_0 & l_1 & 0 \\ -l_0 & 0 & l_2 \end{bmatrix}.$$

As earlier, set $\mathfrak{l}_i = \gcd(l_{i1}, \ldots, l_{n_i})$*. Then, with* $\mathfrak{l}_{ij} = \gcd(\mathfrak{l}_i, \mathfrak{l}_j)$ *and* $\mathfrak{l} = \gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$*, we obtain*

$$K_0^{\mathrm{tors}} \;=\; (\mathbb{Z}^n/\mathrm{im}(P_0^*))^{\mathrm{tors}} \;\cong\; C(\mathfrak{l}) \times C(\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12}).$$

Proof. Suitable elementary column operations to P_0 transform the entries l_i to $(\mathfrak{l}_i, 0, \ldots, 0)$. Thus, $K_0^{\mathrm{tors}} \cong (\mathbb{Z}^3/\mathrm{im}(P_1^*))^{\mathrm{tors}}$ holds with the 2×3 matrix

$$P_1 \;:=\; \begin{bmatrix} -\mathfrak{l}_0 & \mathfrak{l}_1 & 0 \\ -\mathfrak{l}_0 & 0 & \mathfrak{l}_2 \end{bmatrix}.$$

The determinantal divisors of P_0 are $\gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ and $\gcd(\mathfrak{l}_0\mathfrak{l}_1, \mathfrak{l}_0\mathfrak{l}_2, \mathfrak{l}_1\mathfrak{l}_2)$. Thus, the invariant factors of P_0 are $\mathfrak{l}$ and $\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12}$; see [24]. ☐

6. Proof of Theorems 2 and 3

We are ready to prove our first main results. The proof of Theorem 2 will be in fact constructive in the sense that it allows to compute the defining equations of the Cox ring in every iteration step; see Proposition 6.6.

Remark 6.1. Let $R(A, P)$ be a ring of Type 2. Applying suitable admissible operations, one achieves that P is *ordered* in the sense that $l_{i1} \geq \ldots \geq l_{in_i}$ for all $i = 0, \ldots, r$ and $l_{01} \geq \ldots \geq l_{r1}$ hold. For an ordered P, the ring $R(A, P)$ is platonic if and only if (l_{01}, l_{11}, l_{21}) is a platonic triple and $l_{i1} = 1$ holds for $i \geq 3$.

Definition 6.2. The *leading platonic triple* of a ring $R(A, P)$ of Type 2 is the triple (l_{01}, l_{11}, l_{21}) obtained after ordering P.

Lemma 6.3. *Let $R(A, P_0)$ be of Type 2 and platonic such that $l_{i1} \geq \ldots \geq l_{in_i}$ holds for all i and $l_{i1} = 1$ for $i \geq 3$. Moreover, assume* $\gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \mathfrak{l}$*. Then, with $K_0 = \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$, the kernel of $\mathbb{Z}^{n+m} \to K_0/K_0^{\mathrm{tors}}$ is generated by the rows of the matrix*

$$P_1 \;:=\; \begin{bmatrix} \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} l_0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} l_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)} l_0 & 0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)} l_2 & 0 & 0 & & & \\ -l_0 & 0 & & \mathbf{1} & 0 & \vdots & & \vdots \\ \vdots & & & \vdots & \ddots & \vdots & & \\ -l_0 & 0 & \ldots & 0 & \mathbf{1} & 0 & \ldots & 0 \end{bmatrix},$$

where, as before, the symbols $\mathbf{1}$ indicate vectors of length n_i with all entries equal to one.

Proof. Observe that the rows of P_0 generate a sublattice of finite index in the row lattice P_1. Thus, we have a commutative diagram

$$\begin{tikzcd} K_0 \arrow[rr] \arrow[dr] & & K_0/K_0^{\mathrm{tors}} \\ & \mathbb{Z}^{n+m}/\mathrm{im}(P_1^*) \arrow[ur] & \end{tikzcd}$$

It suffices to show, that $\mathbb{Z}^{n+m}/\mathrm{im}(P_1^*)$ is torsion free. Applying suitable elementary column operations to P_1, reduces the problem to showing that for the 2×3 matrix

$$\begin{bmatrix} \frac{\mathfrak{l}_0}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} & \frac{\mathfrak{l}_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} & 0 \\ \frac{\mathfrak{l}_0}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)} & 0 & \frac{\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} \end{bmatrix},$$

all determinantal divisors equal one. The entries of the above matrix are coprime and its 2×2 minors are

$$\frac{\mathfrak{l}_0\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}, \qquad \frac{\mathfrak{l}_1\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}, \qquad \frac{\mathfrak{l}_0\mathfrak{l}_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}.$$

up to sign. By assumption, we have $\gcd(\mathfrak{l}_1,\mathfrak{l}_2) = \mathfrak{l}$. Consequently, we obtain

$$\gcd(\mathfrak{l}_0\mathfrak{l}_2,\mathfrak{l}_0\mathfrak{l}_1,\mathfrak{l}_1\mathfrak{l}_2) \ = \ \gcd(\mathfrak{l}_0\mathfrak{l},\mathfrak{l}_1\mathfrak{l}_2) \ = \ \gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)$$

and therefore the second determinantal divisor equals one. As remarked, the first one equals one as well and the assertion follows. □

Lemma 6.4. *Let $R(A,P_0)$ be of Type 2 and $\overline{X} = \operatorname{Spec} R(A,P_0)$. Then, for any generator T_{01} of $R(A,P_0)$, we have*

$$V(\overline{X},T_{01}) \ \cong \ V(T_{01}) \cap V(T_1^{l_1} - T_i^{l_i};\ i = 2,\ldots,r) \ \subseteq \ \mathbb{C}^{n+m}.$$

In particular, the number of irreducible components of $V(\overline{X},T_{01})$ equals the product of the invariant factors of the matrix

$$\begin{bmatrix} -\mathfrak{l}_1 & \mathfrak{l}_2 & & 0 \\ \vdots & & \ddots & \\ -\mathfrak{l}_1 & 0 & & \mathfrak{l}_r \end{bmatrix}.$$

Proof. First observe that the ideal $\langle T_{01}, g_0, \ldots, g_{r-2}\rangle \subseteq \mathbb{C}[T_{ij},S_k]$ is generated by binomials which can be brought into the above form by scaling the variables appropriately. Now consider the homomorphism of tori

$$\pi\colon \mathbb{T}^{n_1+\ldots+n_r} \ \to \ \mathbb{T}^{r-1}, \qquad (t_1,\ldots,t_r) \ \mapsto \ \left(\frac{t_2^{l_2}}{t_1^{l_1}},\ldots,\frac{t_r^{l_r}}{t_1^{l_1}}\right).$$

Then the number of connected components of $\ker(\pi)$ equals the product of the invariant factors of the above matrix. Moreover, $\mathbb{T}^{n_0-1} \times \ker(\pi) \times \mathbb{T}^m$ is isomorphic to $V(\overline{X},T_{01}) \cap \mathbb{T}^{n+m}$. Finally, one directly checks that $V(\overline{X},T_{01})$ has no further irreducible components outside $\mathbb{T}^{n+m}$. □

Lemma 6.5. *Let $R(A,P_0)$ be of Type 2 and platonic. Assume that P_0 is ordered. Then the number $c(i)$ of irreducible components of $V(\overline{X},T_{ij})$ is given as*

i	0	1	2	≥ 3
$c(i)$	$\gcd(\mathfrak{l}_1,\mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0,\mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0,\mathfrak{l}_1)$	$\mathfrak{l}^2\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12}$

Proof. Suitable admissible operations turn T_{ij} to T_{01}. Then the number of components is computed via Lemma 6.4. □

Proposition 6.6. *Let $R(A,P_0)$ be of Type 2, platonic and non-factorial. Assume that P_0 is ordered and let P_1 be as in Lemma 6.3. Set*

$$n_{i,1},\ldots,n_{i,c(i)} \ := \ n_i, \qquad l_{ij,1},\ldots,l_{ij,c(i)} \ := \ \gcd((P_1)_{1,ij},\ldots,(P_1)_{r,ij})$$

The $l_{i,\alpha} := (l_{i1,\alpha},\ldots,l_{in_i,\alpha}) \in \mathbb{Z}^{n_{i,\alpha}}$ build up an $(n'+m)\times(r'+s)$ matrix P_0', where $n' := c(0)n_0 + \ldots + c(r)n_r$. With a suitable matrix A', the following holds.

(i) *The affine variety* $\operatorname{Spec} R(A',P_0')$ *is the total coordinate space of the affine variety* $\operatorname{Spec} R(A,P_0)$,
(ii) *The leading platonic triple (l.p.t.) of $R(A',P')$ can be expressed in terms of that of $R(A,P)$ as*

l.p.t. of $R(A,P)$	*l.p.t. of* $R(A',P')$
$(4,3,2)$	$(3,3,2)$
$(3,3,2)$	$(2,2,2)$
$(y,2,2)$	$(z,z,1)$ *or* $\left(\frac{y}{2},2,2\right)$
$(x,y,1)$	$\left(\frac{x}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)},\frac{y}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)},1\right)$

Proof. We compute the Cox ring of $\overline{X} = \operatorname{Spec} R(A,P_0)$ according to [2, Thm. 4.4.1.6]; use Corollary 1.9 [19] to obtain the statement given there also in the affine case. That means that we have to figure out which invariant divisors are identified under the rational map onto the curve Y with function field $\mathbb{C}(\overline{X})^{H_0^0}$ and we have to determine the orders of isotropy groups of invariant divisors.

Let P_1 be as in Lemma 6.3. Then the torus H_0^0 acts diagonally on $\mathbb{C}^{n+m}$ with weights provided by the projection $Q_1\colon \mathbb{Z}^{n+m} \to K_0^0$, where $K_0^0 = \mathbb{Z}^{n+m}/\mathrm{im}(P_1^*)$ equals the character group of H_0^0. Consider the commutative diagram

$$\begin{tikzcd} \overline{X}_0 \arrow[d] & \subseteq & \mathbb{C}_0^{n+m} \arrow[d] \\ \overline{X}_0/H_0^0 \arrow[d] & \subseteq & \mathbb{C}_0^{n+m}/H_0^0 \arrow[d] \\ Y & \subseteq & \mathbb{P} \end{tikzcd}$$

where $\overline{X}_0 \subseteq \overline{X}$ and $\mathbb{C}_0^{n+m} \subseteq \mathbb{C}^{n+m}$ denote the open H_0^0-invariant subsets obtained by removing all coordinate hyperplanes $V(S_k)$ and all intersections $V(T_{i_1j_1},T_{i_2j_2})$ with $(i_1,j_1) \neq (i_2,j_2)$ from $\mathbb{C}^{n+m}$. Moreover, the geometric quotient spaces in the middle row are possibly non-separated and the maps to the lower row are separation morphisms.

We determine the orders of isotropy groups. Every point in $\mathbb{T}^{n+m}$ has trivial H_0^0-isotropy. Thus, we only have to look what happens on the sets $V(T_{ij}) \cap \mathbb{C}_0^{n+m}$. According to [2, Prop. 2.1.4.2], the order of isotropy group of H_0^0 at any point $x \in V(T_{ij}) \cap \mathbb{C}_0^{n+m}$ equals the greatest common divisor of the entries of the ij-th column of P_1:

$$|H_{0,x}^0| \ = \ l'_{ij} \ := \ \gcd((P_1)_{1,ij},\ldots,(P_1)_{r,ij}) \quad \text{for all } x \in V(T_{ij}) \cap \mathbb{C}_0^{n+m}.$$

Now we figure out which H_0^0-invariant divisors of $\overline{X}_0$ are identified under the map $\overline{X}_0 \to Y$. Lemma 6.5 provides us explicit numbers $c(0),\ldots,c(r)$ such that for fixed i and $j = 1,\ldots,n_i$, we have the decomposition into prime divisors

$$V(\overline{X},T_{ij}) \ = \ D_{ij,1} \cup \ldots \cup D_{ij,c(i)},$$

in particular, the number $c(i)$ does not depend on the choice of j. The components $D_{ij,1},\ldots,D_{ij,c(i)}$ lie in the common affine chart $W_0 \subseteq \overline{X}_0$ obtained by localizing at all $T_{i'j'}$ different from T_{ij}. Their images thus lie in the affine chart $W_0/H_0^0 \subseteq \overline{X}_0/H_0^0$. Consequently, the $D_{ij,1},\ldots,D_{ij,c(i)}$ have pairwise disjoint images under the composition $\overline{X}_0 \to \overline{X}_0/H_0^0 \to Y$.

On the other hand, $V(\overline{X},T_{ij})$ and $V(\overline{X},T_{ij'})$ are identified isomorphically under the separation map $\overline{X}_0/H_0^0 \to Y$ Thus, suitably numbering, we obtain for every i, and $\alpha = 1,\ldots,c(i)$ a chain

$$D_{i1,\alpha},\ldots,D_{in_i,\alpha},$$

of divisors identified under the morphism $\overline{X}_0/H_0^0 \to Y$. The order of isotropy for any $x \in D_{ij,\alpha}$ equals l'_{ij}. Now, using [2, Thm. 4.4.1.6], we can compute the defining relations of the Cox ring of $\overline{X}$, which establishes the two assertions. $\square$

Remark 6.7. Let $R(A, P_0)$ be a non factorial platonic ring with ordered P_0 and leading platonic triple (l_{01}, l_{11}, l_{21}). Denote by $R(A', P_0')$ the Cox ring of $\operatorname{Spec} R(A, P_0)$. Then the exponents of the defining relations of $R(A', P_0')$ are listed in the following table, where $\mathbf{1}_{n_1}$ denotes the vector of length n_i with all entries equal to one.

leading plat. triple	exponents in $R(A', P')$
$(4,3,2)$	l_1, l_1, $l_0/2$, $\mathbf{1}_{n_2}$ and $2 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(3,3,2)$	l_2, l_2, l_2, $\mathbf{1}_{n_0}$, $\mathbf{1}_{n_1}$ and $3 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(x,2,2)$ and $\mathfrak{l} = 2$	$l_0/2$, $l_0/2$, $2 \times \mathbf{1}_{n_1}$ and $\mathbf{1}_{n_2}$, and $4 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(x,2,2)$ and $2 \nmid \mathfrak{l}_0$	l_0, l_0, $\mathbf{1}_{n_1}$, $\mathbf{1}_{n_2}$ and $2 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(x,2,2)$ and $\mathfrak{l}_2 = 1$	$l_0/2$, l_2, l_2, $\mathbf{1}_{n_1}$ and $2 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(x,y,1)$	$\frac{l_0}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}$, $\frac{l_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}$, $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) \times \mathbf{1}_{n_i}$ for $i \geq 2$.

Proof of Theorem 2. We start with a rational, normal, affine, log terminal X_1 of complexity one. According to Theorem 1, the Cox ring R_2 of X_1 is a platonic ring. If the greatest common divisors of pairs $\mathfrak{l}_i, \mathfrak{l}_j$ of R_2 all equal one, then R_2 is factorial by [17, Thm. 1.1] and we are done. If not, then we pass to the Cox ring R_3 of $X_2 := \operatorname{Spec} R_2$ and so on. Proposition 6.6 ensures that this procedure terminates with a factorial platonic ring R_p. $\square$

Proof of Theorem 3. Let X_1 be any rational, normal, affine variety with a torus action of complexity one of Type 2 and at most log terminal singularities. Then Theorem 2 provides us with a chain of quotients

$$X_p \xrightarrow{/\!\!/ H_{p-1}} X_{p-1} \xrightarrow{/\!\!/ H_{p-2}} \quad \ldots \quad \xrightarrow{/\!\!/ H_3} X_3 \xrightarrow{/\!\!/ H_2} X_2 \xrightarrow{/\!\!/ H_1} X_1 ,$$

such that $X_i = \operatorname{Spec}(R_i)$ holds with a platonic ring R_i when $i \geq 2$, the ring R_p is factorial and each $X_{i+1} \to X_i$ is the total coordinate space. The idea is to construct stepwise solvable linear algebraic groups $G_i \subseteq \operatorname{Aut}(X_{i+1})$ acting algebraically on X_{i+1} such that the unit component $G_i^0 \subseteq G_i$ is a torus, G_i contains H_i as a normal subgroup, $G_{i-1} = G_i / H_i$ holds and we have $G_1 = H_1$.

Start with $G_1 := H_1$, acting on X_2. According to [2, Thm. 2.4.3.2], there exists an (effective) action of a torus $\mathcal{G}_1$ on X_3 lifting the action of G_1^0 on X_2 and commuting with the action of H_2 on X_3. Moreover, [3, Thm. 5.1] provides us with an exact sequence of groups

$$1 \longrightarrow H_2 \longrightarrow \operatorname{Aut}(X_3, H_2) \xrightarrow{\pi} \operatorname{Aut}(X_2) \longrightarrow 1 ,$$

where $\operatorname{Aut}(X_3, H_2)$ denotes the group of automorphisms of X_3 normalizing the quasitorus H_2. Set $G_2 := \pi^{-1}(G_1)$. Then $H_2^0 \mathcal{G}_1$, as a factor group of the torus $H_2^0 \times \mathcal{G}_1$ by a closed subgroup, is an algebraic torus and it is of finite index in G_2. Thus, G_2 is an affine algebraic group with $G_2^0 = H_2^0 \mathcal{G}_1$ being a torus. By construction, $H_2 \subseteq G_2$ is the kernel of $\alpha_1 := \pi|_{G_2}$ and hence a normal subgroup. Moreover, G_2 is solvable and acts algebraically on X_3. Iterating this procedure gives a sequence

$$G_{p-1} \xrightarrow{\alpha_{p-2}} G_{p-2} \xrightarrow{\alpha_{p-3}} \quad \ldots \quad \xrightarrow{\alpha_2} G_2 \xrightarrow{\alpha_1} G_1 \xrightarrow{\alpha_0} 1$$

of group epimorphisms, where, as wanted, G_i is a solvable reductive group acting algebraically on X_{i+1} such that $H_i = \ker(\alpha_{i-1})$ is the characteristic quasitorus of X_i. In particular, the group $G := G_{p-1} \subseteq \operatorname{Aut}(X_p)$ satisfies the first assertion of the theorem.

We turn to the second assertion. From [2, Prop. 1.6.1.6], we infer that $G_1 = H_1$ acts freely on the preimage $U_2 \subseteq X_2$ of the set of smooth points $U_1 \subseteq X_1$ and moreover, the complement $X_2 \backslash U_2$ is of codimension at least two in X_2. Let $U_3 \subseteq X_3$ be the preimage of $U_2 \subseteq X_2$. Again, the complement of U_3 is of codimension at least two in X_3 and, as U_2 consists of smooth points of X_2, the quasitorus H_2 acts freely on U_3. Because of $G_2/H_2 = G_1$, we conclude that U_3 is G_2-invariant and G_2 acts freely on U_2. Repeating this procedure, we end up with an open set $U_p \subseteq X_p$ having complement of codimension at least two such that G acts freely on U_p. Thus, G acts strongly stably on X_p. Now consider

$$G = \mathcal{D}_0 \;\supseteq\; \mathcal{D}_1 \;\supseteq\; \ldots \;\supseteq\; \mathcal{D}_{p-2} \;\supseteq\; \mathcal{D}_{p-1} = 1, \qquad \mathcal{D}_i := \ker(\alpha_i \circ \ldots \circ \alpha_{p-2}).$$

Then we have $X_i = X_p /\!\!/ \mathcal{D}_{i-1}$ and $H_i = \mathcal{D}_{i-1}/\mathcal{D}_i$. Moreover for each $\mathcal{D}_i$, its action on X_p is strongly stable, as remarked before, and X_p is G-factorial because it is factorial. Using [3, Prop. 3.5], we obtain a commutative diagram

$$\begin{tikzcd}
X_p /\!\!/ [\mathcal{D}_i, \mathcal{D}_i] \arrow[rr, "\beta"] \arrow[dr, "/\!\!/ \mathcal{D}_i/[\mathcal{D}_i,\mathcal{D}_i]"'] & & X_p /\!\!/ \mathcal{D}_{i+1} \arrow[dl, "/\!\!/ \mathcal{D}_i/\mathcal{D}_{i+1}"] \\
& X_p /\!\!/ \mathcal{D}_i &
\end{tikzcd}$$

where the left downward map is a total coordinate space. As $\mathcal{D}_i/\mathcal{D}_{i+1} = H_{i+1}$ is abelian, $[\mathcal{D}_i, \mathcal{D}_i]$ is contained in $\mathcal{D}_{i+1}$ and we have the horizontal morphism β. Since the right hand side is a total coordinate space as well, we infer from [2, Sec. 1.6.4] that β is an isomorphism. This implies $\mathcal{D}_{i+1} = [\mathcal{D}_i, \mathcal{D}_i]$, proving the second assertion. □

7. Compound du Val singularities

Between the Gorenstein terminal and canonical threefold singularities lie the compound du Val singularities, introduced by Miles Reid in [26], see also [27, 23, 21]. We discuss compound du Val singularities in the context of T-varieties of complexity one and provide first constraints on the defining data for affine threefolds, preparing the proof of our classification results.

Definition 7.1. [26, Def. 2.1], [21, Thm. 5.34, Cor. 2.3.2]. A normal, canonical, Gorenstein threefold singularity $x \in X$ is called *compound du Val*, if one of the following equivalent criteria is satisfied:

(i) For a general hypersurface $Y \subseteq X$ with $x \in Y$, the point x is a du Val surface singularity of Y.

(ii) Near x, the threefold X is analytically isomorphic to a hypersurface of the following shape

$$V(f(T_1, T_2, T_3) + g(T_1, T_2, T_3, T_4)\, T_4) \;\subseteq\; \mathbb{C}^4,$$

where f is a defining polynomial for a du Val surface singularity in $\mathbb{C}^3$ and g is any polynomial in T_1, T_2, T_3, T_4.

(iii) For every resolution $\varphi\colon X' \to X$ of singularities and every irreducible exceptional divisor $E \subseteq \varphi^{-1}(x)$, the discrepancy of E is greater than zero.

(iv) There is a resolution $\varphi\colon X' \to X$ of singularities such that every irreducible exceptional divisor $E \subseteq \varphi^{-1}(x)$ is of discrepancy greater than zero.

For an affine toric threefold X, Condition 7.1 (iv) means the following: X is defined by the cone over $\triangle \times \{1\}$ with a hollow lattice polytope $\triangle \subseteq \mathbb{Q}^2$, where hollow means that $\triangle$ has no lattice points in its interior. Based on this characterization, one obtains the list of toric compound du Val singularities provided in [8]:

Proposition 7.2. *Let X be an affine toric variety with a compound du Val singularity. Then $X \cong X(\sigma)$ holds with a cone $\sigma \subseteq \mathbb{Q}^3$ generated by the columns of one of the following matrices*

$$(1)\ \begin{bmatrix} 0 & 0 & k \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ k \in \mathbb{Z}_{\geq 2}, \quad (2)\ \begin{bmatrix} 0 & 0 & k_1 & k_2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},\ k_1, k_2 \in \mathbb{Z}_{\geq 1}, \quad (3)\ \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Proof. After removing the third row from the matrices, we find in their columns the vertices of the hollow polytopes $\triangle \subseteq \mathbb{Q}^2$; see [28]. □

We turn to affine T-varieties X of complexity one. As the toric case is settled, we can concentrate on the varieties $X = X(A,P)$ of Type 2. The basic tool is the anticanonical complex A_X^c, described in Proposition 3.8. The following statement specifies a bit more.

Proposition 7.3. *Let $X = X(A,P)$ be an affine Gorenstein, log-terminal threefold of Type 2 such that P is in the form of Proposition 4.3. Consider the intersections*

$$\partial A_X^c(\lambda) := \partial A_X^c \cap \lambda, \quad \partial A_X^c(\lambda_i) := \partial A_X^c \cap \lambda_i, \quad \partial A_X^c(\lambda_i, \tau) := \partial A_X^c(\lambda_i) \cap \tau,$$

where ∂A_X^c is the relative boundary of the anticanonical complex, $\lambda \subseteq \operatorname{trop}(X)$ the lineality part, $\lambda_0, \ldots, \lambda_r \subseteq \operatorname{trop}(X)$ are the leaves and τ is any P-elementary cone.

(i) *Let $x_1, \ldots, x_{r+2}$ be the standard coordinates on the column space $\mathbb{Q}^{r+2}$ of P and set $x_0 := -x_1 - \ldots - x_r$. Then x_i, x_{r+1}, x_{r+2} are linear coordinates on the three-dimensional vector space $\operatorname{Lin}_{\mathbb{Q}}(\lambda_i)$ and we have*

$$\partial A_X^c(\lambda_i) \ = \ A_X^c \cap \lambda_i \cap \mathcal{H}_i \ \subseteq \ \operatorname{Lin}_{\mathbb{Q}}(\lambda_i)$$

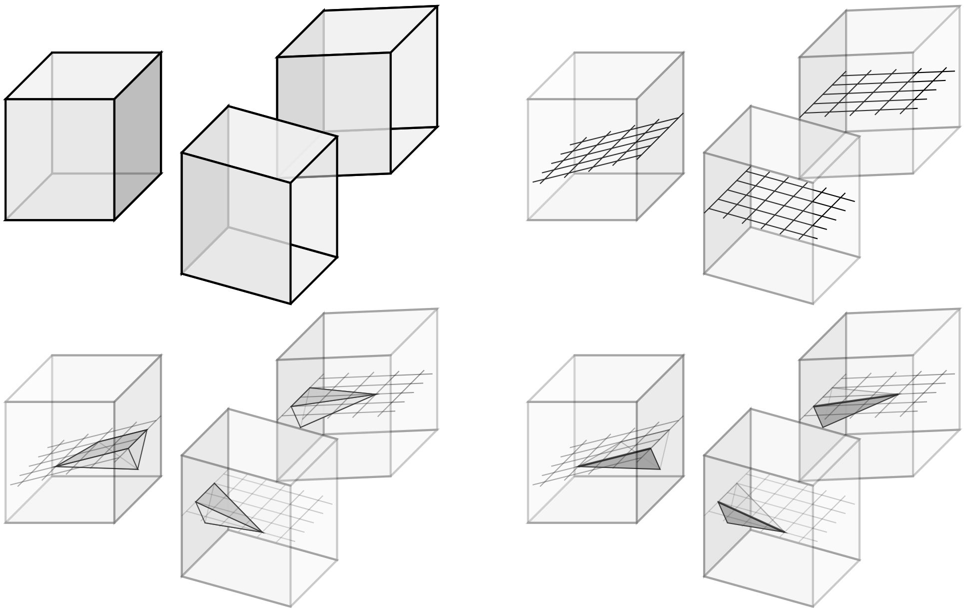
with the plane $\mathcal{H}_i := V(\zeta_X x_{r+2} + \mu_i x_i - \imath_X) \subseteq \operatorname{Lin}_{\mathbb{Q}}(\lambda_i)$, where μ_i is the integer defined in Remark 4.5. In particular, for fixed i, the columns v_{ij} of P lie on the half plane $\lambda_i \cap \mathcal{H}_i$.

(ii) *The set $A_X^c \cap \tau$ is a two-dimensional and $\partial A_X^c \cap \tau$ a one-dimensional polyhedral complex. Furthermore, $\partial A_X^c(\lambda_i, \tau)$ is a line segment.*

Proof. We show (i). Let $\sigma \subseteq \mathbb{Q}^{r+2}$ be the cone over the columns of P. Then the set $\partial A_X^c(\lambda_i)$ equals $\partial A_X^c \cap \sigma \cap \lambda_i$. By the assumption on P, the equation from Proposition 3.8 (ii) gives the assertion.

For (iii), write $\tau = \operatorname{cone}(w_0, \ldots, w_r)$ with $w_i \in \lambda_i$. Observe that $A_X^c \cap \tau \cap \lambda_i$ has the vertices 0, w_i, $v(\tau)'$ and is thus two-dimensional. Only w_i and $v(\tau)'$ satisfy the equation $\zeta_X x_{r+2} + \mu_i x_i = \imath_X$. Thus $A_X^c \cap \tau$ is two-dimensional and $\partial A_X^c \cap \tau$ as well as $\partial A_X^c(\lambda_i, \tau)$ are one-dimensional. □

The following figures visualize the situation of Proposition 7.3 for the case $r = 2$. The first one shows the leaves λ_i, the second one the half planes $\lambda_i \cap \mathcal{H}_i$, the third one all $A_X^c(\lambda_i)$ and the last one all $A_X^c(\lambda_i, \tau)$ for a given P-elementary cone τ.



The following statement shows that the relative boundary ∂A_X^c of the anticanonical complex replaces the lattice polytope $\triangle$ from the toric setting discussed before.

Proposition 7.4. *Let $X = X(A, P)$ be an affine Gorenstein, log-terminal threefold of Type 2. Then X has at most compound du Val singularities if and only if there are no integral points in the relative interior of ∂A_X^c.*

Lemma 7.5. *Let $X = X(A, P, \Phi)$, denote by Σ the fan of the minimal toric ambient variety Z of X and let $\sigma \in \Sigma$ be a big cone.*

(i) *The toric orbit $T_Z \cdot z_\sigma \subseteq Z$ corresponding to the cone $\sigma \in \Sigma$ is contained in $X \subseteq Z$.*
(ii) *If $T_Z \cdot z_\sigma \subseteq X$ contains a singular point of X, then every point of $T_Z \cdot z_\sigma$ is singular in X.*

Proof. We show (i). By the structure of the defining relations g_i, the corresponding statement holds for $\overline{X} \subseteq \overline{Z} = \mathbb{C}^{n+m}$. Passing to the quotient by the characteristic quasitorus H gives the assertion.

We turn to (ii). Let $z \in \widehat{X}$ be a point mapping to $T_Z \cdot z_\sigma$. Using once more the specific shape of the defining relations g_i, we see that if the point $z \in \widehat{X}$ is singular in $\overline{X}$, then every point of $\mathbb{T}^{n+m} \cdot z$ is singular in $\overline{X}$. Thus, the assertion follows from [2, Cor. 3.3.1.12]. □

Proof of Proposition 7.4. Since X is Gorenstein and log terminal, it is canonical. Let Z be the minimal toric ambient variety of X. Recall that Z is the affine toric variety defined by the cone σ over the columns of P and that the toric fixed point $x \in Z$ belongs to X. For any point $x' \in X$ different from x, we infer from Lemma 7.5 and [2, 3.4.4.6] that, if x' is singular in X, then it belongs to a curve consisting of singular points of X. According to [21, Cor. 5.4], the point x' is at most a compund du Val singularity. Thus, X has at most compound du Val singularities if and only if every prime divisor $E \subseteq \varphi^{-1}(x)$ has positive discrepancy; use Condition 7.1 (iv). By Proposition 3.2, the latter holds if and only if there are no integral points in $\partial A_X^c \cap \sigma^\circ$, which in turn is the relative interior of ∂A_X^c. □

Definition 7.6. Let the matrix P be of Type 2 and ordered in the sense of Remark 6.1. By the *leading block* of P, we mean the matrix $[v_{01}, \ldots, v_{r1}]$.

Lemma 7.7. *Let $X = X(A, P)$ be an affine, Gorenstein, log-terminal threefold of canonical multiplicity one of Type 2.*

(i) *By admissible operations one achieves that P is ordered in the sense of Remark 6.1, in the form of Corollary 4.6 and the entry $\mathfrak{d}_i$ sitting in column v_{i1} and row number $r+1$ of P satisfies $\mathfrak{d}_i = 0$ whenever $i \ge 3$.*

(ii) *In the situation of (i), the leading block of the matrix P is fully determined by the data $(l_{01}, l_{11}, l_{21}; \mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2)$.*

Proof. The leading block contains the leading platonic triple (l_{01}, l_{11}, l_{21}). All other l_{i1} must be equal to one. Due to Corollary 4.6, the last row of P is determined by these data. Subtracting the $\mathfrak{d}_i$-fold of the i-th from the $r+1$-th row, we obtain $\mathfrak{d}_i = 0$ for $i \ge 3$. Thus apart from l_{01}, l_{11}, l_{21}, the only free parameters in the leading block are $\mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2$. ☐

Definition 7.8. In the situation of Lemma 7.7 (i), we call $(l_{01}, l_{11}, l_{21}; \mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2)$ the *leading block data* of P.

Proposition 7.9. *Let $X = X(A, P)$ be an affine Gorenstein log-terminal threefold of Type 2 and canonical multiplicity one in the form of Lemma 7.7. By admissible operations, keeping the form of Lemma 7.7, we achieve that the leading block has one of the following data:*

$$
\begin{array}{ccc}
(i)\,(5,3,2;0,0,0) & (ii)\,(4,3,2;0,0,0) & (iii)\,(4,3,2;1,0,0) \\
(iv)\,(3,3,2;0,0,0) & (v)\,(3,3,2;1,0,0) & (vi)\,(l_{01},2,2;0,0,0) \\
(vii)\,(l_{01},2,2;1,0,0) & (viii)\,(l_{01},2,2;0,1,0) & (ix)\,(l_{01},l_{11},1;\mathfrak{d}_0,0,0)
\end{array}
$$

Proof. We go through all possible leading platonic triples and explicitly list the admissible operations on P that produce the desired leading block data. First, we modify P by subtracting the i-th row from the last for $i \ge 3$. Then we have

$$
\nu_{01} \ = \ 1 - l_{01}, \qquad \nu_{11} \ = \ \nu_{21} \ = \ 1, \qquad \nu_{i1} \ = \ \mathfrak{d}_i \ = \ 0, \ i = 3, \ldots, r.
$$

In the sequel, by "applying $a = (a_1, a_2, a_3)$" we mean performing the following sequence of admissible operations on P: add the a_1-fold of the first, the a_2-fold of the second and the a_3-fold of the last to the penultimate row of P.

Case 1: The leading platonic triple is $(5,3,2)$. We arrive at Case (i) by applying

$$
a \ = \ (2\mathfrak{d}_0 + 3\mathfrak{d}_1 + 5\mathfrak{d}_2, \ 3\mathfrak{d}_0 + 5\mathfrak{d}_1 + 7\mathfrak{d}_2, \ -6\mathfrak{d}_0 - 10\mathfrak{d}_1 - 15\mathfrak{d}_2).
$$

Case 2: The leading platonic triple is $(4,3,2)$. If $\mathfrak{d}_0 \equiv \mathfrak{d}_2 \mod 2$ holds, then we arrive at Case (ii) by applying

$$
a \ = \ \left(\mathfrak{d}_0 + \mathfrak{d}_1 + 2\mathfrak{d}_2, \ \frac{3}{2}\mathfrak{d}_0 + 2\mathfrak{d}_1 + \frac{5}{2}\mathfrak{d}_2, \ -3\mathfrak{d}_0 - 4\mathfrak{d}_1 - 6\mathfrak{d}_2\right).
$$

If $\mathfrak{d}_0 \equiv \mathfrak{d}_2 + 1 \mod 2$ holds, then we arrive at Case (iii) by applying

$$
a \ = \ \left(\mathfrak{d}_0 + \mathfrak{d}_1 + 2\mathfrak{d}_2 - 1, \ \frac{3}{2}\mathfrak{d}_0 + 2\mathfrak{d}_1 + \frac{5}{2}\mathfrak{d}_2 - \frac{3}{2}, \ -3\mathfrak{d}_0 - 4\mathfrak{d}_1 - 6\mathfrak{d}_2 + 3\right).
$$

Case 3: The leading platonic triple is $(3,3,2)$. We distinguish the cases $\mathfrak{d}_0 \equiv \mathfrak{d}_1 \mod 3$ and $\mathfrak{d}_0 \equiv \mathfrak{d}_1 + 1 \mod 3$ (if $\mathfrak{d}_0 \equiv \mathfrak{d}_1 - 1 \mod 3$, then exchange the data of the blocks 0 and 1 of P). We arrive at Cases (iv) and (v) by applying respectively

$$
a \ = \ \left(\frac{2}{3}\mathfrak{d}_0 + \frac{1}{3}\mathfrak{d}_1 + \mathfrak{d}_2, \ \mathfrak{d}_0 + \mathfrak{d}_1 + \mathfrak{d}_2, \ -2\mathfrak{d}_0 - 2\mathfrak{d}_1 - 3\mathfrak{d}_2\right),
$$

$$
a \ = \ \left(\frac{2}{3}\mathfrak{d}_0 + \frac{1}{3}\mathfrak{d}_1 + \mathfrak{d}_2 - \frac{2}{3}, \ \mathfrak{d}_0 + \mathfrak{d}_1 + \mathfrak{d}_2 - 1, \ -2\mathfrak{d}_0 - 2\mathfrak{d}_1 - 3\mathfrak{d}_2 + 2\right).
$$

Case 4: The leading platonic triple is $(l_{01}, 2, 2)$. We distinguish several subcases and will work with

$$a \;=\; \left(\frac{1}{2}\mathfrak{d}_0 + \frac{l_{01}-2}{4}\mathfrak{d}_1 + \frac{l_{0j_0}}{4}\mathfrak{d}_2,\ \frac{1}{2}\mathfrak{d}_0 + \frac{l_{01}}{4}\mathfrak{d}_1 + \frac{l_{01}-2}{4}\mathfrak{d}_2,\ -\mathfrak{d}_0 - \frac{l_{01}}{2}(\mathfrak{d}_1+\mathfrak{d}_2)\right).$$

4.1: We have $l_{01} \equiv 1 \mod 4$.

4.1.1: $\mathfrak{d}_1 \equiv \mathfrak{d}_2 \mod 4$. If $\mathfrak{d}_0$ is even, then applying a, we arrive at Case (vi). If $\mathfrak{d}_0$ is odd, then applying $a + (-1/2,\ -1/2,\ 1)$, we arrive at Case (vii).

4.1.2: $\mathfrak{d}_1 \equiv \mathfrak{d}_2 + 1 \mod 4$. If $\mathfrak{d}_0$ is even, then applying $a + (1/4,\ -1/4,\ 1/2)$ leads to Case (viii). If $\mathfrak{d}_0$ is odd, then applying $a + (-1/4,\ 1/4,\ 1/2)$ and exchanging the data of column blocks 1 and 2 leads to Case (viii).

4.1.3: $\mathfrak{d}_1 \equiv \mathfrak{d}_2 - 1 \mod 4$. Exchanging the data of column blocks 1 and 2, we are in 4.1.2 and thus arrive at Case (viii).

4.1.4: $\mathfrak{d}_1 \equiv \mathfrak{d}_2 + 2 \mod 4$. If $\mathfrak{d}_0$ is odd, then applying a, we arrive at Case (vi). If $\mathfrak{d}_0$ is even, then applying $a + (-1/2,\ -1/2,\ 1)$ leads to Case (vii).

4.2: We have $l_{01} \equiv 2 \mod 4$.

4.2.1: $\mathfrak{d}_0 \equiv \mathfrak{d}_1 \equiv \mathfrak{d}_2 \mod 2$. Applying a, we arrive at Case (vi).

4.2.2: $\mathfrak{d}_0 \equiv \mathfrak{d}_1 \not\equiv \mathfrak{d}_2 \mod 2$. Applying $a + (0,\ -1/2,\ 1)$, we arrive at Case (viii).

4.2.3: $\mathfrak{d}_0 \equiv \mathfrak{d}_2 \not\equiv \mathfrak{d}_1 \mod 2$. Exchanging the data of column blocks 1 and 2, we are in 4.2.2 and thus arrive at Case (viii).

4.2.4: $\mathfrak{d}_0 \not\equiv \mathfrak{d}_1 \equiv \mathfrak{d}_2 \mod 2$. Applying $a + (-1/2,\ -1/2,\ 1)$, we arrive at Case (vii).

4.3 and 4.4: $l_{01} \equiv 3 \mod 4$ or $l_{01} \equiv 3 \mod 4$, respectively. These cases are settled by similar arguments as 4.1 and 4.2. That means that the same admissible operations are applied after, if necessary exchanging the data of column blocks 1 and 2.

Case 5: The leading platonic triple is $(l_{01}, l_{11}, 1)$. Applying $(0,\ \mathfrak{d}_1 - \mathfrak{d}_2,\ -\mathfrak{d}_1)$, we arrive at Case (ix).

Finally, in each of the cases (i) to (ix), we modify the matrix P obtained so far by adding the i-th row to the last one for $i = 3, \ldots, r$. This brings P again into the form of Lemma 7.7 (i). □

8. Proof of Theorems 4 and 5

In Propositions 8.1, 8.3 and 8.4, we classify the compound du Val singularities admitting a torus action of complexity one and list their defining matrices P, numerated according to their appearance in Theorem 4. We begin with the case of $\mathbb{Q}$-factorial non-toric threefolds of canonical multiplicity one.

Proposition 8.1. *Let X be a non-toric affine threefold of Type 2. Assume that X is $\mathbb{Q}$-factorial, of canonical multiplicity one and has at most compound du Val singularities. Then X, for suitable A, is isomorphic to $X(A, P)$, where P is one of the following matrices:*

$$\text{(8)} \begin{bmatrix} -5 & 3 & 0 & 0 \\ -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 1 & 1 & 1 \end{bmatrix} \qquad \text{(7)} \begin{bmatrix} -4 & 3 & 0 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 1 & 1 & 1 \end{bmatrix} \qquad \text{(18)} \begin{bmatrix} -4 & -1 & 3 & 0 \\ -4 & -1 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ -3 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{(6)} \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 1 & 1 \end{bmatrix} \qquad \text{(15)} \begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix} \qquad \text{(17)} \begin{bmatrix} -3 & -2 & 3 & 0 \\ -3 & -2 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 1 \end{bmatrix}$$

$$\text{(4)} \begin{bmatrix} -k & 2 & 0 & 0 \\ -k & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1-k & 1 & 1 & 1 \end{bmatrix} \qquad \text{(12-e-e)} \begin{bmatrix} -k_1 & -k_2 & 2 & 0 \\ -k_1 & -k_2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1-k_1 & 1-k_2 & 1 & 1 \end{bmatrix} \qquad \text{(5-o)} \begin{bmatrix} -k & 2 & 0 & 0 \\ -k & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 \end{bmatrix}$$

$$\text{(11)} \begin{bmatrix} -k & 2 & 1 & 0 \\ -k & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 \end{bmatrix} \qquad \text{(12-o-e/o)} \begin{bmatrix} -2k_1 & -2k_2-1 & 2 & 0 \\ -2k_1 & -2k_2-1 & 0 & 2 \\ 0 & \frac{k_1-k_2+1}{2} & 1 & 0 \\ 1-2k_1 & -2k_2 & 1 & 1 \end{bmatrix} \qquad \text{(16)} \begin{bmatrix} -4 & 2 & 1 & 0 \\ -4 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ -3 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{(5-e)} \begin{bmatrix} -2k-1 & 2 & 0 & 0 \\ -2k-1 & 0 & 2 & 0 \\ 0 & 1 & 0 & k+1 \\ -2k & 1 & 1 & 1 \end{bmatrix} \qquad \text{(10-o)} \begin{bmatrix} -2k-1 & 2 & 1 & 0 \\ -2k-1 & 0 & 0 & 2 \\ 0 & 1 & \left\lceil \frac{2k+1}{4} \right\rceil & 0 \\ -2k & 1 & 1 & 1 \end{bmatrix}$$

where the parameters k, k_1, k_2 are positive integers and in (4), (5-o) and (11), we have $k \geq 2$. Moreover, (12-e-e) indicates that the two exponents in the defining equation of Theorem 4 (12) are even, in (5-o) the exponent is odd etc..

Proof. We may assume that P is irredundant and in the form of Proposition 7.9. As X is $\mathbb{Q}$-factorial, the matrix P has precisely $r+2$ columns. Since we assume P to be irredundant and $l_{ij} = 1$ holds for $i \geq 3$, we must have $n_i \geq 2$ for $i \geq 3$. This forces $r \leq 3$. The strategy is now to compute suitable parts of $\partial A_X^{\mathrm{c}}$ explicitly according to Proposition 3.8 and to use the fact that they don't contain interior lattice points, as guaranteed by Proposition 7.4.

Consider the case $r = 3$. Here, we have $n_0 = n_1 = n_2 = 1$ and $n_3 = 2$. Moreover, (l_{01}, l_{11}, l_{21}) is a platonic triple with $l_{21} > 1$ and $l_3 = (1, 1)$ holds. The column apart from the leading block of P is $v_{32} = (0, 0, 1, t, 0)$, where we may assume that t is a positive integer. The vertices of $\partial A_X^{\mathrm{c}}(\lambda)$ thus are

$$\left(0, 0, 0, \frac{\alpha}{\beta}, 1\right), \qquad \left(0, 0, 0, \frac{\alpha + t l_{01} l_{11} l_{21}}{\beta}, 1\right),$$

where

$$\alpha := \mathfrak{d}_0 l_{11} l_{21} + \mathfrak{d}_1 l_{01} l_{21} + \mathfrak{d}_2 l_{01} l_{11}, \quad \beta := l_{11} l_{21} + l_{01} l_{21} + l_{01} l_{11} - l_{01} l_{11} l_{21}$$

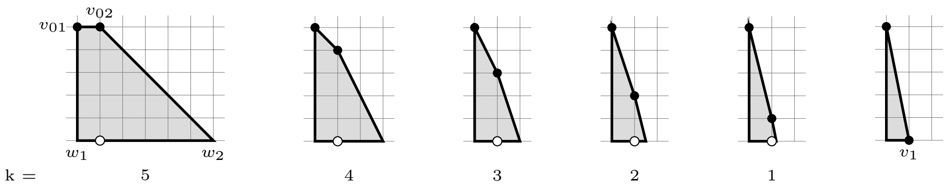
Since l_{01}, l_{11}, l_{21} all differ from one, $t l_{01} l_{11} l_{21} / \beta \geq 2$ holds and thus $\partial A_X^{\mathrm{c}}(\lambda)$ contains an integral point in its relative interior. Consequently $r = 3$ is impossible.

We are left with the case $r = 2$. Here, P is a 4×4 matrix, the leading block columns are v_{01}, v_{11}, v_{21} and the column v of P apart from these three is one of

$$v_{02} = (-k, -k, t, 1-k), \quad v_{12} = (k, 0, 0, t, 1), \quad v_{22} = (0, k, t, 1), \quad v_1 = (0, 0, t, 1).$$

We now go through the list of all possible leading block data provided by Proposition 7.9. We will often compute the line segment $\partial A_X^{\mathrm{c}}(\lambda) \subseteq \mathbb{Q}^4$ from Proposition 7.3 explicitly. According to Proposition 3.8, the P-elementary cone spanned by the columns of the leading block produces the first vertex w_1 of $\partial A_X^{\mathrm{c}}(\lambda)$ and the second vertex w_2 either arises from a (unique) second P-elementary cone or one has $w_2 = v = v_1$.

Let P have the leading block data $(5, 3, 2; 0, 0, 0)$. Then the first vertex of $\partial A_X^{\mathrm{c}}(\lambda)$ is $w_1 = (0, 0, 0, 1)$. Consider the case that the additional column v lies in the relative interior $\lambda_0^\circ \subseteq \lambda_0$. Then $v = v_{02} = (-k, -k, t, 1-k)$ with $1 \leq k \leq 5$, where we may assume $t > 0$. We compute $w_2 = (0, 0, 6t/(6-k), 1)$. The following figures show $\partial A_X^{\mathrm{c}}(\lambda_0) \subseteq \mathcal{H}_0$ with the lower edge being $\partial A_X^{\mathrm{c}}(\lambda)$:

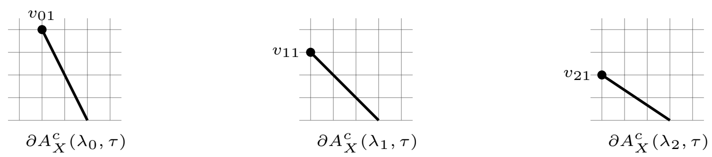


where the last figure indicates the case of the additional column lying in λ, treated below. Now, because of $6t/(6-k) \geq 6/5$, we find the point $(0,0,1,1)$ in the relative interior of $\partial A_X^c(\lambda)$ and hence in the relative interior of ∂A_X^c. According to Proposition 7.4, we leave the compound du Val case here.

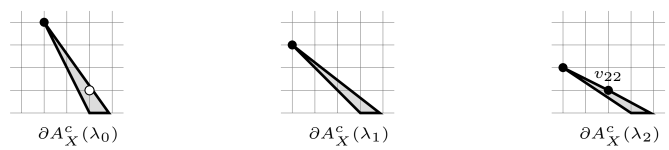
We proceed in a more condensed way. Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k, 0, t, 1)$ with $1 \leq k \leq 3$ and we can assume $t > 1$. We obtain $w_2 = (0, 0, 10t/(10-3k), 1)$. We find again $(0,0,1,1)$ in $\partial A_X^c(\lambda)^\circ$ and thus leave the compound du Val case. Assume $v \in \lambda_2^\circ$. Then $v = v_{22} = (0, k, t, 1)$ with $1 \leq k \leq 2$ and we can assume $t > 1$. We obtain $w_2 = (0, 0, 15t/(15-7k, 1)$. Once more, $(0,0,1,1)$ shows up in $\partial A_X^c(\lambda)^\circ$ and we leave the compound du Val case. Finally, assume $v = v_1 = (0,0,t,1)$. We may assume $t > 0$. Only for $t = 1$ there are no lattice points in $\partial A_X^c(\lambda)^\circ$. Moreover, if $t = 1$, then all $\partial A_X^c(\lambda_i)$ are hollow polytopes of the first type of Proposition 7.2 and we arrive at matrix (8) from the assertion defining the compund du Val singularity $E_8 \times \mathbb{C}$.

Let P have the leading block data $(4,3,2;0,0,0)$. Also here, the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0,0,0,1)$. Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k, -k, t, 1-k)$ with $1 \leq k \leq 4$, where we may assume $t > 0$. We obtain $w_2 = (0,0,6t/(6-k),1)$. Thus, $(0,0,1,1)$ lies in $\partial A_X^c(\lambda)^\circ$ and we leave the compound du Val case. Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k, 0, t, 1)$, where $1 \leq k \leq 3$ and we can assume $t > 0$. We obtain $w_2 = (0,0,4t/(4-k),1)$ and find $(0,0,1,1)$ in the relative interior of $\partial A_X^c(\lambda)$ and thus leave the compound du Val case. Assume $v \in \lambda_2^\circ$. Then $v = v_{22} = (0,k,t,1)$ with $k = 1,2$ and we can assume $t > 0$. We obtain $w_2 = (0,0,12t/(12-5k),1)$ and see that $(0,0,1,1)$ lies in $\partial A_X^c(\lambda)^\circ$. Thus, we leave the compound du Val case. Finally, assume $v = v_1 = (0,0,t,1)$. For $t > 1$, we find $(0,0,1,1)$ in $\partial A_X^c(\lambda)^\circ$. The case $t = 1$ gives matrix (7), defining the compound du Val singularity $E_7 \times \mathbb{C}$.

Let P have the leading block data $(4,3,2;1,0,0)$. Here, the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0,0,3,1)$. To visualize the setting, consider the P-elementary cone $\tau \subseteq \mathbb{Q}^4$ generated by the columns v_{01}, v_{11}, v_{21} of the leading block and the line segments $\partial A_X^c(\lambda_i, \tau) \subseteq \mathcal{H}_i$, where $i = 0,1,2$, from Proposition 7.3:



Note that the additional column v is represented in the above figures by a lattice point not contained in $\partial A_X^c(\lambda_i, \tau)$, indicated by the black line. Going through the cases, we will also have to look at the polytopes $\partial A_X^c(\lambda_i, \tau)$ and will encounter the following situations:

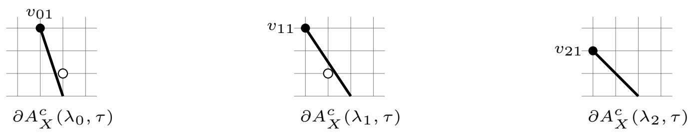


Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k,-k,t,1-k)$ with $1 \le k \le 4$. The second vertex of $\partial A_X^c(\lambda)$ is $w_2 = (0,0,6t/(6-k),1)$. We find one of the points $(0,0,4,1)$ or $(0,0,2,1)$ in $\partial A_X^c(\lambda)^\circ$ for $k = 2,4$. Moreover, for $k = 3$, we find $(-1,-1,3,0)$ in $\partial A_X^c(\lambda_0)^\circ$. Thus, we end up with non compound du Val singularities for $k = 2,3,4$. In the case $k = 1$, we may assume $t > 2$. Only for $t = 3$, no lattice points are inside $\partial A_X^c{}^\circ$. For $t > 3$, the point $(-1,-1,3,0)$ lies in $\partial A_X^c(\lambda_0)^\circ$. So with $t = 3$, we obtain matrix (18), defining a compound du Val singularity.

We show that the remaining possible locations of v all lead to non compound du Val singularities. Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k,0,t,1) \in \lambda_1^\circ$ with $1 \le k \le 3$. The second vertex of $\partial A_X^c(\lambda)$ is $w_2 = (0,0,(k+4t)/(4-k),1)$. Thus, either $(0,0,2,1)$ or $(0,0,4,1)$ lies in $\partial A_X^c(\lambda)^\circ$. Assume $v \in \lambda_2^\circ$. Then $v = v_{22} = (0,k,t,1)$ with $k = 1,2$, where we can assume $t > 1$ or $t > 0$ accordingly. The second vertex of $\partial A_X^c(\lambda)$ is $w_2 = (0,0,(3k+12t)/(12-5k),1)$. For $k = 2$, we find $(0,0,4,1)$ in $\partial A_X^c(\lambda)^\circ$. For $k = 1$, the segment $\partial A_X^c(\lambda)$ is of length $(12t-18)/7$. Thus, for $t \ge 3$, we find a lattice point in $\partial A_X^c(\lambda)^\circ$. For $t = 2$, we look at $\partial A_X^c(\lambda_0)^\circ$ and see that it contains $(-1,-1,3,0)$; see the figure above. Finally, if $v = v_1 \in \lambda$, one finds $(-1,-1,3,0)$ in $\partial A_X^c(\lambda_0)^\circ$.

Let P have the leading block data $(3,3,2;0,0,0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0,0,0,1)$. Assume $v = (-k,-k,t,1-k) \in \lambda_0^\circ$ or $v = (k,0,t,1) \in \lambda_1^\circ$ with $k = 1,2,3$. Then we can assume $t > 0$. We obtain $w_2 = (0,0,6t/(6-k),1)$, find $(0,0,1,1)$ in $\partial A_X^c(\lambda)^\circ$ and thus leave the compound du Val case. If $v = (0,k,t,1) \in \lambda_2^\circ$, with $k = 1,2$, we can assume $t > 0$. We obtain $w_2 = (0,0,3t/(3-k),1)$ and find $(0,0,1,1)$ in $\partial A_X^c(\lambda)^\circ$. Thus also here, we leave the compound du Val case. Finally, if $v = (0,0,t,1) \in \lambda$, then we end up with $t = 1$ and the matrix (6), defining the compound du Val singularity $E_6 \times \mathbb{C}$.

Let P have the leading block data $(3,3,2;1,0,0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0,0,2,1)$. We will take a look at the leaves:



Assume $v \in \lambda_0^\circ$. Then $v = v_{01} = (-k,-k,t,1-k)$ with $k = 1,2,3$. We obtain $w_2 = (0,0,6t/(6-k),1)$. In the case $k = 3$ as well as in the case $k = 2$ with $t \ne 1$, we find one of $(0,0,1,1)$ and $(0,0,3,1)$ in $\partial A_X^c(\lambda)^\circ$ and leave the compound du Val case. For $k = 2$ and $t = 1$, there are no lattice points in ∂A_X^c and the resulting matrix is (15), defining a compound du Val singularity. If $k = 1$ and $t \ne 2$, we find $(0,0,1,1)$ or $(0,0,3,1)$ in $\partial A_X^c(\lambda)^\circ$. The case $t = 2$ leads to the matrix (7), defining a compound du Val singularity. The case of $v \in \lambda_1^\circ$ can be reduced by means of admissible operations to the previous case. We show that for the remaining possible locations of v, we leave the compound du Val case. If $v = (0,k,t,1) \in \lambda_2^\circ$, then $w_2 = (0,0,(3t+k)/(3-k),1)$ and we find $(0,0,1,1)$ or $(0,0,3,1)$ in $\partial A_X^c(\lambda)^\circ$. If $v = (0,0,t,1) \in \lambda$, then $(-1,-1,2,0)$ or $(1,0,1,1)$ lies in $\partial A_X^c(\lambda)^\circ$.

Let P have the leading block data $(l_{01},2,2;0,0,0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $(0,0,0,1)$. Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k,-k,t,1-k)$ with $1 \le k \le l_{01}$, where we can assume $t > 0$. We have $w_2 = (0,0,t,1)$. For $t > 1$, we obtain $(0,0,1,1) \in \partial A_X^c(\lambda)^\circ$ and thus leave the compound du Val case. For $t = 1$, the resulting singularity is compound du Val for every k and has defining matrix (12-e-e) with $k_1 \ge k_2$. Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k,0,t,1)$ with $k = 1,2$. We can assume $l_{01} > 2$ and $t > 0$. For $k = 1$ we have $w_2 = (0,0,2tl_{01}/(2+l_{01}),1)$ and for $k = 2$, we have $w_2 = (0,0,tl_{01}/2,1)$. In both cases, $\partial A_X^c(\lambda)^\circ$ contains $(0,0,1,1)$ and we obtain a non compound du Val singularity. The case of $v \in \lambda_2^\circ$

can be transformed via exchanging the data of blocks 1 and 2 into the previous one. Finally, if $v = (0, 0, t, 1) \in \lambda$, then we must have $t = 1$ and this gives the compound du Val singularity $D_{l_{01}+2} \times \mathbb{C}$, defined by the matrix (4).

Let P have the leading block data $(l_{01}, 2, 2; 1, 0, 0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $(0, 0, 1, 1)$. Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k, -k, t, 1-k)$ with $1 \le k \le l_{01}$. We can assume $t < 1$. For $t < 0$, we have $(0, 0, 0, 1) \in \partial A_X^c(\lambda)^\circ$. For $t = 0$, we obtain a matrix (12-e-e) as in the case of leading block data $(l_{01}, 2, 2; 0, 0, 0)$, now with $k_1 \le k_2$. Assume $v \in \lambda_1^\circ$. The case $l_{01} = 2$ can be transformed via admissible operations into the case of leading block data $(l_{01}, 2, 2; 0, 1, 0)$ and an additional column in λ_0°, which is discussed below. So, let $l_{01} > 2$. Then $v = (k, 0, t, 1)$, where $k = 1, 2$. For $k = 2$, we can assume $t > 0$. We obtain $w_2 = (0, 0, 1 + tl_{01}/2, 1)$ and $(0, 0, 2, 1) \in \partial A_X^c(\lambda)^\circ$ and thus leave the compound du Val case. Now let $k = 1$. Here, t may be any integer and we obtain $w_2 = (0, 0, 2(1 + l_{01}t)/(2 + l_{01}), 1)$. Only for $t = 0, 1$ there are no lattice points in $\partial A_X^c(\lambda)^\circ$. Both cases lead by admissible operations to the compound du Val singularity with defining matrix (11). The case of $v \in \lambda_2^\circ$ can be transformed to the previous one by exchanging the data of column blocks 1 and 2. Finally, if $v \in \lambda$, then it equals either $(0, 0, 0, 1)$ or $(0, 0, 2, 1)$. Both cases lead to the compound du Val singularity with defining matrix (5o).

Let P have leading block data $(l_{01}, 2, 2; 0, 1, 0)$. Then the first vertex of $\partial A_X^c(\lambda)$ is $w_1 = (0, 0, l_{01}/2, 1)$.

Case 1: The exponent l_{01} is even. Assume $v \in \lambda$. Then $v = v_1 = w_2 = (0, 0, t, 1)$. Exchanging the data of blocks 0 and 1 transforms the case $l_{01} = 2$ into the corresponding case with leading block data $(l_{01}, 2, 2; 1, 0, 0)$ treated before. So, let $l_{01} > 2$. Having no lattice points in $\partial A_X^c(\lambda)^\circ$ implies $t = l_{01}/2 \pm 1$. But then, there are integer points in $\partial A_X^c(\lambda_0)^\circ$: for $t = l_{01}/2 + 1$ we find

$$\left(-1, -1, \frac{l_{01}}{2}, 0\right) \;=\; \frac{1}{l_{01}} v_{01} + \frac{1}{2} w_1 + \left(\frac{1}{2} - \frac{1}{l_{01}}\right) w_2$$

and for $t = l_{01}/2 - 1$ we find

$$\left(-1, -1, \frac{l_{01}}{2} - 1, 0\right) \;=\; \frac{1}{l_{01}} v_{01} + \left(\frac{1}{2} - \frac{2}{l_{01}}\right) w_1 + \left(\frac{1}{l_{01}} + \frac{1}{2}\right) w_2.$$

Assume $v \in \lambda_0^\circ$. Then $v = v_{02} = (-k, -k, t, 1-k)$ with $1 \le k \le l_{01}$. The second vertex of $\partial A_X^c(\lambda)$ is $w_2 = (0, 0, t + k/2, 1)$. If k is even, then having no lattice points in $\partial A_X^c(\lambda)^\circ$ implies $t = (l_{01} - k)/2 \pm 1$. Again there are integer points in $\partial A_X^c(\lambda_0)^\circ$: for $t = (l_{01} - k)/2 + 1$ we find

$$\left(-1, -1, \frac{l_{01}}{2}, 0\right) \;=\; \frac{1}{k} v_{02} + \frac{1}{2} w_1 + \left(\frac{1}{2} - \frac{1}{k}\right) w_2$$

and for $t = (l_{01} - k)/2 - 1$ we find

$$\left(-1, -1, \frac{l_{01}}{2} - 1, 0\right) \;=\; \frac{1}{k} v_{02} + \frac{1}{2} w_1 + \left(\frac{1}{2} - \frac{1}{k}\right) w_2.$$

If k is odd, then having no lattice points in $\partial A_X^c(\lambda)^\circ$ implies $t = (l_{01} - k \pm 1)/2$. For both choices of t, this setting produces a compound du Val singularity with matrix (12-o-e/o) and parameters $k_1 \ge k_2$.

Before entering the discussion of the cases $v \in \lambda_i^\circ$ with $i = 1, 2$, the parameter k occurring in v might be $k = 1, 2$ and the vertex w_2 is given by

$$w_2 \;=\; \begin{cases} \left(0, 0, \frac{2tl_{01}}{2l_{01} + 2k - kl_{01}}, 1\right), & v = (k, 0, t, 1) \in \lambda_1^\circ, \\[2ex] \left(0, 0, \frac{2tl_{01} + kl_{01}}{2l_{01} + 2k - kl_{01}}, 1\right), & v = (0, k, t, 1) \in \lambda_2^\circ. \end{cases}$$

Case 1.1: We have $l_{01} \equiv 0 \mod 4$. If $v \in \lambda_2^\circ$ or $v = (2,0,t,1) \in \lambda_1^\circ$, then we find one of $(0,0,l_{01}/2 \pm 1, 1)$ in $\partial A_X^c(\lambda)^\circ$. Thus, we are left with $v \in \lambda_1^\circ$ and $k = 1$. For any $t \neq l_{01}/4 + 1$, we find the lattice point $(1,0,l_{01}/4+1,1)$ in $\partial A_X^c(\lambda_1)^\circ$. Thus, we end up with

$$v \;=\; (k,0,t,1) \;=\; (1,0,l_{01}/4+1,1), \qquad w_2 \;=\; (0,0,l_{01}(l_{01}+4)/(2l_{01}+4),1).$$

Note that the segment $\partial A_X^c(\lambda)$ contains no lattice points, because its length equals $l_{01}/(l_{01}+2) < 1$. Taking a look at λ_0, we observe

$$(-1,-1,l_{01}/2,0) \in \partial A_X^c(\lambda_0)^\circ \iff \frac{l_{01}}{l_{01}+2} > \frac{l_{01}}{2(l_{01}-1)} \iff l_{01} > 4.$$

Thus, to obtain compound du Val singularities, we must have $l_{01} \leq 4$. As $l_{01} \equiv 0 \mod 4$ holds, only $l_{01} = 4$ is left and, indeed, this leads to the compound du Val singularity with defining matrix (16).

Case 1.2: We have $l_{01} \equiv 2 \mod 4$. If $v \in \lambda_1^\circ$ or $v = (0,2,t,1) \in \lambda_2^\circ$, then we find one of $(0,0,l_{01}/2 \pm 1,1)$ in $\partial A_X^c(\lambda)^\circ$. Thus, we are left with $v \in \lambda_2^\circ$ and $k = 1$. For any $t \neq l_{01}/4 + 1/2$, we find the lattice point $(0,1,l_{01}/4+1/2,1)$ in $\partial A_X^c(\lambda_2)^\circ$. We end up with

$$v \;=\; (0,k,t,1) \;=\; (0,1,l_{01}/4+1/2,1) \;\in\; \lambda_2^\circ.$$

Similar to Case 1.1, we obtain that $(-1,-1,l_{01}/2,0) \in \partial A_X^c(\lambda_0)^\circ$ as soon as $l_{01} > 4$. Thus, only $l_{01} = 2$ might lead to a compound du Val singularity. In this case, we exchange the data of blocks 0 and 2 and land in case of leading block data $(l_{01},2,2;0,1,0)$ and an additional column in λ_0°.

Case 2: The exponent l_{01} is odd. If $v \in \lambda$, then $v = v_1 = w_2 = (0,0,(l_{01}+1)/2,1)$ holds and we arrive at the compound du Val singularity with defining matrix (5e). If $v \in \lambda_0^\circ$ holds, then the arguing runs similar as in Case 1. Only for k odd and $v = v_{02} = (-k,-k,(l_{01}-k+1)/2,1-k)$, there are no lattice points in the relative interior of $\partial A_X^c(\lambda)^\circ$ and we end up with the matrix (12-o-e/o) as in Case 1, but now with parameters $k_1 \leq k_2$.

Assume $v \in \lambda_1^\circ$. Then $v = v_{12} = (k,0,t,1)$ with $k = 1,2$. The case $k = 2$ gives $w_2 = (0,0,tl_{01}/2,1)$, the point $(0,0,(l_{01} \pm 1)/2,1)$ lies $\partial A_X^c(\lambda)^\circ$ and thus we leave the compound du Val case. So, let $k = 1$. Then we have $v = (1,0,t,1) \in \lambda_1^\circ$. Moreover, $w_1 = (0,0,l_{01}/2,1)$ and $w_2 = (0,0,2tl_{01}/(2+l_{01}),1)$. Now, as l_{01} is odd, we see that $\partial A_X^c(\lambda)$ to have no lattice points in the relative interior means

$$\frac{1}{2} \;\geq\; \left| \frac{2tl_{01}}{2+l_{01}} - \frac{l_{01}}{2} \right|.$$

If $l_{01} \equiv 1 \mod 4$, this is only fulfilled for $t = (l_{01}+3)/4$. If $l_{01} \equiv 3 \mod 4$, it is only fulfilled for $t = (l_{01}+1)/4$. Altogether, it is fulfilled for $t = \lceil l_{01}/4 \rceil$. This leads to the compound du Val singularity with defining matrix (10o).

The case $v \in \lambda_2^\circ$ can be transformed by suitable admissible operations to the case $v \in \lambda_1^\circ$ just discussed.

Let P have leading block data $(l_{01},l_{11},1;\mathfrak{d}_0,0,0)$. As P is irredundant, the additional column is forced to be $(0,1,t,1) \in \lambda_2^\circ$ and we have $l_{01},l_{11} \geq 2$. The vertices of $\partial A_X^c(\lambda)$ turn out to be

$$w_1 \;=\; \left(0,0,\frac{\mathfrak{d}_0 l_{11}}{l_{01}+l_{11}},1\right), \qquad w_2 \;=\; \left(0,0,\frac{\mathfrak{d}_0 l_{11} + tl_{01}l_{11}}{l_{01}+l_{11}},1\right).$$

We have $0 < tl_{01}l_{11}/(l_{01}+l_{11}) \leq 1$ only for $t = 1$ and $l_{01} = l_{11} = 2$. In this case, the second inequality becomes an equality and thus w_1 is integral w_1 which implies $\mathfrak{d}_0 = 0$. We arrive at the compound du Val singularity with matrix (12-e-e) and parameters $k_1 = k_2 = 1$. □

We turn to the non-toric non-$\mathbb{Q}$-factorial threefolds, still of canonical multiplicity one. The following observation provides the link to the $\mathbb{Q}$-factorial case. Given defining data A, P for a ring $R(A,P)$ of Type 2, we will have to deal with quadratic submatrices P' of P, obtained by erasing columns and rows from P. The corresponding submatrix A' of A gathers all columns a_i of A such that at least one column v_{ij} is not erased from P when passing to P'.

Lemma 8.2. *Let $X = X(A,P)$ be a compound du Val threefold of Type 2 and canonical multiplicity ζ_X with P irredundant in the form of Proposition 4.3 and ordered in the sense of Remark 6.1.*

- (i) *Let P' be an $(r+2) \times (r+2)$ submatrix of P such that for any $i = 0, \ldots, r$ at least one v_{ij} is not erased from P.*
  - (a) *$A' = A$ and P' are defining data of Type 2 in the sense of Construction 2.2; moreover, P' is in the form of Proposition 4.3.*
  - (b) *$X' = X(A', P')$ is a $\mathbb{Q}$-factorial threefold with at most compound du Val singularities of canonical multiplicity $\zeta_{X'} = \zeta_X$.*

  Moreover, one always finds a submatrix P' as above being ordered and having the same leading block as P.
- (ii) *Every P' as in (i) admits a 4×4 submatrix P'' with the same leading block as P' such that*
  - (a) *A'' and P'' are defining data of Type 2 in the sense of Construction 2.2, the matrix P'' is ordered and the form of Proposition 4.3.*
  - (b) *The varieties $X' = X(A', P')$ and $X'' = X(A'', P'')$ are equivariantly isomorphic to each other.*
- (iii) *If the leading platonic triple of P is different from $(x, y, 1)$, then $r = 2$ holds.*
- (iv) *One always finds P' and P'' as in (ii) with the same leading block as P such that*
  - (a) *in case of the leading platonic triple of P differing from $(x, y, 1)$, up to admissible operations, P'' is one of the matrices from Proposition 8.1.*
  - (b) *in case of the leading platonic triple of P equal to $(x, y, 1)$, we have $n''_2 = 2$ for P''.*

Proof. We verify (i). Note that each column of P' is as well a column of P. By Proposition 2.11, the columns of P generate the extremal rays of a full dimensional cone $\sigma \subseteq \mathbb{Q}^{r+2}$. Thus, also the columns of P' generate the extremal rays of a cone $\sigma' \subseteq \mathbb{Q}^{r+2}$. We show that σ' is full dimensional. If P' has a column $v_1 \in \lambda$, then, using Proposition 3.8 (iii) we see that the remaining $r+1$ columns of P' are linearly independent and v_1 does not lie in their linear span. If P' has no column inside λ, then we can form two different P-elementary cones τ_1 and τ_2 out of columns of P'. The corresponding $v_{\tau_i} \in \tau_i^\circ$ generate the pointed two-dimensional cone $\sigma' \cap \lambda$ and we see that the columns of P generate $\mathbb{Q}^{r+2}$. Thus, we can conclude that P' satisfies the conditions of Type 2 of Construction 2.2 and, together with $A' = A$ gives defining data. Observe that $X' = X(A', P')$ is $\mathbb{Q}$-factorial by construction. Using Remark 4.4, we obtain $\zeta_{X'} = \zeta_X$ and see that P' still is in the form of Proposition 4.3. Using Remark 4.5, conclude $\imath_{X'} = \imath_X = 1$. Moreover, according to Proposition 3.8, the anticanonical complex $A^c_{X'}$ is a subcomplex of A^c_X and the same holds for $\partial A^c_{X'}$ and ∂A^c_X. Thus, Proposition 7.4 shows that X' inherits from X the property of having at most compound du Val singularities. The supplement is obvious.

We prove (ii). For $r = 2$, there is nothing to show. So, assume $r \geq 3$. If P' has a column $v_k \in \lambda$, then we have $n_i = l_{i1} = 1$ for $i \geq 3$ and Remark 2.4, applied $r-2$ times, yields the desired 4×4 matrix P''. We turn to the case that P' has no column in λ. Then $n_k = 2$ for some $0 \leq k \leq r$ and all other n_i equal one. If $k \leq 2$

holds, then we have $n_i = l_{i1} = 1$ for $i \geq 3$ and proceed as before to obtain P''. We discuss $k = 3$. First assume that the leading platonic triple of P' equals $(x, y, 1)$. Then, exchanging the data of column blocks 3 and 2 of P', we are in the case $k \leq 2$ just treated. If the leading platonic triple of P' differs from $(x, y, 1)$ then, applying $r - 3$ times Remark 2.4, we arrive at an irredundant 5×5 matrix P'' defining a variety $X'' = X(A'', P'')$ isomorphic to $X' = X(A', P')$; a contradiction to Proposition 8.1. Finally, if $k \geq 4$, then we exchange the data of column blocks k and 3 of P' and are in the case $k = 3$. This proves (ii).

We turn to (iii). Assume $r \geq 3$. Since P is irredundant and ordered in the sense of Remark 6.1, we have $n_i \geq 2$ and $l_{ij} = 1$ for $i \geq 3$. Consider the submatrices

$$P' \ := \ [v_{01}, v_{11}, v_{21}, v_{31}, v_{32}, v_{41}, \ldots, v_{r1}], \qquad P^{\sim} \ := \ [v_{01}, v_{11}, v_{21}, v_{31}, v_{32}].$$

Let P'' be the matrix obtained by erasing from $P^{\sim}$ erasing all but the first three and the last two rows. Then P'' is an irredundant 5×5 matrix and $X'' = X(A'', P'')$ is isomorphic to $X' = X(A', P')$; a contradiction to Proposition 8.1.

Finally, we show (iv). For (a), observe that because of $\imath_{X''} = \imath_X = 1$, Proposition 4.7 gives $\zeta_{X''} = \zeta_X = 1$. Thus X'' is $\mathbb{Q}$-factorial compound du Val and P'' must, up to admissible operations, be one of the matrices from Proposition 8.1. We turn to (b). For any $i \geq 2$, we have $n_i \geq 2$, because P is irredundant. Consider the submatrices

$$P' \ := \ [v_{01}, v_{11}, v_{21}, v_{22}, v_{31}, \ldots, v_{r1}], \qquad P^{\sim} \ := \ [v_{01}, v_{11}, v_{21}, v_{22}].$$

Then we obtain the desired P'' from $P^{\sim}$ by erasing all but the first two and the last two rows. $\square$

Proposition 8.3. *Let $X = X(A, P)$ be a non-toric affine threefold of Type 2. Assume that X is not $\mathbb{Q}$-factorial, of canonical multiplicity one and has at most compound du Val singularities. Then P can be assumed to be the matrix*

$$\text{(10-e)} \quad \begin{bmatrix} -k & 2 & 1 & 0 & 0 \\ -k & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad k \in \ \mathbb{Z}_{\geq 2}.$$

Proof. The strategy is to look first for not necessarily irredundant matrices P'' with $r'' = 2$ defining a $\mathbb{Q}$-factorial $X'' = X(A'', P'')$ of canonical multiplicity one with at most compound du Val singularities. Then we obtain, up to admissible operations, all matrices P with $X(A, P)$ satisfying the assumptions of the proposition by enlarging the P'' in the sense of Lemma 8.2. We organize the subsequent discussion according to the possible leading block data, as listed in Proposition 7.9, and treat pairs P'', P sharing the same leading block data. Note that we have $r = 2$ for P whenever the leading platonic triple differs from $(x, y, 1)$.

Consider the leading block data $(5, 3, 2; 0, 0, 0)$. Proposition 8.1 tells us that after suitable admissible operations, we have

$$P'' \ = \ \begin{bmatrix} -5 & 3 & 0 & 0 \\ -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 1 & 1 & 1 \end{bmatrix}.$$

After performing the corresponding admissible operations on P, we find P'' as a submatrix of P. Moreover, P has at least one further column and thus a submatrix

$$P''' \ = \ \begin{bmatrix} -5 & 3 & 0 & * \\ -5 & 0 & 2 & * \\ 0 & 0 & 0 & * \\ -4 & 1 & 1 & * \end{bmatrix}.$$

Lemma 8.2 (i) says that $X''' = X(A''', P''')$ is $\mathbb{Q}$-factorial, of canonical multiplicity one and with at most compound du Val singularities. Thus, up to admissible operations, P''' occurs in the list of Proposition 8.1. So, the last column must be one of

$$(0,0,1,1), \qquad (0,0,-1,1)$$

The first case is impossible, because the columns of the defining matrix P are pairwise different. For $(0,0,-1,1)$ as last column, the point $(0,0,0,1)$ lies in $\partial A_X^{c}(\lambda)^\circ$; a contradiction to Proposition 7.4.

The case of leading block data $(4,3,2;0,0,0)$ is treated by exactly the same arguments as the preceding case.

Consider the leading block data $(4,3,2;1,0,0)$. Again, Proposition 8.1 tells us that, up to admissible operations, we have

$$P'' = \begin{bmatrix} -4 & -1 & 3 & 0 \\ -4 & -1 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ -3 & 0 & 1 & 1 \end{bmatrix}.$$

Adapting P by admissible operations, it comprises P'' as a submatrix. As before, we obtain a matrix P''' by enhancing the leading block with a further column of P, which this time must be one of

$$(-1,-1,3,0), \qquad (-1,-1,2,0).$$

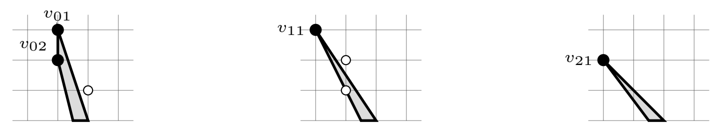
The first leads to two identical columns of P and this is excluded. For the second we find $(0,0,3,1)$ inside $\partial A_X^{c}(\lambda)^\circ$ and leave the compound du Val case.

The case of leading block data $(3,3,2;0,0,0)$ runs exactly as the case of $(5,3,2;0,0,0)$.

Consider the leading block data $(3,3,2;1,0,0)$. Here Proposition 8.1 leaves us with two possibilities for the submatrix P'' of the accordingly adapted P. The first possibility is

$$P'' = \begin{bmatrix} -3 & -2 & 3 & 0 \\ -3 & -2 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 1 \end{bmatrix} \tag{8.3.1}$$

with columns $v_{01}, v_{02}, v_{11}, v_{21}$. Using as above Proposition 8.1, we arrive at three possibilities for submatrices $P''' = [v_{01}, v_{11}, v_{21}, *]$; with $\sigma = \operatorname{cone}(v_{01}, v_{02}, v_{11}, v_{21})$, we find the following situation in the $\partial A_X^{c}(\lambda_i) \cap \sigma$:



$\partial A_X^{c}(\lambda_0) \cap \sigma$ $\qquad$ $\partial A_X^{c}(\lambda_1) \cap \sigma$ $\qquad$ $\partial A_X^{c}(\lambda_2) \cap \sigma$

where the circles indicate the prospective columns $*$ of P''' leading to compound du Val singularities $X(A''', P''')$ of canonical multiplicity one. They are

$$(-1,-1,2,0) \in \lambda_0, \quad (1,0,1,1),\ (2,0,1,1) \in \lambda_1.$$

The lower one in the middle picture is contained in σ which is not possible. The other two force $(0,0,2,1)$ to lie in $\partial A_X^{c}(\lambda)^\circ$ which is as well impossible. So, (8.3.1) does not occur as a submatrix of P. The second possibility is

$$P'' = \begin{bmatrix} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \end{bmatrix}.$$

Here we proceed analogously as with (8.3.1) and see the only possible additional column in P is $(1,0,1,1)$. In this case again $(0,0,2,1)$ lies in $\partial A_X^c(\lambda)^\circ$ and we leave the compound du Val case.

Consider the leading block data $(l_{01},2,2;0,0,0)$. Here Proposition 8.1 tells us that the submatrix P'' of the accordingly adapted P is

$$P'' = \begin{bmatrix} -k_1 & -k_2 & 2 & 0 \\ -k_1 & -k_2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1-k_1 & 1-k_2 & 1 & 1 \end{bmatrix},$$

where we allow $k_2 = 0$ here and in this case change the second and fourth column to have a proper defining matrix. A possible further column for P''' must have the form $(-k_3,-k_3,t,1-k_3)$ with $t = \pm 1$. For $t = 1$, one of $(-k_2,-k_2,1,1-k_2)$ or $(-k_3,-k_3,1,1-k_3)$ does not give an extremal ray of the cone spanned by the columns of P. For $t = -1$, the point $(0,0,0,1)$ lies in $\partial A_X^c(\lambda)^\circ$ and we leave the compound du Val case.

Consider the leading block data $(l_{01},2,2;1,0,0)$. Proposition 8.1 allows two choices for the submatrix P'' of the accordingly adapted P. The first one is

$$P'' = \begin{bmatrix} -k & 2 & 0 & 0 \\ -k & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 \end{bmatrix}.$$

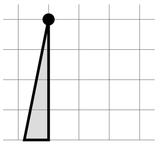
We check the possible further columns of P. A column in λ would lead to $(0,0,1,1) \in \partial A_X^c(\lambda)^\circ$ and this is impossible. For any P''' sharing the first three columns with P'', the additional column, due to Proposition 8.1, must be $(1,0,t,1)$ or $(0,1,t,1)$, where $t = 0,1$. For $t = 0$, such column would not generate an extremal ray of the cone spanned by the columns of P. For $t = 1$, we obtain $(0,0,1,1) \in \partial A_X^c(\lambda)^\circ$ and we leave the compound du Val case. The second choice is

$$P'' = \begin{bmatrix} -k & 2 & 1 & 0 \\ -k & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1-k & 1 & 1 & 1 \end{bmatrix}.$$

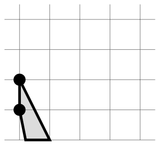
Proposition 8.1 tells us that $(1,0,t,1)$ or $(0,1,t,1)$ with $t = 0,1$ are the only possible further columns of P. But $(1,0,0,1)$ is impossible, since this column already exists in P and for $(1,0,1,1)$, we obtain $(0,0,1,1) \in \partial A_X^c(\lambda)^\circ$. The same holds for $(0,1,1,1)$. For $(0,1,0,1)$, the line segment $\partial A_X^c(\lambda)$ has, in addition to $w_1 = (0,0,1,1)$, the vertex

$$w_2 = \left(0,0,\frac{1}{1+k},1\right).$$

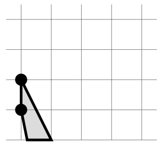
If we have a look at the leaves, we see that we get a compound du Val singularity with defining matrix (10-e):



$\partial A_X^c(\lambda_0)$



$\partial A_X^c(\lambda_1)$



$\partial A_X^c(\lambda_2)$

Consider the leading block data $(l_{01}, 2, 2; 0, 1, 0)$. Proposition 8.1 allows four possible submatrices P'' of the suitably adapted P. We distinguish the following cases.

Case 1: The exponent l_{01} is odd. First assume P has after suitable admissible operations a submatrix

$$P'' = \begin{bmatrix} -2k_1-1 & -2k_2 & 2 & 0 \\ -2k_1-1 & -2k_2 & 0 & 2 \\ 0 & \frac{k_1-k_2+1}{2} & 1 & 0 \\ -2k_1 & 1-2k_2 & 1 & 1 \end{bmatrix}.$$

Assume the matrix P has a further column $(-k, -k, t, 1-k)$ in λ_0. We regard the submatrix containing this further column as well as the last two columns of P'' and either the first (if k odd) or the second (if k even) of P''. This matrix does not show up in Proposition 8.1 and we leave the compound du Val case. So P can have no further column $(-k, -k, t, 1-k)$.

Also an additional column $(0, 0, t, 1)$ in the lineality part is impossible, because due to Proposition 8.1, the only possibilities are $t = k_1$ and $t = k_1 + 1$. But these would either not give an extremal ray of the cone spanned by the columns of P (for $t = k_1 + 1$) or $(-1, -1, k_1, 0)$ would show up in $\partial A_X^c(\lambda_0)^\circ$. Now the last possibility is an additional column $(1, 0, t, 1)$ in λ_1 or $(0, 1, t, 1)$ in λ_2. But the possible values of t, i.e. those giving a compound du Val submatrix of type (10-o) from Proposition 8.1, either generate no extremal ray of the cone spanned by the columns of P or $(-1, -1, k_1, 0)$ is an interior point of $\partial A_X^c(\lambda_0)$. Thus assume P has, after suitable admissible operations, no submatrix of the above form and one

$$P'' = \begin{bmatrix} -2k-1 & 2 & 1 & 0 \\ -2k-1 & 0 & 0 & 2 \\ 0 & 1 & \lceil \frac{2k+1}{4} \rceil & 0 \\ -2k & 1 & 1 & 1 \end{bmatrix}.$$

Now, the submatrix of P given by the first, second and third column of this submatrix and one further column must as well be of this form after suitable admissible operations. So the only possible additional column is $(0, 1, \lceil (2k+1)/4 \rceil - 1, 1)$ in λ_2, but then $(-1, -1, k_1, 0)$ is an inner point of $\partial A_X^c(\lambda_0)$ and we leave the compound du Val case.

Case 2: The exponent l_{01} equals 4. After suitable admissible operations, the matrix P has a submatrix

$$P'' = \begin{bmatrix} -4 & 2 & 1 & 0 \\ -4 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ -3 & 1 & 1 & 1 \end{bmatrix}.$$

A further column must, together with the first two and the last row of P'', give a compound du Val submatrix P''' of P as well. So due to Proposition 8.1, the only possible further column is $(1, 0, 1, 1)$. But with this, the point $(0, 0, 2, 1)$ is an inner point of $\partial A_X^c(\lambda)$ and we leave the compound du Val case.

Consider the leading block data $(l_{01}, l_{11}, 1; \mathfrak{d}_0, 0, 0)$. Note that here, we also have to take care about redundant matrices P''. Proposition 8.1 provides us with one irredundant matrix

$$P'' \;=\; \begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}.$$

The only possible further columns of P are of the form $(-2,-2,t_0,-1)$, $(2,0,t_1,1)$ or $(0,1,t_2,0)$. Each of them would stretch the segment $\partial A_X^{\mathrm{c}}(\lambda)$ which already has the vertices $(0,0,0,1)$ and $(0,0,1,1)$.

Now we treat the redundant P'', which means to deal with $l_{11} = 1$. Due to Lemma 8.2 (iv) (b), after suitable admissible operations, the matrix P has a submatrix

$$P'' = \begin{bmatrix} -l_{01} & 1 & 0 & 0 \\ -l_{01} & 0 & 1 & 1 \\ \mathfrak{d}_0 & 0 & 0 & t_2 \\ 1-l_{01} & 1 & 1 & 1 \end{bmatrix}.$$

But since P is irredundant, it must have a further submatrix

$$P''' = \begin{bmatrix} -l_{01} & 1 & 1 & 0 & 0 \\ -l_{01} & 0 & 0 & 1 & 1 \\ \mathfrak{d}_0 & 0 & t_1 & 0 & t_2 \\ 1-l_{01} & 1 & 1 & 1 & 1 \end{bmatrix}$$

comprising P'' and one further column in λ_1. For this matrix and the vertices of the respective $\partial A_{X'''}^{\mathrm{c}}(\lambda)$, we have

$$w_1 \;=\; \left(0, 0, \frac{\mathfrak{d}_0}{l_{01}+1}, 1\right), \qquad w_2 \;=\; \left(0, 0, \frac{\mathfrak{d}_0 + (t_1+t_2)l_{01}}{l_{01}+1}, 1\right),$$

But $(t_1+t_2)l_{01}/(l_{01}+1) \le 1$ only for $t_1 = t_2 = l_{01} = 1$. But as P is irredunbdant, it must have a sixth column $(-1,-1,\mathfrak{d}_0+t_0,0)$ in P. The distance between then the vertices of $\partial A_X^{\mathrm{c}}(\lambda)$ becomes

$$\frac{t_0+t_1+t_2}{2} \;\ge\; \frac{3}{2}.$$

Thus, $\partial A_X^{\mathrm{c}}(\lambda)^\circ$ contains an integral point. So we obtain no compound du Val singularity in this case. $\square$

Finally, we have to deal with the non-toric threefolds of canonical multiplicity greater than one.

Proposition 8.4. *Let $X = X(A,P)$ be a non-toric affine threefold of Type 2. Assume that X is of canonical multiplicity greater than one and has at most compound du Val singularities. Then one may assume P to be one of the following matrices:*

$$\text{(9)} \quad \begin{bmatrix} -k & -k & \zeta_X - k & \zeta_X - k & 0 & 0 & \cdots & 0 & 0 \\ -k & -k & 0 & 0 & 1 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & & \\ -k & -k & 0 & 0 & 0 & 0 & & 1 & 1 \\ 0 & \mathfrak{d}_0 & 0 & \mathfrak{d}_1 & 0 & \mathfrak{d}_2 & \cdots & 0 & \mathfrak{d}_r \\ \frac{1-\mu k}{\zeta_X} & \frac{1-\mu k}{\zeta_X} & \frac{1-\mu k}{\zeta_X} + \mu & \frac{1-\mu k}{\zeta_X} + \mu & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\text{(13-e)} \begin{bmatrix} -2\zeta_X+1 & 1 & 1 & 0 & 0 \\ -2\zeta_X+1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(13-o)} \begin{bmatrix} -2\zeta_X+2 & 2 & 0 & 0 \\ -2\zeta_X+2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ \zeta_X & -1 & 0 & 0 \end{bmatrix} \quad \text{(14)} \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \end{bmatrix}.$$

In (9), $r \ge 2$ holds, the integers $\zeta_X \ge 2$ and $k \ge 1$ are coprime and μ is the unique integer $1 \le \mu < \zeta_X$ with $\zeta_X \mid (1-\mu k)$. Moreover $\mathfrak{d}_i \in \mathbb{Z}_{\ge 1}$ holds for $i \ge 0$ and if $k \ge 2$ ($\zeta_X - k \ge 2$), then one may erase the second (fourth) column of the matrix.

In (13-e), we have $\zeta_X \geq 2$. *In (13-o), we have* $\zeta_X \geq 3$ *odd. In (14), we have* $\zeta_X = 2$.

Proof. The strategy is similar to that of the proof of Proposition 8.3. We look first for not necessarily irredundant matrices P'' with $r = 2$ and $n''_2 = 2$ defining a $\mathbb{Q}$-factorial $X'' = X(A'', P'')$ with at most compound du Val singularities and of canonical multiplicity bigger than one. Lemma 8.2 then ensures that for $X = X(A, P)$ satisfying the assumptions of the proposition, the matrix P contains, after suitable admissible operations, one of our P'' as a submatrix with the same leading platonic triple as P. In other words, we can construct the possible P by suitably enlarging P''.

The matrix P'' we are looking for is 4×4. Since $\zeta_{X''} > 1$ holds, we are in the setting of Proposition 4.7 and because of $\imath_{X''} = 1$, we end up in Case 4.7 (vi). In addition to the leading block, we have the extra column v_{22} in P''. Moreover, the integer $\mu := (1 - \nu_{01}\zeta_{X''})/l_{01}$ as well as l_{01} and l_{11} must all be coprime to $\zeta_{X''}$, since we have the integer entries $\nu_{01} = (1 - \mu l_{01})/\zeta_{X''}$ and $\nu_{11} = (1 + \mu l_{11})/\zeta_{X''}$. We also see that $\zeta_{X''}$ divides $l_{01} + l_{11}$ by subtracting ν_{01} and ν_{11} from each other. Now let

$$k_0 := \lfloor l_{01}/\zeta_{X''} \rfloor, \qquad k_1 := \lceil l_{11}/\zeta_{X''} \rceil, \qquad \delta := l_{01} - k_0\zeta_{X''}.$$

Furthermore, let in this proof $\mathfrak{d}_{ij}$ be the third entry of the column v_{ij} of P''. With these definitions, our matrix has the following shape

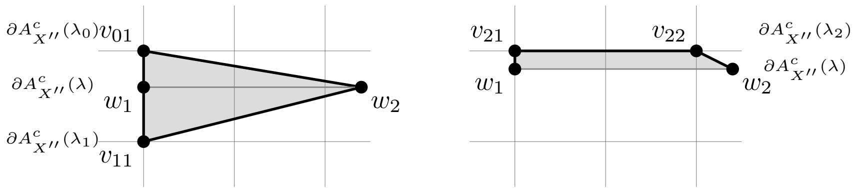
$$P'' = \begin{bmatrix} -(k_0\zeta_{X''} + \delta) & k_1\zeta_{X''} - \delta & 0 & 0 \\ -(k_0\zeta_{X''} + \delta) & 0 & 1 & 1 \\ \mathfrak{d}_{01} & \mathfrak{d}_{11} & 0 & \mathfrak{d}_{22} \\ \frac{1-\mu\delta}{\zeta_{X''}} - \mu k_0 & \frac{1-\mu\delta}{\zeta_{X''}} + \mu k_1 & 0 & 0 \end{bmatrix}, \tag{8.4.1}$$

where we achieve $1 \leq \mu < \zeta_{X''}$ by subtracting the $\lfloor \mu/\zeta_{X''} \rfloor$-fold of the first from the last row, simultaneously. Moreover, we achieve $\mathfrak{d}_{01} = 0$ by subtracting the $\mathfrak{d}_{01}\zeta_{X''}$-fold of the last and the $\mathfrak{d}_{01}\mu$-fold of the first from the penultimate row. Exchanging, if necessary, the data of column blocks 0 and 1, we achieve $k_1 > k_0 \geq 0$. We now figure out those P'' defining a compound du Val singularity. For this, we consider several constellations of k_0 and k_1.

Case 1: We have $k_0 = 0$ and $k_1 = 1$. Here we can also achieve $\mathfrak{d}_{11} = 0$ by subtracting the $\mathfrak{d}_{11}(1 - \mu\delta)/\zeta_{X''}$-fold of the first and the $\mathfrak{d}_{11}\delta$-fold of the last from the penultimate row. The vertices of $\partial A^c_{X''}(\lambda)$ are

$$w_1 = \left(0, 0, 0, \frac{1}{\zeta_{X''}}\right), \qquad w_2 = \left(0, 0, \frac{\mathfrak{d}_{22}\delta(\zeta_{X''} - \delta)}{\zeta_{X''}}, \frac{1}{\zeta_{X''}}\right).$$

We illustrate the situation for the case $\delta = 2$, $\zeta_{X''} = 5$, $\mathfrak{d}_{22} = 2$ below; observe that the lineality part λ contains no integer points and the union of the $\lambda_i \cap \mathcal{H}_i \cap \mathbb{Z}^4$ for $i = 0, 1$ is a sublattice



The polytope $\partial A^c_{X''}(\lambda_0)$ does not contain integer points $(-k, -k, t, (1-\mu k)/\zeta_{X''})$ in its relative interior as for such integer points $k < \delta$ and $(1-\mu k)/\zeta_{X''}$ integral must hold, but δ is minimal with the second property. The same holds for $\partial A^c_{X''}(\lambda_1)$ and $\partial A^c_{X''}(\lambda_2)$ respectively. All points in $\partial A^c_{X''}(\lambda)$ have $1/\zeta_{X''}$ as last coordinate, thus are not integral. So, there is no integral point in the relative interior of $\partial A^{\mathrm{c}}_{X''}$. Thus P'' defines a $\mathbb{Q}$-factorial compound du Val singularity and meanwhile looks as follows:

$$\begin{bmatrix} -\delta & \zeta_{X''}-\delta & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ 0 & 0 & 0 & \mathfrak{d}_{22} \\ \frac{1-\mu\delta}{\zeta_{X''}} & \frac{1-\mu\delta}{\zeta_{X''}}+\mu & 0 & 0 \end{bmatrix}, \qquad \gcd(\delta,\zeta_{X''})=1, \quad \mathfrak{d}_{22}\in\mathbb{Z}_{>0}. \tag{8.4.2}$$

Now we check the possibilities of enlarging P'' in the sense of Lemma 8.2 to a matrix P defining a non-$\mathbb{Q}$-factorial $X(A,P)$ as in the proposition. As further columns we can insert one or both of

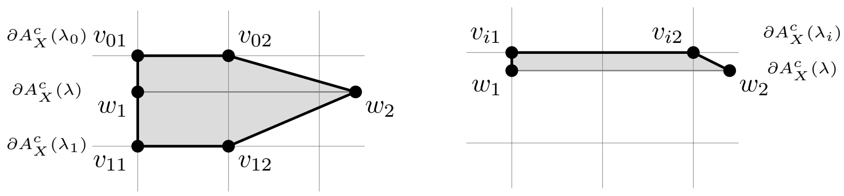
$$v_{02} = \left(-\delta, -\delta, \mathfrak{d}_{02}, \frac{1-\mu\delta}{\zeta_{X''}}\right), \qquad v_{12} = \left(\zeta_{X''}-\delta, 0, \mathfrak{d}_{12}, \frac{1-\mu\delta}{\zeta_{X''}}+\mu\right),$$

with $\mathfrak{d}_{i2}\in\mathbb{Z}_{>0}$ arbitrary. We can not add other columns $(-k,-k,0,(1-\mu k)/\zeta_{X''})$ in λ_0. This is because first, $k\leq\delta$ must hold since $(\delta,\zeta_{X''}-\delta,1)$ is the leading platonic triple. Second, $k=k'\zeta_{X''}+\delta$ with $k'\geq 0$ must hold. So we get $k=\delta$. But then one of the columns

$$\left(-\delta, -\delta, \mathfrak{d}_{01}, \frac{1-\mu\delta}{\zeta_{X''}}\right), \qquad \left(-\delta, -\delta, \mathfrak{d}_{02}, \frac{1-\mu\delta}{\zeta_{X''}}\right), \qquad \left(-\delta, -\delta, \mathfrak{d}_{03}, \frac{1-\mu\delta}{\zeta_{X''}}\right)$$

lies in the cone spanned by the other two. It can give no extremal ray of the cone spanned by the columns of P; a contradiction. Exactly the same argument shows that no more columns can be added in λ_1 and λ_2.

Moreover, we can increase r from two to arbitrary to get P from P''. The leaves $\lambda_0,\dots,\lambda_2$ stay untouched, we add new columns in leaves $\lambda_3,\dots,\lambda_r$. First we have $l_{ij}=1$, $n_i\geq 2$ for $i\geq 3$ due to log-terminality and irredundancy. Second, by the same argument as above for $\lambda_0,\dots,\lambda_2$, we have $n_i\leq 2$. Thus $n_i=2$ holds for $i\geq 3$. So λ_i for $i\geq 3$ must have the same structure as λ_2 with two columns e_i and $e_i+\mathfrak{d}_{i2}e_{r+1}$. Here $\mathfrak{d}_{i2}\in\mathbb{Z}_{>0}$ arbitrary and e_j denotes the j-th basis vector. The distances $\mathfrak{d}_{i2}$ between v_{i1} and v_{i2} for $0\leq i\leq r$ and in consequence between w_1 and w_2 may vary. Nevertheless, all polytopes $\partial A^{\mathrm{c}}_X(\lambda_i)$ are subsets of polytopes of the second type of Proposition 7.2 as also the following exemplary picture shows:



$i \geq 2$

So for any P of this form, there are no integral points in the relative interior of $\partial A^{\mathrm{c}}_X$. Furthermore, as we have seen above, no more columns can be added in any leaf. In total, we get the series (9) of defining matrices P of compound du Val singularities.

Case 2: We have $k_1\geq 2$. Recall that we have P'' of shape (8.4.1) with $\mathfrak{d}_{01}=0$. Let $x_1,\dots,x_4$ be the standard coordinates on the column space $\mathbb{Q}^4$ of P''. Consider the line segments $\partial A^{\mathrm{c}}_{X''}(\lambda)$ and

$L_{0,X''} := \partial A^{\mathrm{c}}_{X''}(\lambda_0)\cap\{x_1=x_2=-\delta\}$, $L_{1,X''} := \partial A^{\mathrm{c}}_{X''}(\lambda_1)\cap\{x_1=\zeta_X-\delta, x_2=0\}$, Let w_1, w_2 denote the vertices of $\partial A^c_{X''}(\lambda)$. Moreover, let ω_{01}, ω_{02} be the vertices of $L_{0,X''}$ and ω_{11}, ω_{12} the vertices of $L_{1,X''}$. Then we have

$$\begin{aligned}
w_1 &= \left(0, 0, \frac{\mathfrak{d}_{11}(k_0\zeta_{X''} + \delta)}{\zeta_{X''}(k_1 + k_0)}, \frac{1}{\zeta_{X''}}\right),\\
w_2 &= w_1 + \mathfrak{d}_{22}\frac{(k_0\zeta_{X''} + \delta)(k_1\zeta_{X''} - \delta)}{\zeta_{X''}(k_1 + k_0)}e_3,\\
\omega_{01} &= \left(-\delta, -\delta, \frac{\mathfrak{d}_{11}k_0\zeta_{X''}}{\zeta_{X''}(k_1 + k_0)}, \frac{1-\mu\delta}{\zeta_{X''}}\right),\\
\omega_{02} &= \omega_{01} + \mathfrak{d}_{22}\frac{(k_1\zeta_{X''} - \delta)k_0}{k_1 + k_0}e_3,\\
\omega_{11} &= \left(\zeta_{X''} - \delta, 0, \mathfrak{d}_{11}\frac{\zeta_{X''}k_1 - \delta k_0 + \zeta_{X''}k_1k_0 - \delta}{(k_1\zeta_{X''} - \delta)(k_1 + k_0)}, \frac{1-\mu\delta}{\zeta_{X''}} + \mu\right),\\
\omega_{12} &= \omega_{11} + \mathfrak{d}_{22}\frac{(k_0\zeta_{X''} + \delta)(k_1 - 1)}{k_1 + k_0}e_3.
\end{aligned}$$

Since there must be no integral point in the relative interior of the line segments $L_{0,X''}$ and $L_{1,X''}$, we at least require

$$\mathfrak{d}_{22}\frac{(k_1\zeta_{X''} - \delta)k_0}{k_1 + k_0} \le 1, \qquad \mathfrak{d}_{22}\frac{(k_0\zeta_{X''} + \delta)(k_1 - 1)}{k_1 + k_0} \le 1. \tag{8.4.3}$$

These inequalities will be observed in the following different cases.

Case 2.1: We have $k_0 = 0$. Here, the inequalities (8.4.3) ease to $\mathfrak{d}_{22}\delta(k_1-1)/k_1 \le 1$. We distinguish between $\delta = 1$ and $\delta > 1$.

Case 2.1.1: We have $\delta = 1$. Here the matrix P'' is redundant. So any matrix P with such submatrix must have an additional column in λ_0. We move on to a matrix P also containing this additional column. Such matrix is of the form

$$P = \begin{bmatrix} -1 & -1 & k_1\zeta_X - 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ 0 & \mathfrak{d}_{02} & \mathfrak{d}_{11} & 0 & \mathfrak{d}_{22} \\ 0 & 0 & k_1 & 0 & 0 \end{bmatrix},$$

where we can assume $\mathfrak{d}_{02} > 0$. But here the length of the line segment $L_{1,X}$ is

$$(\mathfrak{d}_2 + \mathfrak{d}_{02})(k_1 - 1)/k_1,$$

which is less or equal to one - which must hold if it does not contain an integral point - only for $\mathfrak{d}_{02} = \mathfrak{d}_{22} = 1$ and $k_1 = 2$. Thus by adding multiples of the last to the penultimate row, we can assume that $\mathfrak{d}_{11}$ equals one or zero. If $\mathfrak{d}_{11} = 1$, then the line segment $L_{1,X}$ has the vertices

$$\left(\zeta_X - 1, 0, \frac{2\zeta_X + 1}{4\zeta_X - 2}, 1\right), \qquad \left(\zeta_X - 1, 0, \frac{2\zeta_X + 1}{4\zeta_X - 2} + 1, 1\right).$$

So it contains an integer point in its relative interior, since $(2\zeta_X + 1)/(4\zeta_X - 2)$ is not integral. If $\mathfrak{d}_{11} = 0$, then $L_{1,X}$ has the vertices

$$(\zeta_X - 1, 0, 0, 1), \qquad (\zeta_X - 1, 0, 1, 1)$$

and thus contains no integer points. Since $L^\circ_{1,X}$ is the only subset of $\partial A^{c\,\circ}_X$ that may contain integer points, we get the series of defining matrices ($13e$) with arbitrary ζ_X from this.

Such P cannot again be the submatrix of a non-$\mathbb{Q}$-factorial matrix with possibly larger r. This is because for any additional column in $\lambda_0, \dots, \lambda_2$, the line segment $L_{1,X}$ would be stretched and then contain one its the former integral vertices $(\zeta_X - 1, 0, 0, 1)$ and $(\zeta_X - 1, 0, 1, 1)$. The same holds for additional leaves, which by irredundancy must contain at least two columns and also would lead to a stretching of $L_{1,X}$.

Case 2.1.2: We have $\delta > 1$. Here $\mathfrak{d}_{22}\delta(k_1-1)/k_1 \geq 2(k_1-1)/k_1$ holds. Thus (8.4.3) is fulfilled only for $k_1 = \delta = 2$ and $\mathfrak{d}_{22} = 1$. Moreover $\zeta_{X''}$ must be odd since $l_{01} = 2$ is even. Also $\mu = (\zeta_{X''} + 1)/2$ holds, i.e. we have the matrix

$$P'' = \begin{bmatrix} -2 & 2\zeta_{X''} - 2 & 0 & 0 \\ -2 & 0 & 1 & 1 \\ 0 & \mathfrak{d}_{11} & 0 & 1 \\ -1 & \zeta_{X''} & 0 & 0 \end{bmatrix}.$$

By admissible operations, again $\mathfrak{d}_{11}$ can be assumed to be equal to zero or one. For $\mathfrak{d}_{11} = 1$, the line segment $L_{1,X''}$ has the vertices

$$\left(\zeta_{X''} - 2, 0, \frac{\zeta_{X''} - 1}{\zeta_{X''} - 2}, \frac{\zeta_{X''} - 1}{2}\right), \qquad \left(\zeta_{X''} - 2, 0, \frac{\zeta_{X''} - 1}{\zeta_{X''} - 2} + 1, \frac{\zeta_{X''} - 1}{2}\right),$$

which have an integer point inbetween due to $(\zeta_{X''}-1)/(\zeta_{X''}-2)$ not being integral. In case $\mathfrak{d}_{11}$ equals zero, the segment $L_{1,X''}$ has the vertices

$$\left(\zeta_{X''} - 2, 0, 0, \frac{\zeta_{X''} - 1}{2}\right), \qquad \left(\zeta_{X''} - 2, 0, 1, \frac{\zeta_{X''} - 1}{2}\right).$$

Since again $L_{1,X''}^{\circ}$ is the only subset of $\partial A_{X''}^{c\,\circ}$ that may contain integer points, we get a compound du Val series with defining matrices (13-o) and odd $\zeta_{X''}$. With exactly the same argument as in Case 2.1.1, these matrices cannot serve as submatrices for other compound du Val defining matrices.

Case 2.2: We have $k_0 \geq 1$. Here, the first inequality of (8.4.3) leads to

$$1 \leq k_0 \leq \frac{k_1}{k_1\zeta_{X''} - \delta - 1} \;\Rightarrow\; 0 \geq k_1(\zeta_{X''} - 1) - \delta - 1. \tag{8.4.4}$$

Case 2.2.1: We have $k_1 \geq 3$. Remembering $\delta < \zeta_{X''}$, we in total require $\delta < \zeta_{X''} \leq (\delta + 4)/3$ from the above inequality (8.4.4), leading to $1 = \delta < \zeta \leq 5/3$. This gives a contradiction, since $\zeta_{X''}$ is integral.

Case 2.2.2: We have $k_1 = 2$. The inequality (8.4.4) gives $\delta < \zeta_{X''} \leq (\delta+3)/2$ here, leading to $\delta < 3$. While $\delta = 2$ leads to $\zeta_{X''} \leq 5/2$, which contradicts $\delta < \zeta_{X''}$, the case $\delta = 1$ allows $\zeta_{X''} = 2$. The first inequality of (8.4.3) can only be fulfilled for $\mathfrak{d}_2 = k_0 = 1$ here. Furthermore, $\mu = 1$ must hold and inserting everything in (8.4.1), we get a defining matrix

$$P'' = \begin{bmatrix} -3 & 3 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ 0 & \mathfrak{d}_{11} & 0 & 1 \\ -1 & 2 & 0 & 0 \end{bmatrix}.$$

Here in a first step, by admissible operations we can assume $\mathfrak{d}_{11} \in \{0, 1, 2\}$. In a second step, the vertices

$$\omega_{11} = \left(1, 0, \mathfrak{d}_{11}\frac{2}{3}, 1\right), \qquad \omega_{12} = \left(1, 0, \mathfrak{d}_{11}\frac{2}{3} + 1, 1\right)$$

of the line segment $L_{1,X''}$ are integer only for $\mathfrak{d}_{11} = 0$. Exactly the same holds for $L_{0,X''}$. So in this case, P'' itself gives the compound du Val defining matrix (14). By the same arguments as in Case 2.1.1, these matrix cannot serve as submatrix for other compound du Val defining matrices. □

Proof of Theorem 4. Propositions 7.2, 8.1, 8.3, 8.4 provide us with the defining matrices P of the compound du Val threefold singularities of complexity one. The equations thereof are obtained by computing the degree zero Veronese subalgebra of $R(A, P)$ for a suitable matrix A. We used the MDS package [16] for this. □

Proof of Theorem 5. Theorem 4 gives us all compound du Val singularities of complexity one. The respective Cox rings finally can be computed using Remark 6.7. □

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National Research University Higher School of Economics, Faculty of Computer Science, Kochnovskiy Proezd 3, Moscow, 125319 Russia
E-mail address: arjantsev@hse.ru

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail address: juergen.hausen@uni-tuebingen.de

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail address: braun@math.uni-tuebingen.de

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail address: wrobel@math.uni-tuebingen.de