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Effective finite parametrization in phase spaces of parabolic equations

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Abstract. For evolution equations of parabolic type in a Hilbert phase space E, consideration is given to the problem of the effective parametrization (with a Lipschitzian estimate) of the sets $\mathcal{K} \subset E$ by functionals $\varphi_1, \dots, \varphi_m$ in E^* or, in other words, the problem of the linear Lipschitzian embedding of $\mathcal{K}$ in $\mathbb{R}^m$. If $\mathcal{A}$ is the global attractor for the equation, then this kind of parametrization turns out to be equivalent to the finite dimensionality of the dynamics on $\mathcal{A}$. Some tests are established for the parametrization (in various metrics) of subsets in E and, in particular, of manifolds $\mathcal{M} \subset E$ by linear functionals of different classes. We outline a range of physically significant parabolic problems with a fundamental domain $\Omega \subset \mathbb{R}^N$ that admit a parametrization of the elements $u(x) \in \mathcal{A}$ by their values $u(x_i)$ at a finite system of points $x_i \in \Omega$.

Introduction

It is known that under certain conditions, the final regimes of dissipative semilinear parabolic equations

$$\partial_t u = G(u) \tag{0.1}$$

in a Hilbert phase space E are determined by finitely many parameters. We shall consider equations (0.1) with a smooth resolving semiflow $\{\Phi_t\}_{t\geqslant 0}$ in E and a compact global attractor, that is, a set $\mathcal{A} \subset E$ consisting of entire bounded trajectories and attracting the balls E as $t \to +\infty$. In the fundamental investigation in [1] for a class of such problems, a construction was given for linear functionals $\varphi_i \in E^*$, $1 \leqslant i \leqslant m$, for recovering the trajectories on the attractor in the sense that the relations $\varphi_i(u(t)) = \varphi_i(v(t))$, $t \in \mathbb{R}$, for solutions $u(t), v(t) \in \mathcal{A}$ imply the identity $u(t) \equiv v(t)$. Later Mañé [2] realized the possibility of a finite parametrization for invariant compact sets $\mathcal{K} \subset E$ by functionals in E^*, that is, the existence of $\varphi_1, \dots, \varphi_m \in E^*$ such that $\varphi_i(u) = \varphi_i(v)$, $1 \leqslant i \leqslant m$, $u, v \in \mathcal{K}$, only for $u = v$.

In 1984, Foias and Temam [3] put forward the hypothesis that, for the Navier–Stokes system with a bounded fundamental domain $\Omega \subset \mathbb{R}^2$, the elements $u(x)$ of the attractor $\mathcal{A}$ are uniquely determined by their values $u(x_i)$ at a finite system of points (nodes) $x_i \in \Omega$. This conjecture has been confirmed recently (see [4] and [5]) in the case of an external force belonging to one of the Gevrey function classes. Interestingly, the number of determining nodes x_i for a periodic domain Ω turned out to be commensurable with the fractal dimension of $\mathcal{A}$. The methods

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described in [4] and [5] can be used for many parabolic partial differential equations with an analytic global attractor (one consisting of real-analytic functions). However, these methods provide no estimates for the E-norm $\|u-v\|$, $u,v\in\mathcal{A}$, in terms of the $|u(x_i)-v(x_i)|$. Lipschitzian estimates of this kind were obtained in [6] for the Kuramoto–Sivashinsky equation and other one-dimensional equations of parabolic type. In essence, this enables one to speak of steady regimes of the related physical system that are "traced" using finitely many point sensors.

In this paper, we discuss the question of subsets $\mathcal{K}\subset E$ on which the relation

$$\|u-v\|\leqslant c\sum_{i=1}^{m}|\varphi_i(u-v)|,\qquad c=\mathrm{const},\tag{0.2}$$

holds with $\mathcal{K}$-dependent functionals $\varphi_i\in E^*$. Because of (0.2), φ_i can be chosen in an *arbitrary* set $\mathcal{F}\subset E^*$ with linear span $\operatorname{sp}\mathcal{F}$ that is dense in E^*. It turns out that, under suitable conditions on the non-linear part of the vector field G in (0.1), every set of the form $\mathcal{K}=\Phi_\tau\mathcal{N}$, where $\mathcal{N}\subset E$, $\tau>0$, and

$$\|\Phi_\tau u-\Phi_\tau v\|\geqslant\rho\|u-v\|,\qquad \rho>0,\tag{0.3}$$

on $\mathcal{N}$ for $\rho=\rho(\mathcal{N})$, admits an *effective finite parametrization* (0.2) (see Theorem 4.2). In the important case $\mathcal{K}=\mathcal{A}$, it is established that the relation (0.2) is equivalent to the property (see [7] and [8]) of finite dimensionality of the dynamics (the limit dynamics) on the attractor. This property implies the possibility of describing the phase motion on $\mathcal{A}$ using an ordinary differential equation (ODE) with a Lipschitzian vector field in $\mathbb{R}^n$.

The parabolic semiflow smoothes the solutions, and therefore $\mathcal{K}=\Phi_\tau\mathcal{N}$ always belongs to some Hilbert space E_1 continuously embedded in E, and the condition (0.3) actually ensures a *strengthened parametrization*, that is, a type (0.2) parametrization for $\mathcal{K}$ not only in E, but also in E_1. In this case, the E-norm in (0.2) is replaced by the stronger E_1-norm $\|\cdot\|_1$, and the *determining functionals* φ_i are chosen from the dual space E_1^*, which is wider than E^*. In addition, if, as is often the case, E_1 consists of class $C(\Omega)$ functions continuous in the closure of a bounded domain $\Omega\subset\mathbb{R}^N$, then it is natural to realize φ_i as linear combinations of Dirac measures on Ω, which leads (see Theorem 5.1) to the estimate

$$\|u-v\|_1\leqslant c\sum_{i=1}^{r}|u(x_i)-v(x_i)|,\qquad c=\mathrm{const},\qquad r\geqslant m,\tag{0.4}$$

on $\mathcal{K}$ for a proper choice of the points $x_i\in\Omega$.

It is clear that the relation (0.2) is equivalent to the possibility of a linear Lipschitzian embedding of $\mathcal{K}$ in $\mathbb{R}^m$, and therefore it has a purely geometric meaning. Some more general type (0.2) tests for the parametrization of subsets in an arbitrary Banach space $\mathcal{X}$ are established in § 3. In particular, if $\mathcal{X}$ possesses a basis, then (0.2) is true for compact finite-dimensional C^1-submanifolds $\mathcal{M}\subset\mathcal{X}$. On the other hand, it is shown (see §§ 4 and 5) that if $E=L^2(\Omega)$, $\Omega\subset\mathbb{R}^N$, then $\mathcal{K}\subset C(\Omega)$ admits strengthened versions of (0.2) and, which is important, a nodal parametrization (0.4) under either of the following two conditions.

(a) The set $\mathcal{K}$ is invariant and lies on a compact finite-dimensional C^1-submanifold $\mathcal{M}\subset E$.

(b) $\mathcal{K}$ lies on an invariant manifold $\mathcal{M}$ that is the graph of a function $\gamma \in \operatorname{Lip}\bigl(PE,(I-P)E\bigr)$, $I=\mathrm{id}$, where P is the finite-dimensional spectral projection for the principal linear component of the vector field G.

A smooth or Lipschitzian finite-dimensional invariant manifold $\mathcal{M}\subset E$ is said to be *inertial* if it contains the global attractor and attracts exponentially the balls E as $t\to+\infty$. For example, such a manifold exists for reaction-diffusion equations in some bounded domains $\Omega\subset\mathbb{R}^N$, $N\leqslant 3$, and also for the Ginzburg–Landau, Cahn–Hillard, Kuramoto–Sivashinsky, and Kolmogorov–Sivashinsky equations. For more detail, see [9]–[14] and the references there.

We see that a graph-type inertial manifold $\mathcal{M}$ admits a parametrization (0.2), (0.4). This strengthens results of a similar form in [6]. In contrast to [6], we consider partial differential equations of arbitrary dimension and parametrize the elements of the manifold $\mathcal{M}$ using not only point values or integral means, but also functionals of an arbitrary nature. On the other hand, a parametrization (0.2), (0.4) is admitted by the global attractor of parabolic equations demonstrating finite-dimensional limit dynamics and thus (according to [8]) by the attractor of a dissipative scalar equation of the form

$$u_t=u_{xx}+f(x,u,u_x),\qquad x\in(0,1), \tag{0.5}$$

with a smooth function f and a suitable function phase-space E.

The paper is organized as follows. §§ 1 and 2 contain some elementary data concerning the abstract parabolic equation (0.1) and certain properties of sets in the Hilbert scale of spaces generated by the principal linear part of the vector field G. These properties will be useful in what follows. The question of validity for the relation (0.2) in Banach spaces is considered from a general point of view in § 3. The key part of the paper, § 4, is mainly devoted to a derivation of sufficiency tests for the finite parametrization of the sets $\mathcal{K}\subset E$ in terms of the phase dynamics (0.1). In § 5, a theorem is proved on the nodal parametrization (0.4) and an outline is given of the range of its applications to problems in mathematical physics.

§ 1. Preliminaries

We consider semilinear parabolic equations

$$\partial_t u=-Au+F(u) \tag{1.1}$$

in a real separable Hilbert space X with norm $|\cdot|$, for which [15] is a classical reference. A closed unbounded linear operator A in X with a dense domain $D(A)$ is said to be *sectorial* if the semigroup $\{e^{-tA}\}_{t\geqslant 0}$ is analytic. The powers A^θ are defined in the usual way and, for the operator A, the corresponding Hilbert scale of spaces X^θ, $\theta\in\mathbb{R}$, with norm $|\cdot|_\theta$ is constructed. Here $X^0=X$ and $X^1=D(A)$.

Everywhere in what follows, we proceed from the following main hypotheses for equation (1.1).

(H1) The linear operator A is sectorial, its resolvent is compact and the spectrum $\sigma(A)$ lies in the half-plane $\operatorname{Re}\lambda>0$.

(H2) For some $\alpha \in [0,1)$, the function F belongs to the class $BC^2(X^\alpha, X)$ consisting of C^2-smooth bounded maps $X^\alpha \to X$, and the inequality

$$|F(u) - F(v)| \leqslant M|u - v|_\alpha, \quad M = \mathrm{const}, \tag{1.2}$$

holds for $u, v \in X^\alpha$.

(H3) Equation (1.1) generates a smooth dissipative semiflow $\{\Phi_t\}_{t\geqslant 0}$ in X^α.

It can be assumed that $|u|_\theta = |A^\theta u|$ for $u \in X^\theta$ and that $A^\theta\colon X^\theta \to X$ is an isometry. For $\beta > \theta$, the identity embedding $X^\beta \subset X^\theta$ is dense and completely continuous. The dissipativity of $\{\Phi_t\}$ means the existence of an (absorbing) ball $\mathcal{B}_0 \subset X^\alpha$ such that $\Phi_t \mathcal{B} \subset \mathcal{B}_0$, $t > \tau(\mathcal{B})$, for an arbitrary ball $\mathcal{B}$ in X^α. It can easily be shown (see [15], §§ 3.3 and 3.4) that (H3) follows from (H1) and (H2). The evolution operators Φ_t, $t > 0$, are compact, and we have $\Phi_t X^\alpha \subset X^1$. The condition (H1) makes it possible to define finite-dimensional spectral projections P_a, $a > 0$, for the operator A in X that correspond to the part of the spectrum $\sigma(A)$ for $\operatorname{Re}\lambda < a$. The projections P_a commute with A^θ and are continuous in X^θ for all $\theta \geqslant 0$.

We shall also assume that the following additional hypothesis holds.

(H4) $P_a u \to u$ on X^α as $a \to \infty$.

Hypothese (H1) and (H4) hold for every self-adjoint positive discrete operator (with a compact resolvent) in X.

In real dissipative problems, the inequality (1.2) usually holds only on balls $\mathcal{B} \subset X^\alpha$. The standard "truncation" procedure (see [9] and [10]) then enables one to pass to the equation $\partial_t u = -Au + F_1(u)$, where $F_1 \in BC^2(X^\alpha, X)$ is a uniformly Lipschitzian function inheriting the final dynamics (1.1). We shall assume that this passage has already been performed.

We say that a set $\mathcal{N} \subset X^\alpha$ is *invariant* if $\Phi_t \mathcal{N} = \mathcal{N}$ for $t > 0$. Under the hypotheses (H1)–(H3), there exists (for example, see [9], [16]) a finite-dimensional compact global attractor, that is, an invariant set $\mathcal{A} \subset X^\alpha$ attracting the balls X^α as $t \to +\infty$. Every bounded invariant subset X^α is contained in $\mathcal{A}$, and in fact $\mathcal{A} \subset X^1$. Note that, in the context of the introduction, $E = X^\alpha$.

The validity of (H2)–(H4) with the chosen values of the parameter $\alpha \in [0,1)$ implies their validity for an arbitrary parameter $\beta \in (\alpha, 1)$. Indeed, if $F \in BC^2(X^\alpha, X)$, then, as a consequence of the continuity of the embedding $X^\beta \subset X^\alpha$, we have $F \in BC^2(X^\beta X)$. For the same reason, the norm $|\cdot|_\alpha$ on the right-hand side of (1.2) can be replaced by $|\cdot|_\beta$. The convergence $P_a u \to u$ in X^β clearly follows from that in X^α since the projections P_a commute with $A^{\beta-\alpha}$. Thus, along with X^α, the space X^β with an arbitrary $\beta \in (\alpha, 1)$ can be regarded as a phase space (1.1).

Let us recall the notion of finite-dimensional dynamics introduced in [7].

Definition 1.1. We say that the phase dynamics of equation (1.1) on an invariant compact set $\mathcal{K} \subset X^\alpha$ is *finite-dimensional* if, for some ODE $\dot{\xi} = h(\xi)$ with a Lipschitzian vector field $h(\xi)$ and resolving flow $\{S_t\}$ in $\mathbb{R}^n$, there is a Lipschitzian embedding $g\colon \mathcal{K} \to \mathbb{R}^n$ such that $g\Phi_t u = S_t g u$ on $\mathcal{K}$ for $t \geqslant 0$. In the case $\mathcal{K} = A$, we shall speak of the finite-dimensional limit dynamics of equation (1.1).

As was shown in [8], the finite dimensionality of the limit dynamics follows from the identical embeddability of the attractor in a finite-dimensional (not necessarily invariant) C^1-submanifold $\mathcal{M} \subset X^\alpha$ and, a fortiori, from the existence a smooth inertial submanifold $\mathcal{M} \subset X^\alpha$. The existence of an inertial manifold of the form

$$\mathcal{M} = \{u \in X^\alpha\colon u = p + \gamma(p),\ p \in P_a X^\alpha\} \tag{1.3}$$

for $a > 0$ and with a uniformly Lipschitzian function $\gamma\colon P_a X^\alpha \to (I - P_a)X^\alpha$, $I = \mathrm{id}$ in X^α, also ensures the finite dimensionality of the dynamics on the attractor for equation (1.1).

§ 2. Some useful lemmas

The constructions in this section are of interest both in their own right and from the point of view of what follows. Here and below, essential use will be made of the results in [7] and [8].

We begin with a discussion of general compactness properties for the scale of spaces $\{X^\theta\}$. We note that a set which is compact in X^θ is also compact in X^β, $\beta < \theta$. Let $G(u)$ denote the vector field $F(u) - Au$ of equation (1.1). Then G can be regarded as being defined not only on X^1, but also on X^α, and therefore we have $G\colon X^1 \to X$ and $G\colon X^\alpha \to X^{\alpha-1}$.

Lemma 2.1. *If the image $G(\mathcal{H})$ of a relatively compact set $H \subset X^\alpha$ is a bounded subset of X^1, then $\mathcal{H} \subset X^1$ and $\mathcal{H}$ is compact or non-compact simultaneously in all X^θ-metrics for $\theta \leqslant 1$.*

Proof. The inclusion $\mathcal{H} \subset X^1$ follows easily from the relation $A = F - G$ and the inclusion $G(\mathcal{H}) \subset X^1$.

Let the set $\mathcal{H}$ be compact in X^θ for some $\theta \leqslant 1$. If $\theta \geqslant \alpha$, then $\mathcal{H}$ is compact in X^α. But if $\theta < \alpha$, then the X^θ-closed set $\mathcal{H}$ is also closed in the stronger metric of X^α, that is, $\mathcal{H}$ is compact in X^α. By Lemma 4.2 in [7], $G\colon \mathcal{H} \to X^\alpha$ is a Hölder function in the X^α-metric, whence, taking (1.2) into account, we obtain the estimate

$$|u - v|_1 \leqslant c|u - v|_\alpha^\varepsilon$$

on $\mathcal{H}$ for $\varepsilon, c > 0$. This together with the X^α-compactness of $\mathcal{H}$ implies the compactness of $\mathcal{H}$ in X^1 and therefore in X^θ for all $\theta \leqslant 1$. The lemma is proved.

According to Lemma 4.3 in [7], $G(\mathcal{A})$ is a bounded set in X^1. We thus arrive at the following corollary.

Corollary 2.2. *Every invariant compact set $\mathcal{K} \subset X^\alpha$ (and, in particular, the attractor $\mathcal{A}$) is compact in the space X^θ for all $\theta \leqslant 1$.*

For $\mathcal{H} \subset X^\theta$, $\theta < 1$, we write

$$\mathcal{H}_\theta = \{w \in X^\theta\colon w = (u - v)/|u - v|_\theta,\ u, v \in \mathcal{H},\ u \neq v\}. \tag{2.1}$$

In what follows, an important role will be played by the sets $\mathcal{K} \subset X^\alpha$ satisfying the following condition.

Condition 2.3. *The relation $\mathcal{K} = \Phi_\tau \mathcal{N}$, $\tau > 0$, is true for $\mathcal{N} \subset X^\alpha$, and the inequality*

$$|\Phi_\tau u - \Phi_\tau v|_\alpha \geqslant \rho|u - v|_\alpha \tag{2.2}$$

holds on $\mathcal{N}$, where $\rho = \rho(\mathcal{N}) > 0$.

It is clear that, in this case, $\mathcal{K} \subset X^1$, and if the set $\mathcal{K}$ is invariant, then $\mathcal{N} = \mathcal{K}$. Furthermore, according to Theorem 3.5.2 in [15], $G(\mathcal{K}) \subset X^\theta$ for all $\theta < 1$. If $\mathcal{K}$ is bounded in X^α, then the compactness of the operator Φ_τ implies relative compactness of $\mathcal{N}$ and $\mathcal{K}$ in X^α.

Lemma 2.4. *If Condition 2.3 holds for a set $\mathcal{K}$ in X^α, then we have the following assertions.*

(a) *The metrics of the spaces X^θ and X^α on $\mathcal{K}$ are equivalent for all $\theta \in (\alpha, 1)$, and the same is true for $\theta = 1$ since $\mathcal{K}$ is bounded in X^α.*

(b) *The set $\mathcal{K}_\theta$ is relatively compact in X^θ for all $\theta \in [\alpha, 1)$.*

(c) *There is a spectral projection P_a of the operator A such that*

$$|u - v|_\theta \leqslant l|P_a(u - v)|_\theta \tag{2.3}$$

on $\mathcal{K}$ for $\theta \in [\alpha, 1)$, $l = l(\mathcal{K}, \theta)$, and if K is bounded in X^α, then the same is true for $\theta = 1$.

Proof. It is known (see [17], Lemma 5.2) that

$$|\Phi_\tau u - \Phi_\tau v|_\theta \leqslant c(\tau, \theta)|u - v|_\alpha$$

on $\mathcal{N}$ for $\alpha < \theta < 1$. In combination with the inequality (2.2), this gives the estimate

$$|u - v|_\theta \leqslant c_1|u - v|_\alpha$$

for $u, v \in \mathcal{K}$ and $c_1 = \text{const}$. The reverse inequality (with a different constant) follows from the continuity of the embedding $X^\theta \subset X^\alpha$, and the equivalence of the metrics on the spaces X^α and X^θ for $\theta \in (\alpha, 1)$ has thus been established. If $\mathcal{K}$ is bounded in X^α, then $\mathcal{N}$ is relatively compact, and we have

$$|G(\Phi_\tau u) - G(\Phi_\tau v)| \leqslant c_2|G(\Phi_\tau u) - G(\Phi_\tau v)|_\alpha \leqslant c_3|u - v|_\alpha \leqslant c_4|\Phi_\tau u - \Phi_\tau v|_\alpha$$

on $\mathcal{N}$, where c_2, c_3 and c_4 are positive constants. Here we have used the continuity of the embedding $X^\alpha \subset X$, then Lemma 3.2 in [7] and, finally, the relation (2.2). Owing to the relation $A = F - G$ and the Lipschitzian estimate (1.2), we now have

$$|u - v|_1 \leqslant |F(u) - F(v)| + |G(u) - G(v)| \leqslant (M + c_4)|u - v|_\alpha$$

for $u, v \in \mathcal{K}$, that is, the metrics of the spaces X^1 and X^α on $\mathcal{K}$ are equivalent.

To prove assertion (b), we take some $\beta \in (\theta, 1)$. The equivalence of the metrics of X^β and X^θ on $\mathcal{K}$ ensures the boundedness of the set $\mathcal{K}_\theta$ in X^β and thus (in view of the compactness of the embedding $X^\beta \subset X^\theta$) the relative compactness of $\mathcal{K}_\theta$ in X^θ.

We proceed to prove assertion (c). The main hypothesis (H4) guarantees the strong convergence of P_a to I in X^α as $a \to +\infty$. By the Ascoli–Arzela theorem, this convergence is uniform on the relatively compact set $\mathcal{K}_\alpha \subset X^\alpha$. Let $w = u - v$ for $u, v \in \mathcal{K}$, $u \neq v$, and $q = 1/2$. Then there is an $a > 0$ such that

$$|w - P_a w|_\alpha \leqslant q|w|_\alpha, \qquad |w|_\alpha \leqslant |P_a w|_\alpha + q|w|_\alpha,$$

whence follows the desired estimate in (2.3) for $\theta = \alpha$ and $l = 2$. In its general form, the inequality (2.3) is implied by the equivalence of the metrics of X^θ and X^α on $\mathcal{K}$ and on the finite-dimensional space $P_a X^\alpha \subset X^1$. The proof of assertion (c) is complete.

Assertion (c) in Lemma 2.4 enables one to strengthen slightly one of the conclusions in [7], Theorem 1.6.

Lemma 2.5. *For an invariant compact set $\mathcal{K} \subset X^{\alpha}$, the estimate* (2.2) *with a fixed $\tau > 0$ is a necessary and sufficient condition for the finite dimensionality of the phase dynamics on $\mathcal{K}$.*

In Definition 1.1, the conjugating map $g\colon \mathcal{K} \to \mathbb{R}^n$ is bi-Lipschitzian in the X^{α}-metric and, in practice, this definition refers to the finite-dimensional X^{α}-dynamics on $\mathcal{K}$.

Remark 2.6. Corollary 2.2 and assertion (a) of Lemma 2.4 imply that the phase X^{β}-dynamics on an invariant compact set $\mathcal{K} \subset X^1$ is finite-dimensional or not finite-dimensional for all $\alpha \leqslant \beta < 1$ simultaneously.

§ 3. General approach

Here we consider the problem of type (0.2) parametrization in an arbitrary Banach space $\mathcal{X}$ with norm $\|\cdot\|$. As usual, the symbol w (w^*) will indicate the weak (weak*) topology on the dual space $\mathcal{X}^*$.

As already mentioned, the $\mathcal{X}$-parametrization (0.2) of a set $\mathcal{K} \subset \mathcal{X}$ simply means the existence of a linear Lipschitzian embedding $g\colon \mathcal{K} \to \mathbb{R}^m$. There is a somewhat stronger result, as follows.

Lemma 3.1. *The parametrization* (0.2) *for a set $\mathcal{K} \subset \mathcal{X}$ is equivalent to the existence of a finite-dimensional continuous linear projection P in $\mathcal{X}$ such that*

$$\|u - v\| \leqslant l\|P(u - v)\| \tag{3.1}$$

on $\mathcal{K}$ for $l = \text{const}$.

Proof. Representing the projection P in (3.1) using the formula

$$Pu = \sum_{i=1}^{m} \varphi_i(u)e_i, \tag{3.2}$$

where $\{e_i\}$ is an arbitrary basis in $P\mathcal{X}$ and $\{\varphi_i\}$ is a unique system of functionals in $\mathcal{X}^*$ with the properties $\varphi_i(e_j) = \delta_{ij}$ and $\varphi_i = 0$ on $\ker P$, we readily obtain the inequality (0.2) for $u, v \in \mathcal{K}$. Conversely, the functionals φ_i in (0.2) can be assumed to be linearly independent. Let $e_1, \dots, e_m$ be a system of linearly independent vectors in $\mathcal{X}$ such that $\varphi_i(e_j) = \delta_{ij}$ and let P be the corresponding projection in (3.2). Since the norm $|\varphi_1(\,\cdot\,)| + \dots + |\varphi_m(\,\cdot\,)|$ in the finite-dimensional space $P\mathcal{X}$ is equivalent to the $\mathcal{X}$-norm, P realizes the estimate (3.1) on $\mathcal{K}$. The lemma is proved.

Of course, the existence of a linear Lipschitzian embedding of $\mathcal{K}$ in $\mathbb{R}^m$ implies that the fractal dimension $\dim_f \mathcal{K}$ is finite. However, an example is known (see [18]) in which a compact set $\mathcal{K}$ lies in a Hilbert space but admits no (even non-linear) embedding of this kind, although the condition $\dim_f \mathcal{K} < \infty$ holds.

The parametrization (0.2) implies some arbitrariness in the choice of the determining functionals.

Lemma 3.2. *If the relation* (0.2) *holds for a set* $\mathcal{K} \subset \mathcal{X}$, *then the determining functionals in* (0.2) *can be chosen from an arbitrary set* $\mathcal{F} \subset \mathcal{X}^*$ *whose linear span* $\operatorname{sp}\mathcal{F}$ *is strongly dense in* $\mathcal{X}^*$.

Proof. In accordance with what has been said above, we proceed from the fact that, for a given $\mathcal{K} \subset \mathcal{X}$, there is a type (3.2) projection P satisfying (3.1).

For every φ_i in (3.2), there is a sequence of functionals $\{\chi_{i,k}\}$ in $\operatorname{sp} F$ that strongly converges to φ_i as $k \to \infty$. If k is sufficiently large, then $B_k = \{b_{ij}\} = \{\chi_{j,k}(e_i)\}$ is an invertible near-identity $m \times m$ matrix. For the finite-dimensional space $P\mathcal{X}$, let T_k denote a linear operator with matrix B_k^{-1} in the basis $\{e_i\}$. It can be easily seen that $\chi_{i,k}(T_k e_j) = \delta_{ij}$, and therefore the functionals $\chi_{1,k}, \dots, \chi_{m,k}$ are linearly independent, and the operators

$$Q_k u = \sum_{i=1}^{m} \chi_{i,k}(u) T_k e_i$$

are continuous linear projections on $P\mathcal{X}$. The convergence of the operators T_k, $T_k \to I$, where $I = \mathrm{id}$ in $P\mathcal{X}$, and Q_k, $Q_k \to P$, is uniform and, consequently,

$$\|(P - Q_k)(u - v)\| \leqslant q\|u - v\| \tag{3.3}$$

on $\mathcal{K}$ for arbitrarily small values of q, $q > 0$, and $k = k(q)$. Using (3.1), we now derive the estimates

$$\|u - v\| \leqslant l\|Q_k(u - v)\| + lq\|u - v\|$$

and (for $lq < 1$)

$$\|u - v\| \leqslant c_1\|Q_k(u - v)\|, \tag{3.4}$$

where $u, v \in \mathcal{K}$ and $c_1 = l/(1 - lq)$. For a fixed $k = k(q)$ with $q < 1/l$, there are linearly independent functionals $\psi_1, \dots, \psi_r \in \mathcal{F}$ such that $\operatorname{sp}(\chi_{i,k}) \subset \operatorname{sp}\{\psi_\nu\}$. If $\|T_k e_i\| \leqslant c_2$, $1 \leqslant i \leqslant m$, then

$$\|Q_k(u - v)\| \leqslant c_2 \sum_{i=1}^{m} |\chi_{i,k}(u - v)| \leqslant c_3 \sum_{\nu=1}^{r} |\psi_\nu(u - v)|$$

on $\mathcal{K}$. By (3.4), we have

$$\|u - v\| \leqslant c \sum_{\nu=1}^{r} |\psi_\nu(u - v)| \tag{3.5}$$

for $u, v \in \mathcal{K}$, where $c = \mathrm{const}$. The lemma is proved.

If the space $\mathcal{X}$ is reflexive, then for $\operatorname{sp}\mathcal{F}$, the w^*-density in $\mathcal{X}^*$ ensures the strong density in $\mathcal{X}^*$. Indeed, in this case, the w^*- and w-topologies in the dual space $\mathcal{X}^*$ are identical, and the w-closure of the convex set $\operatorname{sp}\mathcal{F}$ coincides with its strong closure.

With every set $\mathcal{K} \subset \mathcal{X}$, we associate (see (2.1) and also [8]) a subset in the unit sphere,

$$\mathcal{K}^0 = \{w \in \mathcal{X} \colon w = (u - v)/\|u - v\|,\ u, v \in \mathcal{K},\ u \neq v\}.$$

The geometry of the subset $\mathcal{K}^0$ turns out to be closely related to the parametrization problem (0.2) for $\mathcal{K}$ or, which is the same, the problem of a finite-dimensional Lipschitzian Cartesian structure of the set $\mathcal{K}$. Let $\omega(\mathcal{K}^0)$ be the Hausdorff measure of non-compactness, that is, the infimum of the values of ε, $\varepsilon > 0$, such that $\mathcal{K}^0$ belongs to the ε-neighbourhood of some compact set $\mathcal{N}_\varepsilon \subset \mathcal{X}$. Usually a somewhat different definition is used (see [19], § 32), in which the values of ε, $\varepsilon > 0$, for which a finite ε-net exists for $\mathcal{K}^0$ are taken. The equivalence of these definitions can easily be established using the construction (see [19], § 18) known as the "Schauder projection". If $\mathcal{K}^0$ is relatively compact, then $\omega(\mathcal{K}^0) = 0$, and vice versa. In the Hilbert case, we always have $\omega(\mathcal{K}^0) \leqslant 1$ because, for $\varepsilon > 1$, the closure of the image $P\mathcal{K}^0$, where P is an arbitrary finite-dimensional orthoprojection, can be taken as $\mathcal{N}_\varepsilon$.

Lemma 3.3. *In a separable Hilbert space $\mathcal{X}$, an estimate of the form* (3.1) *with a finite-dimensional orthoprojection P for $\mathcal{K} \subset \mathcal{X}$ is equivalent to the condition $\omega(\mathcal{K}^0) < 1$.*

Proof. Let $\omega(\mathcal{K}^0) < 1$. Then for some $\varepsilon < 1$, the set $\mathcal{K}^0$ lies in the ε-neighbourhood of a compact set $\mathcal{N}_\varepsilon \subset \mathcal{X}$. Let P_n be a sequence of orthoprojections that strongly converges to the identity operator I and set $Q_n = I - P_n$. By the Ascoli–Arzela theorem, the convergence $Q_n \to 0$ is uniform on $\mathcal{N}_\varepsilon$, and therefore $\|Q_n w\| \leqslant \delta < 1 - \varepsilon$ on $\mathcal{N}_\varepsilon$ for sufficiently large n. For every $w \in \mathcal{K}^0$, there is a $w' \subset \mathcal{N}_\varepsilon$ such that $\|w - w'\| < \varepsilon$, and therefore

$$\|Q_n w\| \leqslant \|Q_n(w - w')\| + \|Q_n w'\| < \varepsilon + \delta.$$

Since $w = (u - v)/\|u - v\|$, where $u, v \in \mathcal{K}$, the desired estimate in (3.1) for $P = P_n$ and $l = (1 - \varepsilon - \delta)^{-1}$ now follows from the inequality $\|w\| \leqslant \|Q_n w\| + \|P_n w\|$.

Conversely, the condition (3.1) with $l > 1$ gives the estimate $\|Pw\| \geqslant 1/l$ on $\mathcal{K}^0$ and, since $\|Pw\|^2 + \|Qw\|^2 = 1$ for $Q = I - P$, we have $\|Qw\| \leqslant \varepsilon$, where $\varepsilon^2 = 1 - 1/l^2$. The closure $\mathcal{N}_\varepsilon$ of the image $P\mathcal{K}^0$ is compact, and we have $\|w - Pw\| = \|Qw\| \leqslant \varepsilon$ for $w \in \mathcal{K}^0$, that is, $\omega(\mathcal{K}^0) \leqslant \varepsilon < 1$. The lemma is proved.

The relative compactness of $\mathcal{K}^0$ leads to a strengthened version of the parametrization (0.2) for $\mathcal{K} \subset \mathcal{X}$.

Lemma 3.4. *If there is a basis in the space $\mathcal{X}$, $\mathcal{K} \subset \mathcal{X}$, and $\omega(\mathcal{K}^0) = 0$, then the relations* (3.1) *and* (0.2) *hold for $\mathcal{K}$. In this case, the determining functionals in* (0.2) *can be taken from an arbitrary set $\mathcal{F} \subset \mathcal{X}^*$ whose linear span $\operatorname{sp}\mathcal{F}$ is w^*-dense in $\mathcal{X}^*$.*

Proof. The existence of a basis ensures that of bounded n-dimensional linear projections P_n in $\mathcal{X}$ that strongly converge to $I = \mathrm{id}$ as $n \to \infty$. As in the proof of assertion (c) of Lemma 2.4, we conclude that the inequality (3.1) with a finite-dimensional projection P of the form (3.2) holds for $\mathcal{K}$. According to Lemma 3.1, this implies the property (0.2) for $\mathcal{K}$.

Further considerations are related to the choice of an adequate approximation for the functionals $\varphi_i \in \mathcal{X}^*$ in (3.2) using elements $\chi \in \operatorname{sp}\mathcal{F}$, and they closely follow the proof of Lemma 3.2. The distinction is that, instead of the strong convergence of sequences of functionals $\chi_{i,k} \to \varphi_i$ for $i = 1, \dots, m$, we assume the w^*-convergence of *generalized sequences* $\chi_{i,d} \to \varphi_i$, where $\{d\}$ is the fundamental system of w^*-neighbourhoods of zero in $\mathcal{X}^*$ with inclusion as a partial ordering (see [20], Ch. 1). We then construct (as in the proof of Lemma 3.2) the corresponding operators T_d in $P\mathcal{X}$ and projections Q_d in $\mathcal{X}$. It is clear that $Q_d u \to Pu$ on $\mathcal{X}$. This convergence is uniform on $\mathcal{K}^0$ since $\mathcal{K}^0$ is relatively compact and the norms of the projections Q_d are jointly bounded. In this way, type (3.3) and (3.4) estimates can be obtained for Q_d, after which the desired relation in (3.5) with functionals $\psi_\nu \in \mathcal{F}$ can be derived.

Furthermore, let $\mathcal{M}$ be a compact finite-dimensional C^1-submanifold in $\mathcal{X}$. Then, as can be seen from the proof of Lemma 2.3 in [7], the set $\mathcal{M}^0$ is relatively compact. This together with Lemma 3.4 allows us to establish a sufficient test for a parametrization (0.2).

Lemma 3.5. *If there is a basis in the space $\mathcal{X}$, then every compact finite-dimensional C^1-submanifold $\mathcal{M} \subset \mathcal{X}$ admits a parametrization* (0.2).

Note that in the case $\mathcal{M} \subset C^2$, the above assertion (without the assumption about the basis) follows from the infinite-dimensional version of Lemma 9.2.1 in [15], Whitney's embedding theorem and Lemma 3.1.

To conclude this section, we give a very useful necessary condition for a parametrization (0.2).

Lemma 3.6. *Let $\mathcal{X}$ and $\mathcal{X}_0$ be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_0$ respectively, and let there be dense continuous embeddings $\mathcal{X} \subset \mathcal{X}_0$ and $\mathcal{X}_0^* \subset \mathcal{X}^*$. Then an $\mathcal{X}$-parametrization* (0.2) *of a set $\mathcal{K} \subset \mathcal{X}$ implies that the metrics of $\mathcal{X}$ and $\mathcal{X}_0$ are equivalent on $\mathcal{K}$.*

Proof. It follows from Lemma 3.2 and the proof of Lemma 3.1 that there is a projection of the form (3.2) in $\mathcal{X}$ with functionals $\varphi_i \in \mathcal{X}_0^*$. In this case, if the $\mathcal{X}$-norms of the vectors e_i and the $\mathcal{X}_0^*$-norms of the functionals φ_i in (3.2) have an upper bound b, then (3.1) implies the inequality $\|u-v\| \leqslant lb^2 m\|u-v\|_0$ for $u, v \in \mathcal{K}$. The reverse inequality (with a different constant) follows from the continuity of the embedding $\mathcal{X} \subset \mathcal{X}_0$.

§ 4. Main results

We return to the parabolic equation (1.1) in the scale of Hilbert spaces $\{X^\theta\}$. Let us set $Y_\theta = (X^\theta)^*$ for $\theta \in \mathbb{R}$. Since the embedding of spaces $X^\theta \subset X^\beta$ is continuous and dense for $\beta < \theta$, the same is true for the embedding of dual spaces $Y_\beta \subset Y_\theta$. The strong topology in the Hilbert scale $\{Y_\theta\}$ weakens as the parameter θ increases. A set $\mathcal{F} \subset Y_\theta$ is said to be *generating* if the linear span $\operatorname{sp}\mathcal{F}$ is strongly dense in Y_θ. As the topology weakens, the number of dense sets increases and, moreover, in a fixed topology the density relation for sets is transitive. Therefore for $\beta < \theta$, every generating set $\mathcal{F} \subset Y_\beta$ preserves this property in Y_θ.

We shall be interested in the question of the parametrization property (0.2) in the spaces X^θ. It is preserved as the parameter θ decreases: this follows from Lemma 3.2 and the density of the embedding $Y_\beta \subset Y_\theta$ for $\beta < \theta$. By the same lemma, the determining functionals in an X^θ-parametrization (0.2) for $\mathcal{K} \subset X^\theta$

can be taken from an arbitrary generating subset $\mathcal{F} \subset Y_\theta$. As in §§1 and 2, we assume the validity of main hypotheses (H1)–(H4) for the coefficients of the parabolic equation (1.1).

We give a necessary condition for a parametrization (0.2) in the case of arbitrary sets in the Hilbert scale $\{X^\theta\}$.

Theorem 4.1. *Let $\theta \in \mathbb{R}$ and let a set $\mathcal{K} \subset X^\theta$ admit an X^θ-parametrization* (0.2). *Then for every $\beta < \theta$, the metrics of the spaces X^β and X^θ on $\mathcal{K}$ are equivalent.*

This follows from Lemma 3.6 and the continuity and density of the embeddings of spaces $X^\theta \subset X^\beta$ and $Y_\beta \subset Y_\theta$.

All sufficient tests for a parametrization (0.2) of the sets $\mathcal{K} \subset X^\alpha$ will be stated in terms of the phase dynamics of equation (1.1).

Theorem 4.2 (main theorem). *Let a set $\mathcal{K} \subset X^\alpha$ satisfy Condition* 2.3. *Then for all $\theta \in [\alpha, 1)$, every generating set $\mathcal{F} \subset Y_\theta$ contains linearly independent functionals $\psi_1, \dots, \psi_r$ such that*

$$|u - v|_\theta \leqslant c \sum_{i=1}^{r} |\psi_i(u - v)| \tag{4.1}$$

for $u, v \in \mathcal{K}$, where $c =$ const. If $\mathcal{K}$ is bounded in X^α, then the same is true for $\theta = 1$.

Proof. Condition 2.3 enables us to use assertion (c) of Lemma 2.4 for $\mathcal{K}$ to derive the estimate (2.3) with $\theta \in [\alpha, 1)$, and also with $\theta = 1$ if $\mathcal{K}$ is bounded in X^α. The assertions of the theorem then follow from Lemmas 3.1 and 3.2.

There is a natural relationship between the notions of effective finite parametrization and finite dimensionality of the phase dynamics in the sense of Definition 1.1.

Theorem 4.3. *For every invariant compact set $\mathcal{K} \subset X^\alpha$, the following properties are equivalent.*

(a) *The dynamics on $\mathcal{K}$ is finite-dimensional.*
(b) *$\mathcal{K}$ admits a parametrization* (0.2) *in X^α.*
(c) *$\mathcal{K}$ admits a parametrization* (0.2) *in X^1.*

Proof. It is clear that property (c) implies property (b), and therefore it suffices to establish the implications (b) $\to$ (a) $\to$ (c).

Let us prove the implication (b) $\to$ (a). According to Lemma 3.1, there is a continuous finite-dimensional projection $P\colon X^\alpha \to X^\alpha$ satisfying (3.1) on $\mathcal{K}$. Since the base space X is reflexive, it then follows from Theorem 1.6 in [7] that the dynamics on $\mathcal{K}$ is finite-dimensional.

We prove the implication (a) $\to$ (c). Lemma 2.5 ensures the estimate (2.2) with some $\tau > 0$ on the invariant compact set $\mathcal{K}$, and therefore $\mathcal{K}$ satisfies Condition 2.3 and, by assertion (c) of Lemma 2.4, the inequality (2.3) with $\theta = 1$ holds for $u, v \in \mathcal{K}$. This together with Lemma 3.1 gives an X^1-parametrization (0.2) of $\mathcal{K}$.

We proceed to consider type (0.2) parametrizations on finite-dimensional submanifolds $\mathcal{M} \subset X^\alpha$.

Proposition 4.4. *Every invariant set $\mathcal{K} \subset \mathcal{M}$, where $\mathcal{M}$ is a compact finite-dimensional C^1-manifold in X^α, admits an X^1-parametrization* (0.2).

Proof. The closure of $\mathcal{K}$ in X^α is compact and invariant. Without loss of generality, we assume that the set $\mathcal{K}$ itself possesses these properties. According to Theorem 1.5 in [8], the phase dynamics on $\mathcal{K}$ is finite-dimensional, and therefore the desired assertion follows from Theorem 4.3.

For an arbitrary subset $\mathcal{K}$ of a compact finite-dimensional C^1-manifold $\mathcal{M}$ in X^α, the general Lemma 3.5 guarantees only a parametrization in X^α for it. Much more can be asserted in the case of an invariant graph type manifold $\mathcal{M}$.

Proposition 4.5. *A finite-dimensional invariant Lipschitzian manifold $\mathcal{M}$ of the form* (1.3) *in X^α admits an X^θ-parametrization* (0.2) *for every $\theta \in [\alpha, 1)$. Furthermore, X^α-bounded submanifolds $\mathcal{M}$ admit an X^1-parametrization* (0.2).

Proof. The phase dynamics on $\mathcal{M}$ is Lipschitz conjugate to the dynamics of the equation

$$\dot{p} = -Ap + P_a F\bigl(P + \gamma(p)\bigr) \tag{4.2}$$

in the finite-dimensional subspace $P_a X^\alpha \subset X^1$. The projection P_a is continuous in X, $P_a X^\alpha = P_a X$, and all the norms are equivalent in $P_a X$. The functions $\gamma\colon P_a X^\alpha \to (I - P_a)X^\alpha$ and $F\colon X^\alpha \to X$ satisfy the Lipschitz condition, and therefore the same is true for the right-hand side of (4.2). The elementary properties of ODE solutions now ensure the validity of Condition 2.3 for $\mathcal{M}$, and it remains to use assertion (c) of Lemma 2.4 and Lemma 3.1. The proof of the proposition is complete.

We note that inertial manifolds are usually constructed for semilinear parabolic equations precisely in the form (1.3). In this connection, it is advisable to mention the results in [17], § 2, according to which, under certain conditions imposed on equation (1.1), every smooth invariant submanifold $\mathcal{M} \subset X^\alpha$ turns out to be an embedded manifold in X^1.

The linear functionals ψ_i in (4.1) can be continuous in an arbitrarily weak topology of X^θ as $\theta \to -\infty$. We have not yet considered the undoubtedly interesting questions on the minimal number of functionals of a given class that are necessary for the effective parametrization of the sets $\mathcal{K} \subset X^\alpha$ and on the relationship between this number and the fractal or Hausdorff dimension of $\mathcal{K}$.

§ 5. Nodal parametrization

In this section, we consider dissipative equations (1.1) with base space $X = L^2(\Omega)$, where Ω is a bounded "sufficiently regular" (see [15], Chapter 1) domain in $\mathbb{R}^N$. Assuming that the main hypotheses (H1)–(H4) hold, we shall try to realize the estimate (4.1) in terms of Dirac measures. We shall need the space $C = C(\Omega)$ of functions continuous in the closure of Ω and the Sobolev L^2-spaces $H^s(\Omega)$, $s \in \mathbb{Z}^+$. All embeddings of function spaces are assumed to be continuous. If $0 < \theta \leqslant 1$, then (see [15], § 1.6) the embedding $X^1 \subset H^s(\Omega)$ with $\theta s > N/2$ implies the embedding $X^\theta \subset C$ and, therefore, the inclusion $\mathcal{K} \subset C$ for sets $\mathcal{K} \subset X^\alpha$ satisfying Condition 2.3.

Theorem 5.1. *Let $X = L^2(\Omega)$, where Ω is a bounded "sufficiently regular" domain in $\mathbb{R}^N$, and let there be an embedding $X^1 \subset H^s(\Omega)$, $s > N/2$. In this case, if a set $\mathcal{K} \subset X^\alpha$ satisfies Condition 2.3 and $\theta < 1$, then there are points $x_1, \dots, x_r \in \Omega$ such that*

$$|u - v|_\theta \leqslant c \sum_{i=1}^{r} |u(x_i) - v(x_i)| \tag{5.1}$$

on $\mathcal{K}$ for $c = \mathrm{const}$. The same is also true for $\theta = 1$ since $\mathcal{K}$ is bounded in X^α.

Proof. Without loss of generality, we assume that $\theta s > N/2$, that is, $X^\theta \subset C$. Let C_θ denote the closure of X^θ in the C-norm. Then C_θ is a closed (not necessarily proper) subspace in C. The embedding $X^\theta \subset C_\theta$ is dense and, consequently, the embedding $C_\theta^* \subset Y_\theta$ of dual spaces is w^*-dense. If $\mathcal{F}$ is a set of Dirac measures supported in Ω, then $\mathcal{F} \subset C^*$ and the linear span $\operatorname{sp}\mathcal{F}$ is w^*-dense in C^*. By the Hahn–Banach theorem, every element of C_θ^* is the restriction of a continuous linear functional in C^* to C_θ and, therefore, the embedding $\operatorname{sp}\mathcal{F} \subset C_\theta^*$ in dense in the w^*-topology of C_θ^*, and hence in the weaker w^*-topology of Y_θ. Thus, the embeddings $\operatorname{sp}\mathcal{F} \subset C_\theta^*$ and $C_\theta^* \subset Y_\theta$ are dense in the w^*-topology of Y_θ and consequently, the embedding $\operatorname{sp}\mathcal{F} \subset Y_\theta$ is also w^*-dense. Since $Y_\theta = (X^\theta)^*$ is a Hilbert space, the convex set $\operatorname{sp}\mathcal{F}$ is in fact strongly dense in Y_θ. The desired assertion now follows from Theorem 4.2.

Remark 5.2. Since the assumptions of Theorem 5.1 imply a continuous embedding $X^\theta \subset C$, the norm $|u - v|_\theta$ in the inequality (5.1) can be replaced by the C-norm $\|u - v\|_C$.

Corollary 5.3. *Let $X^1 \subset H^s(\Omega)$, $s > N/2$. Then the assertions of Theorem 5.1 for a set $\mathcal{K} \subset X^\alpha$ hold in each of the following cases.*

(a) *The set $\mathcal{K}$ is invariant and lies on a compact finite-dimensional C^1-submanifold $\mathcal{M} \subset X^\alpha$.*

(b) *$\mathcal{K}$ lies on an invariant finite-dimensional Lipschitzian manifold $\mathcal{M}$ of the form (1.3).*

Proof. We only need to verify that Condition 2.3 holds for $\mathcal{K}$. In case (b), this was in fact done in the proof of Proposition 4.5. In case (a), according to Theorem 1.5 in [8], the phase dynamics on the closure of $\mathcal{K}$ is finite-dimensional, and it only remains to use Lemma 2.5.

Let us briefly discuss some applications of the above results. It is clear that the main constraint in Theorem 5.1 is Condition 2.3 on $\mathcal{K}$, $\mathcal{K} \subset X^\alpha$. As can be seen from Lemma 2.5 and the proof of Proposition 4.5, this condition holds for the global attractor $\mathcal{A} \subset X^\alpha$ if the dynamics on $\mathcal{A}$ is finite-dimensional, and for an inertial manifold $\mathcal{M} \subset X^\alpha$ of the form (1.3) provided it exists. Below we present a list (not, of course, final) of parabolic problems possessing graph type inertial manifolds (see [9]–[14]) for (1*)–(4*) or demonstrating finite-dimensional limit dynamics (see [8]) for (5*).

(1*) Reaction-diffusion systems and the Ginzburg–Landau equations in $\Omega = (0,1)^N$, $N \leqslant 2$.

(2*) The Kuramoto–Sivashinsky and Kolmogorov–Sivashinsky equations on the interval $(0,1)$.

(3*) The generalized Cahn–Hillard equations in $\Omega = (0,1)^N$, $N \leqslant 2$.
(4*) Scalar reaction-diffusion equations in some domains $\Omega \subset \mathbb{R}^N$, $N = 2, 3$.
(5*) The one-dimensional equation of the form (0.5).

For (4*), we refine the results in [13] and [14]. Namely, a rectangle, a cube, regular polygons and other domains with various kinds of symmetry should be taken as Ω.

To establish the possibility of a finite nodal parametrization (5.1) for a global attractor $\mathcal{A}$ or an inertial manifold $\mathcal{M}$, it is necessary to verify the main hypotheses (H1)–(H4) and also the embedding conditions $X^1 \subset H^s(\Omega)$, $s > N/2$, in Theorem 5.1. We shall show in outline how this can be done. For the necessary details, see [8]–[14].

Let $I = \mathrm{id}$ in $X = L^2(\Omega)$. If we proceed from the abstract form of the representation (1.1), then in cases (1*)–(5*) we always have $A = kI - \Delta$ or $A = kI + \Delta^2$, where $k \geqslant 0$, Δ is the Laplace operator and Δ^2 the biharmonic operator in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \leqslant 3$. Here the chosen boundary conditions must ensure that the linear operator A is self-adjoint and positive in X, and therefore the hypotheses (H1) and (H4) hold. We have the embedding $X^1 \subset H^s(\Omega)$ when $s = 2$ or $s = 4$ for $A = kI - \Delta$ or $A = kI + \Delta^2$, and therefore, in accordance with the assumptions of Theorem 5.1, the estimate $s > N/2$ holds. The required smoothness of the non-linear function $F\colon X^\alpha \to X$ (as part of the hypothesis (H2)) can readily be established for the examples in question using embedding theorems and well-known properties of the Nemytskii operator. For the various dissipativity conditions imposed on the phase dynamics in the parabolic problems (1*)–(5*) and the possibility of treating bounded uniformly Lipschitzian non-linearity, see [8]–[13].

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