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CROSS-SECTIONS, QUOTIENTS, AND REPRESENTATION RINGS OF SEMISIMPLE ALGEBRAIC GROUPS

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To T. A. Springer on his 85th birthday

"*Is Steinberg's theorem* [...] *only true for simply connected groups* [...]? *What happens for* GP(1), *for instance*? *Is there a rational section of* G *over* $I(G)$ ("*invariants*") *in this case*? [...] *Is it true that* $I(G)$ *is a rational variety* [...]?"

A. Grothendieck, *Letter to J.-P. Serre,* January 15, 1969 [GS, pp. 240–241]

Abstract. Let G be a connected semisimple algebraic group over an algebraically closed field k. In 1965 Steinberg proved that if G is simply connected, then in G there exists a closed irreducible cross-section of the set of closures of regular conjugacy classes. We prove that in arbitrary G such a cross-section exists if and only if the universal covering isogeny $\tau\colon \widehat{G} \to G$ is bijective; this answers Grothendieck's question cited in the epigraph. In particular, for $\operatorname{char} k = 0$, the converse to Steinberg's theorem holds. The existence of a cross-section in G implies, at least for $\operatorname{char} k = 0$, that the algebra $k[G]^G$ of class functions on G is generated by $\operatorname{rk} G$ elements. We describe, for arbitrary G, a minimal generating set of $k[G]^G$ and that of the representation ring of G and answer two Grothendieck's questions on constructing generating sets of $k[G]^G$. We prove the existence of a rational (i.e., local) section of the quotient morphism for arbitrary G and the existence of a rational cross-section in G (for $\operatorname{char} k = 0$, this has been proved earlier); this answers the other question cited in the epigraph. We also prove that the existence of a rational section is equivalent to the existence of a rational W-equivariant map $T \dashrightarrow G/T$ where T is a maximal torus of G and W the Weyl group.

1. Introduction

Below, all algebraic varieties are taken over an algebraically closed field k. We use the standard notation and conventions of [Bor] and [Spr].

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Let G be a connected semisimple algebraic group, $G \neq \{e\}$. Let $(G/\!\!/G, \pi_G)$ be a categorical quotient for the conjugating action of G on itself, i.e., $G/\!\!/G$ is an affine variety (quotient variety) and

$$\pi_G \colon G \longrightarrow G/\!\!/G \tag{1}$$

a surjective morphism (quotient morphism) such that $\pi_G^*\big(k[G/\!\!/G]\big)$ is the algebra $k[G]^G$ of class functions on G. Every fiber of π_G is then the closure of a regular conjugacy class (i.e., that of the maximal dimension) and such classes in general position are closed [Ste_1, Theorem 6.11, Cor. 6.13, and Sect. 2.14].

Definition 1.1. A closed irreducible subvariety S of G is called a *cross-section* (of the collection of fibers of π_G) in G if S intersects every fiber of π_G at a single point.

The elements of S are the "canonical forms" of the elements of a dense constructible subset of G with respect to conjugation. The image of any *section* of π_G (i.e., a morphism $\sigma \colon G/\!\!/G \to G$ such that $\pi_G \circ \sigma = \mathrm{id}_{G/\!\!/G}$) is an example of such S; moreover, this S has the property that $\pi_G|_S \colon S \to G/\!\!/G$ is an isomorphism. For $\operatorname{char} k = 0$, every cross-section in G is obtained in this manner (see Subsection 6.A).

In 1965 Steinberg gave an explicit construction of a section of π_G for every simply connected semisimple group G (see his celebrated paper [Ste_1]). Its image is a cross-section that intersects every regular conjugacy class and does not intersect other conjugacy classes.

In this paper we explore what happens in the general case, i.e., when G is not necessarily simply connected. In this case the following two facts about cross-sections in G for $\operatorname{char} k = 0$ are known.

First, by [CTKPR, Theorem 0.3] in every connected semisimple algebraic group G there is a *rational section* of π_G, i.e., a section over a dense open subset of $G/\!\!/G$ (local section).

Second, by Kostant's theorem [Kos, Theorem 0.10] there is an infinitesimal counterpart of Steinberg's cross-section: for the adjoint action of G on its Lie algebra $\operatorname{Lie} G$, there is a closed irreducible subvariety in $\operatorname{Lie} G$ that intersects every regular G-orbit at a single point.

In order to formulate our result consider the universal covering of G, i.e., a central isogeny

$$\tau \colon \widehat{G} \longrightarrow G$$

such that $\widehat{G}$ is a simply connected semisimple algebraic group (by [BT, Prop. (2.24)(ii)] it exists and is unique up to G-isomorphism).

We prove the following:

Theorem 1.2. *Let G be a connected semisimple algebraic group.*

(i) *The following properties are equivalent:*

(a) *there is a cross-section in G;*
(b) *the isogeny τ is bijective.*

(ii) *If $\sigma\colon G/\!\!/G \to G$ is a section of π_G, then the cross-section $\sigma(G/\!\!/G)$ in G intersects every regular conjugacy class and does not intersect other conjugacy classes.*

Remark 1.3. The isogeny τ is bijective if and only if it is either an isomorphism or purely inseparable (radical). The latter holds if and only if $\operatorname{char} k = p > 0$ and p divides the order of the fundamental group of G.

The next corollary answers the first of Grothendieck's questions cited in the epigraph and the question posed in [CTKPR, p. 4].

Corollary 1.4. *Let G be a connected semisimple algebraic group.*

(i) *If a section of π_G exists, then τ is bijective.*
(ii) *For* $\operatorname{char} k = 0$*, the following properties are equivalent:*
  (a) *there is a section of π_G;*
  (b) *there is a cross-section in G;*
  (c) *G is simply connected.*

Theorem 1.2 is proved in Section 2.

One can show (see below Lemma 3.1) that if a cross-section in G exists, then, at least for $\operatorname{char} k = 0$, the variety $G/\!\!/G$ is smooth (the converse is not true). The known criterion of smoothness of $G/\!\!/G$ (Theorem 3.2) may be interpreted as that of the existence of $\operatorname{rk} G$ generators of $k[G]^G$. In Section 3 we consider the general case and describe a minimal generating set of $k[G]^G$ and singularities of $G/\!\!/G$ for any G. This is based on the property that actually $G/\!\!/G$ is a toric variety of a maximal torus T of G. In particular, it also implies the affirmative answer to the last of Grothendieck's questions cited in the epigraph:

Corollary 3.9. *$G/\!\!/G$ and T/W are rational varieties.*

Here $W = N_G(T)/T$, where $N_G(T)$ is the normilizer of T in G, is the Weyl group of G. It acts on T via conjugation.

Parallel to this we describe a minimal generating set of the representation ring $R(G)$ of G. Note that finding generators of $R(G)$ attracted people's attention for a long time, in particular, because of the bearing on the K-theory (cf., e.g., [Hus, Chap. 13] where the generators of $R(G)$ are found for some classical G's utilizing the ad hoc bulky arguments; see also [Ada]). Singularities of $G/\!\!/G$ attracted people's attention as well (see [Slo, Sects. 3.15, 4.5]).

The precise formulations of these results are given below in Theorems 3.5 and 3.12 and Lemma 3.14.

Constructing generating sets of $k[G]^G$ is the topic of two further questions of Grothendieck asked in [GS, p. 241]. In Section 4 we answer the first question in the negative and the second in the positive.

In Section 5 we consider rational (i.e., local) sections of π_G and *rational cross-sections* in G, i.e., irreducible closed subsets S of G that intersect at a single point every fiber of π_G over a point of a dense open subset of $G/\!\!/G$ (depending on S). The closure of the image of a rational section of π_G is an example of such S; moreover, this S has the property that $\pi_G|_S\colon S \to G/\!\!/G$ is a birational isomorphism. For $\operatorname{char} k = 0$, every rational cross-section in G is obtained in this way.

First, we show that the existence of a rational section of π_G is equivalent to another property. Namely, note that W also acts on G/T as follows:

$$w \cdot gT := g\dot{w}^{-1}T, \tag{2}$$

where $\dot{w} \in N_G(T)$ is a representative of w. We prove:

Theorem 1.5. *Let G be a connected semisimple algebraic group. The following properties are equivalent:*

(i) *there is a rational section of π_G;*
(ii) *there is a W-equivariant rational map $T \dashrightarrow G/T$.*

Then we consider the existence problem and prove the following.

First, the next theorem answers the third of Grothendieck's questions cited in the epigraph.

Theorem 1.6. *For every connected semisimple algebraic group G, there is a rational section of π_G.*

For $\operatorname{char} k = 0$, this theorem has been proved earlier in [CTKPR, Theorem 0.3]. In our proof we use the relevant characteristic free results from [CTKPR], but bypass Theorem 2.12 from this paper (whose proof is based on the assumption $\operatorname{char} k = 0$) by exploring properties of π_G and proving that versality of G holds in arbitrary characteristic (Lemma 5.8); this permits us to use Steinberg's section of $\pi_{\widehat{G}}$ in place of Kostant's cross-section in $\operatorname{Lie} G$ used in [CTKPR].

Corollary 1.7. *In every connected semisimple algebraic group G there is a rational cross-section S such that $\pi_G|_S\colon S \to G/\!\!/G$ is a birational isomorphism.*

Second, Theorems 1.5 and 1.6 yield the following:

Theorem 1.8. *For every connected semisimple algebraic group G, there is a W-equivariant rational map $T \dashrightarrow G/T$.*

Section 6 contains some remarks, questions, and an example of a cross-section S in G such that $\pi_G|_S$ is not separable (hence S is not the image of a section of π_G).

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2. Cross-sections in G

Given a torus S, below we denote by $\mathrm{X}(S)$ the character lattice of S in additive notation. To avoid confusion between the additions in $\mathrm{X}(S)$ and $k[S]$, an element

$\lambda \in \mathrm{X}(S)$ considered as that of $k[S]$ is denoted by χ^λ. The value of χ^λ at $s \in S$ is denoted by s^λ.

Fix a choice of Borel subgroup $\widehat{B}$ of $\widehat{G}$ and maximal torus $\widehat{T} \subset \widehat{B}$. Let

$$\varpi_1, \ldots, \varpi_r \in \mathrm{X}(\widehat{T})$$

be the system of fundamental weights of $\widehat{T}$ regarding $\widehat{B}$.

Let $\varrho_i \colon \widehat{G} \to \mathbf{GL}(V_i)$ be an irreducible representation of $\widehat{G}$ with ϖ_i as the highest weight. Let $\mathrm{ch}_{\varpi_i} \in k[\widehat{G}]^{\widehat{G}}$ be the character of ϱ_i.

Let $\widehat{C}$ be the center of $\widehat{G}$; it is a finite subgroup of $\widehat{T}$. The conjugating action of $\widehat{G}$ on itself commutes with the action of $\widehat{C}$ on $\widehat{G}$ by left translations. Therefore the latter action descends to $\widehat{G}/\!\!/\widehat{G}$ and

$$\pi_{\widehat{G}} \colon \widehat{G} \longrightarrow \widehat{G}/\!\!/\widehat{G}$$

becomes a $\widehat{C}$-equivariant morphism.

Endow the r-dimensional affine space $\mathbf{A}^r$ with the linear action of $\widehat{T}$ by the formula

$$t \cdot (a_1, \ldots, a_r) := (t^{\varpi_1} a_1, \ldots, t^{\varpi_r} a_r), \qquad t \in \widehat{T}, \ (a_1, \ldots, a_r) \in \mathbf{A}^r. \tag{3}$$

Lemma 2.1.

(i) *The $\widehat{T}$-stabilizer of the point $(1, \ldots, 1) \in \mathbf{A}^r$ is trivial. In particular, the considered action of $\widehat{T}$ on $\mathbf{A}^r$ is faithful.*

(ii) *There is a $\widehat{C}$-equivariant isomorphism*

$$\lambda \colon \widehat{G}/\!\!/\widehat{G} \xrightarrow{\simeq} \mathbf{A}^r.$$

Proof. Since $\varpi_1, \ldots, \varpi_r$ generate $\mathrm{X}(\widehat{T})$, we have

$$\bigcap_{i=1}^{r} \{t \in T \mid t^{\varpi_i} = 1\} = \{e\}. \tag{4}$$

But (3) entails that the $\widehat{T}$-stabilizer of the point $(1, \ldots, 1)$ coincides with the left-hand side of equality (4). Hence (i) follows from this equality.

By [Ste$_1$, Theorems 6.1, 6.16] the k-algebra $k[\widehat{G}]^{\widehat{G}}$ is freely generated by $\mathrm{ch}_{\varpi_1}, \ldots$ $\ldots, \mathrm{ch}_{\varpi_r}$ and the morphism

$$\theta \colon \widehat{G} \longrightarrow \mathbf{A}^r, \qquad \theta(g) = (\mathrm{ch}_{\varpi_1}(g), \ldots, \mathrm{ch}_{\varpi_r}(g)),$$

is surjective. Hence there is an isomorphism $\lambda \colon \widehat{G}/\!\!/\widehat{G} \longrightarrow \mathbf{A}^r$ such that the following diagram is commutative:

$$\begin{array}{ccc} & \widehat{G} & \\ {}^{\pi_{\widehat{G}}}\swarrow & & \searrow^{\theta} \\ \widehat{G}/\!\!/\widehat{G} & \xrightarrow{\lambda} & \mathbf{A}^r \end{array} \quad . \tag{5}$$

The morphism θ is $\widehat{C}$-equivariant. Indeed, let $c \in \widehat{C}$. Since ϱ_i is irreducible, SCHUR's lemma entails that $\varrho_i(c) = \mu_{i,c}\,\mathrm{id}_{V_i}$ for some $\mu_{i,c} \in k$. On the other hand, since $c \in \widehat{T}$, any highest vector in V_i regarding $\widehat{B}$ is an eigenvector of c with the eigenvalue c^{ϖ_i}. Hence $\mu_{i,c} = c^{\varpi_i}$. Therefore, for every $g \in \widehat{G}$, by (3) we have

$$\begin{aligned}\theta(cg) &= \big(\mathrm{ch}_{\varpi_1}(cg), \ldots, \mathrm{ch}_{\varpi_r}(cg)\big)\\ &= \big(\mathrm{trace}\,\big(\varrho_1(cg)\big), \ldots, \mathrm{trace}\,\big(\varrho_r(cg)\big)\big)\\ &= \big(\mathrm{trace}\,\big(\varrho_1(c)\varrho_1(g)\big), \ldots, \mathrm{trace}\,\big(\varrho_1(c)\varrho_r(g)\big)\big)\\ &= \big(\mathrm{trace}\,\big(c^{\varpi_1}\varrho_1(g)\big), \ldots, \mathrm{trace}\,\big(c^{\varpi_r}\varrho_r(g)\big)\big)\\ &= \big(c^{\varpi_1}\mathrm{trace}\,\big(\varrho_1(g)\big), \ldots, c^{\varpi_r}\mathrm{trace}\,\big(\varrho_r(g)\big)\big)\\ &= \big(c^{\varpi_1}\mathrm{ch}_{\varpi_1}(g), \ldots, c^{\varpi_r}\mathrm{ch}_{\varpi_r}(g)\big)\\ &= c \cdot \theta(g),\end{aligned}$$

as claimed.

Since both θ and $\pi_{\widehat{G}}$ are $\widehat{C}$-equivariant and $\pi_{\widehat{G}}$ is surjective, commutativity of diagram (5) entails that λ is $\widehat{C}$-equivariant as well. This proves (ii). □

Corollary 2.2. *Let g be a nonidentity element of $\widehat{C}$. Then there is no g-stable cross-section in $\widehat{G}$.*

Proof. Assume the contrary and let $\widehat{S}$ be a g-stable cross-section in $\widehat{G}$. Since $\pi_{\widehat{G}}$ is $\widehat{C}$-equivariant, $\pi_{\widehat{G}}|_{\widehat{S}}\colon \widehat{S} \to \widehat{G}/\!\!/\widehat{G}$ is a bijective g-equivariant morphism. As, by Lemma 2.1(ii), there is a point of $\widehat{G}/\!\!/\widehat{G}$ fixed by $\widehat{C}$, hence by g, this implies that there is a point of $\widehat{S}$ fixed by g. But for the action of $\widehat{C}$ on $\widehat{G}$ by left translations, the stabilizer of every point is trivial, a contradiction with $g \neq e$. □

Given an element h of an algebraic group H, we shall denote its conjugacy class in H by $H(h)$:

$$H(h) := \{shs^{-1} \mid s \in H\}. \tag{6}$$

Lemma 2.3. *Let H and $\widetilde{H}$ be connected algebraic groups and let $\sigma\colon \widetilde{H} \to H$ be an isogeny. Then the following properties hold:*

(i) *σ is a finite morphism;*

(ii) *$\sigma\big(\widetilde{H}(h)\big) = H\big(\sigma(h)\big)$ and $\dim \widetilde{H}(h) = \dim H\big(\sigma(h)\big)$ for every $h \in \widetilde{H}$;*

(iii) *if $\widetilde{H}(h)$ is a regular conjugacy class in $\widetilde{H}$ (i.e., that of the maximal dimension), then $\sigma\big(\widetilde{H}(h)\big)$ is a regular conjugacy class in H;*

(iv) *if H and $\widetilde{H}$ are semisimple, then for every $h \in \widetilde{H}$,*

$$\sigma\big(\pi_{\widetilde{H}}^{-1}\big(\pi_{\widetilde{H}}(h)\big)\big) = \pi_H^{-1}\big(\pi_H\big(\sigma(h)\big)\big).$$

Proof. The varieties H and $\widetilde{H}$ are normal (even smooth) and the fiber of σ over every point of H is a finite set whose cardinality does not depend on this point. Hence (cf. [Gro_1, Sect. 2, Cor. 3]) $\widetilde{H}$ is the normalization of H in the field of rational functions on $\widetilde{H}$ and σ is the normalization map. This proves (i).

The first equality in (ii) holds as σ is an epimorphism of groups. The second follows from the first and the theorem on dimension of fibers [Bor, AG 10.1]. This proves (ii).

As σ is surjective, (iii) follows from (ii).

Since the fibers of $\pi_{\widetilde{H}}$ and π_H are the closures of regular conjugacy classes and, by (i), the map σ is closed, (iv) follows from (iii). □

Corollary 2.4. *Let $\widetilde{G}$ be a connected semisimple algebraic group and let $\sigma\colon \widetilde{G} \to G$ be a bijective isogeny.*

(i) *If $\widetilde{S}$ is a cross-section in $\widetilde{G}$, then $\sigma(\widetilde{S})$ is a cross-section in G.*
(ii) *If S is a cross-section in G, then $\sigma^{-1}(S)$ is a cross-section in $\widetilde{G}$.*

The same holds if "cross-section" is replaced with "rational cross-section".

Proof. By Lemma 2.3(i) the bijective map σ is closed. Hence it is a homeomorphism. Both claims follow from this, the definitions of cross-section and rational cross-section, and Lemma 2.3(iv). □

Lemma 2.5. *Assume that there is a subgroup Z of $\widehat{C}$ such that $G = \widehat{G}/Z$ and τ is the quotient morphism $\widehat{G} \to \widehat{G}/Z$. Then there is a morphism*

$$\varphi\colon \widehat{G}/\!\!/\widehat{G} \longrightarrow G/\!\!/G \tag{7}$$

such that

(i) *$(G/\!\!/G, \varphi)$ is a categorical quotient for the action of Z on $\widehat{G}/\!\!/\widehat{G}$;*
(ii) *the following diagram is commutative:*

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\ \tau\ } & G \\ {\scriptstyle \pi_{\widehat{G}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_G} \\ \widehat{G}/\!\!/\widehat{G} & \xrightarrow{\ \varphi\ } & G/\!\!/G \end{array} ; \tag{8}$$

(iii) *for every point $x \in \widehat{G}/\!\!/\widehat{G}$, the following equality holds:*

$$\tau\big(\pi_{\widehat{G}}^{-1}(x)\big) = \pi_G^{-1}\big(\varphi(x)\big). \tag{9}$$

Proof. As τ^*, $\pi_{\widehat{G}}^*$, and π_G^* are injections, there is a unique morphism (7) such that $\tau^* \circ \pi_G^* = \pi_{\widehat{G}}^* \circ \varphi^*$, i.e., diagram (8) is commutative.

Consider the action of $\widehat{G}$ on G via the isogeny τ and the conjugating action of G on itself. The isogeny τ is then $\widehat{G}$-equivariant and $\widehat{G}$-orbits in G are G-conjugacy classes, so we have $k[G]^G = k[G]^{\widehat{G}}$. Since the conjugating action of $\widehat{G}$ on itself commutes with the action of Z by left translations, we have

$$\begin{aligned} \pi_{\widehat{G}}^*\big(\varphi^*(k[G/\!\!/G])\big) &= \tau^*\big(\pi_G^*(k[G/\!\!/G])\big) = \tau^*\big(k[G]^G\big) = \tau^*\big(k[G]^{\widehat{G}}\big) = \big(\tau^*(k[G])\big)^{\widehat{G}} \\ &= \big(k[\widehat{G}]^Z\big)^{\widehat{G}} = \big(k[\widehat{G}]^{\widehat{G}}\big)^Z = \big(\pi_{\widehat{G}}^*(k[\widehat{G}/\!\!/\widehat{G}])\big)^Z = \pi_{\widehat{G}}^*\big(k[\widehat{G}/\!\!/\widehat{G}]^Z\big). \end{aligned}$$

Thus, $\varphi^*(k[G/\!\!/G]) = k[\widehat{G}/\!\!/\widehat{G}]^Z$. This proves (i) and (ii). Lemma 2.3(iv) and commutativity of diagram (8) imply (iii). □

Below, given a variety Y, we denote by $\mathrm{T}_{y,Y}$ the tangent space of Y at a point y.

Proof of Theorem 1.2. First, we shall prove criterion (i).

Step 1. By STEINBERG's theorem, $\widehat{G}$ has a cross-section. Hence, by Corollary 2.4, if τ is bijective, then there exists a cross-section in G as well.

So we may assume that τ is not bijective and we then have to prove that there is no cross-section in G. Solving this problem, we may assume that τ is separable. Indeed, if this is not the case, then by [Bor, Prop. 17.9] there exist a connected semisimple algebraic group $\widetilde{G}$ and a commutative diagram of isogenies

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\ \tau\ } & G \\ & {}_{\mu}\searrow \quad \nearrow_{\sigma} & \\ & \widetilde{G} & \end{array}, \tag{10}$$

where μ is separable and σ is purely inseparable. As σ is bijective, Corollary 2.4 then reduces the problem to proving that there is no cross-section in $\widetilde{G}$, i.e., we may replace G by $\widetilde{G}$ and τ by μ.

So from now on we may (and shall) assume that τ is a separable isogeny of degree $\geqslant 2$. This means that there is a nontrivial subgroup Z of $\widehat{C}$ such that $G = \widehat{G}/Z$ and τ is the quotient morphism $\widehat{G} \to \widehat{G}/Z$.

Step 2. Now, arguing on the contrary, assume that there is a cross-section S in G.

Claim 1.

(i) *For every point $x \in \widehat{G}/\!\!/\widehat{G}$, the intersection*

$$\pi_{\widehat{G}}^{-1}(x) \cap \tau^{-1}(S) \tag{11}$$

is a nonempty subset of a single Z-orbit; in particular, it is finite.

(ii) *There is a nonempty open subset U of $\widehat{G}/\!\!/\widehat{G}$ such that, for every $x \in U$, intersection* (11) *is a single point.*

Proof of Claim 1. Consider diagram (8). Since $S \cap \pi_G^{-1}\big(\varphi(x)\big)$ is a single point g, we deduce from (9) that intersection (11) is contained in $\tau^{-1}(g)$. This proves (i) as the fibers of τ are Z-orbits.

By Lemma 2.1(i) there is a nonempty open subset U in $\widehat{G}/\!\!/\widehat{G}$ such that the $\widehat{C}$-stabilizer of every point of U is trivial. Take a point $x \in U$. Assume that intersection (11) contains two points g_1 and $g_2 \neq g_1$. By (i) there exists an element $z \in Z$ such that $g_2 = zg_1$. As $\pi_{\widehat{G}}$ is $\widehat{C}$-equivariant, $x = \pi_{\widehat{G}}(g_2) = \pi_{\widehat{G}}(zg_1) = z \cdot \pi_{\widehat{G}}(g_1) = z \cdot x$. Thus, z belongs to the $\widehat{C}$-stabilizer of x. The definition of U then implies that $z = e$. Hence $g_1 = g_2$, a contradiction. This proves (ii). □

Step 3. Since all the fibers of τ are finite, every irreducible component of $\tau^{-1}(S)$ has dimension $\leqslant \dim S = r$ and at least one of them has dimension r.

Claim 2.

(i) *There is a unique r-dimensional irreducible component $\widehat{S}$ of $\tau^{-1}(S)$.*
(ii) $\tau(\widehat{S}) = S$.

Proof of Claim 2. Let $\widehat{S}$ be an r-dimensional irreducible component of $\tau^{-1}(S)$. Then $\tau(\widehat{S})$ contains an open subset of S. Since τ is closed, this proves (ii).

From (ii) we conclude that

$$\pi_G\big(\tau(\widehat{S})\big) = G/\!\!/G. \tag{12}$$

But by Lemma 2.5 the fibers of φ in commutative diagram (8) are finite. This and (12) imply that $\pi_{\widehat{G}}(\widehat{S})$ contains a nonempty open subset of $\widehat{G}/\!\!/\widehat{G}$.

Now let $\widehat{S}'$ be another r-dimensional irreducible component of $\tau^{-1}(S)$. Then, as above, $\pi_{\widehat{G}}(\widehat{S}')$ contains a nonempty open subset of $\widehat{G}/\!\!/\widehat{G}$ as well. Therefore, $\pi_{\widehat{G}}(\widehat{S}) \cap \pi_{\widehat{G}}(\widehat{S}')$ contains a nonempty open subset V of $\widehat{G}/\!\!/\widehat{G}$. We may assume that $V \subseteq U$ for U from Claim 1(ii). The latter then yields that $\pi_{\widehat{G}}^{-1}(V) \cap \widehat{S} = \pi_{\widehat{G}}^{-1}(V) \cap \widehat{S}'$. As both sides of this equality are open subsets of respectively $\widehat{S}$ and $\widehat{S}'$, we infer that $\widehat{S} = \widehat{S}'$. This proves (i). □

Step 4. As $\widehat{S}$ is a unique r-dimensional irreducible component of the Z-stable variety $\tau^{-1}(S)$, we conclude that $\widehat{S}$ is Z-stable. We shall now show that $\widehat{S}$ is a cross-section in $\widehat{G}$. As this property contradicts Corollary 2.2, the proof of (i) will be then completed.

Step 5. Let x be a point of $\widehat{G}/\!\!/\widehat{G}$. As S is a section of G, the intersection $S \cap \pi_G^{-1}\big(\varphi(x)\big)$ is a single point $g \in G$. By Claim 2(ii) there is a point $\widehat{g} \in \widehat{S}$ such that $\tau(\widehat{g}) = g$. Commutativity of diagram (8) then entails that x and $\widehat{x} := \pi_{\widehat{G}}(\widehat{g})$ are in the same fiber of φ. Since the fibers of φ are Z-orbits, there is an element $z \in Z$ such that $x = z \cdot \widehat{x}$. As $\pi_{\widehat{G}}$ is Z-equivariant, this yields the equality $\pi_{\widehat{G}}(z\widehat{g}) = x$. But $z\widehat{g} \in \widehat{S}$ as $\widehat{S}$ is Z-stable and $\widehat{g} \in \widehat{S}$. Hence $\pi_{\widehat{G}}^{-1}(x) \cap \widehat{S} \neq \varnothing$, i.e.,

$$\pi_{\widehat{G}}(\widehat{S}) = \widehat{G}/\!\!/\widehat{G}. \tag{13}$$

Step 6. It follows from Claims 1(i),(ii) and (13) that $\pi_{\widehat{G}}|_{\widehat{S}}$ is a surjective morphism with finite fibers, bijective over an open subset of $\widehat{G}/\!\!/\widehat{G}$. As $\widehat{G}$ is normal, $\widehat{G}/\!\!/\widehat{G}$ is normal as well. Let $\nu\colon \widetilde{S} \to \widehat{S}$ be the normalization. Then the surjective morphism $\pi_{\widehat{G}}|_{\widehat{S}} \circ \nu\colon \widetilde{S} \to \widehat{G}/\!\!/\widehat{G}$ of normal varieties has finite fibers and is bijective over an open subset of $\widehat{G}/\!\!/\widehat{G}$. Hence $\pi_{\widehat{G}}|_{\widehat{S}} \circ \nu$ is bijective (see [Gro$_1$, Sect. 2, Cor. 2]); whence $\pi_{\widehat{G}}|_{\widehat{S}}$ is bijective as well, i.e., $\widehat{S}$ is a cross-section in $\widehat{G}$. This completes the proof of (i).

We now turn to the proof of (ii).

Let $S := \sigma(G/\!\!/G)$. Take a point $x \in S$ and put $y := \pi_G(x)$. As $\pi_G|_S\colon S \to G/\!\!/G$ is an isomorphism (σ is its inverse), $d(\pi_G|_S)_x$ is an isomorphism as well. Hence $(d\pi_G)_x$ is surjective. As $\dim \mathrm{T}_{y,G/\!\!/G} \geqslant \dim G/\!\!/G = r$, this implies that there are functions $f_1, \dots, f_r \in k[G]^G$ such that $(df_1)_x, \dots, (df_r)_x$ are linearly independent. By [Ste$_1$, Theorem 8.7] this yields that x is regular. As S intersects every fiber of π_G at a single point and every such fiber contains a unique regular orbit, this proves (ii). Thus, the proof of Theorem 1.2 comes to a close. $\square$

3. Singularities of $G/\!\!/G$ and generators of $k[G]^G$ and $R(G)$

The following statement, whose role for us is solely heuristic, shows that there is a connection between the existence of a cross-section in G and the smoothness of $G/\!\!/G$.

Lemma 3.1. *Let* $\operatorname{char} k = 0$*. If a surjective morphism* $\alpha\colon X \to Y$ *of irreducible varieties admits a section* $\sigma\colon Y \to X$*, then smoothness of* X *implies smoothness of* Y*.*

Proof. Arguing on the contrary, assume that y is a singular point of Y, i.e.,

$$\dim \mathrm{T}_{y,Y} > \dim Y. \tag{14}$$

Put $x = \sigma(y) \in X$. Since $\alpha \circ \sigma = \mathrm{id}_Y$, the composition $d\alpha_x \circ d\sigma_y$ is the identity map of $\mathrm{T}_{y,Y}$. Hence $d\alpha_x$ is surjective, i.e., $\operatorname{rk} d\alpha_x = \dim \mathrm{T}_{y,Y}$. By (14) this yields the inequality

$$\operatorname{rk} d\alpha_x > \dim Y. \tag{15}$$

As $\operatorname{char} k = 0$, there is a dense open subset U of X such that $\operatorname{rk} d\alpha_z = \dim Y$ for every point $z \in U$; see [Har, 14.4]. As $z \mapsto \dim\ker d\alpha_z$ is the upper semi-continuous function [Har, 14.6], we conclude that the smoothness of X implies that $\operatorname{rk} d\alpha_z \leqslant \dim Y$ for every point $z \in X$. This contradicts (15). $\square$

This prompts us to explore the smoothness of $G/\!\!/G$. The answer is known:

Theorem 3.2 ([Ste$_3$, §3], [Rich$_1$, Prop. 4.1], [Rich$_2$, Prop.13.3], Remark 3.16 below)**.** *The following properties are equivalent*:

(i) *$G/\!\!/G$ is smooth*;
(ii) *$G/\!\!/G$ is isomorphic to the affine space $\mathbf{A}^r$*;
(iii) *$G = G_1 \times \cdots \times G_s$ where every G_i is either a simply connected simple algebraic group or isomorphic to $\mathbf{SO}_{n_i}$ for an odd n_i.*

This criterion of smoothness of $G/\!\!/G$ may be also interpreted as that of the existence of r generators of the algebra of class functions on G. Below we describe a minimal system of generators of this algebra and singularities of $G/\!\!/G$ in the general case. This also yields a minimal system of generators of the representation ring of G.

Let $B := \tau(\widehat{B})$ and $T := \tau(\widehat{T})$. They are respectively a Borel subgroup and a maximal torus of G. We naturally identify $\mathrm{X}(\widehat{T})$ with the lattice $\mathrm{X}(\widehat{T}) \otimes 1$ in $\mathrm{X}(\widehat{T}) \otimes_{\mathbf{Z}} \mathbf{R}$ and view $\mathrm{X}(T)$ as a sublattice of $\mathrm{X}(\widehat{T})$ identifying χ^μ with $\big(\tau|_{\widehat{T}}\big)^*(\chi^\mu)$ for $\mu \in \mathrm{X}(T)$.

Let $N_{\widehat{G}}(\widehat{T})$ be the normalizer of $\widehat{T}$ in $\widehat{G}$. As τ is a central isogeny, the Weyl group W of T is naturally identified with $N_{\widehat{G}}(\widehat{T})/\widehat{T}$; see [Bor, Prop. 11.20]. As W is finite, a categorical quotient for the conjugating action of W on T is, in fact, geometric, so we denote the corresponding quotient variety by T/W. Let

$$\pi_{W,T}\colon T \to T/W \tag{16}$$

be the corresponding quotient morphism.

The root system Φ of $\widehat{G}$ regarding $\widehat{T}$ (respectively, its positive part Φ_+ regarding $\widehat{B}$) coincides with that of G regarding T (respectively, its positive part regarding B) [Bor, Theorem 22.6(iii)]. Let

$$\alpha_1, \ldots, \alpha_r \in \Phi_+$$

be the set of all simple roots. The weight lattice of Φ is $\mathrm{X}(\widehat{T})$.

The monoid of highest weights of simple $\widehat{G}$-modules (regarding $\widehat{T}$ and $\widehat{B}$) is

$$\widehat{\mathcal{D}} := \mathbf{N}\varpi_1 + \cdots + \mathbf{N}\varpi_r, \quad \text{where } \mathbf{N} = \{0, 1, 2, \ldots\}. \tag{17}$$

and that of simple G-modules (regarding T and B) is

$$\mathcal{D} := \widehat{\mathcal{D}} \cap \mathrm{X}(T). \tag{18}$$

If $\varpi \in \mathcal{D}$ and $E(\varpi)$ is a simple G-module with ϖ as the highest weight, we denote by $\mathrm{ch}_{\varpi} \in k[G]^G$ the character of $E(\varpi)$.

Given a commutative monoid M, we denote by $\mathbf{Z}[M]$ the semigroup ring of M over $\mathbf{Z}$. If S is a submonoid of the multiplicative monoid of $\mathbf{Z}[M]$ whose elements are linearly independent over $\mathbf{Z}$, then the subring of $\mathbf{Z}[M]$ generated by S is naturally identified with $\mathbf{Z}[S]$. In particular, we consider $\mathbf{Z}[\mathrm{X}(T)]$ and $\mathbf{Z}[\mathcal{D}]$ as the subrings of $\mathbf{Z}[\mathrm{X}(\widehat{T})]$. The former is stable with respect to the natural action of W on $\mathbf{Z}[\mathrm{X}(\widehat{T})]$. Following the notation and terminology of [Bou_2, VI.3.1], we denote by e^{μ} the element of $\mathbf{Z}[\mathrm{X}(\widehat{T})]$ corresponding to $\mu \in \mathrm{X}(\widehat{T})$ and, for any element

$$x = \sum_{\mu \in \mathrm{X}(\widehat{T})} a_{\mu} e^{\mu} \in \mathbf{Z}[\mathrm{X}(\widehat{T})], \qquad a_{\mu} \in \mathbf{Z}, \tag{19}$$

call $\{\mu \in \mathrm{X}(\widehat{T}) \mid a_{\mu} \neq 0\}$ the *support* of x. The nonzero summands $a_{\mu}e^{\mu}$ in (19) are called the *terms* of x.

Given an algebraic group H, we denote by $R(H)$ the *representation ring* of H: its additive group is the Grothendieck group of the category of finite dimensional algebraic H-modules with respect to exact sequences and the multiplication is induced by the tensor product of modules. Using τ, we identify $R(G)$ in the natural way with the subring of $R(\widehat{G})$.

If E is a finite dimensional algebraic G-module and E_μ is its weight space of a weight $\mu \in \mathrm{X}(T)$, then the formal character of E,

$$\mathrm{ch}_G[E] := \sum_{\mu \in \mathrm{X}(T)} (\dim E_\mu) e^\mu, \tag{20}$$

is an element of $\mathbf{Z}[\mathrm{X}(T)]^W$ depending only on the class $[E]$ of E in $R(G)$. Clearly,

$$\mathrm{ch}_G[E \otimes E'] = \mathrm{ch}_G[E]\,\mathrm{ch}_G[E']. \tag{21}$$

According to [Ser_1, 3.6], the homomorphism of $\mathbf{Z}$-modules

$$\mathrm{ch}_G\colon R(G) \longrightarrow \mathbf{Z}[\mathrm{X}(T)]^W, \qquad [E] \mapsto \mathrm{ch}_G[E], \tag{22}$$

is an isomorphism. By (21) it is an isomorphism of rings.

Next, we fix on $\mathrm{X}(\widehat{T}) \otimes_{\mathbf{Z}} \mathbf{R}$ the following partial order $\geqslant$:

$$\mu \geqslant \nu \iff \mu - \nu \in \mathbf{N}\alpha_1 + \cdots + \mathbf{N}\alpha_r; \tag{23}$$

cf. [Hum_1, 10.1], [Hum_2, 31.2]. If μ is a maximal (with respect to $\geqslant$) element of the support of element (19), then $a_\mu e^\mu$ is called the *maximal term* of x; cf. [Bou_2, VI.3.2].

Definition 3.3. Let $\varpi \in \widehat{\mathcal{D}}$. We say that an element $x \in \mathbf{Z}[\mathrm{X}(\widehat{T})]^W$ is *ϖ-sharp* if the following property (M) holds:

(M) e^ϖ is the unique maximal term of x.

Example 3.4. The elements $\mathrm{ch}_{\widehat{G}}[E(\varpi)]$ and

$$S(e^\varpi) := \sum_{\mu \in W\cdot\varpi} e^\mu \tag{24}$$

are ϖ-sharp (this follows, e.g., from [Hum_2, 31.2, 31.3]; cf. also [Bou_2, VI.3.4]). $\square$

By (23) property (M) implies that the support of a ϖ-sharp element x lies in $\varpi + \mathrm{X}(T)$. This and [Bou_2, VI.3.4, formula (6)] yield the equality

$$x = S(e^\varpi) + \text{sum of some of the elements } \pm S(e^{\varpi'}) \text{ with } \varpi' \in \widehat{\mathcal{D}},\ \varpi' < \varpi. \tag{25}$$

By [Bou_2, VI.3.2, Lemma 2] if x is ϖ-sharp and x' is ϖ'-sharp, then xx' is $(\varpi+\varpi')$-sharp.

Now, fix a ϖ_i-sharp element $x_{\varpi_i} \in \mathbf{Z}[\mathrm{X}(\widehat{T})]^W$, $i = 1, \ldots, r$, and put

$$x_\varpi := x_{\varpi_1}^{m_1} \cdots x_{\varpi_r}^{m_r} \qquad \text{for} \qquad \varpi = m_1\varpi_1 + \cdots + m_r\varpi_r \in \widehat{\mathcal{D}}.$$

By [Bou$_2$, VI.3.4, Theorem 1] the set $\{x_\varpi \mid \varpi \in \widehat{\mathcal{D}}\}$ is then a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(\widehat{T})]^W$. As $\{e^\mu \mid \mu \in \mathrm{X}(T)\}$ is a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(T)]$ and the support of x_ϖ lies in $\varpi + \mathrm{X}(T)$, we deduce from this and (18) that $\{x_\varpi \mid \varpi \in \mathcal{D}\}$ is a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(T)]^W$. Hence the homomorphism of the $\mathbf{Z}$-modules

$$\vartheta\colon \mathbf{Z}[\mathrm{X}(T)]^W \longrightarrow \mathbf{Z}[\mathcal{D}], \qquad \vartheta(x_\varpi) = e^\varpi \quad \text{for} \quad \varpi \in \mathcal{D}, \tag{26}$$

is an isomorphism. Since $x_{\varpi+\varpi'} = x_\varpi x_{\varpi'}$, it is, in fact, an isomorphism of rings.

As $\{\chi^\mu \mid \mu \in \mathrm{X}(T)\}$ is a k-basis of $k[T]$ (cf. [Spr, 3.2.3]) and $\chi^\mu\chi^\nu = \chi^{\mu+\nu}$, the k-linear map

$$k[T] \longrightarrow k \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{X}(T)], \qquad \chi^\mu \mapsto 1 \otimes e^\mu, \tag{27}$$

is an isomorphism of k-algebras. As this isomorphism is W-equivariant, its restriction to $k[T]^W$ is an isomorphism of k-algebras

$$\eta\colon k[T]^W \longrightarrow \big(k \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{X}(T)]\big)^W = k \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{X}(T)]^W \tag{28}$$

(regarding the latter equality, see, e.g., [Bou$_2$, VI.3.4] or [Lor$_2$, Prop. 3.3.1]).

Finally, take into account that by [Ste$_1$, 6.4] the map

$$\mathrm{res}\colon k[G]^G \longrightarrow k[T]^W, \qquad f \mapsto f|_T, \tag{29}$$

is an isomorphism of k-algebras.

Summing up, we obtain the following

Theorem 3.5.

(i) *All the maps in the diagram*

$$k[G/\!\!/G] \xrightarrow{\pi_G^*} k[G]^G \xrightarrow{\mathrm{res}} k[T]^W \xrightarrow{\eta} k \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{X}(T)]^W \xrightarrow{\mathrm{id}\otimes\vartheta} k \otimes_{\mathbf{Z}} \mathbf{Z}[\mathcal{D}]$$

(*see* (1), (29), (28), (26)) *are isomorphisms of k-algebras.*

(ii) *Let F be a subring of k. Then the image of $F \otimes_{\mathbf{Z}} R(G)$ in $k[G/\!\!/G]$ under the composition of the following ring isomorphisms*

$$k \otimes_{\mathbf{Z}} R(G) \xrightarrow{\mathrm{id}\otimes\mathrm{ch}_G} k \otimes_{\mathbf{Z}} \mathbf{Z}[\mathrm{X}(T)]^W \xrightarrow{\eta^{-1}} k[T]^W \xrightarrow{\mathrm{res}^{-1}} k[G]^G \xrightarrow{(\pi_G^*)^{-1}} k[G/\!\!/G] \tag{30}$$

(*see* (22), (28), (29), (1)) *is an F-form of $k[G/\!\!/G]$. In particular, if* $\operatorname{char} k = 0$*, then $R(G)$ is a $\mathbf{Z}$-form of $k[G/\!\!/G]$.*

Recall (see, e.g., [Ful]) that there is a bijection between T-isomorphism classes of affine toric varieties of T and r-dimensional rational convex polyhedral cones in $\mathrm{X}(T) \otimes_{\mathbf{Z}} \mathbf{R}$ (i.e., convex cones with apex at the origin generated by a finite number of elements of $\mathrm{X}(T)$): it juxtaposes to a cone $\mathcal{C}$ the affine variety $Y_{\mathcal{C}}$

such that $k[Y_{\mathcal{C}}] := k \otimes_{\mathbf{Z}} \mathbf{Z}[\mathcal{C} \cap \mathrm{X}(T)]$ endowed with the natural action of T. As $w \cdot \mathrm{X}(T) = \mathrm{X}(T)$ for any element $w \in W$, the varieties $Y_{\mathcal{C}}$ and $Y_{w\cdot\mathcal{C}}$ are isomorphic (but not T-isomorphic if $w \neq e$). As W acts transitively on the set of Weyl chambers of Φ, Theorem 3.5(i) and formula (18) then yield the following:

Corollary 3.6. *$G/\!\!/G$ and T/W are isomorphic to the (underlying variety of) affine toric variety $Y_{\mathcal{C}}$, where $\mathcal{C}$ is any Weyl chamber of the root system Φ.*

Corollary 3.7. *There are actions of T on $G/\!\!/G$ and T/W with dense open orbits.*

Remark 3.8. This action of T on T/W cannot be lifted to T making quotient morphism (16) equivariant, see below Subsection 6.C.

Since toric varieties are rational, Corollary 3.6 yields

Corollary 3.9. *$G/\!\!/G$ and T/W are rational varieties.*

Remark 3.10. The fact that "multiplicative invariants" of finite reflection groups are semigroup algebras is already in the literature, first implicitly, then explicitly; see the historical account in [Lor_1, Introduction]. Essentially, the main ingredients date back to [Ste_1, §6] and [Bou_2,VI, §3].

In the next statement Theorem 3.5 is applied to finding a minimal system of generators of the algebra $k[G]^G$ and that of the ring $R(G)$.

Let $\mathcal{H}$ be the Hilbert basis of the monoid $\mathcal{D}$, i.e., the set of all its indecomposable elements:

$$\mathcal{H} = \mathcal{D}_+ \setminus 2\mathcal{D}_+ \quad \text{where} \quad \mathcal{D}_+ := \mathcal{D} \setminus \{0\}, \quad 2\mathcal{D}_+ := \mathcal{D}_+ + \mathcal{D}_+. \tag{31}$$

The set $\mathcal{H}$ is finite, generates $\mathcal{D}$, and every generating set of $\mathcal{D}$ contains $\mathcal{H}$ (see, e.g., [Lor_2, 3.4]).

Remark 3.11. There is an algorithm for efficient computing $\mathcal{H}$, see [Stu, 13.2] (see also Example 3.15 below).

Theorem 3.12.

(i) *The cardinality of every generating set of the algebra $k[G]^G$ of class functions on G is not less than the cardinality of $\mathcal{H}$. The same holds for every generating set of the representation ring $R(G)$ of G.*
(ii) *$\{[E(\varpi)] \mid \varpi \in \mathcal{H}\}$ is a generating set of the ring $R(G)$.*
(iii) *$\{\mathrm{ch}_{\varpi} \mid \varpi \in \mathcal{H}\}$ is a generating set of the algebra $k[G]^G$.*

Proof. (i) Let Y be the affine toric variety of T with $k[Y] = k \otimes_{\mathbf{Z}} \mathbf{Z}[\mathcal{D}]$. The linear span I of $\{1 \otimes e^{\varpi} \mid \varpi \in \mathcal{D}_+\}$ over k is a maximal T-invariant ideal in $k[Y]$. Hence I/I^2 is the cotangent space of Y at the T-fixed point y where I vanishes. As I^2 is the linear span of $\{1 \otimes e^{\varpi} \mid \varpi \in 2\mathcal{D}_+\}$ over k, this and (31) yield the equalities

$$\dim \mathrm{T}_{y,Y} = \dim I/I^2 = |\mathcal{H}|. \tag{32}$$

Now take into account that, given an affine algebraic variety X, the algebra $k[X]$ can be generated by d elements if and only if X admits a closed embedding in $\mathbf{A}^d$. Hence $d \geqslant \dim \mathrm{T}_{x,X}$ for every point $x \in X$. This, Theorem 3.5, and (32) prove (i).

(ii) Let $\mu \in \mathcal{D}$. As $\mathcal{H}$ generates $\mathcal{D}$, there is a decomposition

$$\mu = \sum_{\varpi \in \mathcal{H}} a_\varpi \varpi, \qquad \text{where} \quad a_\varpi \in \mathbf{N}.$$

Hence, by Example 3.4,

$$M^\mu := \prod_{\varpi \in \mathcal{H}} \big(\mathrm{ch}_G[E(\varpi)]\big)^{a_\varpi} \tag{33}$$

is a μ-sharp element of $\mathbf{Z}[\mathrm{X}(T)]^W$. By (25) we have

$$M^\mu = S(e^\mu) + \text{sum of some of the elements } \pm S(e^{\mu'}) \text{ with } \mu' \in \mathcal{D},\ \mu' < \mu. \tag{34}$$

But $\{S(e^\mu) \mid \mu \in \mathcal{D}\}$ is a basis of the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(T)]^W$ (see [Bou$_2$, VI.3.4, Lemma 3]). By [Bou$_2$, VI.3.4, Lemma 4] we then deduce from (34) that the set $\{M^\mu \mid \mu \in \mathcal{D}\}$ generates the $\mathbf{Z}$-module $\mathbf{Z}[\mathrm{X}(T)]^W$. This and (33) imply that the ring $\mathbf{Z}[\mathrm{X}(T)]^W$ is generated by the set $\{\mathrm{ch}_G[E(\varpi)] \mid \varpi \in \mathcal{H}\}$. As (22) is an isomorphism of rings, this proves (ii).

(iii) It follows from (ii) that the set $\{1 \otimes [E(\varpi)] \mid \varpi \in \mathcal{H}\}$ generates the ring $k \otimes_{\mathbf{Z}} R(G)$. But formula (20) shows that ch_ϖ is the image of $1 \otimes E[(\varpi)]$ under the isomorphism $\mathrm{res}^{-1} \circ \eta^{-1} \circ (\mathrm{id} \otimes \mathrm{ch}_G)$ (see diagram (30)). This proves (iii). □

The proof of Theorem 3.12 and formula (32) yield the following:

Corollary 3.13. *The maximum of the function* $x \mapsto \dim \mathrm{T}_{x,G/\!\!/G}$ *is equal to* $|\mathcal{H}|$.

In line with the general theory of toric varieties, as the Weyl chambers are simplicial cones, Corollary 3.6 implies, at least for $\mathrm{char}\, k = 0$, that $G/\!\!/G$ and T/W are isomorphic to the quotient of $\mathbf{A}^r$ by a linear action of a certain finite abelian group and hence, in particular, $G/\!\!/G$ and T/W may have only finite quotient singularities [Oda, Prop. 1.25]. Below, for arbitrary $\mathrm{char}\, k$ and separable τ, we prove the existence of such a finite group and its action on $\mathbf{A}^r$ by means of their explicit description. This yields an explicit description of singularities of $G/\!\!/G$ and that of the minimal generating sets of the algebra of class functions on G and of the representation ring of G.

The assumption that τ is separable means that there is a subgroup Z of $\widehat{C}$ such that $G = \widehat{G}/Z$ and τ is the quotient morphism $\widehat{G} \to \widehat{G}/Z$. In this situation we have

$$\mathrm{X}(T) = \{\mu \in \mathrm{X}(\widehat{T}) \mid c^\mu = 1 \text{ for every } c \in Z\}. \tag{35}$$

For the action of $\widehat{T}$ on $\mathbf{A}^r$ defined by formula (3), consider the $\widehat{T}$-orbit map of the point $(1, \dots, 1) \in \mathbf{A}^r$:

$$\iota\colon \widehat{T} \longrightarrow \mathbf{A}^r, \qquad \iota(t) = t \cdot (1, \dots, 1), \tag{36}$$

and identify $k[\widehat{T}]$ with $k\otimes_{\mathbf{Z}}\mathbf{Z}[\mathrm{X}(\widehat{T})]$ by means of the isomorphism

$$k[\widehat{T}]\longrightarrow k\otimes_{\mathbf{Z}}\mathbf{Z}[\mathrm{X}(\widehat{T})],\qquad \chi^{\mu}\mapsto 1\otimes e^{\mu}.$$

The map $\iota^*\colon k[\mathbf{A}^r]\to k\otimes_{\mathbf{Z}}\mathbf{Z}[\mathrm{X}(\widehat{T})]$ is an embedding as ι is dominant by Lemma 2.1(i). Let

$$y_1,\dots,y_r$$

be the standard coordinate functions on $\mathbf{A}^r$. Then (3) and (36) yield the equality

$$\iota^*(y_i)=1\otimes e^{\varpi_i}. \tag{37}$$

From (3) we deduce that $k[\mathbf{A}^r]^Z$ is the linear span over k of all the monomials $y^{m_1}\cdots y^{m_r}$ with $m_1,\dots,m_r\in\mathbf{N}$ such that $c^{m_1\varpi_1+\cdots+m_r\varpi_r}=1$ for every $c\in Z$. By (35) the latter condition is equivalent to the inclusion $m_1\varpi_1+\cdots+m_r\varpi_r\in\mathrm{X}(T)$. This, (37), (17), and (18) imply the equality

$$\iota^*\big(k[\mathbf{A}^r]^Z\big)=k\otimes_{\mathbf{Z}}\mathbf{Z}[\mathcal{D}].$$

Since Z is finite, a categorical quotient for the action of Z on $\mathbf{A}^r$ is geometric and so we denote the corresponding quotient variety by $\mathbf{A}^r/Z$. Thus, taking into account Theorem 3.5, we obtain the following isomorphisms of k-algebras:

$$\begin{array}{ccccccccc} & & & & k[T/W] & & & & \\ & & & & \big\downarrow{\scriptstyle \pi^*_{W,T}} & & & & \\ k[\mathbf{A}^r]^Z & \xrightarrow{\ \iota^*\ } & k\otimes_{\mathbf{Z}}\mathbf{Z}[\mathcal{D}] & \xleftarrow{(\mathrm{id}\otimes\vartheta)\circ\eta} & k[T]^W & \xleftarrow{\ \mathrm{res}\ } & k[G]^G & \xleftarrow{\ \pi^*_G\ } & k[G/\!\!/G]. \end{array}$$

They, in turn, induce the following isomorphisms of varieties

$$G/\!\!/G\xrightarrow{\simeq}T/W\xrightarrow{\simeq}\mathbf{A}^r/Z.$$

By means of a special parametrization of $\widehat{T}$ one can obtain an explicit description of the elements of $\widehat{C}$ well adapted for computing $k[\mathbf{A}^r]^Z$. Since $\widehat{G}=\widehat{G}_1\times\cdots\times\widehat{G}_s$ and $\widehat{C}=\widehat{C}_1\times\cdots\times\widehat{C}_s$ where every $\widehat{G}_i$ is a nontrivial normal simply connected simple subgroup of $\widehat{G}$ and $\widehat{C}_i$ is the center of $\widehat{G}_i$, it suffices to describe this parametrization for simple groups $\widehat{G}$. The answer is given below in Lemma 3.14.

Namely, let $\alpha_i^\vee\colon\mathbf{G}_m\to\widehat{T}$ be the coroot corresponding to α_i. Then, for every $s\in\mathbf{G}_m$, we have

$$\big(\alpha_i^\vee(s)\big)^{\varpi_j}=\begin{cases}s & \text{if } i=j,\\ 1 & \text{if } i\neq j.\end{cases} \tag{38}$$

If $\langle\ ,\ \rangle$ is the natural pairing between the lattices of characters and cocharacters of $\widehat{T}$, we put

$$n_{ij}:=\langle\alpha_i,\alpha_j^\vee\rangle.$$

So $(n_{ij})_{i,j=1}^r$ is the Cartan matrix of $\widehat{G}$.

By [Ste$_2$, Lemma 28(b),(d) and its Cor. (a)] the map

$$\nu\colon \mathbf{G}_m^r \longrightarrow \widehat{T}, \qquad \nu(s_1,\ldots,s_r)=\alpha_1^\vee(s_1)\cdots\alpha_r^\vee(s_r), \tag{39}$$

is an isomorphism of groups and

$$\widehat{C}=\{\alpha_1^\vee(s_1)\cdots\alpha_r^\vee(s_r)\mid s_1^{n_{i1}}\cdots s_r^{n_{ir}}=1 \text{ for every } i=1,\ldots,r\}. \tag{40}$$

By (38) and (39) we have

$$\big(\nu(s_1,\ldots,s_r)\big)^{\varpi_i}=\big(\alpha_1^\vee(s_1)\big)^{\varpi_i}\cdots\big(\alpha_r^\vee(s_r)\big)^{\varpi_i}=s_i.$$

This and (3) imply that, for every $s=(s_1,\ldots,s_r)\in\mathbf{G}_m^r$ and $(a_1,\ldots,a_r)\in\mathbf{A}^r$, the following equality holds:

$$\nu(s)\cdot(a_1,\ldots,a_r)=(s_1a_1,\ldots,s_ra_r). \tag{41}$$

Lemma 3.14. *For every simple simply connected group* $\widehat{G}$, *the subgroup* $\nu^{-1}(\widehat{C})$ *of the torus* $\mathbf{G}_m^r$ *is described in the following Table* 1 (*simple roots in* (39) *are numbered as in* [Bou$_2$]):

Table 1.

type of $\widehat{G}$	$\nu^{-1}(\widehat{C})$
A_r	$\{(t,t^2,t^3,\ldots,t^r)\mid t^{r+1}=1\}$
B_r	$\{(1,\ldots,1,t)\mid t^2=1\}$
C_r	$\{(t,1,t,1,\ldots,t^{r \bmod 2})\mid t^2=1\}$
D_r, r odd	$\{(t^2,1,t^2,1,\ldots,t^2,t,t^{-1})\mid t^4=1\}$
D_r, r even	$\{(t_1,1,t_1,1,\ldots,t_1,1,t_1t_2,t_2)\mid t_1^2=t_2^2=1\}$
E_6	$\{(t,1,t^{-1},1,t,t^{-1})\mid t^3=1\}$
E_7	$\{(1,t,1,1,t,1,t)\mid t^2=1\}$
E_8	$\{(1,1,1,1,1,1,1,1)\}$
F_4	$\{(1,1,1,1)\}$
G_2	$\{(1,1)\}$

Proof. By (40) an element $(s_1,\ldots,s_r)\in\mathbf{G}_m^r$ lies in $\nu^{-1}(\widehat{C})$ if and only if $(s_1,\ldots,s_r)$ is a solution of the following system of equations:

$$\left.\begin{array}{c} x_1^{n_{11}}\cdots x_r^{n_{1r}}=1,\\ \cdots\cdots\cdots\cdots\\ x_1^{n_{r1}}\cdots x_r^{n_{rr}}=1.\end{array}\right\} \tag{42}$$

Let, for instance, $\widehat{G}$ be of type D_r for even r. Using the explicit form of the Cartan matrix [Bou$_2$, Planche IV], one immediately verifies that every element of $C' := \{(t_1, 1, t_1, 1, \ldots, t_1, 1, t_1t_2, t) \mid t_1^2 = t_2^2 = 1\}$ is a solution of (42). Hence, $C' \subseteq \nu^{-1}(\widehat{C})$. On the other hand, the fundamental group of the root system of type D_r is isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ (see [Spr, 8.1.11] and [Bou$_2$, Planche IV]). Hence, the SMITH normal form of $(n_{ij})_{i,j=1}^r$ is $\operatorname{diag}(1, \ldots, 1, 2, 2)$. Therefore, there is a basis $\beta_1, \ldots, \beta_r$ of the coroot lattice of $\widehat{T}$ such that, for $(s_1, \ldots, s_r) \in \mathbf{G}_m^r$, we have $\beta_1(s_1) \cdots \beta_r(s_r) \in \widehat{C}$ if and only if $(s_1, \ldots, s_r)$ is a solution of the following system of equations:

$$x_1 = 1, \ldots, x_{r-2} = 1, x_{r-1}^2 = 1, x_r^2 = 1.$$

This yields the equality $|C'| = |\widehat{C}|$; whence $C' = \nu^{-1}(\widehat{C})$.

For the groups of the other types the proofs are similar. □

The following examples illustrate how this can be applied to exploring singularities of $G/\!\!/G$ and finding the minimal generating sets $\{\mathrm{ch}_\varpi \mid \varpi \in \mathcal{H}\}$ and $\{[E(\varpi)] \mid \varpi \in \mathcal{H}\}$ of, respectively, the algebra of class functions on G and the representation ring of G.

Examples 3.15.

(1) If Z is trivial, i.e., $G = \widehat{G}$, then we have

$$\mathcal{H} = \{\varpi_1, \ldots, \varpi_r\}$$

and $G/\!\!/G$ is isomorphic to $\mathbf{A}^r$.

(2) Let $\widehat{G}$ be of type A_r. If $\operatorname{char} k > 0$, let $(\operatorname{char} k)^d$ be the maximal power of $\operatorname{char} k$ dividing $r + 1$. Put

$$m := \begin{cases} r+1 & \text{if } \operatorname{char} k = 0, \\ (r+1)/(\operatorname{char} k)^d & \text{if } \operatorname{char} k > 0. \end{cases}$$

Then there are precisely m different mth roots of unity in k and from Table 1 we deduce that $\widehat{C}$ is a cyclic group of order m. Assume that $\widehat{C}$ is nontrivial, i.e., $m \geqslant 2$, and consider the case $Z = \widehat{C}$, i.e.,

$$G = \widehat{G}/\widehat{C} = \mathbf{PGL}_{r+1}.$$

Take an element $z \in \widehat{C}$, $g \neq e$. As $z = \nu((t, t^2, t^3, \ldots, t^r))$ where $t \neq 1$, $t^{r+1} = 1$, formula (41) implies that z acts on $\mathbf{A}^r$ as a pseudo-reflection (i.e., $\dim_k(\mathbf{A}^r)^z = r-1$) if and only if $r = 1$. As is known [Ser$_2$] (cf. also [Ben, Theorem 7.2.1]) if a quotient variety of $\mathbf{A}^r$ by a finite linear group is smooth, then this group is generated by pseudo-reflections (and this quotient variety is isomorphic to $\mathbf{A}^r$). Hence, if $r \geqslant 2$, then $G/\!\!/G$ has singular points (this agrees with Theorem 3.2); if $r = 1$, then $G/\!\!/G$ is smooth; see the next example.

Actually, our analysis provides more precise information. Namely, let $\boldsymbol{\mu}_m$ be the cyclic group $\mathbf{Z}/m\mathbf{Z}$. Fix a choice of generator g of $\boldsymbol{\mu}_m$ and primitive mth root of unity ζ in k. Let L be a one-dimensional $\boldsymbol{\mu}_m$-module on which $\boldsymbol{\mu}_m$ acts by means of the character $g^h \mapsto \zeta^h$. Put

$$V = \bigoplus_{i=1}^{r} L^{\otimes i}$$

(thus, if $\operatorname{char} k \nmid (r+1)$, then the representation of $\boldsymbol{\mu}_{r+1}$ in V is the reduced regular representation of $\boldsymbol{\mu}_{r+1}$, i.e., the quotient of regular representation by the unique one-dimensional trivial subrepresentation). Then $G/\!\!/G$ is isomorphic to $V/\boldsymbol{\mu}_m$. Let $\mathcal{I}_{r,m}$ be the set of all indecomposable elements of the additive monoid

$$\Big\{(a_1, \dots, a_r) \in \mathbf{N}^r \;\Big|\; \sum_{i=1}^{r} i a_i \equiv 0 \bmod m\Big\}.$$

Then $k[V/\boldsymbol{\mu}_m]$ is isomorphic to the subalgebra of $k[y_1, \dots, y_r]$ generated by all the monomials $y_1^{a_1} \cdots y_r^{a_r}$ with $(a_1, \dots, a_r) \in \mathcal{I}_{r,m}$, and the following equality holds:

$$\mathcal{H} = \Big\{\sum_{i=1}^{r} a_i \varpi_i \;\Big|\; (a_1, \dots, a_r) \in \mathcal{I}_{r,m}\Big\}.$$

For instance, let $r = 2$. If $\operatorname{char} k = 3$, then $\widehat{C} = \{e\}$, and if $\operatorname{char} k \neq 3$, then the order of $\widehat{C}$ is 3. In the latter case $\mathcal{H} = \{3\varpi_1, \varpi_1 + \varpi_2, 3\varpi_2\}$ for $G = \mathbf{PGL}_3$, and $G/\!\!/G$ is isomorphic to the surface $\{(c_1, c_2, c_3) \in \mathbf{A}^3 \mid c_1 c_2 = c_3^3\}$.

To illustrate the dependence of $\mathcal{H}$ on $\operatorname{char} k$, consider the case $r = 5$. Then $\widehat{C} \neq \{e\}$ and, for $G = \mathbf{PGL}_6$, the following holds: If $\operatorname{char} k \neq 2, 3$, then $(a, 0, 0, 0, 0) \in \mathcal{H}$ only for $a = 6$, but if $\operatorname{char} k = 2$ or 3, then $(6/(\operatorname{char} k), 0, 0, 0, 0) \in \mathcal{H}$.

Note that $|\mathcal{H}|$ ($= \max_{x \in G/\!\!/G} \dim \mathrm{T}_{x, G/\!\!/G}$, see Corollary 3.13) grows very rapidly when $r \to \infty$. Indeed, a simple observation from [Kac, p. 105] shows that $|\mathcal{I}_{r,r+1}| \geqslant p(r+1) + \varphi(r+1) - 1$ where p and φ are, respectively, the classical partition function and the Euler function, and, as is known, $p(s) \sim \big(\exp(\pi\sqrt{2s/3})\big)/4s\sqrt{3}$ when $s \to +\infty$, see [HR].

(3) Let $\widehat{G}$ be of type B_r (where $\mathsf{B}_1 := \mathsf{A}_1$). Table 1 implies that $\nu^{-1}(\widehat{C})$ is generated by $(1, \dots, 1, -1)$. Hence $\widehat{C} \neq \{e\}$ (and then $|\widehat{C}| = 2$) if and only if $\operatorname{char} k \neq 2$. Assume that this inequality holds. As $1 \neq -1$, we then have $k[\mathbf{A}^r]^{\widehat{C}} = k[y_1, \dots, y_{r-1}, y_r^2]$. Therefore, for $G = \widehat{G}/\widehat{C} = \mathbf{SO}_{2r+1}$, we have

$$\mathcal{H} = \{\varpi_1, \dots, \varpi_{r-1}, 2\varpi_r\}$$

and $G/\!\!/G$ is isomorphic to $\mathbf{A}^r$ (the latter agrees with Theorem 3.2).

(4) Let $\widehat{G}$ be of type D_r, $r \geqslant 3$, and let $Z := \{t \in \widehat{C} \mid t^{\varpi_1} = 1\}$. Table 1 implies that $\nu^{-1}(Z)$ is generated by $(1, \dots, 1, -1, -1)$. Hence $Z \neq \{e\}$ if and only if $\operatorname{char} k \neq 2$. Assume that this inequality holds. As $1 \neq -1$, we then have $k[\mathbf{A}^r]^Z = k[y_1, \dots, y_{r-2}, y_{r-1}^2, y_r^2, y_{r-1}y_r]$. Therefore, for $G := \widehat{G}/Z = \mathbf{SO}_{2r}$, we have

$$\mathcal{H} = \{\varpi_1, \dots, \varpi_{r-2}, 2\varpi_{r-1}, 2\varpi_r, \varpi_{r-1} + \varpi_r\}$$

and $G/\!\!/G$ is isomorphic to $\mathbf{A}^{r-2} \times X$ where X is a nondegenerate quadratic cone in $\mathbf{A}^3$.

(5) Let $\widehat{G}$ be of type D_r with even $r = 2d \geqslant 4$ and let $Z := \{t \in \widehat{C} \mid t^{\varpi_r} = 1\}$. Table 1 implies that $\nu^{-1}(Z)$ is generated by $(-1, 1, -1, 1, \dots, -1, 1)$. Hence $Z \neq \{e\}$ if and only if $\operatorname{char} k \neq 2$. Assume that this inequality holds. As $1 \neq -1$, the algebra $k[\mathbf{A}^r]^Z$ is then minimally generated by all y_i's with even i and all the monomials of degree 2 in y_j's with odd j. Therefore, for $G := \widehat{G}/Z = \mathbf{Spin}_{2r}^{1/2}$ (the half-spinor group), we have

$$\mathcal{H} = \{\varpi_i \mid i \text{ is even}\} \cup \{\varpi_l + \varpi_m \mid l, m \text{ are odd}\}$$

and $G/\!\!/G$ is isomorphic to $\mathbf{A}^d \times Y$ where Y is the affine cone over the Veronese variety $\nu_2(\mathbf{P}^{d-1})$ in $\mathbf{P}^{(d-1)(d+2)/2}$. Note that if $r = 4$, then, up to the renumbering of simple roots, we obtain the same answer as in the previous example.

(6) Let $\widehat{G}$ be of type E_7. Table 1 implies that $\nu^{-1}(\widehat{C})$ is generated by the element $(1, -1, 1, 1, -1, 1, -1)$. Hence $\widehat{C} \neq \{e\}$ (and then $|\widehat{C}| = 2$) if and only if $\operatorname{char} k \neq 2$. Assume that this inequality holds. Then, as $1 \neq -1$, the algebra $k[\mathbf{A}^7]^{\widehat{C}}$ is minimally generated by y_1, y_3, y_4, y_6 and all the monomials of degree 2 in y_2, y_5, y_7. Therefore, for $G = \widehat{G}/\widehat{C}$, we have

$$\mathcal{H} = \{\varpi_1, \varpi_3, \varpi_4, \varpi_6, 2\varpi_2, 2\varpi_5, 2\varpi_7, \varpi_2 + \varpi_5, \varpi_2 + \varpi_7, \varpi_5 + \varpi_7\}$$

and $G/\!\!/G$ is isomorphic to $\mathbf{A}^4 \times Y$ where Y is the affine cone over the Veronese variety $\nu_2(\mathbf{P}^2)$ in $\mathbf{P}^5$ (in particular, the maximum of dimensions of tangent spaces of the 7-dimensional variety $G/\!\!/G$ is 10.) □

Remark 3.16. Considering in the same way the remaining types of simple groups, one obtains the proof of Theorem 3.2.

4. Two further questions of Grothendieck

Theorem 3.12 describes a minimal generating set of the algebra $k[G]^G$ of class functions on G. Constructing generating sets of $k[G]^G$ is the topic of two further questions of Grothendieck in [GS, p. 241]:

> "[...] *When G is an adjoint group, is it possible to generate the affine ring of I(G) with coefficients of the Killing polynomial? In the general case, is it enough to take the coefficients of analogous polynomials for certain linear representations* (*perhaps arbitrary faithful representations*)? [...]"

Below we answer these questions.

Let $\varrho\colon G \to \mathbf{GL}(V)$ be a finite dimensional linear representation of G. Define the set

$$C_\varrho := \{c_{\varrho,i} \in k[G] \mid i = 1, \ldots, \dim V\}$$

by the equality

$$\det(xI - \varrho(g)) = \sum_{i=0}^{\dim V} c_{\varrho,i}(g) x^{\dim V - i} \quad \text{for every } \ g \in G, \tag{43}$$

where x is a variable. If $V = E(\varpi)$ (here and below we use the notation of Section 3) and ϱ determines the G-module structure of $E(\varpi)$, we put $C_\varpi := C_\varrho$.

Clearly, $c_{\varrho,i} \in k[G]^G$ and $c_{\varrho,1}$ is the character of ϱ. Hence by Theorem 3.12(iii)

$$\bigcup_{\varpi \in \mathcal{H}} C_\varpi$$

is a generating set of the algebra $k[G]^G$. This answers the second of GROTHENDIECK's questions in the affirmative.

In order to answer the first one in the negative it is sufficient to find an adjoint G and two elements $z_1, z_2 \in T$ such that

(i) z_1 and z_2 are not in the same W-orbit;
(ii) the spectra of the linear operators $\mathrm{Ad}_G\, z_1$ and $\mathrm{Ad}_G\, z_2$ on the vector space $\mathrm{Lie}\, G$ coincide.

Indeed, property (i) implies that there is a function $f \in k[T]^W$ such that $f(z_1) \neq f(z_2)$. Given isomorphism (29), this means that there is a function $\widetilde{f} \in k[G]^G$ such that $\widetilde{f}(z_1) \neq \widetilde{f}(z_2)$. On the other hand, (43) and property (ii) imply that

$$c_{\mathrm{Ad}_G, i}(z_1) = c_{\mathrm{Ad}_G, i}(z_2) \qquad \text{for every } \ i.$$

Therefore, $\widetilde{f}$ is not in the subalgebra of $k[G]^G$ generated by $C_{\mathrm{Ad_G}}$, i.e., the latter is not a generating set of $k[G]^G$.

The following two examples show that one indeed can find G, z_1, and z_2 sharing properties (i) and (ii).

Examples 4.1.

(1) Let $G = H \times H$ where H is a connected adjoint semisimple algebraic group. Let $T = S \times S$ where S is a maximal torus of H. Let W_S be the Weyl group of H naturally acting on S. Take any two elements $a, b \in S$ that are not in the same W_S-orbit and put $z_1 := (a, b)$, $z_2 := (b, a) \in T$. As $W = W_S \times W_S$, property (i) holds. On the other hand, clearly, for every $i = 1, 2$, the spectrum of $\mathrm{Ad}_G\, z_i$ is the union of the spectra of $\mathrm{Ad}_H\, a$ and $\mathrm{Ad}_H\, b$; whence property (ii) holds.

(2) In this example G is simple, namely, $G = \mathbf{PGL}_3$. Let $\alpha_1, \alpha_2 \in \mathrm{X}(T)$ be the simple roots of T regarding B. As the map $T \to \mathbf{G}_{\mathrm{m}}^2$, $t \mapsto (t^{\alpha_1}, t^{\alpha_2})$, is surjective (in fact, an isomorphism), for every $u, v \in k$, $uv \neq 0$, there are $z_1, z_2 \in T$ such that $z_1^{\alpha_1} = u$, $z_1^{\alpha_2} = v$ and $z_2^{\alpha_1} = v$, $z_2^{\alpha_2} = u$. For these z_1, z_2, property (ii) holds as the set of roots of G regarding T is $\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$. Now take u and v such that all the elements u, u^{-1}, v, v^{-1}, uv, $u^{-1}v^{-1}$ are pairwise distinct. Then property (i) holds as there are no $w \in W$ such that $w(\alpha_1) = \alpha_2$ and $w(\alpha_2) = \alpha_1$.

5. Rational cross-sections

Recall from [Ste$_1$, 2.14, 2.15] that an element $x \in G$ is called *strongly regular* if its centralizer G_x is a maximal torus. Such x is regular and semisimple. Strongly regular elements form a dense open subset G_0 of G stable with respect to the conjugating action of G. Every G-orbit in G_0 is regular and closed in G. We put

$$(G/\!\!/G)_0 := \pi_G(G_0) \qquad \text{and} \qquad T_0 := T \cap G_0.$$

Abusing the notation, we denote $\pi_G|_{G_0}$ still by π_G:

$$\pi_G\colon G_0 \longrightarrow (G/\!\!/G)_0. \tag{44}$$

Lemma 5.1.
(i) $(G/\!\!/G)_0$ *is an open smooth subset of* $G/\!\!/G$.
(ii) $\pi_G|_{T_0}\colon T_0 \to (G/\!\!/G)_0$ *is a surjective étale map.*
(iii) $((G/\!\!/G)_0, \pi_G)$ *is the geometric quotient for the action of* G *on* G_0.

Proof. Since $G/\!\!/G$ is normal and all fibers of π_G are of constant dimension and irreducible, π_G is an open map (see [Bor, AG.18.4]). Hence $(G/\!\!/G)_0$ is open in $G/\!\!/G$.

As every element of G_0 is semisimple, it is conjugate to an element of T_0; whence $\pi_G|_{T_0}$ is surjective.

The set T_0 is open in T and W-stable. For every point $t \in T_0$, we have $G_t = T$, hence the W-stabilizer of t is trivial. Thus, the action of W on T_0 is set theoretically free. Since T is smooth, $G/\!\!/G$ is normal, and $(G/\!\!/G, \pi_G|_T)$ is the quotient for the action of W on T (see [Ste$_1$, 6.4]), we deduce from [Gro$_3$, Exp. I, Théorème 9.5(ii)] and [Bou$_1$, V.2.3, Cor. 4] that $\pi_G|_{T_0}$ is étale and hence $(G/\!\!/G)_0$ is smooth. This proves (i) and (ii).

By (ii) the map $\pi_G\colon G_0 \to (G/\!\!/G)_0$ is separable and surjective. As its fibers are G-orbits and $(G/\!\!/G)_0$ is normal, (iii) follows from [Bor, 6.6]. $\square$

The group W acts on $G/T \times T_0$ diagonally with the action on the first factor defined by formula (2). The group G acts on $G/T \times T_0$ via left translations of the first factor. These two actions commute with each other.

Consider the G-equivariant morphism

$$\psi\colon G/T \times T_0 \longrightarrow G_0, \qquad (gT, t) \mapsto gtg^{-1}. \tag{45}$$

The proofs of Lemma 5.2 and Corollary 5.4 reproduce that from my letter [Pop$_2$].

Lemma 5.2. *ψ is a surjective étale map.*

Proof. As every G-orbit in G_0 intersects T_0, surjectivity of ψ follows from (45).

Take a point $z \in G/T \times T_0$. We shall prove that $d\psi_z$ is an isomorphism. As $G/T \times T_0$ and G_0 are smooth, this is equivalent to proving that ψ is étale at z. Using that ψ is G-equivariant, we may assume that $z = (eT, s)$, $s \in T_0$.

Let U_α be the one-dimensional unipotent root subgroup of G corresponding to a root α with respect to T and let $\theta_\alpha\colon \mathbf{G}_a \to U_\alpha$ be the isomorphism of groups such that

$$t\theta_\alpha(x)t^{-1} = \theta_\alpha(t^\alpha x) \qquad \text{for all } \ t \in T, \ x \in \mathbf{G}_a;$$

see [Bor, IV.13.18]. Put

$$\begin{aligned} C_\alpha &:= \{(\theta_\alpha(x)T, s) \in G/T \times T_0 \mid x \in \mathbf{G}_a\}, \\ D &:= \{(eT, t) \in G/T \times T_0 \mid t \in T_0\}. \end{aligned}$$

The linear span of all tangent spaces T_{z,C_α} and $\mathrm{T}_{z,D}$ is $\mathrm{T}_{z,G/T\times T_0}$. We have

$$\begin{aligned} \psi(\theta_\alpha(x)T, s) &= \theta_\alpha(x)s\theta_\alpha(x)^{-1} = \theta_\alpha(x)s\theta_\alpha(-x) \\ &= \theta_\alpha(x)\theta_\alpha(-s^\alpha x)s = \theta_\alpha((1-s^\alpha)x)s. \end{aligned} \tag{46}$$

Since s is regular, $s^\alpha \neq 1$. Hence (46) shows that ψ maps the curve C_α isomorphically onto the curve

$$\psi(C_\alpha) = \{\theta_\alpha((1-s^\alpha)x)s \mid x \in \mathbf{G}_\alpha\}.$$

Clearly, $\psi(D) = T_0$ and $\psi|_D\colon D \to T_0$ is an isomorphism. But $\mathrm{T}_{e,G}$ is the linear span of $\mathrm{T}_{e,T}$ and the tangent spaces to the curves $\{\theta_\alpha(x) \mid x \in \mathbf{G}_\alpha\}$ at e. Hence $\mathrm{T}_{s,G}$ is the linear span of $\mathrm{T}_{s,T}$ and the tangent spaces at s to the right translates of these curves by s. This implies the claim of the lemma. □

Corollary 5.3. *ψ is separable.*

Corollary 5.4. *(G_0, ψ) is the quotient for the action of W on $G/T \times T_0$.*

Proof. By [Bor, Prop. II.6.6], as G_0 is normal and ψ is surjective and separable, it suffices to prove that the fibers of ψ are W-orbits.

Using (2) and (45) one immediately verifies that the fibers of ψ are W-stable. On the other hand, let $\psi(g_1T, t_1) = \psi(g_2T, t_2)$. By (45) this equality is equivalent to $(g_1^{-1}g_2)t_2(g_1^{-1}g_2)^{-1} = t_1$. By [Ste$_1$, 6.1] the latter, in turn, implies that there is an element $w \in W$ such that

$$\dot{w}t_2\dot{w}^{-1} = (g_1^{-1}g_2)t_2(g_1^{-1}g_2)^{-1}.$$

Hence $g_1^{-1}g_2 = \dot{w}z$ for $z \in G_{t_2}$. As $t_2 \in T$ is strongly regular, this yields that $z \in T$. Therefore,

$$(g_2T, t_2) = (g_1\dot{w}T, \dot{w}^{-1}t_1\dot{w}) = w^{-1} \cdot (g_1T, t_1).$$

Thus, (g_1T, t_1) and (g_2T, t_2) are in the same W-orbit. This completes the proof. □

Let $\pi_2\colon G/T\times T_0\to T_0$ be the second projection. Clearly, (T_0,π_2) is the geometric quotient for the action of G on $G/T\times T_0$. As ψ is G-equivariant, this implies that there is a morphism $\phi\colon T_0\to G/\!\!/G$ such that the following diagram is commutative:

$$\begin{array}{ccc} G/T\times T_0 & \xrightarrow{\ \psi\ } & G_0 \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi_G} \\ T_0 & \xrightarrow{\ \phi\ } & (G/\!\!/G)_0 \end{array} . \tag{47}$$

Lemma 5.5.

(i) $\phi=\pi_G|_{T_0}$.

(ii) *For every point $t\in T_0$, the restriction of ψ to $\pi_2^{-1}(t)$ is a G-equivariant isomorphism $\pi_2^{-1}(t)\to\pi_G^{-1}(\phi(t))$.*

Proof. Take a point $t\in T_0$. Commutativity of diagram (47) and formula (45) yield that $\pi_G(t)=\pi_G(\psi(eT,t))=\phi(\pi_2(eT,t))=\phi(t)$. This proves (i).

Commutativity of diagram (47) implies that the restriction of ψ to $\pi_2^{-1}(t)$ is a G-equivariant morphism $\pi_2^{-1}(t)\to\pi_G^{-1}(\phi(t))$. As both $\pi_2^{-1}(t)$ and $\pi_G^{-1}(\phi(t))$ are G-orbits and the stabilizers of their points are conjugate to T, this morphism is bijective. By Lemma 5.2 it is separable. Then, as $\pi_G^{-1}(\phi(t))$ is normal, it is an isomorphism. This proves (ii). $\square$

Proof of Theorem 1.5. Assume that (i) holds. Let $\sigma\colon G/\!\!/G\dashrightarrow G$ be a rational section of π_G, i.e., a section of π_G over a dense open subset U of $(G/\!\!/G)_0$. Let S be the closure of $\sigma(U)$. Put $\rho:=\pi_G|_S\colon S\to(G/\!\!/G)_0$. Since $\pi_G\circ\sigma=\mathrm{id}$, shrinking U if necessary, we may assume that, for every point $x\in U$, the following properties hold:

(a) $S\cap\pi_G^{-1}(x)$ is a single point s;

(b) $d\rho_s$ is an isomorphism.

Since ψ is an isomorphism on the fibers of π_2, property (a) implies that, for every point $t\in\phi^{-1}(U)$, the W-stable closed set $\psi^{-1}(S)$ intersects $\pi_2^{-1}(t)$ at a single point. From this we infer that $\psi^{-1}(S)$ has a unique irreducible component $\widetilde{S}$ whose image under π_2 is dense in T_0 — the argument is the same as that in the proof of Claim 2(i) in Section 2. Due to the uniqueness, this $\widetilde{S}$ is W-stable.

Let $V\subseteq\pi_2(\widetilde{S})\cap\phi^{-1}(U)$ be an open subset of T_0. Replacing it, if necessary, by $\bigcap_{w\in W}w(V)$, we may assume that V is W-stable. Let $\widetilde{\rho}\colon\pi_2^{-1}(V)\cap\widetilde{S}\to V$ be the restriction of π_2 to $\pi_2^{-1}(V)\cap\widetilde{S}$. Then $\widetilde{\rho}$ is a bijective W-equivariant morphism. We claim that it is separable and hence, by Zariski's Main Theorem, an isomorphism (as V is normal). Indeed, take a point $\widetilde{s}\in\pi_2^{-1}(V)\cap\widetilde{S}$ and put $\pi_2(\widetilde{s})=t$. Then property (b), Lemma 5.2, and commutativity of diagram (47) imply that $d\widetilde{\rho}_s\colon \mathrm{T}_{\widetilde{s},\widetilde{S}}\to\mathrm{T}_{t,V}$ is an isomorphism; whence the claim by [Bor, AG.17.3].

Thus, $\widetilde{\rho}^{-1}\colon V\to\pi_2^{-1}(V)\cap\widetilde{S}$ is a rational W-equivariant section of π_2. Its composition with the first projection $G/T\times T_0\to G/T$ is then a W-equivariant rational map $T\dashrightarrow G/T$. This proves (i)⇒(ii).

Conversely, assume that (ii) holds. Let $\gamma\colon T \dashrightarrow G/T$ be a W-equivariant rational map. Then $\varsigma := (\gamma, \mathrm{id})\colon T_0 \dashrightarrow G/T \times T_0$ is a W-equivariant rational section of π_2, i.e., a section of π_2 over a dense open subset V of T_0. We may assume that $\varsigma(V)$ and $S := \psi(\varsigma(V))$ are open in their closures, $\varsigma\colon V \to \varsigma(V)$ is an isomorphism, and the subsets $\phi(V)$, $\pi_G(S)$ of $G/\!\!/G$ are open and coincide. As above, we may also assume that V is W-stable.

Taking into account that ς is W-equivariant, $\varsigma(V) \cap \pi_2^{-1}(t)$ is a single point for every $t \in V$, and ψ is an isomorphism on the fibers of π_2, we conclude that property (a) holds for every $x \in \varsigma(V)$. Thus, $\rho := \pi_G|_S\colon S \to \phi(V)$ is a bijection.

We claim that ρ is separable, hence an isomorphism as $\phi(V)$ is normal by Lemma 5.1(i). Indeed, $d\phi_t$ is an isomorphism by Lemma 5.5(i) and Lemma 5.1(ii). Let $s = \psi(\varsigma(t)) \in S$. Since the restriction of $(d\pi_2)_{\varsigma(t)}$ to $\mathrm{T}_{\varsigma(t),\varsigma(V)}$ is an isomorphism with $\mathrm{T}_{t,V}$, commutativity of diagram (47) and Lemma 5.2 imply that property (b) holds; whence the claim.

Thus, the composition of $\rho^{-1}\colon \phi(V) \to S$ and the identical embedding $S \hookrightarrow G$ is a rational section of π_G. This proves (ii)$\Rightarrow$(i) and completes the proof of the theorem. $\square$

Recall some definitions from [CTKPR, Sects. 2.2, 2.3, and 3].

Let P be a linear algebraic group acting on a variety X and let Q be its closed subgroup. X is called a (P, Q)-*variety* if in X there is a dense open P-stable subset U, called *a friendly subset*, such that a geometric quotient $\pi_U\colon U \to U/P$ exists and π_U becomes the second projection $P/Q \times \widehat{U/P} \to \widehat{U/P}$ after a surjective étale base change $\widehat{U/P} \to U/P$. If there is a rational section of π_U, one says that X *admits a rational section*. X is called a *versal* (P, Q)*-variety* if U/P is irreducible and, for every dense open subset $(U/P)_0$ of U/P and every (P, Q)-variety Y, there is a friendly subset V of Y such that π_V is induced from π_U by a base change $V \to (U/H)_0$.

Now we shall give the characteristic free proofs of the following two statements proved in [CTKPR] for char = 0.

Lemma 5.6. *Let X be an irreducible variety endowed with a faithful action of a finite algebraic group H. Then*

(i) *X is an $(H, \{e\})$-variety;*
(ii) *X is a versal $(H, \{e\})$-variety in each of the following cases:*
  (a) *X is a free H-module;*
  (b) *X is a linear algebraic torus and H acts by its automorphisms.*

Proof. (i) Replacing X by its smooth locus, we may assume that X is smooth.

As H is finite, for any nonempty open affine subset U of X, the set $\bigcap_{h\in H} h(U)$ is H-stable, affine, and open in X. So, replacing X by it, we may assume that X is affine. Then, as is well known, for the action of H on X there is a geometric quotient $\pi\colon X \to X/H$ (see, e.g., [Bor, Prop. 6.15]). As X is normal, X/H is normal as well.

Since H is finite and the action is faithful, the points with trivial stabilizer form an open subset of X. Replacing X by it, we may also assume that the action is set-theoretically free, i.e., the H-stabilizer of every point of X is trivial. As X and

X/G are normal, by [Gro$_3$, Exp. I, Théorème 9.5(ii)] and [Bou$_1$, V.2.3, Cor. 4] this property implies that π is étale and hence X/H is smooth.

For every base change $\beta\colon Y \to X/H$ of π, the group H acts on $X \times_{X/H} Y$ via X. As the action of H on X is set-theoretically free, taking $Y = X$ and $\beta = \pi$, we obtain

$$X \times_{X/H} X = \bigsqcup_{h \in H} h(D) \qquad \text{where} \quad D := \{(x, x) \mid x \in X\}.$$

From this we deduce that in the commutative diagram

$$\begin{array}{ccc} H \times X & \xrightarrow{\ \alpha\ } & X \times_{X/H} X \\ & \searrow \quad \swarrow & \\ & X & \end{array},$$

where $\alpha(h, x) := (h(x), x)$ and two other maps are the second projections, α is an H-equivariant isomorphism. This proves (i).

The proofs of (ii)(a) and (ii)(b) are the same as that of (b) and (d) in [CTKPR, Lemma 3.3] if one replaces in them the references to [CTKPR, Theorem 2.12] (whose proof is based on the assumption char $k = 0$) by the references to statement (i) of the present lemma. $\square$

Remark 5.7. The proof of (i) shows that, for finite group actions, set-theoretical freeness coincides with that in the sense of GIT, [MF, Def. 0.8].

Lemma 5.8. *G is a versal (G, T)-variety.*

Proof. First, we shall give a characteristic free proof of the fact that G is a (G, T)-variety (the proof given in [CTKPR] is based on the assumption char $k = 0$). By Lemma 5.1(iii) this is equivalent to proving the existence of a dense open subset U of $(G/\!\!/G)_0$ such that after a surjective étale base change $U' \to U$ morphism (44) becomes the second projection $G/T \times U' \to U'$.

Consider the base change of π_G in (47) by means of ϕ. Lemma 5.5(i) implies that

$$F := G_0 \times_{(G/\!\!/G)_0} T_0 = \{(g, t) \in G_0 \times T_0 \mid G(g) = G(t)\} \tag{48}$$

(see (6)). We have the canonical map corresponding to commutative diagram (47):

$$\gamma := \psi \times \mathrm{id}\colon G/T \times T_0 \longrightarrow F, \qquad (gT, t) \mapsto (gtg^{-1}, t). \tag{49}$$

It follows from (48) that γ is surjective; whence F is irreducible. But if for $t \in T_0$ and $g_1, g_2 \in G$ we have $g_1tg_1^{-1} = g_2tg_2^{-1}$, then $g_1T = g_2T$ since $G_t = T$. Therefore, γ is bijective. Lemma 5.2 and (49) show that $d\gamma_x$ is injective for every $x \in G/T \times T_0$. Hence if $\gamma(x)$ lies in the smooth locus F_{sm} of F, then $d\gamma_x$ is the isomorphism. This implies that γ is separable and then, by Zariski's Main Theorem, that the restriction of γ to $\gamma^{-1}(F_{\mathrm{sm}})$ is an isomorphism $\gamma^{-1}(F_{\mathrm{sm}}) \to F_{\mathrm{sm}}$.

As $F_{\rm sm}$ is G-stable and γ is G-equivariant, $\gamma^{-1}(F_{\rm sm})$ is a G-stable open subset of $G/T{\times}T_0$. Hence it is of the form $G/T{\times}U'$ for an open subset U' of T_0. But Lemmas 5(ii) and 5.5(i) imply that $U := \phi(U')$ is open in $(G/\!\!/G)_0$ and $\phi|_{U'}\colon U' \to U$ is étale. This proves that after the étale base change $\phi|_{U'}\colon U' \to U$ morphism (44) becomes the second projection $G/T \times U' \to U'$. Hence G is a (G,T)-variety.

By Lemma 5.6(b), T is a versal $(W, \{e\})$-variety. The characteristic free arguments from [CTKPR, proof of Prop. 4.3(c)] then show that this fact implies versality of the (G,T)-variety G. This completes the proof of the lemma. □

Proof of Theorem 1.6. Since the isogeny τ is central, the natural morphism $\widehat{G}/\widehat{T} \to G/T$ is an isomorphism by [Bor, Props. 6.13, 22.5].

Using τ, every action of G naturally lifts to an action of $\widehat{G}$ on the same variety. In particular, G is endowed with an action of $\widehat{G}$. But G is a (G,T)-variety by Lemma 5.8(i). As $\widehat{G}/\widehat{T}$ and G/T are isomorphic, this means that G is a $(\widehat{G},\widehat{T})$-variety. But $\widehat{G}$ is a versal $(\widehat{G},\widehat{T})$-variety (by Lemma 5.8) that admits a rational section (by Lemma 5.1(iii) and [Ste_1, Theorem 1.4]). Hence by [CTKPR, Theorem 3.6(a)] (the proof of this result is characteristic free) every $(\widehat{G},\widehat{T})$-variety admits a rational section. In particular, this is so for G. This proves (ii) and completes the proof of the theorem. □

Proof of Corollary 1.7. By Theorem 1.6 there is a rational section $\sigma\colon G/\!\!/G \dashrightarrow G$ of π_G. The closure of the image of σ is then the desired cross-section S (see Subsection 6.A). □

6. Complements

6.A. Cross-sections versus sections

If there is a section $\sigma\colon G/\!\!/G \to G$ of π_G, then $\sigma(G/\!\!/G)$ is a cross-section in G. Indeed, as $\mathrm{id}_{k[G/\!\!/G]}$ is the composition of the homomorphisms

$$k[G/\!\!/G] \xrightarrow{\pi_G^*} k[G] \xrightarrow{\sigma^*} k[G/\!\!/G],$$

π_G^* is surjective; by [Gro_2, Cor. 4.2.3] this means that σ is a closed embedding.

The cross-section $\sigma(G/\!\!/G)$ has the property that the restriction of π_G to $\sigma(G/\!\!/G)$ is an isomorphism $\sigma(G/\!\!/G) \to G/\!\!/G$. Conversely, let S be a cross-section in G. If $\pi_G|_S\colon S \to G/\!\!/G$ is separable, then, since $\pi_G|_S$ is bijective and $G/\!\!/G$ is normal, ZARISKI's Main Theorem implies that $\pi_G|_S$ is an isomorphism (cf. [Bor, AG 18.2]). So in this case the composition of $(\pi_G|_S)^{-1}$ with the identity embedding $S \hookrightarrow G$ is a section of π_G whose image is S. In particular, if $\operatorname{char} k = 0$, then every cross-section in G is the image of a section of π_G. If $\operatorname{char} k > 0$, then in the general case this is not true.

Example 6.1. Let $G = \mathbf{SL}_3$ and $\operatorname{char} k = p > 0$. Then for every integer $d > 0$,

$$S := \{s(a_1,a_2) \mid a_1, a_2 \in k\}, \qquad \text{where} \quad s(a_1,a_2) := \begin{pmatrix} a_1 & a_2 & 1 \\ 1 & a_1^{p^d} - a_1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

is a cross-section in G such that $\pi_G|_S$ is not separable. Indeed, as $\mathrm{ch}_{\varpi_i}(g)$ is the sum of principal i-minors of $g \in G$, we have (see Lemma 2.1(ii))

$$(\lambda \circ \rho)\big(s(a_1, a_2)\big) = \big(a_1^{p^d}, a_1\big(a_1^{p^d} - a_1\big) - a_2\big). \qquad \square$$

Similarly, if $\sigma\colon G/\!\!/G \dashrightarrow G$ is a rational section of π_G and S is the closure of its image, then S is a rational cross-section in G such that the restriction of π_G to it is a birational isomorphism with $G/\!\!/G$.

6.B. Group action on the set of cross-sections

Let $\mathrm{Mor}(G/\!\!/G, G)$ be the group of morphisms $G/\!\!/G \to G$. If S is a cross-section in G and $\gamma \in \mathrm{Mor}(G/\!\!/G, G)$, then

$$\gamma(S) := \{\gamma(s)s\gamma(s)^{-1} \mid s \in S\}$$

is a cross-section in G. This defines an action of $\mathrm{Mor}(G/\!\!/G, G)$ on the set of cross-sections in G. If $\mathrm{char}\, k = 0$, then by [FM] this action is transitive. If $\mathrm{char}\, k > 0$, then in the general case this is not true: in Example 6.1, STEINBERG's section and S are not in the same $\mathrm{Mor}(G/\!\!/G, G)$-orbit since, for the former, the restriction of π_G is separable [Ste$_1$, Theorem 1.5], but, for the latter, it is not.

6.C. Lifting T-action

By Corollary 3.7 there is an action of T on T/W determining a structure of a toric variety. This action cannot be lifted to T making $\pi_{W,T}\colon T \to T/W$ equivariant. This follows from the fact that the automorphism group of the underlying variety of T is $\mathbf{GL}_r(\mathbf{Z}) \ltimes T$.

6.D. Image of a rational cross-section in G under π_G

Assume that τ is not bijective (for $\mathrm{char}\, k = 0$, this means that G is not simply connected). Let S be a rational cross-section in G such that $\varphi := \pi_G|_S\colon S \to G/\!\!/G$ is a birational isomorphism (S exists by Corollary 1.7). Let D be the closure of the complement of $\pi_G(S)$ in $G/\!\!/G$.

The following shows that D cannot be "too small".

Theorem 6.2. $\mathrm{codim}_{G/\!\!/G} D = 1$.

Proof. Assume the contrary. Take a function $f \in k[S]$. Since φ is a birational isomorphism, $f = \varphi^*(h)$ for some function $h \in k(G/\!\!/G)$. As $G/\!\!/G$ is normal, h is regular at every point of $(G/\!\!/G) \setminus D$; see [Pop$_1$, Sect. 2, Lemma]. Using again that $G/\!\!/G$ is normal, we then deduce from $\mathrm{codim}_{G/\!\!/G} D > 1$ that $h \in k[G/\!\!/G]$. As G and $G/\!\!/G$ are affine and S is closed in G, this shows that φ is an isomorphism. Hence S is a (global) cross-section in G. As τ is not bijective, the latter contradicts Theorem 1.2(i). $\square$

6.E. Questions

Given Theorem 1.8, it would be interesting to construct explicitly an example of a W-equivariant rational map $T \dashrightarrow G/T$.

— Is there such a map defined on T_0?

— Is there a rational section of π_G defined on $(G/\!\!/G)_0$?

References

[Ada] J. F. Adams, *Lectures on Lie Groups*, Benjamin, New York, 1969. Russian transl.: Дж. Адамс, *Лекции по группам Ли*, Наука, М., 1979.

[Ben] D. J. Benson, *Polynomial Invariants of Finite Groups*, London Mathematical Society Lecture Note Series, Vol. 190, Cambridge University Press, Cambridge, 1993.

[Bor] A. Borel, *Linear Algebraic Groups*, 2nd enlarged ed., Graduate Texts in Mathematics, Vol. 126, Springer-Verlag, 1991.

[BT] A. Borel, J. Tits, *Compléments à l'article* "*Groupes réductifs*", Publ. math. IHES **41** (1972), 253–276.

[Bou_1] N. Bourbaki, *Algèbre Commutative*, Chap. V, VI, Hermann, Paris, 1964. Russian transl.: Н. Бурбаки, *Коммутативная алгебра*, Мир, М., 1971.

[Bou_2] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap. IV, V, VI, Hermann, Paris, 1968. Russian transl.: Н. Бурбаки, *Группы и алгебры Ли. Группы Кокстера и системы Титса. Группы, порожденные отражениями. Системы корней*, Мир, М., 1972.

[CTKPR] J.-L. Colliot-Thélène, B. Kunyavskiĭ, V. L. Popov, Z. Reichstein, *Is the function field of a reductive Lie algebra purely transcendental over the field of invariants for the adjoint action*?, `arXiv:0901.4358v1` (27 January, 2009).

[FM] R. Friedman, J. W. Morgan, *Automorphism sheaves, spectral covers, and the Kostant and Steinberg sections*, in: *Vector Bundles and Representation Theory* (Columbia, MO, 2002), Contemp. Math., Vol. 322, Amer. Math. Soc., Providence, RI, 2003, pp. 217–244.

[Ful] W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, Princeton, New Jersey, 1993.

[Gro_1] A. Grothendieck, *Compléments de géométrie algébrique. Espaces de transformations*, in: *Séminaire C. Chevalley,* 1956–1958. *Classification de groupes de Lie algébriques*, Vol. 1, Exposé no. 5, Secr. math. ENS, Paris, 1958.

[Gro_2] A. Grothendieck, *EGA* I, Publ. Math. IHES **4** (1960), 5–228.

[Gro_3] A. Grothendieck et al., *Revêtements Etales et Groupe Fondamental*, Lecture Notes in Mathematics, Vol. 224, Springer-Verlag, Berlin, 1991.

[GS] *Grothendieck–Serre Correspondence*, Bilingual Edition, P. Colmez, J.-P. Serre, eds., American Mathematical Society, Société Mathématique de France, 2004.

[HR] *G. H. Hardy, S. Ramanujan*, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. **17** (1918), 75–115.

[Har] J. Harris, *Algebraic Geometry. A First Course*, Graduate Texts in Mathematics, Vol. 133, Springer-Verlag, New York, 1995. Russian transl.: Дж. Харрис, Алгебраическая геометрия. Начальный курс, МЦНМО, М., 2006.

[Hum_1] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag, New York, 1972. Russian transl.: Дж. Хамфрис, *Введение в теорию алгебр Ли и их представлений*, МЦНМО, М., 2003.

[Hum_2] J. E. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics, Vol. 21, Springer-Verlag, New York, 1975. Russian transl.: Дж. Хамфрис, *Линейные алгебраические группы*, Наука, М., 1980.

[Hum_3] J. E. Humphreys, *Conjugacy Classes in Semisimple Algebraic Groups*, Mathematical Surveys and Monographs, Vol. 43, American Mathematical Society, Providence, RI, 1995.

[Hus] D. Husemoller, *Fibre Bundles*, McGraw-Hill Book Company, New York, 1966. Russian transl.: Д. Хьюзмоллер, *Расслоенные пространства*, Мир, М., 1970.

[Kac] V. G. Kac, *Root systems, representations of quivers and invariant theory*, in: *Invariant Theory*, Proceedings, Montecatini 1982, Lecture Notes in Mathematics, Vol. 996, Springer-Verlag, Berlin, 1983, pp. 74–108.

[Kos] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[Lor_1] M. Lorenz, *Multiplicative invariants and semigroup algebras*, Algebras and Representation Theory **4** (2001), 293–304.

[Lor_2] M. Lorenz, *Multiplicative Invariant Theory*, Encyclopaedia of Mathematical Sciences, Vol. 135, Subseries *Invariant Theory and Algebraic Transformation Groups*, Vol. VI, Springer, Berlin, 2005.

[MF] D. Mumford, J. Fogarty, *Geometric Invariant Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 34, Springer-Verlag, Berlin, 1982.

[Oda] T. Oda, *Convex Bodies and Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Band 15, Springer-Verlag, Berlin, 1988.

[Pop_1] V. L. Popov, *On the "Lemma of Seshadri"*, in: *Lie Groups, their Discrete Subgroups, and Invariant Theory*, Advances in Soviet Mathematics, Vol. 8, Amer. Math. Soc., Providence, RI, 1992, 167–172.

[Pop_2] V. L. Popov, *Letter to A. Premet*, July 5, 2009.

[$Rich_1$] R. W. Richardson, *The conjugating representation of a semisimple group*, Invent. Math. **54** (1979), 229–245.

[$Rich_2$] R. W. Richardson, *Orbits, invariants, and representations associated to involutions of reductive groups*, Invent. Math. **66** (1982), 287–312.

[Ser_1] J.-P. Serre, *Groupes de Grothendieck des schémas en groupes réductifs déployés*, Publ. Math. IHES **34** (1968), 37–52.

[Ser_2] J.-P. Serre, *Groupes finis d'automorphismes d'anneaux locaux réguliers*, in: *Colloque d'Algèbre*, Secrétariat matheématique, Paris, 1968, pp. 8-01–8-11.

[Slo] P. Slodowy, *Simple Singularities and Simple Algebraic Groups*, Lecture Notes in Mathematics, Vol. 815, Springer-Verlag, Berlin, 1980.

[Spr] T. A. Springer, *Linear Algebraic Groups*, 2nd ed., Birkhäuser, Boston, 1998.

[Ste_1] R. Steinberg, *Regular elements of semi-simple algebraic groups*, Publ. Math. IHES **25** (1965), 49–80.

[Ste_2] R. Steinberg, *Lectures on Chevalley Groups*, Yale University, New Haven, Conn., 1968.

[Ste_3] R. Steinberg, *On a theorem of Pittie*, Topology **14** (1975), 173–177.

[Stu] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, University Lecture Series, Vol. 8, American Mathematical Society, Providence, Rhode Island, 1996.