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KINK WAVES IN AN EXTENDED NONLINEAR SCHRÖDINGER EQUATION WITH ALLOWANCE FOR STIMULATED SCATTERING AND NONLINEAR DISPERSION

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We study the stationary wave solutions of an extended nonlinear Schrödinger equation with nonlinear dispersion and stimulated scattering. New classes of the kink waves with nonlinear phase modulation, the existence of which is stipulated by the balance of nonlinear dispersion and stimulated scattering, have been found.

1. INTRODUCTION

Interest in the high-frequency stationary waves is stipulated by their possibility of propagating over long distances with the shape preservation and transmitting energy/information without significant losses. Such waves appear in many models of propagation of the intense wave fields in nonlinear dispersive media, e.g., optical pulses in the fiber communication lines, electromagnetic and Langmuir waves in plasmas, surface waves in deep water, etc. [1–7]. Propagation of the sufficiently long high-frequency pulses is well described by the nonlinear Schrödinger equation [8–10], which allows for the second-order linear dispersion and cubic nonlinearity. Within the framework of this equation, the stationary waves result from the balance of the dispersion expansion of the wave packet, on the one hand, and the nonlinear compression, on the other hand.

A decrease in the high-frequency wave-pulse length leads to the necessity of allowance for the higher (third) order infinitesimal terms in the nonlinear Schrödinger equation. These terms correspond to the nonlinear effects of steepening [11] and stimulated scattering [12], and to the linear effect of an aberrational distortion corresponding to the third-order linear dispersion. The equation resulting from this allowance is conventionally called the extended nonlinear Schrödinger equation earlier, the stationary waves were studied in [13, 14] within the framework of this equation with allowance only for stimulated scattering, disregarding nonlinear dispersion and the third-order linear dispersion. As a result, the solutions in the form of the stationary shock waves stipulated by the balance of stimulated scattering and the second-order linear dispersion were found. The stationary waves with allowance for nonlinear dispersion and the third-order linear dispersion were studied in [15–17] within the framework of an extended nonlinear Schrödinger equation for nonlinear and linear phase modulation in [15] and [16, 17], respectively, disregarding the stimulated scattering. In this case, the stationary waves result from the balance of nonlinear dispersion and the third-order linear dispersion. In [18], the stationary waves are considered within the framework of an extended nonlinear Schrödinger equation with allowance for the nonlinear dispersion but disregarding stimulated scattering and the third-order linear dispersion. In [19], the stationary kink (shock) wave is described within the framework of an extended nonlinear Schrödinger equation with allowance for the stimulated scattering, nonlinear dispersion, and nonlinearity. However, only one kink-wave type (monotonic, oscillationless transition from zero

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amplitude to the finite one) is considered in [19] for a particular relationship among the equation parameters. The shock-wave existence conditions for an arbitrary relationship among the parameters of stimulated scattering and nonlinear dispersion are not determined in [19].

The dynamics of nonstationary waves of the envelope is numerically considered in [20] within the framework of an extended nonlinear Schrödinger equation with allowance for the stimulated scattering, nonlinear dispersion, and nonlinearity up to the fifth order inclusively. It is shown that nonstationary shock waves of the envelope (both limited and unlimited in space) propagate with varying velocity and amplitude.

It should be noted that the currently widely used notion “shock waves” was initially used only for describing low-frequency waves with an infinite derivative in the presence of dissipation [21]. Such waves are nonstationary and result from the dynamic balance of nonlinearity and dissipation. The features of such waves are the nonuniqueness of the exact solution of the model equation (which is interpreted as the wave-front tilting and the appearance of an infinite derivative of the wave field) and their spatial boundedness, i.e., their energy finiteness. At present, the notion “shock waves” for the envelope waves means smallness of the characteristic length of the wave front compared with the wavelength of the wave-packet filling, rather than the wave-front tilting, i.e., the infinite derivative of the wave field is not assumed. On the other hand, the nonzero wave field at infinity, i.e., its energy unboundedness is permitted. In the classical meaning, such envelope waves should be called “kink waves.” However, in many recent papers [13, 14, 18, 19], the “kink waves” are called “shock waves.” The existence of such kink waves can be determined by the balance (of, e.g., the second-order linear dispersion and nonlinear dispersion) leading to appearance of the stationary kink waves, rather than by the presence of dissipation. The stationary kink waves propagate with constant velocity.

In this work, we analyze the stationary waves of the envelope within the framework of an extended nonlinear Schrödinger equation with allowance for stimulated scattering and nonlinear dispersion. New classes of the stationary kink waves both on the pedestal (i.e., having nonzero values for the coordinate tending to positive and negative infinity) and with oscillations on the “tails” (i.e., with oscillations either in front of or after the main kink front) have been found. One part of the obtained stationary kink waves results from the balance of the effects of stimulated scattering and nonlinear dispersion, while the other part results from the balance of the stimulated scattering, on the one hand, and nonlinearity and the second-order linear dispersion, on the other hand. All the obtained solutions are unbounded in space. The relationship among the parameters of stimulated scattering and nonlinearity dispersion for which the existence of the shock waves is possible has been determined.

2. INITIAL EQUATION

Let us consider the dynamics of the envelope $U(\xi,t)$ of the wave field $U(\xi,t)\exp(i\omega t - ik\xi)$ within the framework of an extended nonlinear Schrödinger equation with allowance for nonlinear dispersion and stimulated scattering:

$$2i\frac{\partial U}{\partial t} + q\frac{\partial^2 U}{\partial \xi^2} + 2\alpha U\,|U|^2 + 2i\beta\frac{\partial(U\,|U|^2)}{\partial \xi} + \mu U\frac{\partial(|U|^2)}{\partial \xi} = 0, \tag{1}$$

where q is the second-order linear-dispersion coefficient, α is the cubic-nonlinearity coefficient, β is the nonlinear-dispersion coefficient, and μ is the stimulated-scattering coefficient. To solve Eq. (1), we use the reference frame moving with velocity V, i.e., introduce new independent variables $\eta = \xi - Vt$ and $t' = t$. The solution of the new obtained equation is represented in the form of the stationary wave $U(\eta,t) = A(\eta)\exp[i\Omega t + i\varphi(\eta)]$. To determine the amplitude $A(\eta)$ and the phase $\varphi(\eta)$, we obtain the following system of equations:

$$q\frac{\mathrm{d}K}{\mathrm{d}\eta}A + 2qK\frac{\mathrm{d}A}{\mathrm{d}\eta} + 2\,(3\beta A^2 - V)\frac{\mathrm{d}A}{\mathrm{d}\eta} = 0, \tag{2}$$

$$q\frac{\mathrm{d}^2A}{\mathrm{d}\eta^2}+(\alpha-2\beta K)\,A^3-2\mu A^2\frac{\mathrm{d}A}{\mathrm{d}\eta}+(2VK-qK^2-2\Omega)\,A=0, \tag{3}$$

where $K\equiv \mathrm{d}\varphi/\mathrm{d}\eta$ is the additional wave number. Integrating Eq. (2), we obtain $(qK-V+3\beta A^2/2)\,A^2=C$, where C is the integration constant which is determined by the stationary-wave parameters for $\eta\to-\infty$. Assuming $C=0$ in what follows, we obtain the following expression for the additional wave number: $K=(V-3\beta A^2/2)/q$. With allowance for this expression, the equation for the packet envelope takes the form

$$q^2\frac{\mathrm{d}^2A}{\mathrm{d}\eta^2}+2\mu qA^2\frac{\mathrm{d}A}{\mathrm{d}\eta}+(V^2-2q\Omega)\,A+(q\alpha-2V\beta)\,A^3+\frac{3}{4}\beta^2A^5=0. \tag{4}$$

Note that in the simplified versions, the equations in the form of Eq. (4) for the stationary waves of the extended Schrödinger equation were more than once obtained, analyzed, and solved in some cases. For example, an analog of Eq. (4) was developed in [22] for $\beta=0$ and $\mu=0$ (i.e., for the classical Schrödinger equation) and different types of its solutions were indicated, namely, periodic waves of the envelope and envelope solitons. The process of appearance of the nonstationary shock wave of the envelope is also described in [22]. In [18], the equation of the type of Eq. (4) is obtained for $\mu=0$ and its solutions are represented.

However, the presence of stimulated scattering (i.e., the nonzero coefficient μ) essentially complicates the structure of Eq. (4). As a result, it is impossible to integrate it and obtain its explicit solution in the general case. It is also impossible to obtain the spatially-bounded stationary solutions within the framework of such an equation. Therefore, we qualitatively analyze Eq. (4) and its solutions for different signs of the differences $V^2-2q\Omega$ and $q\alpha-2V\beta$ by considering the phase space of this equation.

3. STATIONARY WAVES

If the inequality $V^2-2q\Omega>0$ is fulfilled, then, after the change of the variable $\rho=\eta\,\sqrt{V^2-2q\Omega}/q$, Eq. (4) takes the form

$$\frac{\mathrm{d}^2A}{\mathrm{d}\rho^2}-\frac{2\mu}{\sqrt{V^2-2q\Omega}}A^2\frac{\mathrm{d}A}{\mathrm{d}\rho}+A+\frac{q\alpha-2V\beta}{V^2-2q\Omega}A^3+\frac{3}{4}\,\frac{\beta^2}{V^2-2q\Omega}A^5=0. \tag{5}$$

If $q\alpha-2V\beta<0$, after the substitution of the amplitude $A=B\,\sqrt{(V^2-2q\Omega)/(2V\beta-q\alpha)}$, Eq. (5) is reduced to

$$\frac{\mathrm{d}^2B}{\mathrm{d}\rho^2}+pB^2\frac{\mathrm{d}B}{\mathrm{d}\rho}+B-B^3+rB^5=0, \tag{6}$$

where $p=2\mu\,\sqrt{V^2-2q\Omega}/(2V\beta-q\alpha)$ and $r=(3/4)\beta^2\,(V^2-2q\Omega)/(2V\beta-q\alpha)^2$ are the parameters. The parameters $p\propto\propto\mu$ and $r\propto\beta^2$ characterize the stimulated scattering and the nonlinear dispersion, respectively. If $r>1/4$, Eq. (6) has only one equilibrium state $B=0$ (focus), i.e., the stationary waves with limited amplitudes do not exist in this case. In the opposite case where $r\leq 1/4$, Eq. (6) has five equilibrium states, namely, $B=0$ (stable focus), $B=\pm B_+$ (stable foci or nodes), and $B=\pm B_-$ (saddles), where $B_\pm^2=(1\pm\sqrt{1-4r})/(2r)$. Figures 1*a* and 1 *b* show the phase planes of Eq. (6) for $r\leq 1/4$ and various relationships between the parameters p and r. The separatrices going from the saddles (trajectories *1*–*4*) correspond to the stationary kink waves with limited amplitude. Figures 1*c* and 1*d* show the plots of some solutions of Eq. (6), namely, the stationary kink waves corresponding to trajectories *1* and *3* depicted in Figs. 1*a* and 1*b*. Figures 1*a* and 1*c* correspond to the condition $p<p_1=4\sqrt{r\,\sqrt{1-4r}/(1+\sqrt{1-4r})}$ (weak stimulated scattering), while Figs. 1*b* and 1*d* correspond to $p\geq p_1$ (strong stimulated scattering).

For a sufficiently weak stimulated scattering (Figs. 1*a* and 1*c*), only the shock waves with the damping spatial oscillations, both with the pedestal (trajectories *3* and *4*) and without it (trajectories *1* and *2*), are realized. It should be noted that as the nonlinear dispersion disappears($r\to 0$), the equilibrium states $\pm B_+$

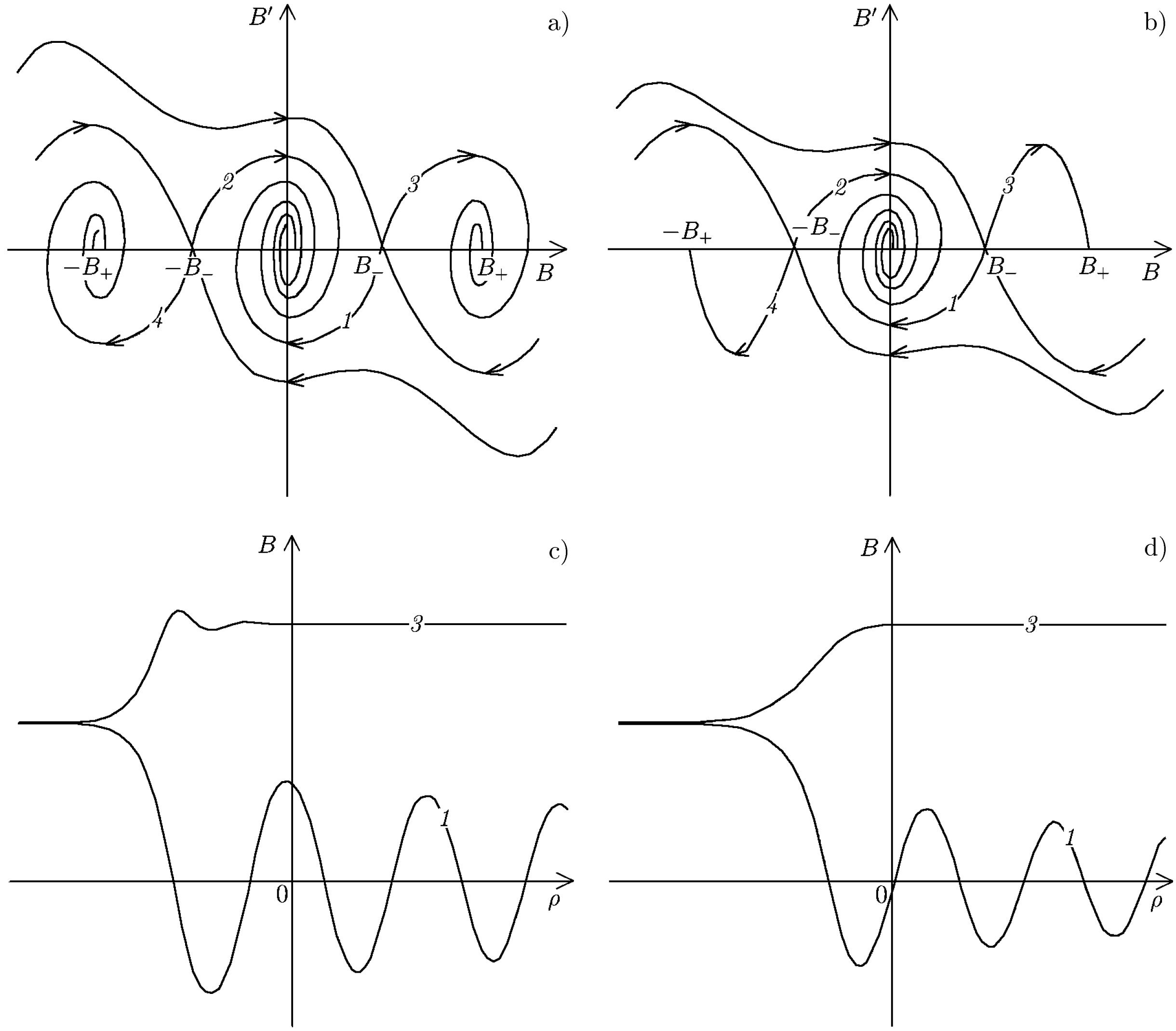

Fig. 1. Phase planes of Eq. (6) (panels *a* and *b*), and the corresponding kink waves for different values of p (panels *c* and *d*). Panels *a* and *c* correspond to the inequality $p < p_1$ and panels *b* and *d* to $p \geq p_1$.

of Eq. (6) go to infinity, while the equilibrium states $\pm B_-$ tend to 1 in magnitude. Therefore, the shock waves without the pedestal (trajectories *1* and *2*) are preserved if the linear dispersion disappears, and their existence is stipulated by the balance of the stimulated-scattering effects, on the one hand, and nonlinearity and the second-order linear dispersion, on the other hand [13, 14]. On the contrary, the shock waves on the pedestal (trajectories *3* and *4*) disappear if the linear dispersion disappears, i.e., their existence is stipulated by the balance of linear dispersion and stimulated scattering.

For a sufficiently strong stimulated scattering (Figs. 1*b* and 1*d*), the shock waves on the pedestals without oscillations are realized (trajectories *3* and *4*).

If the inequality $q\alpha - 2V\beta > 0$ is fulfilled, then Eq. (6) has no solutions in the form of stationary waves with finite amplitude.

In the case $V^2 - 2q\Omega < 0$, Eq. (5) is reduced to

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\rho^2} + \frac{2\mu A^2}{\sqrt{2q\Omega - V^2}} \frac{\mathrm{d}A}{\mathrm{d}\rho} - A + \frac{q\alpha - 2V\beta}{2q\Omega - V^2} A^3 + \frac{3}{4} \frac{\beta^2}{2q\Omega - V^2} A^5 = 0 \tag{7}$$

by using the substitution of the variable $\rho = \eta \sqrt{2q\Omega - V^2}/q$. In the case $q\alpha - 2V\beta < 0$, using the substitution of the amplitude $A = B\sqrt{(2q\Omega - V^2)/(2V\beta - q\alpha)}$, Eq. (7) yields the equation

$$\frac{\mathrm{d}^2B}{\mathrm{d}\rho^2} + pB^2\frac{\mathrm{d}B}{\mathrm{d}\rho} - B - B^3 + rB^5 = 0, \tag{8}$$

where $p = 2\mu\sqrt{2q\Omega - V^2}/(2V\beta - q\alpha)$ and $r = (3/4)\beta^2\,(2q\Omega - V^2)/(q\alpha - 2V\beta)^2$ are the parameters. Equation (8) has three equilibrium states, i.e., $B = 0$ (saddle) and $B = \pm B_0$ (stable foci or nodes), where $B_0^2 = (1 + \sqrt{1 + 4r})/(2r)$. Figures 2*a* and 2*b* show the phase planes of system (8) for different relationships between the parameters p and r. The outgoing separatrices of the saddle (trajectories *1* and *2*) correspond to the amplitude-limited stationary kink waves without pedestal. Figures 2*c* and 2*d* show the plots of some solutions of Eq. (8), i.e., the stationary kink waves corresponding to the trajectories *1* in Figs. 2*a* and 2*b*. Figures 2*a* and 2*c* correspond to the condition $p < p_2 = 4\sqrt{r\sqrt{1+4r}/(1+\sqrt{1+4r})}$ (weak stimulated scattering). In this case, the kink waves have oscillations on the "tails." Figures 2*b* and 2*d* correspond to the inequality $p \geq p_2$ (strong stimulated scattering). In this case, the kink waves without oscillations are realized.

In the special case where $r = (p-1)\,(p+3)/16$, Eq. (8) has an explicit solution in the form of a shock wave $B = \pm\sqrt{2\,(p-1)\,(1-\tanh\rho)}$, which coincides with that obtained in [19].

It should be noted that as nonlinear dispersion disappears ($r \to 0$), the equilibrium states $\pm B_0$ of Eq. (8) go to infinity, which leads to disappearance of the stationary kink waves. Therefore, the existence of the shock waves (trajectories *1* and *2* in Fig. 2) is stipulated by the balance of nonlinear dispersion and stimulated scattering.

If the inequality $q\alpha - 2V\beta > 0$ is fulfilled, then Eq. (7) yields the following equation by substituting the amplitude $A = B\sqrt{(2q\Omega - V^2)/(2V\beta - q\alpha)}$:

$$\frac{\mathrm{d}^2B}{\mathrm{d}\rho^2} + pB^2\frac{\mathrm{d}B}{\mathrm{d}\rho} - B + B^3 + rB^5 = 0, \tag{9}$$

where $p = 2\mu\sqrt{2q\Omega - V^2}/(q\alpha - 2V\beta)$ and $r = (3/4)\beta^2\,(2q\Omega - V^2)/(q\alpha - 2V\beta)^2$ are the parameters. Equation (9) has three equilibrium states, namely, $B = 0$ (saddle) and $B = \pm B_1$ (stable foci or nodes), where $B_1^2 = (\sqrt{1+4r} - 1)/(2r)$. Figures 3*a* and 3*b* demonstrate the phase planes of Eq. (9) for various relationships between the parameters p and r. The separatrices going from the saddle correspond to the amplitude-limited stationary kink waves without pedestal (trajectories *1* and *2*). Figures 3*c* and 3*d* show the plots of some solutions of Eqs. (9), which correspond to the trajectories *1* in Figs. 3*a* and 3*b*. Figures 3*a* and 3*c* correspond to the condition $p < p_3 = 4\sqrt{r\sqrt{1+4r}/(\sqrt{1+4r}-1)}$ (weak stimulated scattering), and Figs. 3*b*, and 3*d* to $p \geq p_3$ (strong stimulated scattering).

For a sufficiently weak stimulated scattering ($p < p_3$; Figs. 3*a* and 3*c*), shock waves with the damping spatial oscillations exist (trajectories *1* and *2*). In the limiting case corresponding to the stimulated-scattering disappearance ($p \to 0$), the ingoing and outgoing separatrices of the saddle close each other and the equilibrium states at the foci $B = \pm B_1$ become the centers. In this case, Eq. (9) has the well-known soliton solution [13] $B^2 = 4/[1 + \sqrt{1 - 16r/3}\cosh(2\rho)]$.

For a sufficiently strong stimulated scattering ($p \geq p_3$; Figs. 3*b* and 3*d*), shock waves without oscillations exist (trajectories *1* and *2*). In the special case where the condition $r = (1+p)\,(p-3)/16$ is fulfilled, Eq. (9) has the explicit analytical solution $B = \pm\sqrt{2\,(p+1)\,(1-\tanh\rho)}$.

It should be noted that on disappearance of nonlinear dispersion ($r \to 0$), the phase-plane structure of Eq. (9) is preserved, i.e., in this case, the kink waves result from the balance of the stimulated-scattering effects, on the one hand, and nonlinearity and the second-order linear dispersion, on the other hand.

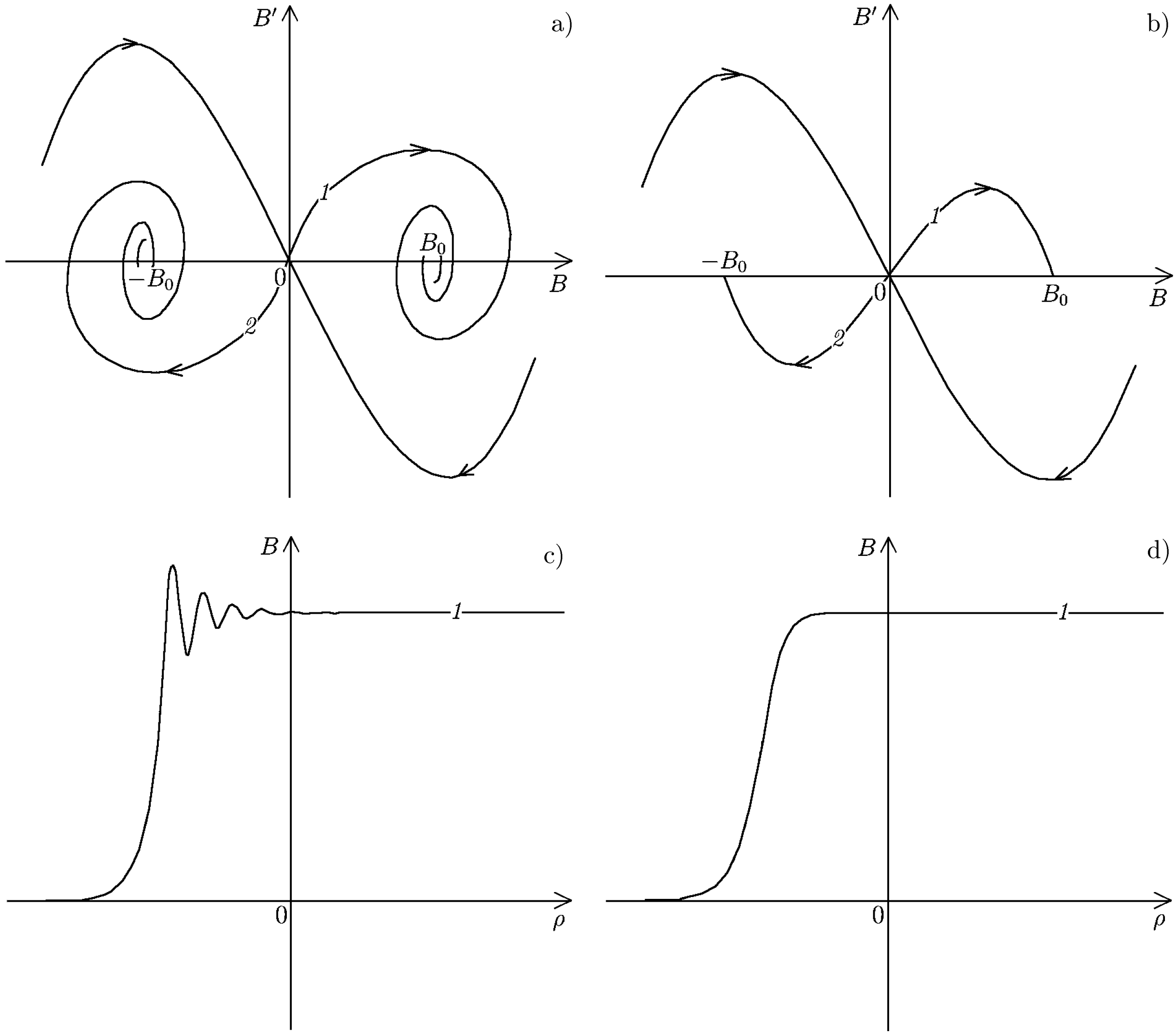

Fig. 2. Phase planes of Eq. (8) (panels *a* and *b*) and the corresponding kink waves (panels *c* and *d*). Panels *a* and *c* correspond to the inequality $p < p_2$, and panels *b* and *d*, to $p \geq p_2$.

4. CONCLUSIONS

In this work, the stationary waves with nonlinear phase modulation were analytically studied within the framework of the nonlinear Schrödinger equation allowing for nonlinear dispersion and stimulated scattering. The kink (shock) waves existing due to the balance of nonlinear dispersion and stimulated scattering were found. The kink waves resulting from the balance of the stimulated-scattering effects, on the one hand, and nonlinearity and the second-order linear dispersion, on the other hand, were also found. The found kink waves include the waves with and without pedestal and, as well as the kink waves with and without oscillations on one "tail." The equation-parameter relationships for which the obtained kink waves exist are indicated. The role of the third-order linear dispersion during the shock-wave formation will be studied in the forthcoming works.

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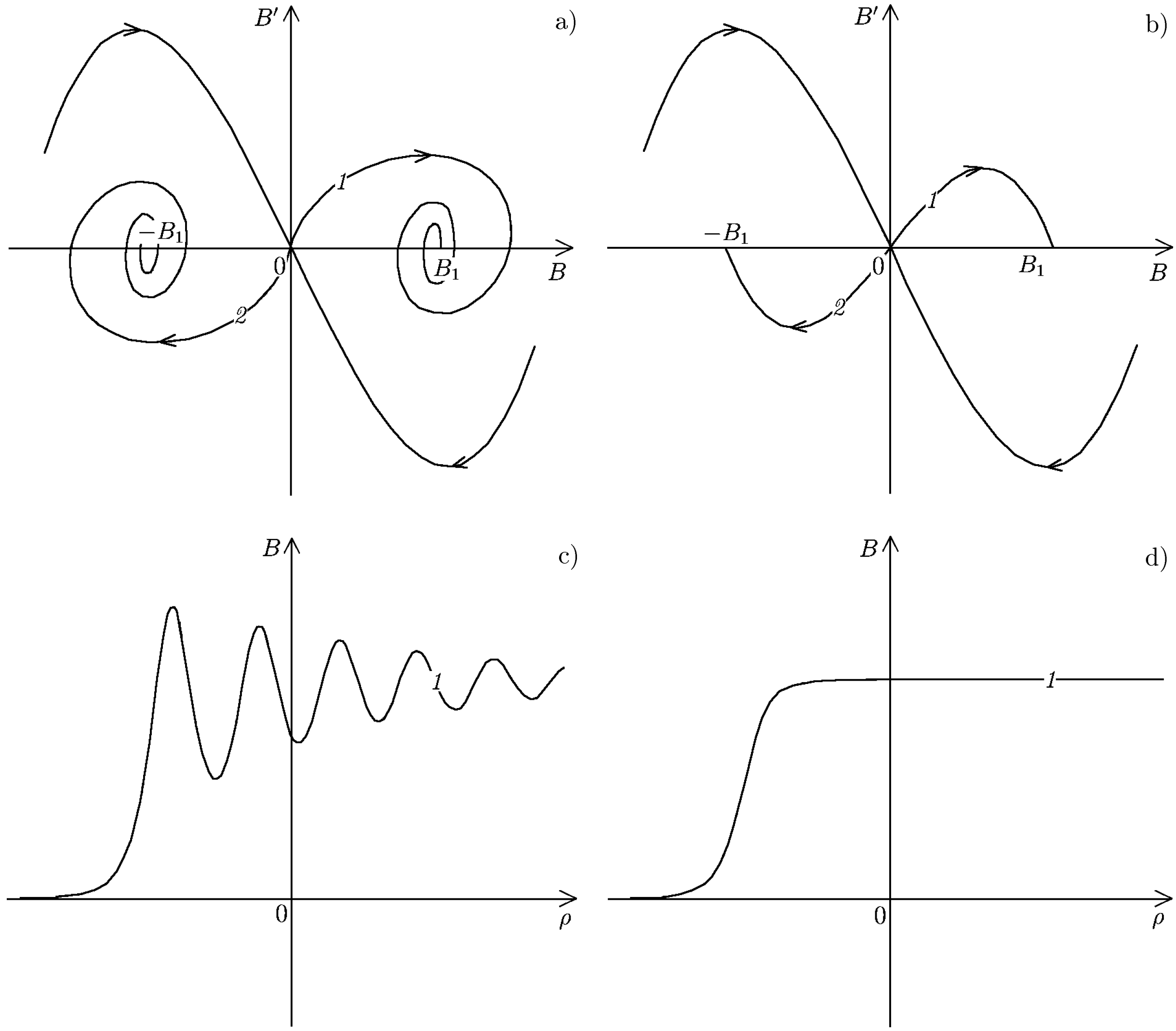

Fig. 3. Phase planes of Eq. (9) (panels *a* and *b*) and the corresponding kink waves (panels *c* and *d*). Panels *a* and *c* correspond to the inequality $p < p_2$ and panels *b* and *d*, to $p \geq p_3$.

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