The description of extreme 2-monotone measures

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Abstract

The paper is devoted to the description of extreme points in the set of 2-monotone measures. We describe them using lattices on which an extreme 2-monotone measure is additive. We also propose the way of generation extreme monotone measures based on the aggregation of extreme measures with the help of multilinear extension. We describe also the class of extreme 2-monotone measures that are additive on the filter on which a 2-monotone measure has positive values.

Keywords. 2-monotone measures, extreme points, additivity on lattices, filters, partially ordered sets, multilinear extension.

1 Introduction

2-monotone measures play an important role in the theory of imprecise probabilities [17], because for imprecise probabilities represented by 2-monotone measures it is possible to find analytical solutions for many problems and, therefore, such models are more attractive in a computational point of view. Meanwhile, some unsolved problems concerning 2-monotone measures can be solved [5] if we know the structure of extreme points of the set of all 2-monotone measures. It is worth to mention that finding description of extreme points of a convex set is usually a hard problem. This problem is solved for the set of all monotone measures [13,15], *p*-symmetrical measures [7,8], but for some convex families, e.g. k-additive measures [7], is far from the final solution.

The aim of this paper is to make one step forward in this direction, providing some general necessary and sufficient conditions that a 2-monotone measure is an extreme point and giving descriptions of some special families of them.

The paper has the following structure. We remind first some results concerning monotone measures and criteria of 2-monotonicity. After that we provide general necessary and sufficient conditions that a 2-monotone measure is an extreme point through lattices on which it is additive. After that we remind the multilinear extension of monotone measures and using it we define the composition of monotone measures. We show that the composition of extreme 2-monotone measures is an extreme 2-monotone measure again. The paper is ended by describing a special class of 2-monotone measures which are additive on the filter of sets on which a 2-monotone measure has positive values.

2 Monotone measures

Let X be a finite set and let $\mu : 2^X \to [0,1]$ be a set function on the powerset 2^X. Then μ is called a *monotone measure* [9] if the following conditions hold:

1) $\mu(\varnothing) = 0$ and $\mu(X) = 1$;

2) $A \subseteq B$ for $A, B \in 2^X$ implies $\mu(A) \le \mu(B)$.

Let us denote the set of all monotone measures on 2^X by $M_{mon}(X)$ or briefly M_{mon} if the set X is clearly defined from the context. For monotone measures $\mu_1, \mu_2 \in M_{mon}$ we define their convex sum as $\mu(A) = a\mu_1(A) + (1-a)\mu_2(A)$, where $a \in [0,1]$ and $A \in 2^X$. Clearly, $\mu \in M_{mon}$, i.e. the set M_{mon} is convex and it is possible to show [13,15] that extreme points of M_{mon} are $\{0,1\}$-valued monotone measures, i.e. monotone measures with values in $\{0,1\}$. Let the algebra 2^X be considered as a partially ordered set w.r.t. inclusion of sets. By definition, a filter $\mathbf{f}$ in 2^X is a nonempty subset of 2^X such that $A \in \mathbf{f}$, $A \subseteq B$ implies $B \in \mathbf{f}$. Any filter can be uniquely defined by the set of its minimal elements $\{A_1, ..., A_m\}$. This fact is denoted by $\mathbf{f} = \langle A_1, ..., A_m \rangle$. The connection between filters of algebra 2^X and $\{0,1\}$-valued monotone measures is shown in the following lemma [13].

Lemma 1. *Any* $\{0,1\}$*-valued monotone measure* η *defines a filter* $\mathbf{f} = \{A \in 2^X \mid \eta(A) > 0\}$ *such that* $\varnothing \notin \mathbf{f}$. *Conversely, any filter* $\mathbf{f}$ *with* $\varnothing \notin \mathbf{f}$ *defines a* $\{0,1\}$*-valued monotone measure* η *by*

$$\eta(A) = \begin{cases} 1, & A \in \mathbf{f}, \\ 0, & A \notin \mathbf{f}. \end{cases} \quad (1)$$

In the sequel we denote a $\{0,1\}$-valued measure as $\eta_{\mathbf{f}}$ if it corresponds to a filter $\mathbf{f}$.

Remark 1. Clearly, a set $\mathbf{f}(t) = \{A \in 2^X \mid \mu(A) > t\}$ for any given $\mu \in M_{mon}$ and $t \in [0,1)$ is a filter in algebra

2^X , moreover, $\mu(A)=\int_0^1 \eta_{\mathbf{f}(t)}(A)dt$ and if $\{t_1,t_2,...,t_k\}$ is the set of all values of μ and $0=t_1<t_2<...<t_k=1$, then $\mu=\sum_{i=1}^{k-1}(t_{i+1}-t_i)\eta_{\mathbf{f}(t_i)}$.

3 2-monotone measures

A monotone measure μ is called *2-monotone* [9] if the following inequality

$$\mu(A)+\mu(B)\le\mu(A\cap B)+\mu(A\cup B) \qquad (2)$$

is fulfilled for any $A,B\in 2^X$. We denote the set of all 2-monotone measures on the algebra 2^X by $M_{2-mon}(X)$. The condition (2) can be simplified [3]. It is sufficient to check inequalities of the following type:

$$\mu(A\cup\{x_i\})+\mu(A\cup\{x_j\})\le\mu(A)+\mu(A\cup\{x_i,x_j\}) \text{ , (3)}$$

for all $A\in 2^X$ and $x_i,x_j\in 2^X$ such that $|X\setminus A|\ge 2$, $x_i,x_j\in X\setminus A$ and $x_i\ne x_j$.

In the next we can also consider nonnegative set functions μ on 2^X with $\mu(\varnothing)=0$. Such set functions are called 2-monotone if they are monotone and inequalities (2) or equivalently inequalities (3) are fulfilled. The next proposition shows that the monotonicity of μ is not necessary to check.

Proposition 1. Let μ *be a nonnegative set function on* 2^X *with* $\mu(\varnothing)=0$. *Then it is 2-monotone iff inequalities* (3) *are fulfilled* for all $A\in 2^X$ and $x_i,x_j\in 2^X$ such that $|X\setminus A|\ge 2$, $x_i,x_j\in X\setminus A$ and $x_i\ne x_j$.

Let us consider how Proposition 1 can be strengthened if we know that the sets on which a nonnegative set function is positive, form a filter. In this case we say that μ is *2-monotone on the filter* $\mathbf{f}=\{A\in 2^X\mid\mu(A)>0\}$, if inequalities (3) are fulfilled, when $A\cup\{x_i\}\in\mathbf{f}$ and $A\cup\{x_j\}\in\mathbf{f}$. In addition, we say that a set function μ, which is 2-monotone on the filter $\mathbf{f}$, is also *2-monotone on its borders* if inequalities (3) are fulfilled if at least $A\cup\{x_i\}\in\mathbf{f}$ or $A\cup\{x_j\}\in\mathbf{f}$. Next proposition is the direct consequence of Proposition 1.

Proposition 2. *Given a nonnegative set function* μ *such that* $\mathbf{f}=\{A\in 2^X\mid\mu(A)>0\}$ *is a filter. Then* μ *is 2-monotone iff it is 2-monotone on the filter* $\mathbf{f}$ *and its borders.*

In some cases the 2-monotonicity on the filter can imply the 2-monotonicity on its borders. The description of such a case is given in the following proposition.

Proposition 3. *Let a nonnegative set function be 2-monotone on the filter* $\mathbf{f}\supseteq\{A\in 2^X\mid\mu(A)>0\}$ *and let* $\mathcal{C}=\{C_1,...,C_m\}$ *be the set of its minimal elements. Then* μ *is 2-monotone on borders of* $\mathbf{f}$, *if for every* $C_k\in\mathcal{C}$ *and every* $x_i\notin C_k$ *there exists a* $C_l\in\mathcal{C}$, *such that* $\{x_i\}=C_l\setminus C_k$.

4 Additivity properties of 2-monotone measures on lattices

We denote by $M_{pr}(X)$ the set of all probability measures on the algebra 2^X. Let $\mu\in M_{2-mon}$, then the *core* of μ is the set of probability measures defined by $core(\mu)=\{P\in M_{pr}\mid P\ge\mu\}$. It is well known [16] that $core(\mu)$ is a nonempty convex set for any $\mu\in M_{2-mon}$ and its extreme points are probability measures P_γ, where $\gamma:\{1,2,...,n\}\to\{1,2,...,n\}$ is a permutation of the set $\{1,2,...,n\}$ and any P_γ is constructed with the help of the chain of sets $B_1=\{x_{\gamma(1)}\}$, $B_2=\{x_{\gamma(1)},x_{\gamma(2)}\}$, …, $B_n=\{x_{\gamma(1)},...,x_{\gamma(n)}\}$ by the rule: $P_\gamma(B_i)=\mu(B_i)$, $i=1,...,n$. Let us remind the result from [2,4], that can be also found in [10].

Proposition 4. *Let* $\mu\in M_{2-mon}$, *then the system of sets* $\mathcal{L}_\gamma(\mu)=\{A\in 2^X\mid\mu(A)=P_\gamma(A)\}$ *is a lattice w.r.t. operations* $\cap$ *and* $\cup$, *i.e.* $A,B\in\mathcal{L}_\gamma(\mu)$ *implies* $A\cap B, A\cup B\in\mathcal{L}_\gamma(\mu)$ *and it is a maximal lattice, on which* μ *is additive.*

Remark 2. Additivity of μ on $\mathcal{L}_\gamma(\mu)$ means that if $A,B\in\mathcal{L}_\gamma(\mu)$, then

$$\mu(A)+\mu(B)=\mu(A\cap B)+\mu(A\cup B).$$

As we will see in the next such maximal lattices play an important role for the extreme points description of 2-monotone measures. Therefore, we also present here some results showing the connections between such lattices and partially ordered sets.

Let us assume that a maximal lattice $\mathfrak{L}$, on which a 2-monotone measure μ is additive, contains maximal chains described by a set of permutations $\Gamma=\{\gamma_i\}$. We put into correspondence the linear order ρ_γ on X to each permutation $\gamma\in\Gamma$ in a way that $x_i\rho_\gamma x_j$ if $\gamma(i)\le\gamma(j)$. Then the following theorem is valid.

Theorem 1. *Let* $\mathfrak{L}$ *be a maximal lattice, on which a 2-monotone measure* μ *is additive, and let the maximal chains in* $\mathfrak{L}$ *be described by a a set of permutations* $\Gamma=\{\gamma_i\}$. *Consider the partial order* $\rho_\Gamma=\bigcap_{\gamma\in\Gamma}\rho_\gamma$ [1], *where linear orders* ρ_γ *are defined as above. Then* $\{\rho_\gamma\}_{\gamma\in\Gamma}$ *is the set of all linear orders satisfying* $\rho_\gamma\supseteq\rho_\Gamma$.

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[1] Here is used the usual intersection of relations, i.e. if $\rho_1,\rho_2\in\{1,...,n\}\times\{1,...,n\}$, then $\rho_1\cap\rho_2$ is the usual intersection defined for sets.

Next result can be considered as a corollary of more general result that can be found in [10].

Theorem 2. *Let* ρ *be a partial order on* X *and let* $\{\rho_\gamma\}_{\gamma\in\Gamma}$ *be the set of all linear extensions* ρ*, i.e. each* ρ_γ *is a linear order and* $\rho_\gamma \supseteq \rho$*. Then any* ρ_γ*,* $\gamma\in\Gamma$*, induces a chain of sets* $B_0=\varnothing$*,* $B_1=\{x_{\gamma(1)}\}$*,* $B_2=\{x_{\gamma(1)},x_{\gamma(2)}\}$*, ...,* $B_n=\{x_{\gamma(1)},x_{\gamma(2)},...,x_{\gamma(n)}\}$ *and the union of all such chains is a lattice of sets w.r.t. union and intersection.*

5 The description of extreme 2-monotone measures through lattices

Proposition 5. *Let us consider the set of all maximal lattices* $\mathfrak{L}_\gamma(\mu)$*, on which a 2-monotone measure* μ *is additive, and let* $\mathbf{f}_\mu=\{A\in 2^X \mid \mu(A)>0\}$*. Then* μ *is not an extreme point in* M_{2-mon} *iff there exists a 2-monotone measure* ν *(*$\nu\neq\mu$*) such that*

1) $\mathfrak{L}_\gamma(\mu)\subseteq\mathfrak{L}_\gamma(\nu)$ *for any permutation* γ*;*

2) $\mathbf{f}_\nu\subseteq\mathbf{f}_\mu$.

Corollary 1. *Let* μ *be an extreme point in* M_{2-mon}*. Then the filter* $\mathbf{f}_\mu$ *and the system of lattices* $L_\gamma(\mu)$ *define* μ *uniquely.*

6 Multilinear extension and composition of monotone measures

In this section we will use the notion of pseudo-Boolean functions [12]. Any pseudo-Boolean function is a mapping $\varphi:\{0,1\}^n\to\mathbb{R}$. For our purpose, it is sufficient to consider pseudo-Boolean functions taking their values in $[0,1]$, i.e. we assume that $\varphi:\{0,1\}^n\to[0,1]$. It is easy to see that there is a one-to-one correspondence between pseudo-Boolean functions and set functions. For this purpose, we consider set functions defined on the algebra 2^Z, where $Z=\{1,...,n\}$, and consider vectors $\mathbf{1}_A=(x_1,...,x_n)$, where $A\in 2^Z$ and $x_i=1$ if $i\in A$ and $x_i=0$ otherwise. Then obviously $\mu(A)=\varphi(\mathbf{1}_A)$, where $A\in 2^Z$ is a set function on 2^Z. If we consider the class of monotone pseudo-Boolean functions $\varphi:\{0,1\}^n\to[0,1]$ with $\varphi(\mathbf{0})=0$ and $\varphi(\mathbf{1})=1$, where $\mathbf{0}=(0,...,0)$ and $\mathbf{1}=(1,...,1)$, then it corresponds to the class of monotone measures on 2^Z.

Any pseudo-Boolean function can be uniquely represented as a multilinear polynomial [14] as

$$\varphi(\mathbf{x})=\sum_{A\in 2^Z} m(A)\prod_{i\in A} x_i\,, \qquad (4)$$

where m is the Möbius transform m of the set function $\mu(A)=\varphi(\mathbf{1}_A)$, defined by

$$m(A)=\sum_{B\subseteq A}(-1)^{|A\setminus B|}\mu(B)\,.$$

We see that there is a one-to-one correspondence between multilinear polynomials and pseudo-Boolean functions. In addition, we can assume that the vector $\mathbf{x}$ in formula (4) can take values in $[0,1]^n$. In this case, the function $\tilde{\varphi}:[0,1]^n\to[0,1]$ is called [14] the *multilinear extension* of φ.

The next proposition [3] shows how to check monotonicity and 2-monotonicity of a set function using its multilinear extension.

Proposition 6. *Let* $\mu:2^Z\to[0,1]$ *and let* φ *be its corresponding pseudo-Boolean function. Then* μ *is a monotone measure iff the multilinear extension* $\tilde{\varphi}$ *of* φ *has the following properties:*

1) $\tilde{\varphi}(\mathbf{0})=0$ *and* $\tilde{\varphi}(\mathbf{1})=1$*;*

2) $\dfrac{\partial\tilde{\varphi}(\mathbf{x})}{\partial x_i}\geq 0$ *for any* x_i *and at any point* $\mathbf{x}\in[0,1]^n$.

In addition, μ *is 2-monotone iff*

3) $\dfrac{\partial^2\tilde{\varphi}(\mathbf{x})}{\partial x_i\partial x_j}\geq 0$ *for any* x_i,x_j *and at any point* $\mathbf{x}\in[0,1]^n$.

Proposition 6 shows that the multilinear extension of a monotone measure is an aggregation function. Let us remind that, by definition [11], an aggregation function $\tilde{\varphi}$ is a mapping $\tilde{\varphi}:[0,1]^n\to[0,1]$ such that

1) $\tilde{\varphi}(\mathbf{0})=0$ and $\tilde{\varphi}(\mathbf{1})=1$;

3) $\tilde{\varphi}(\mathbf{x})\leq\tilde{\varphi}(\mathbf{y})$ for $\mathbf{x},\mathbf{y}\in[0,1]^n$ if $\mathbf{x}\leq\mathbf{y}$ ($\mathbf{x}\leq\mathbf{y}$ means for $\mathbf{x}=(x_1,...,x_n)$ and $\mathbf{y}=(y_1,...,y_n)$ that $x_i\leq y_i$, $i=1,...,n$).

We can generate monotone measures using aggregation functions as follows. Let $\tilde{\varphi}:[0,1]^n\to[0,1]$ be an aggregation function and $X_1,...,X_n$ be mutually disjoint finite nonempty sets and let μ_i, $i=1,..,n$, be monotone measures on 2^{X_i}. Then a set function μ on 2^X, where $X=X_1\cup...\cup X_n$, defined by

$$\mu(A)=\tilde{\varphi}\left(\mu_1(A\cap X_1),...,\mu_n(A\cap X_n)\right),\ A\in 2^X, \qquad (5)$$

is also a monotone measure. For the measure μ, defined by formula (5), we will use the notation $\mu=\tilde{\varphi}\circ\boldsymbol{\mu}$, where $\boldsymbol{\mu}=(\mu_1,...,\mu_n)$.

In this section, we will use multilinear polynomials as aggregation functions. It can be shown [3] that if φ is a multilinear extension of a 2-monotone measure and μ_i, $i=1,...,n$, be 2-monotone measures on 2^{X_i}, then $\mu=\varphi\circ\boldsymbol{\mu}$ is also a 2-monotone measure.

Obviously, we can introduce the same representation like (5) for pseudo-Boolean functions. Let $\tilde{\varphi}:[0,1]^n\to[0,1]$ be an aggregation function and let $\mu_i(\mathbf{x}^{(i)})$, $i=1,...,n$, be pseudo-Boolean functions. Then the aggregation of these functions is defined as

$\mu(\mathbf{x})=\tilde{\varphi}\left(\mu_1(\mathbf{x}^{(1)}),...,\mu_n(\mathbf{x}^{(n)})\right)$, where $\mathbf{x}=\left(\mathbf{x}^{(1)},...,\mathbf{x}^{(n)}\right)$.

Proposition 7. *Let* $\tilde{\varphi}:[0,1]^n\to\mathbb{R}$ *be the multilinear extension of a pseudo-Boolean function* φ *and let*

$\mu_i(\mathbf{x}^{(i)})$, $i=1,...,n$, *be pseudo-Boolean functions. Let us consider the pseudo-Boolean function* $\mu(\mathbf{x})=\tilde{\varphi}\left(\mu_1(\mathbf{x}^{(1)}),...,\mu_n(\mathbf{x}^{(n)})\right)$, *where* $\mathbf{x}=\left(\mathbf{x}^{(1)},...,\mathbf{x}^{(n)}\right)$. *Then the multilinear extension of* μ *can be computed as*

$$\tilde{\mu}(\mathbf{x})=\tilde{\varphi}\left(\tilde{\mu}_1(\mathbf{x}^{(1)}),...,\tilde{\mu}_n(\mathbf{x}^{(n)})\right).$$

Remark 3. Proposition 7 allows us to represent the aggregation (5) using more simple aggregations as follows. Let $\tilde{\varphi}(t_1,...,t_n)$ be a multilinear extension of a pseudo-Boolean function $\varphi:\{0,1\}^n\to[0,1]$. Then we can consider the following sequence of pseudo-Boolean functions

$$\varphi_0=\tilde{\varphi}(t_1,...,t_n),\ \varphi_1=\tilde{\varphi}\left(\mu_1(\mathbf{x}^{(1)}),t_2,...,t_n\right),$$
$$\varphi_2=\tilde{\varphi}\left(\mu_1(\mathbf{x}^{(1)}),\mu_2(\mathbf{x}^{(2)}),t_3,...,t_n\right),$$
$$\varphi_n(\mathbf{x})=\tilde{\varphi}\left(\mu_1(\mathbf{x}^{(1)}),...,\mu_n(\mathbf{x}^{(n)})\right),$$

where $t_i\in\{0,1\}$, $i=1,...,n$, and corresponding aggregation functions:

$$\tilde{\varphi}_0=\tilde{\varphi}(t_1,...,t_n),\ \tilde{\varphi}_1=\tilde{\varphi}\left(\tilde{\mu}_1(\mathbf{x}^{(1)}),t_2,...,t_n\right),$$
$$\tilde{\varphi}_2=\tilde{\varphi}\left(\tilde{\mu}_1(\mathbf{x}^{(1)}),\tilde{\mu}_2(\mathbf{x}^{(2)}),t_3,...,t_n\right),$$
$$\tilde{\varphi}_n(\mathbf{x})=\tilde{\varphi}\left(\tilde{\mu}_1(\mathbf{x}^{(1)}),...,\tilde{\mu}_n(\mathbf{x}^{(n)})\right),$$

that have to be obviously multilinear extensions of corresponding pseudo-Boolean functions. Each φ_i is generated from φ_{i-1} by replacing variable t_i with the pseudo-Boolean function $\mu_i(\mathbf{x}^{(i)})$.

The interpretation of simple aggregations, considered in Remark 3, through set functions is given in the following lemma.

Lemma 2. *Let* $\varphi_1:\{0,1\}^n\to[0,1]$ *and* $\varphi_2:\{0,1\}^m\to[0,1]$ *be pseudo-Boolean functions and let* $\tilde{\varphi}_i$, $i=1,2$, *be their multilinear extensions. Consider their aggregation of the following type:*

$$\varphi(x_1,...,x_{n-1},x_{n+1},....,x_{n+m})=\tilde{\varphi}_1(x_1,...,x_{n-1},\varphi_2(x_{n+1},....,x_{n+m})),$$

and corresponding set functions on 2^Z, *where* $Z=\{1,...,n+m\}$*:*

$$\mu_1(A)=\varphi_1(\mathbf{1}_A),\ \textit{where}\ A\subseteq\{1,2,...,n\};$$
$$\mu_2(B)=\varphi_2(\mathbf{1}_B),\ \textit{where}\ B\subseteq\{n+1,...,n+m\};$$
$$\mu(C)=\varphi(\mathbf{1}_C),\ \textit{where}\ C\subseteq\{1,...,n-1,n+1,...,n+m\}.$$

Then

$$\mu(A\cup B)=\mu_1(A)+(\mu_1(A\cup\{n\})-\mu_1(A))\mu_2(B),$$

where $A\subseteq\{1,...,n-1\}$ *and* $B\subseteq\{n+1,...,n+m\}$.

Like in the theory of Boolean functions, let us introduce the notion of essential variable for pseudo-Boolean functions. Let $\varphi:\{0,1\}^n\to[0,1]$ be a pseudo-Boolean function. The variable x_i is called *essential* for φ if there are vectors $\mathbf{x}_1=\left(x_1,...,x_{i-1},0,x_{i+1},...,x_n\right)$ and $\mathbf{x}_2=\left(x_1,...,x_{i-1},1,x_{i+1},...,x_n\right)$ in $\{0,1\}^n$ such that $\varphi(\mathbf{x}_1)\neq\varphi(\mathbf{x}_2)$. It is easy to express such a property using set functions. Let $\mu(A)=\varphi(\mathbf{1}_A)$, where $A\in 2^Z$. Then the variable x_i is essential if the set function $\nu(A)=\mu(A\cup\{i\})-\mu(A)$, where $A\in 2^Z$, is not identical to zero.

Proposition 8. *Let* $\varphi:\{0,1\}^n\to[0,1]$ *be a pseudo-Boolean function and let* $\tilde{\varphi}:[0,1]^n\to[0,1]$ *be its multilinear extension. Then the variable* x_i *is essential for* φ *iff there is a* $\mathbf{x}\in[0,1]^n$ *such that* $\frac{\partial\tilde{\varphi}(\mathbf{x})}{\partial x_i}\neq 0$.

Proposition 9. Let $\mu=\tilde{\varphi}\circ\boldsymbol{\mu}$ *be the aggregation defined by formula (5), and let* $\tilde{\varphi}:[0,1]^n\to[0,1]$ *be a multilinear extension of a monotone measure* φ. *Then representation* $\mu=\tilde{\varphi}\circ\boldsymbol{\mu}$ *for fixed sets* $X_1,...,X_n$ *is defined uniquely iff each variable in* φ *is essential.*

In this section we will prove the following result.

Theorem 3. *Let* $\tilde{\varphi}:[0,1]^n\to[0,1]$ *be a multilinear extension of a 2-monotone measure* φ *on* 2^Z *and let all variables of* $\tilde{\varphi}$ *be essential. Let us assume that* μ_i *are 2-monotone measures on* 2^{X_i}, *where* $X_1,...,X_n$ *are mutually disjoint finite nonempty sets. Then* $\mu=\tilde{\varphi}\circ\boldsymbol{\mu}$ *is an extreme point iff 2-monotone measures* φ, $\mu_1,...,\mu_n$ *are extreme points too.*

7 Examples of extreme 2-monotone measures

Let μ be an extreme 2-monotone measure. Then we call it *perfect* if it is uniquely defined by a filter $\mathbf{f}=\left\{A\in 2^X\mid\mu(A)>0\right\}$, in other words, an extreme measure is not perfect if there is another extreme 2-monotone measure with the same filter $\mathbf{f}$, on which it has positive values. We will describe next the class of such extreme 2-monotone measures.

Let μ be a set function on 2^X. We say that μ is *additive on a filter* $\mathbf{f}$ if

a) $\mu(A)=0$ for any $A\notin\mathbf{f}$;

b) $\mu(A\cup\{x_i\})+\mu(A\cup\{x_j\})=\mu(A)+\mu(A\cup\{x_i,x_j\})$ for any sets $A\cup\{x_i\},A\cup\{x_j\}\in\mathbf{f}$ such that $x_i,x_j\notin A$ and $x_i\neq x_j$.

Lemma 3. *Let a set function* μ *be additive on a filter* $\mathbf{f}$. *Consider any* $A\in\mathbf{f}$ *and* $x_i\notin A$. *Then* $\mu(A\cup\{x_i\})-\mu(A)=\mu(C\cup\{x_i\})-\mu(C)$ *for any* $C\in\mathbf{f}$ *with* $C\subseteq A$.

Corollary 2. *If the set function* μ *is additive on a filter* $\mathbf{f}$, *then* $\mu(A\cup\{x_i\})-\mu(A)=\mu(C\cup\{x_i\})-\mu(C)$ *for any* $A,C\in\mathbf{f}$ *such that* $A,C\subseteq X\setminus\{x_i\}$.

The results formulated in Lemma 3 and Corollary 2 can be better described by the function

$$\nu(x_i)=\mu(A\cup\{x_i\})-\mu(A),$$

where $A \in \mathbf{f}$ and $x_i \notin A$. Let us notice that $\nu(x_i)$ does not depend on the choice of A. The value $\nu(x_i)$ is called *the weight* of x_i on filter $\mathbf{f}$ for a set function μ.

Proposition 10. *Let a nonnegative set function μ be additive on a filter $\mathbf{f}$, and $\nu(x_i) \geq 0$ for all $x_i \in X$. Then μ is 2-monotone.*

Proposition 11. *Let a nonnegative set function μ be additive on a filter $\mathbf{f}$. Let us consider the system of sets $2^X \setminus \mathbf{f}$ and the set of its maximal elements $\{C_1,...,C_k\}$. Then μ is 2-monotone if $\{\bar{C}_1,...,\bar{C}_k\}$ is a covering of X.*

Let us consider how to construct 2-monotone measures that are additive on a filter. We prove first the following auxiliary lemma.

Lemma 4. *Let $\mathbf{f}$ be a filter of the algebra 2^X. Then the system of sets*

$$\mathbf{f}_0 = \{A \mid A \cup \{x_i\}, A \cup \{x_j\} \in \mathbf{f}, x_i, x_j \notin A\}$$

is also a filter and $\mathbf{f}_0 \supseteq \mathbf{f}$.

Proposition 12. *Let we use the notations from Lemma 4, $A \in \mathbf{f}_0$, $x_j \notin A$, and let a set function μ be additive on the filter $\mathbf{f}$. Then the value $\nu(x_j) = \mu(A \cup \{x_j\}) - \mu(A)$ does not depend on the choice of $A \in \mathbf{f}_0$.*

Corollary 3. *Let $A \in \mathbf{f}_0$, $B \supseteq A$, and let μ be additive on the filter $\mathbf{f}$. Then*

$$\mu(B) = \mu(A) + \sum_{x_i \in B \setminus A} \nu(x_i).$$

Corollary 4. *Let $A \cap B \in \mathbf{f}_0$ for sets A and B, and let μ be additive on the filter $\mathbf{f}$. Then*

$$\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B).$$

Proposition 13. *Let a set function μ be additive on the filter $\mathbf{f}$. Then values of ν obeys the following system of equations*:

$$\sum_{x_i \notin B} \nu(x_i) = \mu(X) \text{ for all } B \in \mathbf{f}_0 \setminus \mathbf{f}, \qquad (6)$$

in addition

$$\mu(B) = \begin{cases} 0, & B \notin \mathbf{f}, \\ \mu(X) - \sum_{x_i \notin B} \nu(x_i), & B \in \mathbf{f}, \end{cases} \qquad (7)$$

Conversely, each set function μ obeying equalities (6) *and* (7) *is additive on the filter $\mathbf{f}$.*

Remark 4. Solving equations (6) and (7) w.r.t. $\nu(x_i)$ we can find all set functions that are additive on the filter $\mathbf{f}$, i.e. it is guaranteed that any such function satisfies

1) $\mu(A \cup \{x_i\}) + \mu(A \cup \{x_j\}) = \mu(A) + \mu(A \cup \{x_i, x_j\})$ for $A \cup \{x_i\}, A \cup \{x_j\} \in \mathbf{f}$ and $A \cap \{x_i, x_j\} = \varnothing$;

2) $\mu(A) = 0$ for $A \notin \mathbf{f}$.

However, we can not guarantee that μ is 2-monotone, because 2-monotonicity of μ in this case is equivalent to $\nu(x_i) \geq 0$ for all $x_i \in X$ by Proposition 10.

Proposition 14. *Let μ be a 2-monotone measure that is additive on the filter $\mathbf{f} = \{A \in 2^X \mid \mu(A) > 0\}$. Then μ is an extreme 2-monotone measure if it is defined uniquely.*

Proposition 15. *Let the filter $\mathbf{f}$ obey the conditions formulated in Proposition 3. Then if an extreme 2-monotone measure μ, which is additive on $\mathbf{f} = \{A \in 2^X \mid \mu(A) > 0\}$, exists, then it is perfect.*

Proposition 16. *Let the filter $\mathbf{f}$ obey the conditions formulated in Proposition 11. Then if an extreme 2-monotone measure μ, which is additive on $\mathbf{f} = \{A \in 2^X \mid \mu(A) > 0\}$, exists, then it is defined uniquely.*

Let us consider examples of perfect 2-monotone measures that are additive on filter. A monotone measure is called *symmetrical* if its values depend only on the cardinality of the corresponding set. The next proposition gives the description of extreme symmetrical 2-monotone measures.

Proposition 17. *Let $X = \{x_1, x_2, ..., x_n\}$. Then any symmetrical monotone measure, defined by*

$$\mu_k(A) = \begin{cases} 0, & |A| < k-1, \\ (m-k+1)/(n-k+1), & |A| = m \geq k-1, \end{cases}$$

where $k = 2,...,n$, is a perfect extreme 2-monotone measure.

Remark 5. It is easy to show that the set of all symmetrical 2-monotone measures on 2^X, where $X = \{x_1, x_2, ..., x_n\}$, is convex and the extreme points of it are measures μ_k, $k = 1,...,n$. Obviously, μ_1 is not an extreme point of M_{2-mon} if $n = 1$, because it is represented as $\mu_1 = (1/n)\sum_{k=1}^{n} \eta_{\langle\{x_k\}\rangle}$.

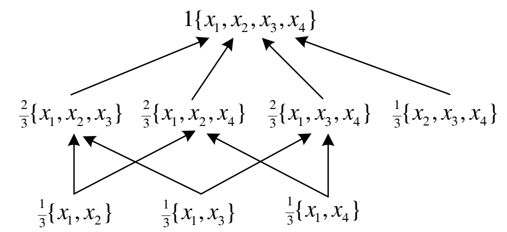

Figure. 1: A perfect extreme 2-monotone measure that is additive on the filter.

Remark 6. Let $X = \{x_1, x_2, x_3\}$, then the extreme points of $M_{2-mon}(X)$ are perfect and they are additive on the filter of their positive values. These measures are $\eta_{\langle A \rangle}$, where $|A| > 0$, and the symmetrical measure μ_2 for $n = 3$. If $X = \{x_1, x_2, x_3, x_4\}$, then extreme points of $M_{2-mon}(X)$ are not necessarily measures described in Proposition 17. For example, let us consider the filter $\mathbf{f} = \langle \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3, x_4\} \rangle$. Let us try to find a 2-monotone measure μ that is additive on

$\mathbf{f}=\left\{A\in 2^X \mid \mu(A)>0\right\}$. In this case, $\mathbf{f}_0=\left\langle \{x_1\},\{x_2,x_3\},\{x_2,x_4\},\{x_3,x_4\}\right\rangle$, and the function $\nu(x_i)$, $x_i\in X$, should obey the following linear system of equations:

$$\begin{cases} \nu(x_2)+\nu(x_3)+\nu(x_4)=1, \\ \nu(x_1)+\nu(x_2)=1, \\ \nu(x_1)+\nu(x_3)=1, \\ \nu(x_1)+\nu(x_4)=1, \end{cases}$$

that has the following unique solution $\nu(x_1)=2/3$, $\nu(x_2)=\nu(x_3)=\nu(x_4)=1/3$. After that we can calculate the values of 2-monotone measure μ by the formula (7). This measure is depicted on Figure 1. Using Proposition 15 it easy to check that μ is a perfect extreme 2-monotone measure.

Let us consider an example of extreme 2-monotone measure μ, depicted on Figure 2, that is not additive on the filter $\mathbf{f}=\left\{A\in 2^X \mid \mu(A)>0\right\}$. Using Corollary 1, it is easy to show that it is an extreme point of M_{2-mon}. In addition, it is possible to show that μ is a perfect extreme 2-monotone measure.

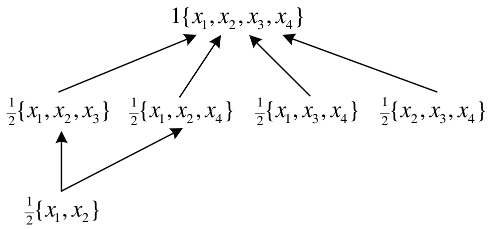

Figure. 2: A perfect extreme 2-monotone measure that is not additive on the filter.

It is easy to find extreme 2-monotone measures that are not perfect. Such measures are depicted on Figure 3, with parameters α,β,γ, and λ, given in Table 1.

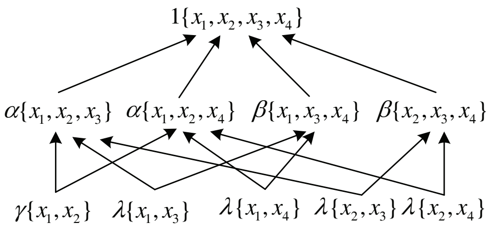

Figure 3: Extreme 2-monotone measures that are not perfect.

No.	α	β	γ	λ
1.	2/3	1/2	1/2	1/6
2.	2/3	1/2	1/3	1/6
3.	1/3	1/2	1/6	1/6
4.	1/3	1/3	1/6	1/6
5.	5/6	1/3	2/3	1/6

Table 1: Values of parameters $\alpha,\beta,\gamma,\lambda$.

8 Conclusion

In this paper we give general necessary and sufficient conditions under which 2-monotone measures are extreme points of M_{2-mon}, describe some important classes of them, and give ways of their generation. As shown by examples, the introduced class of extreme 2-monotone measures, that are additive on filters do not cover all possible extreme 2-monotone measures, and we cannot generate all possible extreme 2-monotone measures based on aggregations with the help of multilinear extension. However, this paper can be considered as the first step to the desirable solution. As one can see from the examples, general extreme 2-monotone measures have structures that are similar to a structure of extreme 2-monotone measures that are additive on filter and there is a possibility to generalize it. This can be the topic for the future research.

Appendix

Proof of Proposition 1. We should prove monotonicity, i.e. $\mu(A\cup\{x_k\})-\ \mu(A)\geq 0$ for $A\in 2^X$ and $x_k\notin A$. Let us consider a chain of sets $B_0=\varnothing$, $B_1=\{y_1\}$, $B_1=\{y_1,y_2\}$,…, $B_m=\{y_1,...,y_m\}$. Then inequalities (2) imply

$$0\leq\mu(B_0\cup\{x_k\})-\mu(B_0)\leq\mu(B_1\cup\{x_k\})-\mu(B_1)\leq...\leq$$
$$\mu(B_m\cup\{x_k\})-\mu(B_m)\text{, i.e. }\mu(A\cup\{x_k\})-\mu(A)\geq 0\text{ . }\blacksquare$$

Proof of Proposition 3. It is necessary to show that (3) is valid for $A\cup\{x_i\}\in\mathbf{f}$ and $A\cup\{x_j\}\notin\mathbf{f}$. Since $\mu(A\cup\{x_j\})=0$ and $\mu(A)=0$, this inequality is transformed to

$$\mu(A\cup\{x_i\})\leq\mu(A\cup\{x_i\}\cup\{x_j\}).$$

Let us prove that $\mu(B\cup\{x_j\})-\mu(B)\geq 0$ for any $B\in\mathbf{f}$ and $x_j\notin B$. Since $B\in\mathbf{f}$, then there exists a minimal element $C_k\in\mathcal{C}$ such that $C_k\subseteq B$. Let us show first that

$$\mu(C_k\cup\{x_j\})-\mu(C_k)\geq 0 \qquad \text{(A1)}.$$

According to the statement of the proposition for $C_k\in\mathcal{C}$ and $x_j\notin C_k$ there exists $C_l\in\mathcal{C}$ such that $\{x_j\}=C_l\setminus C_k$. Since $C_k\setminus C_l\neq\varnothing$, there is some $x_i\in C_k\setminus C_l$, and obviously $C_l\subseteq(C_k\setminus\{x_i\})\cup\{x_j\}$, i.e. $(C_k\setminus\{x_i\})\cup\{x_j\}\in\mathbf{f}$. Because μ is 2-monotone on $\mathbf{f}$, we have $\mu((C_k\setminus\{x_j\})\cup\{x_i\})+\mu(C_k)\leq\mu(C_k\setminus\{x_j\})+\mu(C_k\cup\{x_i\})$. Let us notice that in the last inequality $\mu((C_k\setminus\{x_j\})\cup\{x_i\})>0$ and $\mu(C_k\setminus\{x_j\})=0$, therefore, the inequality (A1) is valid.

Let us show next that this inequality is fulfilled for B if $C_k\subseteq B$. For this purpose, consider the following chain of sets

$$B_0=C_k,\ B_1=C_k\cup\{x_{i_1}\},...,B_r=C_k\cup\{x_{i_1},...,x_{i_r}\}=B.$$

Since μ is 2-monotone on the filter $\mathbf{f}$, the following inequalities are valid:

$$\mu(B_0\cup\{x_j\})-\mu(B_0)\leq\mu(B_1\cup\{x_j\})-\mu(B_1)\leq...$$
$$\leq\mu(B_r\cup\{x_j\})-\mu(B_r),$$

i.e. $\mu(C_k\cup\{x_i\})-\mu(C)\leq\mu(B\cup\{x_i\})-\mu(B)$.■

Proof of Theorem 1. Obviously, $\rho_\gamma\supseteq\rho_\Gamma$ for any order ρ_γ with $\gamma\in\Gamma$. Let us show next that if $\rho_{\gamma'}\supseteq\rho_\Gamma$, then $\gamma'\in\Gamma$. For this purpose, it is necessary to show that sets $B_1=\{y_1\}$, $B_2=\{y_1,y_2\}$, ..., $B_n=\{y_1,y_2,...,y_n\}$, where $y_i=x_{\gamma'(i)}$, $i=1,2,...,n$, are in $\mathfrak{L}$. Let us show first that $B_1\in\mathfrak{L}$. Let us put into correspondence to each permutation $\gamma\in\Gamma$ the set

$B_\gamma(y_1)=\{x_{\gamma(1)},x_{\gamma(2)},...,x_{\gamma(m)}\}$ such that $y_1=x_{\gamma(m)}$. It is easy to see that conditions $\rho_\Gamma=\bigcap_{\gamma\in\Gamma}\rho_\gamma$ and $\rho_{\gamma'}\supseteq\rho_\Gamma$ imply $\bigcap_{\gamma\in\Gamma}B_\gamma(y_1)=\{y_1\}$, i.e. $B_1\in\mathfrak{L}$. We then prove $B_k\in\mathfrak{L}$, $k=2,...,n$, by induction. Let us assume that $B_1,...,B_{k-1}\in\mathfrak{L}$ and show that $B_k\in\mathfrak{L}$. In this case the conditions $\rho_\Gamma=\bigcap_{\gamma\in\Gamma}\rho_\gamma$ and $\rho_{\gamma'}\supseteq\rho_\Gamma$ imply $\bigcap_{\gamma\in\Gamma}B_\gamma(y_k)\subseteq B_{k-1}\cup\{y_k\}$. Therefore, $B_{k-1}\cup\bigcap_{\gamma\in\Gamma}B_\gamma(y_k)=B_k$, i.e. $B_k\in\mathfrak{L}$.■

Proof of Proposition 5. *Necessity*. Let us assume that μ is not an extreme point in M_{2-mon}. Then it can be represented in the form $\mu=a\mu_1+(1-a)\mu_2$, where $a\in(0,1)$, $\mu_1,\mu_2\in M_{2-mon}$ and $\mu_1\neq\mu$. Clearly, μ_1 obeys conditions on ν 1) and 2) in this proposition.

Sufficiency. Let us assume to the contrary that there exists $\nu\in M_{2-mon}$ with properties 1) and 2). We will show that in this case μ is not an extreme point in M_{2-mon}. For this purpose, let us consider a set function $\theta_a(A)=\mu(A)-a\nu(A)$, parametrically depending on $a\in[0,1]$ and also

$$\varepsilon_1=\max\{a\in[0,1]\,|\,\theta_a(A)\geq 0 \text{ for all } A\in 2^X\},$$

$$\varepsilon_2=\max\{a\in[0,1]\,|\,\theta_a(A)+\theta_a(B)\leq \theta_a(A\cap B)+\theta_a(A\cup B) \text{ for all } A,B\in 2^X\}.$$

It is easy to see that conditions 1) and 2) imply that $\varepsilon_1>0$ and $\varepsilon_2>0$. Therefore, a set function θ_b, where $b=\min\{\varepsilon_1,\varepsilon_2\}$, is nonnegative and 2-monotone. Thus, μ is represented as

$$\mu=b\nu(A)+(1-b)\mu_2,$$

where $\mu_2=\theta_b/(1-b)$ and, obviously, $\nu,\mu_2\in M_{2-mon}$, i.e. μ is not an extreme point in M_{2-mon}.■

Proof of Proposition 7. Clearly, $\tilde{\mu}(\mathbf{x})=\tilde{\varphi}(\tilde{\mu}_1(\mathbf{x}^{(1)}),...,\tilde{\mu}_n(\mathbf{x}^{(n)}))$ for every binary vector $\mathbf{x}$ and $\tilde{\varphi}(\tilde{\mu}_1(\mathbf{x}^{(1)}),...,\tilde{\mu}_n(\mathbf{x}^{(n)}))$ is a multilinear polynomial. Therefore, the proposition follows from the uniqueness of such a polynomial for the pseudo-Boolean function μ. ■

Proof of Lemma 2. Using the Taylor decomposition at the point $\mathbf{x}=(x_1,...,x_{n-1},0)$, we get

$$\varphi(x_1,...,x_{n-1},x_{n+1},....,x_{n+m})=\varphi_1(x_1,...,x_{n-1},0)+\frac{\partial\tilde{\varphi}_1(x_1,...,x_{n-1},0)}{\partial x_n}\varphi_2(x_{n+1},...,x_{n+m}).$$

Then we find that if $(x_1,...,x_{n-1},x_{n+1},....,x_{n+m})=\mathbf{1}_{A\cup B}$, then

$$\varphi_1(x_1,...,x_{n-1},0)=\mu_1(A),$$

$$\frac{\partial\tilde{\varphi}_1(x_1,...,x_{n-1},0)}{\partial x_n}=\mu_1(A\cup\{n\})-\mu_1(A),$$

$$\varphi_2(x_{n+1},...,x_{n+m})=\mu_2(B).$$ ■

Proof of Proposition 8. Let $\mu(A)=\varphi(\mathbf{1}_A)$, where $A\in 2^Z$. Then the multilinear extension of φ can be represented as

$$\tilde{\varphi}(\mathbf{x})=\sum_{A\subseteq Z}\mu(A)\prod_{k\in A}x_k\prod_{k\notin A}(1-x_k).$$

Taking partial derivative, we get

$$\frac{\partial\tilde{\varphi}(\mathbf{x})}{\partial x_i}=\sum_{A\subseteq Z\setminus\{i\}}\left(\mu(A\cup\{i\})-\mu(A)\right)\prod_{k\in A}x_k\prod_{k\notin A}(1-x_k).$$

The proposition follows from the last formula. ■

Proof of Proposition 9. Let us show that φ is defined uniquely. Let $\mathbf{x}\in\{0,1\}^n$ and $A=\bigcup_{i=1}^n A_i$, where $A_i\in 2^{X_i}$, is chosen such that $A_i=X_i$ if $x_i=1$ and $A_i=\varnothing$ if $x_i=0$. Then $(\mu_1(A\cap X_1),...,\mu_n(A\cap X_n))=\mathbf{x}$, i.e. $\mu(A)=\varphi(\mathbf{x})$. This means that φ is defined uniquely by μ.

Let us show that vector $\boldsymbol{\mu}$ is defined uniquely if each variable x_i is essential for φ. Let us assume that the variable x_i is essential for φ. Then by definition there are vectors $\mathbf{x}_1=(x_1,...,x_{i-1},0,x_{i+1},...,x_n)$ and $\mathbf{x}_2=(x_1,...,x_{i-1},1,x_{i+1},...,x_n)$ in $\{0,1\}^n$ such that $\varphi(\mathbf{x}_1)\neq\varphi(\mathbf{x}_2)$. Let $A=\bigcup_{k=1}^n A_k$, where $A_k\in 2^{X_k}$, such that $A_k=X_k$ if $x_k=1$ and $k\neq i$; $A_k=\varnothing$ if $x_k=0$ and $k\neq i$; and A_i is chosen arbitrary in 2^{X_i}. Then using the Taylor decomposition, we get

$$\mu(A)=\tilde{\varphi}(x_1,...,x_{i-1},\mu_i(A_i),x_{i+1},...,x_n)=\tilde{\varphi}(\mathbf{x}_1)+\mu_i(A_i)\frac{\partial\tilde{\varphi}(\mathbf{x}_1)}{\partial x_i}.$$

Therefore, we can calculate

$$\mu_i(A_i)=\left(\mu(A)-\tilde{\varphi}(\mathbf{x}_1)\right)\Big/\frac{\partial\tilde{\varphi}(\mathbf{x}_1)}{\partial x_i},$$

because $\frac{\partial\tilde{\varphi}(\mathbf{x}_1)}{\partial x_i}\neq 0$ according to Proposition 8. Thus, each set function μ_i is defined uniquely if every variable x_i is essential. Let us notice that if φ contains a nonessential variable x_i, then $\tilde{\varphi}$ does not depend on x_i. This implies that the representation $\mu=\tilde{\varphi}\circ\boldsymbol{\mu}$ is not defined uniquely, since any μ_i has no influence on the result of aggregation.■

Proof of Theorem 3. *Necessity.* Consider 2 possible cases.

1) Let us assume to the contrary that φ is not an extreme 2-monotone measure, however, μ is an extreme 2-monotone measure. Then $\varphi=a\varphi_1+(1-a)\varphi_2$, where $a\in(0,1)$ and φ_1,φ_2 are different 2-monotone measures on 2^Z. Therefore, $\tilde{\varphi}=a\tilde{\varphi}_1+(1-a)\tilde{\varphi}_2$ and $\mu=\tilde{\varphi}\circ\boldsymbol{\mu}=a\tilde{\varphi}_1\circ\boldsymbol{\mu}+(1-a)\tilde{\varphi}_2\circ\boldsymbol{\mu}$, where $\tilde{\varphi}_1\circ\boldsymbol{\mu},\tilde{\varphi}_2\circ\boldsymbol{\mu}$ are different 2-monotone measures by Proposition 9. But this contradicts our assumption that μ is an extreme 2-monotone measure.

2) Let us assume to the contrary that μ_i is not an extreme 2-monotone measure for some $i\in\{1,...,n\}$, however, μ is an extreme 2-monotone measure. Then μ_i can be represented as a convex sum of two different 2-monotone measures: $\mu_i=a\mu_i^{(1)}+(1-a)\mu_i^{(2)}$, where $a\in(0,1)$, therefore,

$$\mu=\tilde{\varphi}\circ\left(\mu_1,...,\mu_{i-1},a\mu_i^{(1)}+(1-a)\mu_i^{(2)},\mu_{i+1}...,\mu_n\right)=a\tilde{\varphi}\circ\left(\mu_1,...,\mu_{i-1},\mu_i^{(1)},\mu_{i+1}...,\mu_n\right)+(1-a)\tilde{\varphi}\circ\left(\mu_1,...,\mu_{i-1},\mu_i^{(2)},\mu_{i+1}...,\mu_n\right),$$

where

$\tilde{\varphi}\circ\left(\mu_1,...,\mu_{i-1},\mu_i^{(1)},\mu_{i+1}...,\mu_n\right),\tilde{\varphi}\circ\left(\mu_1,...,\mu_{i-1},\mu_i^{(2)},\mu_{i+1}...,\mu_n\right)$, are different 2-monotone measures. But this contradicts the assumption that μ is an extreme 2-monotone measure.

Sufficiency. We will prove sufficiency by induction. According to Remark 3 any aggregation (5) can be represented as a composition of simple aggregations described in Lemma 2. Therefore, if we prove that any simple aggregation of a type $\varphi(x_1,...,x_{n-1},x_{n+1},....,x_{n+m}) = \tilde{\varphi}_1(x_1,...,x_{n-1},\varphi_2(x_{n+1},....,x_{n+m}))$, where the corresponding set functions μ_1 and μ_2 are extreme 2-monotone measures (see notations from Lemma 2), generates the extreme 2-monotone measure μ. Then we can also say that the general aggregation produces the extreme 2-monotone measure if the conditions of the theorem are fulfilled. Let us assume to the contrary that μ is not an extreme 2-monotone measure. Then there are 2 different 2-monotone measures $\mu^{(0)}$ and $\mu^{(1)}$ such that

$$\mu = a\mu^{(0)} + (1-a)\mu^{(1)}, \text{ where } a \in (0,1).$$

Let us consider 2-monotone measures, generated by a mapping

$$\psi(i) = \begin{cases} i, & i \in \{1,...,n-1\}, \\ n, & i \in \{n+1,...,m\}, \end{cases}$$

Obviously, $\mu^{\psi} = a(\mu^{(0)})^{\psi} + (1-a)(\mu^{(1)})^{\psi}$ [2], $\mu^{\psi} = \mu_1$, and $\mu^{(0)}$, $\mu^{(1)}$ are 2-monotone measures. But according to our assumption μ_1 is an extreme 2-monotone measure. Therefore, this implies that $(\mu^{(0)})^{\psi} = \mu_1$.

Our next step is to show that if μ_2 is also an extreme 2-monotone measure, then $\mu^{(0)} = \mu^{(1)} = \mu$.

By Lemma 2, μ can be represented as

$$\mu(A \cup B) = \mu_1(A) + (\mu_1(A \cup \{n\}) - \mu_1(A))\mu_2(B), \qquad \text{(A2)}$$

where $A \subseteq \{1,...,n-1\}$ and $B \subseteq \{n+1,...,n+m\}$. Let us denote $Y = \{n+1,...,n+m\}$. Then, taking in account the correspondence between pseudo-Boolean and set functions, the formula (A2) can be rewritten as

$$\mu(A \cup B) = \mu(A) + (\mu(A \cup Y) - \mu(A))\mu_2(B),$$

and we can calculate

$$\mu_2(B) = \frac{\mu(A \cup B) - \mu(A)}{\mu(A \cup Y) - \mu(A)},$$

for any $A \subseteq \{1,...,n-1\}$ such that $\mu(A \cup Y) - \mu(A) > 0$. Let us consider set functions:

$$\mu_2^{(i)}(B) = \frac{\mu^{(i)}(A \cup B) - \mu^{(i)}(A)}{\mu^{(i)}(A \cup Y) - \mu^{(i)}(A)}, \; i = 1,2,$$

of $B \subseteq Y$ for any $A \subseteq \{1,...,n-1\}$ with $\mu(A \cup Y) - \mu(A) > 0$ and $\mu^{(i)}(A \cup Y) = \mu(A \cup Y)$. It is easy to show that these set functions are 2-monotone. Let us notice that we have proved that $\mu^{(i)}(A \cup Y) = \mu(A \cup Y)$ and $\mu^{(i)}(A) = \mu(A)$. After that we easily derive that

$$a\mu_2^{(0)}(B) + (1-a)\mu_2^{(1)}(B) = \mu_2(B).$$

By our assumption, μ_2 is an extreme 2-monotone measure. This implies that $\mu_2^{(0)} = \mu_2^{(1)} = \mu_2$. Thus, we can write

$$\mu^{(i)}(A \cup B) = \mu(A) + (\mu(A \cup Y) - \mu(A))\mu_2(B) = \mu(A \cup B)$$

for any $A \subseteq \{1,...,n-1\}$ and $B \subseteq \{n+1,...,n+m\}$, i.e. $\mu^{(0)} = \mu^{(1)} = \mu$, but this contradicts our assumption that measures $\mu_2^{(0)}$ and $\mu_2^{(1)}$ are different.■

Proof of Lemma 3. Let us consider the sequence of sets

$$B_0 = C,\; B_1 = C \cup \{x_{i_1}\},...,B_m = C \cup \{x_{i_1},...,x_{i_m}\} = A.$$

Since μ is additive on the filter $\mathbf{f}$, we can write

$$\mu(C \cup \{x_j\}) - \mu(C) = \mu(B_1 \cup \{x_i\}) - \mu(B_1) = ...$$
$$= \mu(B_m \cup \{x_i\}) - \mu(B_m),$$

i.e. $\mu(C \cup \{x_i\}) - \mu(C) = \mu(A \cup \{x_i\}) - \mu(A)$. Thus, the required equality is valid.■

Proof of Corollary 2. By Lemma 3

$$\mu(X) - \mu(X \setminus \{x_i\}) = \mu(C \cup \{x_i\}) - \mu(C)$$

for any $C \in \mathbf{f}$ with $C \subseteq X \setminus \{x_i\}$. This implies the result.■

Proof of Proposition 10. Let us check inequality (3), considering the following possible cases:

a) if $A \cup \{x_i\} \in \mathbf{f}$ and $A \cup \{x_j\} \in \mathbf{f}$, then the inequality (3) follows from the additivity of μ on $\mathbf{f}$;

b) if $A \cup \{x_i\} \in \mathbf{f}$ and $A \cup \{x_j\} \notin \mathbf{f}$, then (3) is transformed to

$$\mu(A \cup \{x_i\}) \le \mu(A \cup \{x_i, x_j\}).$$

The last inequality is valid, because according to our assumption $\nu(x_j) \ge 0$;

c) if $A \cup \{x_i\} \notin \mathbf{f}$ and $A \cup \{x_j\} \notin \mathbf{f}$, then inequality (3) is obviously true. ■

Proof of Proposition 11. It is sufficient to show that

$$\mu(A \cup \{x_i\}) - \mu(A) \ge 0 \qquad \text{(A3)}$$

for all $A \in \mathbf{f}$ and any $x_i \in X$. By the assumption $\{\bar{C}_1,...,\bar{C}_k\}$ is a covering of X, therefore, there is a set $\bar{C}_l$ such that $x_i \in \bar{C}_l$. Let us consider 2 possible cases.

If $|\bar{C}_l| = 1$, i.e. $\bar{C}_l = \{x_i\}$, then $x_i \in A$ for all $A \in \mathbf{f}$. Obviously, in this case the inequality (A3) is valid.

If $|\bar{C}_l| \ge 2$, then there is $x_j \in \bar{C}_l$ such that $x_j \ne x_i$. Since C_l is a maximal element in $2^X \setminus \mathbf{f}$, then $x_i \cup C_l \in \mathbf{f}$, $x_j \cup C_l \in \mathbf{f}$, and additivity of μ on $\mathbf{f}$ implies

$$\nu(x_i) = \mu(C_l \cup \{x_i, x_j\}) - \mu(C_l \cup \{x_j\}) =$$
$$\mu(C_l \cup \{x_i\}) - \mu(C_l) = \mu(C_l \cup \{x_i\}) \ge 0,$$

i.e. the inequality (A3) is valid for all $A \in \mathbf{f}$.■

Proof of Lemma 4. Clearly $\mathbf{f}_0 \supseteq \mathbf{f}$. Let us show that $B \in \mathbf{f}_0$ and $B \subseteq C$ implies $C \in \mathbf{f}_0$. It is sufficient to consider the case, when $B \notin \mathbf{f}$ and $C \notin \mathbf{f}$. Then there exist $x_i, x_j \notin B$ such that

$$B = (B \cup \{x_i\}) \cap (B \cup \{x_j\}).$$

By our assumption $C \notin \mathbf{f}$, therefore $x_i, x_j \notin C$. This implies that $C = (C \cup \{x_i\}) \cap (C \cup \{x_j\})$, i.e. $C \in \mathbf{f}_0$. ■

Proof of Proposition 12. It is necessary to show that $\mu(A \cup \{x_i\}) - \mu(A) = \nu(x_i)$ for any $A \in \mathbf{f}_0$ and $x_j \notin A$. Let us show first that if $A \in \mathbf{f}_0$, then

$$\mu(A) + \sum_{x_i \notin A} \nu(x_i) = \mu(X).$$

Let us consider two possible cases. Let $A \in \mathbf{f}$ and $X \setminus A = \{y_1, y_2, ..., y_m\}$. Then

[2] μ^{ψ} denotes a measure on $2^{\{1,...,n\}}$ such that $\mu^{\psi}(A) = \mu(\psi^{-1}(A))$, where $\psi^{-1}(A) = \{i \in \{1,...,n+m\} \mid \psi(i) \in A\}$.

$$\mu(A\cup\{y_1\})=\mu(A)+\nu(y_1),$$
$$\mu(A\cup\{y_1,y_2\})=\mu(A\cup\{y_1\})+\nu(y_2)$$
$$=\mu(A)+\nu(y_1)+\nu(y_2),$$
$$\vdots$$
$$\mu(X)=\mu(A)+\sum_{i=1}^{m}\nu(y_i),$$
i.e. the required equality is valid for $A\in\mathbf{f}$. Let us consider the case, when $A\in\mathbf{f}_0\setminus\mathbf{f}$. Then there exist $x_i,x_j\notin A$ such that $A\cup\{x_i\},A\cup\{x_j\}\in\mathbf{f}$, and
$$\mu(A\cup\{x_i\})+\mu(A\cup\{x_j\})=\mu(A\cup\{x_i,x_j\})+\mu(A),$$
i.e.
$$\mu(A\cup\{x_j\})-\mu(A)=\mu(A\cup\{x_i,x_j\})-$$
$$\mu(A\cup\{x_i\})=\nu(x_j).$$
After that we see that $\mu(A\cup\{x_j\})=\mu(A)+\nu(x_j)$ and
$$\mu(X)=\mu(A\cup\{x_j\})+\sum_{x_i\notin A\cup\{x_j\}}\nu(x_i)=\mu(A)+\sum_{x_i\notin A}\nu(x_i).$$
Thus, we can write
$$\mu(A\cup\{x_j\})-\mu(A)=\mu(X)-\sum_{x_i\notin A\cup\{x_j\}}\nu(x_i)-$$
$$\left(\mu(X)-\sum_{x_i\notin A}\nu(x_i)\right)=\nu(x_j).\blacksquare$$

Proof of Corollary 4.
$$\mu(A)+\mu(B)=\mu(A\cap B)+\sum_{x_i\in A\setminus B}\nu(x_i)+\mu(A\cap B)+$$
$$\sum_{x_i\in B\setminus A}\nu(x_i)=\ \mu(A\cap B)+\mu(A\cap B)+$$
$$\sum_{x_i\in(A\cup B)\setminus(A\cap B)}\nu(x_i)=\mu(A\cap B)+\mu(A\cup B).\ \blacksquare$$

Proof of Proposition 13. The first part of the proposition follows from the results considered above. Let us prove the second part. For this purpose, let us show that any set function μ, obeying (6) and (7) is additive on $\mathbf{f}$, i.e.
$$\mu(A\cup\{x_i\})+\mu(A\cup\{x_j\})=\mu(A)+\mu(A\cup\{x_i,x_j\}),$$
for $A\cup\{x_i\}$, $A\cup\{x_j\}\in\mathbf{f}$ and $A\cap\{x_i,x_j\}=\varnothing$.

Let us consider 2 possible cases. Let $A\in\mathbf{f}$, then
$$\mu(A\cup\{x_i\})+\mu(A\cup\{x_j\})=\mu(X)-$$
$$\sum_{x_k\notin A\cup\{x_i\}}\nu(x_k)+\mu(X)-\sum_{x_k\notin A\cup\{x_i\}}\nu(x_k)=$$
$$\mu(X)-\sum_{x_k\notin A}\nu(x_k)+\mu(X)-\sum_{x_k\notin A\cup\{x_i,x_j\}}\nu(x_k)=$$
$$\mu(A)+\mu(A\cup\{x_i,x_j\}).$$
Let $A\in\mathbf{f}_0\setminus\mathbf{f}$, then
$$\mu(A\cup\{x_i\})+\mu(A\cup\{x_j\})=\mu(X)-$$
$$\sum_{x_k\notin A\cup\{x_i\}}\nu(x_k)+\mu(X)-\sum_{x_k\notin A\cup\{x_i\}}\nu(x_k)=$$
$$\mu(X)-\sum_{x_k\notin A}\nu(x_k)+\mu(X)-\sum_{x_k\notin A\cup\{x_i,x_j\}}\nu(x_k).$$
In the last expression $\sum_{x_k\notin A}\nu(x_k)=\mu(X)$, in addition, $\mu(A)=0$. This implies that
$$\mu(A\cup\{x_i\})+\mu(A\cup\{x_j\})=\mu(A)+\mu(A\cup\{x_i,x_j\}).\ \blacksquare$$

Proof of Proposition 14. Let us assume to the contrary that μ is uniquely defined by the filter, but it is not extreme. Then there are 2 different 2-monotone measures μ_1 and μ_2 such that $\mu=a\mu_1+(1-a)\mu_2$, where $a\in(0,1)$. It easy to check that both measures μ_1 and μ_2 are additive on filter $\mathbf{f}$ but this contradicts our assumption.$\blacksquare$

Proof Proposition 15. Let us assume to the contrary that μ obeys conditions of the proposition, however, there is another extreme 2-monotone measure ν with $\mathbf{f}=\left\{A\in 2^X\,|\,\nu(A)>0\right\}$. Let us consider the set function $\theta_a(A)=\nu(A)-a\mu(A)$, parametrically depending on $a\in[0,1]$ and
$$b=\max\left\{a\in[0,1]\,|\,\theta_a(A)\geq 0\ for\ all\ A\in 2^X\right\}.$$
Clearly, $b>0$ and the set function θ_b is 2-monotone on the filter $\mathbf{f}$. According to Proposition 3 θ_b is 2-monotone on 2^X. Therefore, we can represent ν as
$$\nu=b\mu+(1-b)\mu_2,$$
where $\mu_2=\theta_b/(1-b)$ is a 2-monotone measure, but this contradicts our assumption.$\blacksquare$

Proof of Proposition 16. Let us assume to the contrary that μ obeys conditions of the proposition, however, there is another extreme 2-monotone measure ν, which is additive on $\mathbf{f}=\left\{A\in 2^X\,|\,\nu(A)>0\right\}$. Let us consider the set function $\theta_a(A)=\mu(A)-a\nu(A)$, parametrically depending on $a\in[0,1]$ and
$$b=\max\left\{a\in[0,1]\,|\,\theta_a(A)\geq 0\ for\ all\ A\in 2^X\right\}.$$
Clearly, $b>0$ and the set function θ_b is additive on the filter $\mathbf{f}$. According to Proposition 11, θ_b is 2-monotone on 2^X. Therefore, we can represent μ as
$$\mu=b\nu+(1-b)\mu_2,$$
where $\mu_2=\theta_b/(1-b)$ is a 2-monotone measure, but this contradicts our assumption.$\blacksquare$

Proof of Proposition 17. Let us notice that $\mu_n=\eta_{\langle X\rangle}$ and for this case the proposition is obviously true. Let us check that μ_k, where $k\in\{2,...,n-1\}$, is additive on the filter $\mathbf{f}=\{A\in 2^X\,||A|\geq k\}$, i.e. the following equality holds
$$\mu(A\cup\{x_i\})+\mu(A\cup\{x_j\})=\mu(A)+\mu(A\cup\{x_i,x_j\}),\ \text{(A4)}$$
for $A\cup\{x_i\}$, $A\cup\{x_j\}\in\mathbf{f}$ and $A\cap\{x_i,x_j\}=\varnothing$. Let $|A|=m\geq k-1$, then the equality (A4) is transformed to
$$\frac{m-k+2}{n-k+1}+\frac{m-k+2}{n-k+1}=\frac{m-k+1}{n-k+1}+\frac{m-k+3}{n-k+1},$$
i.e. (A4) is valid for this case.

Let us show that μ_k is an extreme 2-monotone measure. In this case $\mathbf{f}_0=\{A\in 2^X\,||A|\geq k-1\}$ and by Proposition 13 all possible 2-monotone measures that are additive on $\mathbf{f}$ can be found by solving the following linear system of equations:
$$\sum_{x_i\notin A}\nu(x_i)=1\ \text{for all}\ A\in 2^X\ \text{with}\ |A|=k-1.$$
It is easy to check that the solution is uniquely defined by $\nu(x_i)=1/(n-k+1)$, $i=1,...,n$. This implies that μ_k is an extreme 2-monotone measure. It is easy to check that μ_k is a perfect extreme 2-monotone measure by Proposition 15. $\blacksquare$

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References

[1] G. Birkhoff. *Lattice Theory*. American Mathematical Society Colloquium Publications, v. 25 (3rd ed.), Providence, R.I, 1979.

[2] Bronevich A.G. Canonical sequences of fuzzy measures. In Proc. of International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU-2004), Perugia-Italy, 2004, 8 pp.

[3] Bronevich A.G. On the closure of families of fuzzy measures under eventwise aggregations. *Fuzzy sets and systems,* v. 153, 2005, 45-70.

[4] Bronevich A.G., Augustin T. Approximation of coherent lower probabilities by 2-monotone measures. Proc. of the 6th International Symposium on Imprecise Probability: Theories and Their Applications, Durham, United Kingdom, 2009, 9 pp.

[5] Bronevich A.G., Klir G.J. Measures of uncertainty for imprecise probabilities: An axiomatic approach. International *Journal of Approximate Reasoning*, v. 51, 2010, 365-390.

[6] Chateauneuf A., Jaffray J.Y. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion*. Mathematical Social Sciences*, v. 17, 1989, 263-283.

[7] Combarro E., Miranda P. Extreme points of some families of non-additive measures. *European Journal of Operational Research*, v. 174, 2006, 1865-1884.

[8] Combarro E., Miranda P. On the polytope of non-additive fuzzy measures*. Fuzzy Sets and Systems*, v. 159, 2008, 2145-2162.

[9] Denneberg D. *Non-additive Measure and Integral*. Kluwer, Dordrecht, 1997.

[10] Fujishige S. *Submodular functions and optimization.* Series: Annals of Discrete Mathematics, v. 58, second edition, Elsevier, Amsterdam, 2005.

[11] Grabisch M., Marichal J.-L., Mesiar R., Pap E. *Aggregation Functions*, Cambridge University Press, Cambridge. UK, 2009.

[12] Hammer P.L., Rudeanu S. *Boolean Methods in Operational Research and Related Arreas*. Springer, Berlin, Germany, 1968.

[13] Karkishchenko A.N. Invariant fuzzy measures on a finite algebra. Proc. of the North American Fuzzy Information society, NAFIPS'96, USA, Berkeley, June 20-22, 1996, v. 1, pp. 588- 592.

[14] Owen G. Multilinear extensions of games. In A.E. Roth (ed.) *The Shapley Value. Essays in Honor of Lloyd S. Shapley*. Cambridge University Press. Cambridge. UK, 1988, 139-151.

[15] Radojevic D. The logical representation of the discrete Choquet integral*. Belgian Journal of Operations Research, Statistics and Computer Science*, v. 38, 1998, 67-89.

[16] Shapley L.S. Cores of convex games. *International Journal of Game Theory*, v. 1, 1971, 11-26.

[17] Walley P. *Statistical reasoning with imprecise probabilities.* Chapman & Hall, London, 1991.