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Complex geometry of moment-angle manifolds

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Abstract Moment-angle manifolds provide a wide class of examples of non-Kähler compact complex manifolds. A complex moment-angle manifold $\mathcal{Z}$ is constructed via certain combinatorial data, called a complete simplicial fan. In the case of rational fans, the manifold $\mathcal{Z}$ is the total space of a holomorphic bundle over a toric variety with fibres compact complex tori. In general, a complex moment-angle manifold $\mathcal{Z}$ is equipped with a canonical holomorphic foliation $\mathcal{F}$ which is equivariant with respect to the $(\mathbb{C}^\times)^m$-action. Examples of moment-angle manifolds include Hopf manifolds of Vaisman type, Calabi–Eckmann manifolds, and their deformations. We construct transversely Kähler metrics on moment-angle manifolds, under some restriction on the combinatorial data. We prove that any Kähler submanifold (or, more generally, a Fujiki class $\mathcal{C}$ subvariety) in such a moment-angle manifold is contained in

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a leaf of the foliation $\mathcal{F}$. For a generic moment-angle manifold $\mathcal{Z}$ in its combinatorial class, we prove that all subvarieties are moment-angle manifolds of smaller dimension and there are only finitely many of them. This implies, in particular, that the algebraic dimension of $\mathcal{Z}$ is zero.

Keywords Moment-angle manifold · Simplicial fan · Non-Kähler complex structure · Holomorphic foliation · Transversely Kähler metric

Mathematics Subject Classification 32J18 · 32L05 · 32M05 · 32Q55 · 37F75 · 57R19 · 14M25

Contents

1 Introduction

1.1 Non-Kähler geometry

The class of non-Kähler complex manifolds is by several orders of magnitude bigger than that of Kähler ones. For instance, any finitely presented group can be realized as a fundamental group of a compact complex 3-manifold.[1] However, fundamental groups of compact Kähler manifolds are restricted by many different constraints, and apparently belong to a minuscule subclass of the class of all finitely presented groups [1].

Despite the abundance of non-Kähler complex manifolds, there are only few explicit constructions of them. Historically the first example was the *Hopf surface*, obtained as the quotient of $\mathbb{C}^2 \setminus \{\mathbf{0}\}$ by the action of $\mathbb{Z}$ generated by the transformation $z \mapsto 2z$. This construction can be generalised by considering quotients of $\mathbb{C}^n \setminus \{\mathbf{0}\}$, $n \geqslant 2$, by the action of $\mathbb{Z}$ generated by a linear operator with all eigenvalues α_i satisfying $|\alpha_i| > 1$. The resulting quotients are called *(generalised) Hopf manifolds*. Every Hopf manifold is diffeomorphic to $S^1 \times S^{2n-1}$. It is clearly non-Kähler, as its second cohomology group is zero. A particular class of generalised Hopf manifolds also features in the work of Vaisman [41], where he

[1] By a result of C. Taubes. See [37] for a simpler proof and references to earlier works.

proved that any compact locally conformally Kähler manifold is non-Kähler. We discuss these in more detail in Sect. 1.3.

Another generalisation of the Hopf construction is due to Calabi and Eckmann. A Calabi–Eckmann manifold is diffeomorphic to a product of even number of odd-dimensional spheres: $E = S^{2n_1+1} \times S^{2n_2+1} \times \cdots \times S^{2n_{2k}+1}$. It is the total space of a principal T^{2k}-bundle over $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_{2k}}$.

Any principal T^l-bundle E over a base B is determined topologically by the Chern classes $\nu_i \in H^2(B, \mathbb{Z})$, $i = 1, \dots, l$. These Chern classes can be obtained as the cohomology classes of the curvature components of a connection on E. Now, suppose that B is complex and these curvature components are forms of type $(1, 1)$. In this case, for each complex structure on the fibre T^l with even l, the total space becomes a complex manifold, which is easy to see if one expresses the commutators of vector fields tangent to E through the connection.

For the Calabi–Eckmann fibration,

$$E = S^{2n_1+1} \times S^{2n_2+1} \times \cdots \times S^{2n_{2k}+1} \xrightarrow{T^{2k}} B = \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_{2k}},$$

the natural connection on fibres has the respective Fubini-Study forms as its curvatures. This makes E into a complex manifold, holomorphically fibred over B.

Applying this to a T^2-fibration $S^1 \times S^{2l-1} \to \mathbb{C}P^{l-1}$, we obtain the classical Hopf manifold, which is also the quotient of $\mathbb{C}^n \backslash \{\mathbf{0}\}$ by an action of $\mathbb{Z}$ given by $z \to \lambda z$, $\lambda \in \mathbb{C}$, $|\lambda| > 1$. By deforming this action we obtain a manifold which does not admit elliptic fibrations. However, there is still a holomorphic *foliation*, and its differential-geometric properties remain qualitatively the same (Sect. 1.3).

In the present paper, we study complex manifolds which arise in a similar fashion.

Consider a projective (or Moishezon) toric manifold B, that is, a complex manifold equipped with an action of $(\mathbb{C}^\times)^n$ which has an open orbit. Fixing an even number of cohomology classes $\eta_1, \dots, \eta_{2k} \in H^2(B; \mathbb{Z})$, one obtains a T^{2k}-fibration $\mathcal{E} \to B$ with the Chern classes η_i. Since the classes η_i are represented by $(1, 1)$-forms, the corresponding connection makes $\mathcal{E}$ into a complex manifold, similar to the Calabi–Eckmann example. Complex moment-angle manifolds $\mathcal{Z}$, which are the main objects of study in this paper, may be thought of as deformations of $\mathcal{E}$, using a combinatorial procedure explained in detail in Sect. 3.

The topology of $\mathcal{Z}$ depends only on a simplicial complex $\mathcal{K}$ on m elements, which is reflected in the notation $\mathcal{Z} = \mathcal{Z}_\mathcal{K}$, while the complex structure comes from a realisation of $\mathcal{Z}_\mathcal{K}$ as the quotient of an open subset $U(\mathcal{K})$ in $\mathbb{C}^m$ by a holomorphic action of a group C isomorphic to $\mathbb{C}^\ell$. In order that the action of C be free and proper one needs to assume that $\mathcal{K}$ is the underlying complex of a complete simplicial fan Σ.

1.2 Positive (1, 1)-forms and currents on non-Kähler manifolds

Let M be a compact complex manifold. A differential $(1, 1)$-form β on M is called *positive* if it is real and $\beta(V, JV) \geqslant 0$ for any real tangent vector $V \in TM$, where J is the operator of the complex structure. Equivalently, a $(1, 1)$-form $\beta = i \sum_{j,k} f_{jk} dz_j \wedge d\bar{z}_k$ is positive if $\sum_{j,k} f_{jk} \xi_j \bar{\xi}_k$ is a Hermitian positive semidefinite form.

A $(1, 1)$-*current* is a linear functional Θ on the space $\Lambda_c^{1,1}(M)$ of forms with compact support which is continuous in one of the C^k-topologies defined by the C^k-norm $|\varphi|_{C^k} = \sup_{m \in M} \sum_{i=0}^k |\nabla^i \varphi|$. A $(1, 1)$-current Θ is called *positive* if $\langle \Theta, \beta \rangle \geqslant 0$ for any positive $(1, 1)$-form β. See [25] and [15] for more details on the notions of positive forms and currents.

The main result of Harvey and Lawson's work [25] states that M does not admit a Kähler metric if and only if M has a nonzero positive $(1, 1)$-current Θ which is the $(1, 1)$-component of an exact current. Any complex surface (2-dimensional manifold) M admits a closed positive $(1, 1)$-current [9,29], and sometimes a closed positive $(1, 1)$-form. The maximal rank of such form is called the *Kähler rank* of M. In [8] and [12], complex surfaces were classified according to their Kähler rank; it was shown that the Kähler rank is equal to 1 for all elliptic, Hopf and Inoue minimal surfaces, and 0 for the rest of class VII surfaces.

One of the most striking advances in complex geometry of the 2000s was the discovery of *Oeljeklaus–Toma manifolds* [34]. These are complex solvmanifolds associated with all number fields admitting complex and real embeddings. When the number field has the lowest possible degree (that is, cubic), the corresponding Oeljeklaus–Toma manifold is an *Inoue surface* of type S^0, hence Oeljeklaus–Toma manifolds can be considered as generalisations of Inoue surfaces. In [36], a positive exact $(1, 1)$-form ω_0 was constructed on any Oeljeklaus–Toma manifold. Using this form, it was proved that an Oeljeklaus–Toma manifold has no curves (see [42]), and that an Oeljeklaus–Toma manifold has no positive dimensional subvarieties when the number field has precisely 1 complex embedding, up to complex conjugation (see [36]). The argument, in both cases, is similar to that used in the present paper: a complex subvariety $Z \subset M$ cannot be transverse to the zero foliation $\mathcal{F}$ of ω_0, as otherwise $\int_Z \omega_0^{\dim_{\mathbb{C}} Z} > 0$, which is impossible by the exactness of ω_0. The rank of $\mathcal{F}$ is equal to the number of real embeddings. For the case considered in [36] it is 1, hence any subvariety $Z \subset M$ must contain a leaf of $\mathcal{F}$. The argument of [36] is finished by applying the "strong approximation theorem" (a result of number theory) to prove that the closure of a leaf of $\mathcal{F}$ cannot be contained in a subvariety of dimension less than $\dim M$.

1.3 Hopf manifolds and Vaisman manifolds

Positive exact $(1, 1)$-forms are also very useful in the study of Vaisman manifolds. The latter form a special class of *locally conformally Kähler manifold* (*LCK manifolds* for short). An LCK manifold is a manifold M whose universal covering $\widetilde{M}$ has a Kähler metric with the monodromy of the covering acting on $\widetilde{M}$ by non-isometric homotheties. An LCK manifold is called *Vaisman* if its Kähler covering $\widetilde{M}$ admits a monodromy-equivariant action of $\mathbb{C}^\times$ by non-isometric holomorphic homotheties.

The original definition of Vaisman manifolds (see [17] for more details, history and references) builds upon an explicit differential-geometric construction, and their description given above is a result of Kamishima–Ornea [27]. The existence of positive exact $(1, 1)$-forms of Vaisman manifolds is shown in [43].

Vaisman called his manifolds "generalised Hopf manifolds"; however, his construction does not give all generalised Hopf manifolds in the sense of Sect. 1.1. Indeed, it is proved in [35, Theorem 3.6] that a Hopf manifold $\mathbb{C}^n \setminus \{\mathbf{0}\}/\{z \sim Az\}$ corresponding to a linear operator A with all eigenvalues >1 is Vaisman if and only if A is diagonalisable (the "if" part follows from [22]).

Hopf manifolds of Vaisman type are particular cases of moment-angle manifolds (see Example 4.14). Therefore, our description of complex submanifolds in moment-angle manifolds generalises the results obtained for Hopf manifolds by Kato [28].

1.4 Moment-angle manifolds and their geometry

A moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is a cellular subcomplex in the unit polydisc $\mathbb{D}^m \subset \mathbb{C}^m$ composed of products of discs and circles [10]. These products are parametrised by a finite simplicial complex $\mathcal{K}$ on a set of m elements. Each moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ carries a natural action of the m-torus T^m. Moment-angle complexes have many interesting homotopical and geometric properties, and have been studied intensively over the last ten years (see [4,10,23,24,38]). When $\mathcal{K}$ is the boundary of a simplicial polytope or, more generally, a triangulated sphere, the moment-angle complex $\mathcal{Z}_\mathcal{K}$ is a manifold, referred to as a *moment-angle manifold*.

Moment-angle manifolds arising from polytopes can be realised by nondegenerate intersections of Hermitian quadrics in $\mathbb{C}^m$. These intersections of quadratic surfaces feature in holomorphic dynamics as transverse spaces for complex foliations (see [7] for a historic account of these developments), and in symplectic toric topology as level sets for the moment maps of Hamiltonian torus actions on $\mathbb{C}^m$ (see [3]). As was discovered by Bosio and Meersseman [7], moment-angle manifolds arising from polytopes admit complex structures as *LVM-manifolds*. This led to establishing a fruitful link between the theory of moment-angle manifolds and complex geometry.

An alternative construction of complex structures on moment-angle manifolds was given in [39]; it works not only for manifolds $\mathcal{Z}_\mathcal{K}$ arising from polytopes, but also when $\mathcal{K}$ is the underlying complex of a complete simplicial fan Σ. The complex structure on $\mathcal{Z}_\mathcal{K}$ comes from its realisation as the quotient of an open subset $U(\mathcal{K}) \subset \mathbb{C}^m$ (the complement to a set of coordinate subspaces determined by $\mathcal{K}$) by an action of $\mathbb{C}^\ell$ embedded as a holomorphic subgroup in the $\mathbb{C}^\times$-torus $(\mathbb{C}^\times)^m$. As in toric geometry, the simplicial fan condition guarantees that the quotient is Hausdorff [38].

The complex moment-angle manifold $\mathcal{Z}_\mathcal{K}$ is non-Kähler, except for the case when $\mathcal{K}$ is an empty simplicial complex and $\mathcal{Z}_\mathcal{K}$ is a compact complex torus. When the simplicial fan Σ is rational, the manifold $\mathcal{Z}_\mathcal{K}$ is the total space of a holomorphic principal (Seifert) bundle with fibre a complex torus and base the toric variety V_Σ. (The polytopal case was studied by Meersseman and Verjovsky [32]; the situation is similar in general, although the base need not to be projective.) We therefore obtain a generalisation of the classical families of Hopf and Calabi–Eckmann manifolds (which are the total spaces of fibrations with fibre an elliptic curve over $\mathbb{C}P^n$ and $\mathbb{C}P^k \times \mathbb{C}P^l$, respectively). In this case, although $\mathcal{Z}_\mathcal{K}$ is not Kähler, its complex geometry is quite rich, and meromorphic functions, divisors and Dolbeault cohomology can be described explicitly [32,39].

The situation is completely different when the data defining the complex structure on $\mathcal{Z}_\mathcal{K}$ is generic (in particular, the fan Σ is not rational). In this case, the moment-angle manifold $\mathcal{Z}_\mathcal{K}$ does not admit global meromorphic functions and divisors, and behaves similarly to the complex torus obtained as the quotient of $\mathbb{C}^m$ by a generic lattice of full rank. Furthermore, $\mathcal{Z}_\mathcal{K}$ has only finitely many complex submanifolds of a very special type: they are all "coordinate", i. e. obtained as quotients of coordinates subspaces in $\mathbb{C}^m$.

Our study of the complex geometry of moment-angle manifolds $\mathcal{Z}_\mathcal{K}$ builds upon two basic constructions: a holomorphic foliation $\mathcal{F}$ on $\mathcal{Z}_\mathcal{K}$ (which may be thought of as the result of deformation of the holomorphic fibration $\mathcal{E} \to B$ described in Sect. 1.1), and a positive closed $(1, 1)$-form ω whose zero spaces define the foliation $\mathcal{F}$ over a dense open subset of $\mathcal{Z}_\mathcal{K}$. Forms with these properties are called *transverse Kähler* for $\mathcal{F}$. The precise result is as follows:

Theorem 4.6 *Assume that* Σ *is a weakly normal fan defined by* $\{\mathcal{K}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$. *Then there exists an exact* $(1, 1)$*-form* $\omega_\mathcal{F}$ *on* $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/C$ *which is transverse Kähler for the foliation* $\mathcal{F}$ *on the dense open subset* $(\mathbb{C}^\times)^m/C \subset U(\mathcal{K})/C$.

Transverse Kähler forms are often used to study the complex geometry (subvarieties, stable bundles) of non-Kähler manifolds [36,44–46]. In our case, Fujiki class $\mathcal{C}$ subvarieties (in particular, Kähler submanifolds) of moment-angle manifolds are described as follows:

Theorem 4.11 *Under the assumptions of Theorem 4.6, any Fujiki class $\mathcal{C}$ subvariety of $\mathcal{Z}_{\mathcal{K}}$ is contained in a leaf of the ℓ-dimensional foliation $\mathcal{F}$.*

An important consequence is that under generic assumption on the complex structure (when a generic leaf of $\mathcal{F}$ is biholomorphic to $\mathbb{C}^{\ell}$) there are actually *no* Fujiki class $\mathcal{C}$ subvarieties of $\mathcal{Z}_{\mathcal{K}}$ through a generic point (Corollary 4.12).

We proceed by studying general subvarieties of $\mathcal{Z}_{\mathcal{K}}$ (not necessarily of Fujiki class $\mathcal{C}$). In the case of codimension one (divisors), generically there are only finitely many of them (Theorem 4.15). The proof does not use a transverse Kähler form (and therefore no geometric restrictions on the fan need to be imposed); instead we reduce the statement to the case of Hopf manifolds, where it is proved by analysing one-dimensional foliations. A corollary is that a generic $\mathcal{Z}_{\mathcal{K}}$ does not have non-constant meromorphic functions, so its algebraic dimension is zero.

The foliation $\mathcal{F}$ comes from a group action (Construction 4.1), so its tangent bundle $T\mathcal{F} \subset T\mathcal{Z}_{\mathcal{K}}$ is trivial. Given an irreducible subvariety $Y \subset \mathcal{Z}_{\mathcal{K}}$, consider the number $k = \dim T_yY \cap T_y\mathcal{F}$ for y a generic point in Y. Since $T\mathcal{F}$ is a trivial bundle, there is a rational map ψ from Y to the corresponding Grassmann manifold of k-planes in $T_y\mathcal{F}$. Using the absence of Fujiki class $\mathcal{C}$ submanifolds in $\mathcal{Z}_{\mathcal{K}}$, we deduce by induction on $\dim Y$ that ψ is constant (see the details in Sect. 4.7). Generically this cannot happen when Y contains a point from an open subset $(\mathbb{C}^{\times})^m/C \subset \mathcal{Z}_{\mathcal{K}}$, whose complement consists of "coordinate" submanifolds. We therefore prove

Theorem 4.18 *Let $\mathcal{Z}_{\mathcal{K}}$ be a complex moment-angle manifold, with linear maps $A\colon \mathbb{R}^m \to N_{\mathbb{R}}$ and $\Psi\colon \mathbb{C}^{\ell} \to \mathbb{C}^m$ described in Construction 3.1. Assume that*

(a) no rational linear function on $\mathbb{R}^m$ vanishes on $\operatorname{Ker} A$*;*
(b) $\operatorname{Ker} A$ *does not contain rational vectors of $\mathbb{R}^m$;*
(c) the map Ψ satisfies the generic condition of Lemma 4.17;
(d) the fan Σ is weakly normal.

Then any irreducible analytic subset $Y \subsetneq \mathcal{Z}_{\mathcal{K}}$ of positive dimension is contained in a coordinate submanifold $\mathcal{Z}_{\mathcal{K}_J} \subsetneq \mathcal{Z}_{\mathcal{K}}$.

Under further assumption on the complex structure of $\mathcal{Z}_{\mathcal{K}}$ we prove that all irreducible analytic subsets are actually coordinate submanifolds (Corollary 4.19).

A similar idea was used by Dumitrescu to classify holomorphic geometries on non-Kähler manifolds [18,19], and also much earlier by Bogomolov in his work [6] on holomorphic tensors and stability.

It has been recently shown by Ishida [26] that complex moment-angle manifolds $\mathcal{Z}_{\mathcal{K}}$ have the following universal property: any compact complex manifold with a holomorphic maximal action of a torus is biholomorphic to the quotient of a moment-angle manifold by a freely acting closed subgroup of the torus. (An effective action of T^k on an m-dimensional manifold M is *maximal* if there exists a point $x \in M$ whose stabiliser has dimension $m-k$; the two extreme cases are the free action of a torus on itself and the half-dimensional torus action on a toric manifold.)

In Sect. 2 we review the definition of moment-angle manifolds $\mathcal{Z}_{\mathcal{K}}$ and their realisation by nondegenerate intersections of Hermitian quadrics in the polytopal case. In Sect. 3 we describe complex structures on moment-angle manifolds. Section 4 contains the main results. We start with defining a holomorphic foliation $\mathcal{F}$ which replaces the holomorphic fibration $\mathcal{Z}_{\mathcal{K}} \to V_{\Sigma}$ over a toric variety. Theorem 4.6 is the main technical result, which may be of independent interest. It provides an exact $(1,1)$-form $\omega_{\mathcal{F}}$ which is transverse Kähler for the foliation $\mathcal{F}$ on a dense open subset. In Theorem 4.11 we show that, under a generic assumption on the data defining the complex structure, any Kähler submanifold (more generally, a Fujiki class $\mathcal{C}$ subvariety) of $\mathcal{Z}_\mathcal{K}$ is contained in a leaf of $\mathcal{F}$. In Theorem 4.15 we show that a generic moment-angle manifold $\mathcal{Z}_\mathcal{K}$ has only coordinate divisors. This result is independent of the existence of a transverse Kähler form. Consequently, a generic $\mathcal{Z}_\mathcal{K}$ does not admit non-constant global meromorphic functions (Corollary 4.16). In Theorem 4.18 and Corollary 4.19 we prove that any irreducible analytic subset of positive dimension in a generic moment-angle manifold is a coordinate submanifold.

2 Basic constructions

Let $\mathcal{K}$ be an abstract *simplicial complex* on the set $[m] = \{1, \dots, m\}$, that is, a collection of subsets $I = \{i_1, \dots, i_k\} \subset [m]$ closed under inclusion. We refer to $I \in \mathcal{K}$ as (abstract) *simplices* and always assume that $\varnothing \in \mathcal{K}$. We do not assume that $\mathcal{K}$ contains all singletons $\{i\} \subset [m]$, and refer to $\{i\} \notin \mathcal{K}$ as a *ghost vertex*.

Consider the closed unit polydisc in $\mathbb{C}^m$,

$$\mathbb{D}^m = \left\{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_i| \leqslant 1\right\}.$$

Given $I \subset [m]$, define

$$B_I = \left\{(z_1, \dots, z_m) \in \mathbb{D}^m : |z_j| = 1 \text{ for } j \notin I\right\},$$

Following [10], define the *moment-angle complex* $\mathcal{Z}_\mathcal{K}$ as

$$\mathcal{Z}_\mathcal{K} = \bigcup_{I \in \mathcal{K}} B_I = \left\{(z_1, \dots, z_m) \in \mathbb{D}^m : \{i : |z_i| < 1\} \in \mathcal{K}\right\}. \tag{2.1}$$

This is a particular case of the following general construction.

Construction 2.1 (*Polyhedral product*) Let X be a topological space, and A a subspace of X. Given $I \subset [m]$, set

$$(X, A)^I = \left\{(x_1, \dots, x_m) \in X^m : x_j \in A \text{ for } j \notin I\right\} \cong \prod_{i \in I} X \times \prod_{i \notin I} A, \tag{2.2}$$

and define the *polyhedral product* of (X, A) as

$$(X, A)^\mathcal{K} = \bigcup_{I \in \mathcal{K}} (X, A)^I \subset X^m.$$

Obviously, if $\mathcal{K}$ has k ghost vertices, then $(X, A)^\mathcal{K} \cong (X, A)^{\mathcal{K}'} \times A^k$, where $\mathcal{K}'$ does not have ghost vertices.

We have $\mathcal{Z}_\mathcal{K} = \mathcal{Z}_\mathcal{K}(\mathbb{D}, \mathbb{T})$, where $\mathbb{T}$ is the unit circle. Another important particular case is the complement of a *complex coordinate subspace arrangement*:

$$U(\mathcal{K}) = (\mathbb{C}, \mathbb{C}^\times)^\mathcal{K} = \mathbb{C}^m \setminus \bigcup_{\{i_1, \dots, i_k\} \notin \mathcal{K}} \{z \in \mathbb{C}^m : z_{i_1} = \dots = z_{i_k} = 0\}, \tag{2.3}$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. We obviously have $\mathcal{Z}_\mathcal{K} \subset U(\mathcal{K})$. Moreover, $\mathcal{Z}_\mathcal{K}$ is a deformation retract of $U(\mathcal{K})$ for every $\mathcal{K}$ [11, Theorem 4.7.5].

According to [11, Theorem 4.1.4], $\mathcal{Z}_{\mathcal{K}}$ is a (closed) topological manifold of dimension $m+n$ whenever $\mathcal{K}$ defines a simplicial subdivision of a sphere S^{n-1}; in this case we refer to $\mathcal{Z}_{\mathcal{K}}$ as a *moment-angle manifold*.

An important geometric class of simplicial subdivisions of spheres is provided by *star-shaped spheres*, or underlying complexes of complete simplicial fans.

Let $N_{\mathbb{R}} \cong \mathbb{R}^n$ be an n-dimensional space. A *polyhedral cone* σ is the set of nonnegative linear combinations of a finite collection of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ in $N_{\mathbb{R}}$:

$$\sigma = \{\mu_1\boldsymbol{a}_1 + \cdots + \mu_k\boldsymbol{a}_k \colon \mu_i \geqslant 0\}.$$

A polyhedral cone is *strongly convex* if it contains no line. A strongly convex polyhedral cone is *simplicial* if it is generated by a subset of a basis of $N_{\mathbb{R}}$.

A *fan* is a finite collection $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of strongly convex polyhedral cones in $N_{\mathbb{R}}$ such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan Σ is *simplicial* if every cone in Σ is simplicial. A fan $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ is called *complete* if $\sigma_1 \cup \cdots \cup \sigma_s = N_{\mathbb{R}}$.

A simplicial fan Σ in $N_{\mathbb{R}}$ is therefore determined by two pieces of data:

– a simplicial complex $\mathcal{K}$ on $[m]$;
– a configuration of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ in $N_{\mathbb{R}}$ such that the subset $\{\boldsymbol{a}_i \colon i \in I\}$ is linearly independent for any simplex $I \in \mathcal{K}$.

Then for each $I \in \mathcal{K}$ we can define the simplicial cone σ_I spanned by $\boldsymbol{a}_i$ with $i \in I$. The "bunch of cones" $\{\sigma_I \colon I \in \mathcal{K}\}$ patches into a fan Σ whenever any two cones σ_I and σ_J intersect in a common face (which has to be $\sigma_{I\cap J}$). Equivalently, the relative interiors of cones σ_I are pairwise non-intersecting. Under this condition, we say that the data $\{\mathcal{K}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ *define a fan* Σ, and $\mathcal{K}$ is the *underlying simplicial complex* of the fan Σ. Note that a simplicial fan Σ is complete if and only if its underlying simplicial complex $\mathcal{K}$ defines a simplicial subdivision of S^{n-1}; this simplicial subdivision is obtained by intersecting Σ with a unit sphere in $N_{\mathbb{R}}$.

Here is an important point in which our approach to fans differs from the standard one adopted in toric geometry: since we allow ghost vertices in $\mathcal{K}$, we do not require that each vector $\boldsymbol{a}_i$ spans a one-dimensional cone of Σ. The vector $\boldsymbol{a}_i$ corresponding to a ghost vertex $\{i\} \in [m]$ may be zero.

In the case when $\{\mathcal{K}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ define a complete fan Σ, the topological manifold $\mathcal{Z}_{\mathcal{K}}$ can be smoothed by means of the following procedure [39]:

Construction 2.2 Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ be a set of vectors. We consider the linear map

$$A \colon \mathbb{R}^m \to N_{\mathbb{R}}, \quad \boldsymbol{e}_i \mapsto \boldsymbol{a}_i, \tag{2.4}$$

where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ is the standard basis of $\mathbb{R}^m$. Let

$$\mathbb{R}^m_{>} = \{(y_1, \dots, y_m) \in \mathbb{R}^m \colon y_i > 0\}$$

be the multiplicative group of m-tuples of positive real numbers, and define

$$\begin{aligned} R = \exp(\operatorname{Ker} A) &= \bigl\{\bigl(e^{y_1}, \dots, e^{y_m}\bigr) \colon (y_1, \dots, y_m) \in \operatorname{Ker} A\bigr\} \\ &= \left\{(t_1, \dots, t_m) \in \mathbb{R}^m_{>} \colon \prod_{i=1}^m t_i^{\langle \boldsymbol{a}_i, \boldsymbol{u}\rangle} = 1 \text{ for all } \boldsymbol{u} \in N^*_{\mathbb{R}}\right\}. \end{aligned} \tag{2.5}$$

We let $\mathbb{R}^m_{>}$ act on $\mathbb{C}^m$ by coordinatewise multiplications. The subspace $U(\mathcal{K})$ (2.3) is $\mathbb{R}^m_{>}$-invariant for any $\mathcal{K}$, and there is an action of $R \subset \mathbb{R}^m_{>}$ on $U(\mathcal{K})$ by restriction.

Theorem 2.3 [39, Theorem 2.2] *Assume that data* $\{\mathcal{K}; \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ *define a complete fan* Σ *in* $N_{\mathbb{R}} \cong \mathbb{R}^n$. *Then*

(a) the group $R \cong \mathbb{R}^{m-n}$ *acts on* $U(\mathcal{K})$ *freely and properly, so the quotient* $U(\mathcal{K})/R$ *is a smooth* $(m+n)$*-dimensional manifold;*
(b) $U(\mathcal{K})/R$ *is homeomorphic to* $\mathcal{Z}_{\mathcal{K}}$.

Remark The group R depends on $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$. However, we expect that the smooth structure on $\mathcal{Z}_{\mathcal{K}}$ as the quotient $U(\mathcal{K})/R$ does not depend on this choice, i.e. smooth manifolds corresponding to different R are diffeomorphic.

An important class of complete fans arises from convex polytopes:

Construction 2.4 (*Normal fan*) Let P be a convex polytope in the dual space $N^*_{\mathbb{R}} \cong \mathbb{R}^n$. It can be written as a bounded intersection of m halfspaces:

$$P = \bigl\{\boldsymbol{u} \in N^*_{\mathbb{R}} \colon \langle \boldsymbol{a}_i, \boldsymbol{u}\rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\bigr\}, \tag{2.6}$$

where $\boldsymbol{a}_i \in N_{\mathbb{R}}$ and $b_i \in \mathbb{R}$.

We assume that the intersection $P \cap \{\langle \boldsymbol{a}_i, \boldsymbol{u}\rangle + b_i = 0\}$ with the ith hyperplane is either empty or $(n-1)$-dimensional. In the latter case, $F_i = P \cap \{\langle \boldsymbol{a}_i, \boldsymbol{u}\rangle + b_i = 0\}$ is a *facet* of P. We allow $P \cap \{\langle \boldsymbol{a}_i, \boldsymbol{u}\rangle + b_i = 0\}$ to be empty for technical reasons explained below, in this case the ith inequality $\langle \boldsymbol{a}_i, \boldsymbol{u}\rangle + b_i \geqslant 0$ is *redundant*. Our assumption also implies that P has full dimension n.

A *face* of P is a nonempty intersection of facets. Given a face $Q \subset P$, define

$$\sigma_Q = \{\boldsymbol{x} \in N_{\mathbb{R}} \colon \langle \boldsymbol{x}, \boldsymbol{u}'\rangle \leqslant \langle \boldsymbol{x}, \boldsymbol{u}\rangle \text{ for all } \boldsymbol{u}' \in Q \text{ and } \boldsymbol{u} \in P\}.$$

Each σ_Q is a strongly convex cone, and the collection $\{\sigma_Q \colon Q \text{ is a face of } P\} \cup \{\boldsymbol{0}\}$ is a complete fan in $N_{\mathbb{R}}$, called the *normal fan* of P and denoted by Σ_P.

An n-dimensional polytope P is called *simple* if exactly n facets meet at each vertex of P. In a simple polytope, any face Q of codimension k is uniquely written as an intersection of k facets $Q = F_{i_1} \cap \cdots \cap F_{i_k}$. Furthermore, the cone σ_Q is simplicial and is generated by the k normal vectors $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_k}$ to the corresponding facets.

Define the *nerve complex* of a polytope P with m facets by

$$\mathcal{K}_P = \left\{ I \subset [m] \colon \bigcap_{i \in I} F_i \neq \varnothing \right\}. \tag{2.7}$$

If P is simple then Σ_P is the simplicial fan defined by the data $\{\mathcal{K}_P; \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$.

Remark Not every complete fan is a normal fan of a polytope (see [21, §3.4] for an example). Even the weaker form of this statement fails: there are complete simplicial fans Σ whose underlying complexes $\mathcal{K}$ are not combinatorially equivalent to the boundary of a convex polytope. The *Barnette sphere* (a non-polytopal triangulation of S^3 with 8 vertices) provides a counterexample [20, §III.5].

In the case when $\mathcal{K} = \mathcal{K}_P$ for a simple polytope P, there is an alternative way to give $\mathcal{Z}_{\mathcal{K}}$ a smooth structure, by writing it as a nondegenerate intersection of Hermitian quadrics:

Construction 2.5 Let P be a simple convex polytope given by (2.6). We consider the affine map

$$i_P \colon N^*_{\mathbb{R}} \to \mathbb{R}^m, \quad \boldsymbol{u} \mapsto \bigl(\langle \boldsymbol{a}_1, \boldsymbol{u}\rangle + b_1, \ldots, \langle \boldsymbol{a}_m, \boldsymbol{u}\rangle + b_m\bigr),$$

or $i_P(\boldsymbol{u}) = A^*\boldsymbol{u} + \boldsymbol{b}$, where A^* is the dual map of (2.4) and $\boldsymbol{b} \in \mathbb{R}^m$ is the vector with coordinates b_i. The image $i_P(N^*_{\mathbb{R}})$ is an affine n-plane in $\mathbb{R}^m$, which can be written by $m-n$ linear equations:

$$\begin{aligned} i_P(N^*_{\mathbb{R}}) &= \{\boldsymbol{y} \in \mathbb{R}^m : \boldsymbol{y} = A^*\boldsymbol{u} + \boldsymbol{b} \quad \text{for some } \boldsymbol{u} \in N^*_{\mathbb{R}}\} \\ &= \{\boldsymbol{y} \in \mathbb{R}^m : \Gamma\boldsymbol{y} = \Gamma\boldsymbol{b}\}, \end{aligned} \tag{2.8}$$

where $\Gamma = (\gamma_{jk})$ is an $(m-n) \times m$-matrix whose rows form a basis of linear relations between the vectors $\boldsymbol{a}_i$. That is, Γ is of full rank and satisfies the identity $\Gamma A^* = 0$.

The map i_P embeds $P \subset \mathbb{R}^n$ into the positive orthant $\mathbb{R}^m_{\geqslant}$. We define the space $\mathcal{Z}_P$ from the commutative diagram

$$\begin{tikzcd} \mathcal{Z}_P \arrow[r, "i_{\mathcal{Z}}"] \arrow[d] & \mathbb{C}^m \arrow[d, "\mu"] \\ P \arrow[r, "i_P"] & \mathbb{R}^m_{\geqslant} \end{tikzcd} \tag{2.9}$$

where $\mu(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$. The torus $\mathbb{T}^m$ acts on $\mathcal{Z}_P$ with quotient P, and $i_{\mathcal{Z}}$ is a $\mathbb{T}^m$-equivariant embedding.

By replacing y_k by $|z_k|^2$ in the Eq. (2.8) defining the affine plane $i_P(N^*_{\mathbb{R}})$ we obtain that $\mathcal{Z}_P$ embeds into $\mathbb{C}^m$ as the set of common zeros of $m-n$ real quadratic equations (*Hermitian quadrics*):

$$i_{\mathcal{Z}}(\mathcal{Z}_P) = \left\{ \boldsymbol{z} \in \mathbb{C}^m : \sum_{k=1}^m \gamma_{jk}|z_k|^2 = \sum_{k=1}^m \gamma_{jk} b_k \quad \text{for } 1 \leqslant j \leqslant m-n \right\}. \tag{2.10}$$

We identify $\mathcal{Z}_P$ with its image $i_{\mathcal{Z}}(\mathcal{Z}_P)$ and consider $\mathcal{Z}_P$ as a subset of $\mathbb{C}^m$.

Theorem 2.6 [10,39] *Let P be a simple polytope given by* (2.6) *and let $\mathcal{K} = \mathcal{K}_P$ be its nerve simplicial complex* (2.7). *Then*

(a) $\mathcal{Z}_P \subset U(\mathcal{K})$;
(b) $\mathcal{Z}_P$ is a smooth submanifold in $\mathbb{C}^m$ of dimension $m+n$;
(c) there is a $\mathbb{T}^m$-equivariant homeomorphism $\mathcal{Z}_{\mathcal{K}} \cong \mathcal{Z}_P$.

Sketch of proof (a) By (2.9), if a point $\boldsymbol{z} = (z_1, \dots, z_m) \in \mathcal{Z}_P$ has its coordinates $z_{i_1}, \dots, z_{i_k}$ vanishing, then $F_{i_1} \cap \dots \cap F_{i_k} \neq \varnothing$. Hence the claim follows from the definition of $U(\mathcal{K})$ and $\mathcal{K}$, see (2.3) and (2.7).

(b) One needs to check that the condition of P being simple translates into the condition of (2.10) being nondegenerate.

(c) We have the quotient projection $U(\mathcal{K}) \to U(\mathcal{K})/R$ by the action of R (2.5), and both compact subsets $\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$, $\mathcal{Z}_P \subset U(\mathcal{K})$ intersect each R-orbit at a single point. For $\mathcal{Z}_{\mathcal{K}}$ this is proved in [39, Theorem 2.2], and for $\mathcal{Z}_P$ in [3, Chapter VI,Proposition 3.1.1]. Furthermore, $\mathcal{Z}_P$ is transverse to each orbit. □

3 Complex structures

Bosio and Meersseman [7] showed that some moment-angle manifolds admit complex structures. They only considered moment-angle manifolds corresponding to convex polytopes, and identified them with *LVM-manifolds* [31,32] (a class of non-Kähler complex manifolds whose underlying smooth manifolds are intersections of Hermitian quadrics in a complex projective space). A modification of this construction was considered in [39], where a complex structure was defined on any moment-angle manifold $\mathcal{Z}_\mathcal{K}$ corresponding to a complete simplicial fan. Similar results were obtained by Tambour [40].

In this section we assume that $m-n$ is even. We can always achieve this by adding a ghost vertex with any corresponding vector to our data $\{\mathcal{K}; \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$; topologically this results in multiplying $\mathcal{Z}_\mathcal{K}$ by a circle. We set $\ell = \frac{m-n}{2}$.

We identify $\mathbb{C}^m$ (as a real vector space) with $\mathbb{R}^{2m}$ using the map

$$(z_1, \ldots, z_m) \mapsto (x_1, y_1, \ldots, x_m, y_m),$$

where $z_k = x_k + iy_k$, and consider the $\mathbb{R}$-linear map

$$\mathrm{Re}\colon \mathbb{C}^m \to \mathbb{R}^m, \qquad (z_1, \ldots, z_m) \mapsto (x_1, \ldots, x_m).$$

In order to define a complex structure on the quotient $\mathcal{Z}_\mathcal{K} \cong U(\mathcal{K})/R$ we replace the action of R by the action of a holomorphic subgroup $C \subset (\mathbb{C}^\times)^m$ by means of the following construction.

Construction 3.1 Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ be a configuration of vectors that span $N_\mathbb{R} \cong \mathbb{R}^n$. Assume further that $m-n = 2\ell$ is even. Some of the $\boldsymbol{a}_i$'s may be zero. Consider the map

$$A\colon \mathbb{R}^m \to N_\mathbb{R}, \quad \boldsymbol{e}_i \mapsto \boldsymbol{a}_i.$$

We choose a complex ℓ-dimensional subspace in $\mathbb{C}^m$ which projects isomorphically onto the real $(m-n)$-dimensional subspace $\mathrm{Ker}\, A \subset \mathbb{R}^m$. More precisely, choose a $\mathbb{C}$-linear map $\Psi\colon \mathbb{C}^\ell \to \mathbb{C}^m$ satisfying the two conditions:

(a) the composite map $\mathbb{C}^\ell \xrightarrow{\Psi} \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m$ is a monomorphism;
(b) the composite map $\mathbb{C}^\ell \xrightarrow{\Psi} \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m \xrightarrow{A} N_\mathbb{R}$ is zero.

Set $\mathfrak{c} = \Psi(\mathbb{C}^l)$. Then the two conditions above are equivalent to the following:

(a′) $\mathfrak{c} \cap \bar{\mathfrak{c}} = \{\mathbf{0}\}$;
(b′) $\mathfrak{c} \subset \mathrm{Ker}(A_\mathbb{C}\colon \mathbb{C}^m \to N_\mathbb{C})$,

where $\bar{\mathfrak{c}}$ is the complex conjugate space and $A_\mathbb{C}\colon \mathbb{C}^m \to N_\mathbb{C}$ is the complexification of the real map $A\colon \mathbb{R}^m \to N_\mathbb{R}$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbb{C}^\ell & \xrightarrow{\Psi} & \mathbb{C}^m & \xrightarrow{\mathrm{Re}} & \mathbb{R}^m & \xrightarrow{A} & N_\mathbb{R} \\
 & & \big\downarrow{\scriptstyle\exp} & & \big\downarrow{\scriptstyle\exp} & & \\
 & & (\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}^m_> & &
\end{array} \tag{3.1}$$

where the vertical arrows are the componentwise exponential maps, and $|\cdot|$ denotes the map $(z_1, \ldots, z_m) \mapsto (|z_1|, \ldots, |z_m|)$. Now set

$$C = \exp \mathfrak{c} = \left\{ (e^{w_1}, \ldots, e^{w_m}) \in (\mathbb{C}^\times)^m \colon \boldsymbol{w} = (w_1, \ldots, w_m) \in \Psi(\mathbb{C}^\ell) \right\}. \tag{3.2}$$

Then $C \cong \mathbb{C}^\ell$ is a complex (but not algebraic) subgroup in $(\mathbb{C}^\times)^m$, and $\mathfrak{c}$ is its Lie algebra. There is a holomorphic action of C on $\mathbb{C}^m$ and $U(\mathcal{K})$ by restriction. Furthermore, condition (a′) above implies that C is closed in $(\mathbb{C}^\times)^m$.

Example 3.2 Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ be the configuration of $m = 2\ell$ zero vectors. We supplement it by the empty simplicial complex $\mathcal{K}$ on $[m]$ (with m ghost vertices), so that the data $\{\mathcal{K}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ define a complete fan in 0-dimensional space. Then $A\colon \mathbb{R}^m \to \mathbb{R}^0$ is a zero map, and condition (b) of Construction 3.1 is void. Condition (a) means that $\mathbb{C}^\ell \xrightarrow{\Psi} \mathbb{C}^{2\ell} \xrightarrow{\mathrm{Re}} \mathbb{R}^{2\ell}$ is an isomorphism of real spaces.

Consider the quotient $(\mathbb{C}^\times)^m/C$ (note that $(\mathbb{C}^\times)^m = U(\mathcal{K})$ in our case). The exponential map $\mathbb{C}^m \to (\mathbb{C}^\times)^m$ identifies $(\mathbb{C}^\times)^m$ with the quotient of $\mathbb{C}^m$ by the imaginary lattice $\Gamma = \mathbb{Z}\langle 2\pi i\boldsymbol{e}_1, \dots, 2\pi i\boldsymbol{e}_m\rangle$. Condition (a) implies that the projection $p\colon \mathbb{C}^m \to \mathbb{C}^m/\mathfrak{c}$ is nondegenerate on the imaginary subspace of $\mathbb{C}^m$. In particular, $p(\Gamma)$ is a lattice of rank $m = 2\ell$ in $\mathbb{C}^m/\mathfrak{c} \cong \mathbb{C}^\ell$. Therefore,

$$(\mathbb{C}^\times)^m/C \cong \big(\mathbb{C}^m/\Gamma\big)/\mathfrak{c} = \big(\mathbb{C}^m/\mathfrak{c}\big)\big/p(\Gamma) \cong \mathbb{C}^\ell/\mathbb{Z}^{2\ell}$$

is a complex compact ℓ-dimensional torus.

Any complex torus can be obtained in this way. Indeed, let $\Psi\colon \mathbb{C}^\ell \to \mathbb{C}^m$ be given by an $2\ell \times \ell$-matrix $\binom{-B}{I}$ where I is the unit matrix and B is a square matrix of size ℓ. Then $p\colon \mathbb{C}^m \to \mathbb{C}^m/\mathfrak{c}$ is given by the matrix $(I\ B)$ in appropriate bases, and $(\mathbb{C}^\times)^m/C$ is isomorphic to the quotient of $\mathbb{C}^\ell$ by the lattice $\mathbb{Z}\langle \boldsymbol{e}_1, \dots, \boldsymbol{e}_\ell, \boldsymbol{b}_1, \dots, \boldsymbol{b}_\ell\rangle$, where $\boldsymbol{b}_k$ is the kth column of B. (Condition (b) implies that the imaginary part of B is nondegenerate.)

For example, if $\ell = 1$, then $\Psi\colon \mathbb{C} \to \mathbb{C}^2$ is given by $w \mapsto (\beta w, w)$ for some $\beta \in \mathbb{C}$, so that the subgroup (3.2) is

$$C = \{(e^{\beta w}, e^w)\} \subset (\mathbb{C}^\times)^2.$$

Condition (a) implies that $\beta \notin \mathbb{R}$. Then $\exp\Psi\colon \mathbb{C} \to (\mathbb{C}^\times)^2$ is an embedding, and

$$(\mathbb{C}^\times)^2/C \cong \mathbb{C}/(\mathbb{Z} \oplus \beta\mathbb{Z})$$

is a complex 1-dimensional torus with the lattice parameter $\beta \in \mathbb{C}$.

Theorem 3.3 [39, Theorem 3.3] *Assume that data $\{\mathcal{K}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ define a complete fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$, and $m - n = 2\ell$. Let $C \cong \mathbb{C}^\ell$ be the group defined by* (3.2)*. Then*

(a) the holomorphic action of C on $U(\mathcal{K})$ is free and proper, and the quotient $U(\mathcal{K})/C$ has the structure of a compact complex manifold;
(b) $U(\mathcal{K})/C$ is diffeomorphic to $\mathcal{Z}_\mathcal{K}$.

Therefore, $\mathcal{Z}_\mathcal{K}$ has a complex structure, in which each element of $\mathbb{T}^m$ acts by a holomorphic transformation.

Besides compact complex tori described in Example 3.2, other examples of $\mathcal{Z}_\mathcal{K}$ include Hopf and Calabi–Eckmann manifolds (see [39] and Example 4.14 below).

Remark The subgroup $C \subset (\mathbb{C}^\times)^m$ depends on the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ and the choice of Ψ in Construction 3.1. Unlike the smooth case, the complex structure on $\mathcal{Z}_\mathcal{K}$ depends in an essential way on all these pieces of data.

As in the real situation of Theorem 2.6, in the case of normal fans we can view $\mathcal{Z}_\mathcal{K}$ as the intersection of quadrics $\mathcal{Z}_P$ given by (2.10):

Theorem 3.4 *Let P be a simple polytope given by* (2.6) *with even $m - n$, and let $\mathcal{K} = \mathcal{K}_P$ be the corresponding simplicial complex* (2.7)*. Then the composition*

$$\mathcal{Z}_P \xrightarrow{i_\mathcal{Z}} U(\mathcal{K}) \to U(\mathcal{K})/C$$

is a $\mathbb{T}^m$-equivariant diffeomorphism. It endows the intersection of quadrics $\mathcal{Z}_P$ with the structure of a complex manifold.

Proof We need to show that the C-orbit $C\boldsymbol{z}$ of any $\boldsymbol{z} \in U(\mathcal{K})$ intersects $\mathcal{Z}_P \subset U(\mathcal{K})$ transversely at a single point. First we show that the C-orbit of any $\boldsymbol{y} \in U(\mathcal{K})/\mathbb{T}^m$ intersects $\mathcal{Z}_P/\mathbb{T}^m = i_P(P)$ at a single point; this follows from Theorem 2.6 and the fact that the induced actions of C and R on $U(\mathcal{K})/\mathbb{T}^m$ coincide. Then we show that $C\boldsymbol{z}$ intersects the full preimage $\mathcal{Z}_P = \mu^{-1}(i_P(P))$ at a single point using the fact that C and $\mathbb{T}^m$ have trivial intersection in $(\mathbb{C}^\times)^m$. The transversality of the intersection $C\boldsymbol{z} \cap \mathcal{Z}_P$ follows from the transversality for R-orbits, because $T_{\boldsymbol{z}}(R\boldsymbol{z}) \oplus T_{\boldsymbol{z}}\mathcal{Z}_P = T_{\boldsymbol{z}}(C\boldsymbol{z}) \oplus T_{\boldsymbol{z}}\mathcal{Z}_P$.

Remark The embedding $i_{\mathcal{Z}}\colon \mathcal{Z}_P \to U(\mathcal{K})$ is not holomorphic. In the polytopal case, the complex structure on $\mathcal{Z}_P$ coming from Theorem 3.4 is equivalent to the structure of an LVM-manifold described in [7].

4 Submanifolds, analytic subsets and meromorphic functions

In this section we consider moment-angle manifolds $\mathcal{Z}_\mathcal{K}$ corresponding to complete simplicial n-dimensional fans Σ defined by data $\{\mathcal{K}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ with $m - n = 2\ell$. The manifold $\mathcal{Z}_\mathcal{K}$ is diffeomorphic to the quotient $U(\mathcal{K})/C$, as described in the previous section. This is used to equip $\mathcal{Z}_\mathcal{K}$ with a complex structure. The complex structure depends on the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ and the choice of a map Ψ in Construction 3.1, but we shall not reflect this in the notation.

4.1 Coordinate submanifolds

For each $J \subset [m]$, we define the corresponding *coordinate submanifold* in $\mathcal{Z}_\mathcal{K}$ by

$$\mathcal{Z}_{\mathcal{K}_J} = \{(z_1, \dots, z_m) \in \mathcal{Z}_\mathcal{K}\colon z_i = 0 \quad \text{for } i \notin J\}.$$

It is the moment-angle manifold corresponding to the *full subcomplex*

$$\mathcal{K}_J = \{I \in \mathcal{K}\colon I \subset J\}.$$

In the situation of Theorem 3.3, $\mathcal{Z}_{\mathcal{K}_J}$ is identified with the quotient of

$$U(\mathcal{K}_J) = \{(z_1, \dots, z_m) \in U(\mathcal{K})\colon z_i = 0 \quad \text{for } i \notin J\}$$

by the group C. In particular, $U(\mathcal{K}_J)/C$ is a complex submanifold in $U(\mathcal{K})/C$.

Observe that the closure of any $(\mathbb{C}^\times)^m$-orbit of $U(\mathcal{K})$ has the form $U(\mathcal{K}_J)$ for some $J \subset [m]$ (in particular, the dense orbit corresponds to $J = [m]$). Similarly, the closure of any $(\mathbb{C}^\times)^m/C$-orbit of $\mathcal{Z}_\mathcal{K} \cong U(\mathcal{K})/C$ has the form $\mathcal{Z}_{\mathcal{K}_J}$.

4.2 Holomorphic foliations

Construction 4.1 (*Holomorphic foliation $\mathcal{F}$ on $\mathcal{Z}_\mathcal{K}$*) Consider the complexified map $A_\mathbb{C}\colon \mathbb{C}^m \to N_\mathbb{C}$, $\boldsymbol{e}_i \mapsto \boldsymbol{a}_i$. Define the following complex $(m-n)$-dimensional subgroup in $(\mathbb{C}^\times)^m$:

$$G = \exp(\operatorname{Ker} A_\mathbb{C}) = \left\{\left(e^{z_1}, \dots, e^{z_m}\right) \in (\mathbb{C}^\times)^m\colon (z_1, \dots, z_m) \in \operatorname{Ker} A_\mathbb{C}\right\}. \tag{4.1}$$

It contains both the real subgroup R (2.5) and the complex subgroup C (3.2).

The group G acts on $U(\mathcal{K})$, and its orbits define a holomorphic foliation on $U(\mathcal{K})$. Since $G \subset (\mathbb{C}^\times)^m$, this action is free on the open subset $(\mathbb{C}^\times)^m \subset U(\mathcal{K})$, so that a generic leaf of the foliation has complex dimension $m-n=2\ell$ (in fact, all leaves have the same dimension, see Proposition 4.2). The ℓ-dimensional closed subgroup $C \subset G$ acts on $U(\mathcal{K})$ freely and properly by Theorem 3.3, so that $U(\mathcal{K})/C$ carries a holomorphic action of the quotient group $D=G/C$.

Recall from Construction 3.1 that $\mathfrak{c} = \operatorname{Lie} C \subset \operatorname{Ker} A_\mathbb{C} = \operatorname{Lie} G \subset \mathbb{C}^m$.

The action of the group D on $\mathcal{Z}_\mathcal{K}$ is holomorphic and nondegenerate, i.e., for any point $x \in \mathcal{Z}_\mathcal{K}$ the differential $T_e D \to T_x \mathcal{Z}_\mathcal{K}$ of the action map $g \mapsto g \cdot x$ is injective. Therefore the orbits of D define a smooth holomorphic foliation $\mathcal{F}$ on $\mathcal{Z}_\mathcal{K}$.

The leaves of the foliations from Construction 4.1 are described as follows (compare Example 3.2):

Proposition 4.2 *For any subset $I \subset [m]$, define the coordinate subspace $\mathbb{C}^I = \mathbb{C}\langle \boldsymbol{e}_k : k \in I\rangle \subset \mathbb{C}^m$ and the subgroup*

$$\Gamma_I = \operatorname{Ker} A_\mathbb{C} \cap \bigl(\mathbb{Z}\langle 2\pi i \boldsymbol{e}_1, \dots, 2\pi i \boldsymbol{e}_m\rangle + \mathbb{C}^I\bigr).$$

(a) Γ_I is discrete whenever $I \in \mathcal{K}$.
(b) We have $G \cong \operatorname{Ker} A_\mathbb{C}/\Gamma$, where $\Gamma = \Gamma_\varnothing$. Any leaf $G\boldsymbol{z}$ of the foliation on $U(\mathcal{K})$ by the orbits of G is biholomorphic to $(\mathbb{C}^\times)^{\operatorname{rk}\Gamma_I} \times \mathbb{C}^{2\ell - \operatorname{rk}\Gamma_I}$, where $I \in \mathcal{K}$ is the set of zero coordinates of $\boldsymbol{z} \in U(\mathcal{K})$.
(c) We have $D = G/C \cong \bigl(\operatorname{Ker} A_\mathbb{C}/\mathfrak{c}\bigr)\big/ p(\Gamma)$, where $p\colon \operatorname{Ker} A_\mathbb{C} \to \operatorname{Ker} A_\mathbb{C}/\mathfrak{c}$ is the projection. Any leaf $\mathcal{F}z$ of the foliation $\mathcal{F}$ on the moment-angle manifold $\mathcal{Z}_\mathcal{K} \cong U(\mathcal{K})/C$ is biholomorphic to $\mathbb{C}^\ell/p(\Gamma_I)$.

Proof (a) If $I \in \mathcal{K}$, then the composite map $\mathbb{C}^I \to \mathbb{C}^m \xrightarrow{A_\mathbb{C}} N_\mathbb{C}$ is monomorphic by the definition of a simplicial fan. Therefore, $\mathbb{C}^I \cap \operatorname{Ker} A_\mathbb{C} = \{\boldsymbol{0}\}$, which implies that Γ_I is discrete.

(b) Since $(\mathbb{C}^\times)^m = \exp(\mathbb{C}^m) \cong \mathbb{C}^m/\mathbb{Z}\langle 2\pi i \boldsymbol{e}_1, \dots, 2\pi i \boldsymbol{e}_m\rangle$, the isomorphism $G \cong \operatorname{Ker} A_\mathbb{C}/\Gamma$ follows from the definition of G, see (4.1). Therefore, to describe the orbit $G\boldsymbol{z}$, we can consider the orbit of the exponential action of $\operatorname{Ker} A_\mathbb{C}$ instead. The stabiliser of $\boldsymbol{z}$ under the action of $\operatorname{Ker} A_\mathbb{C}$ is exactly Γ_I, and the orbit itself is $\operatorname{Ker} A_\mathbb{C}/\Gamma_I \cong (\mathbb{C}^\times)^{\operatorname{rk}\Gamma_I} \times \mathbb{C}^{2\ell - \operatorname{rk}\Gamma_I}$.

(c) By the same argument, the orbit of $z \in \mathcal{Z}_\mathcal{K}$ under the action of $D = G/C$ is $(\operatorname{Ker} A_\mathbb{C}/\mathfrak{c})/p(\Gamma_I)$, and $\operatorname{Ker} A_\mathbb{C}/\mathfrak{c} \cong \mathbb{C}^\ell$. □

A subset $X' \subset X$ of a space with Lebesgue measure (X, μ) is said to contain *almost all* elements of X if its complement has zero measure: $\mu(X \backslash X') = 0$; in this case points of X' are *generic* for X, and the condition specifying X' in X is *generic*.

A vector of $\mathbb{Q}^m \subset \mathbb{R}^m$ is called *rational*. A linear subspace $V \subset \mathbb{R}^m$ is *rational* if it is generated by rational vectors. A linear function $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ is *rational* if it takes rational values on rational vectors (equivalently, if it has rational coefficients when written in the standard basis of $(\mathbb{R}^m)^*$).

Here are two examples of generic conditions for linear maps $A\colon \mathbb{R}^m \to N_\mathbb{R}$ (assuming that $m > \dim N_\mathbb{R}$, which is the case when Σ is a complete fan):

(g1) no rational linear function on $\mathbb{R}^m$ vanishes identically on $\operatorname{Ker} A$ (equivalently, $\operatorname{Ker} A$ is not contained in any rational hyperplane of $\mathbb{R}^m$);
(g2) $\operatorname{Ker} A$ does not contain rational vectors of $\mathbb{R}^m$.

We observe that if the configuration of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ satisfies the generic condition (g2), then the subgroup Γ of Proposition 4.2 is trivial, G is biholomorphic to $\mathbb{C}^{2\ell}$ and $D = G/C$ is biholomorphic to $\mathbb{C}^\ell$.

On the other hand, if $\operatorname{Ker} A \subset \mathbb{R}^m$ is a rational subspace (i.e. it has a basis consisting of vectors with integer coordinates), then $\operatorname{rk}\Gamma = 2\ell$ and the subgroup $G \subset (\mathbb{C}^\times)^m$ is closed and is isomorphic to $(\mathbb{C}^\times)^{2\ell}$. In this case the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ generate a lattice $N_\mathbb{Z} = \mathbb{Z}\langle \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\rangle$, and Σ is a rational (possibly singular) fan with respect to this lattice. If each vector $\boldsymbol{a}_i$ is primitive in $N_\mathbb{Z}$, then the quotient $U(\mathcal{K})/G$ is the *toric variety* V_Σ corresponding to the fan Σ (see e.g. [13]). The foliation $\mathcal{F}$ gives rise to a holomorphic principal Seifert bundle $\pi : \mathcal{Z}_\mathcal{K} \to V_\Sigma$ with fibres compact complex tori G/C (leaves of $\mathcal{F}$), see [32] and [39, Proposition 5.2].

Foliations similar to $\mathcal{F}$ were studied in [5], together with generic conditions.

4.3 Transverse Kähler forms

Definition 4.3 Let $\mathcal{F}$ be a holomorphic ℓ-dimensional foliation on a complex manifold M. A $(1, 1)$-form $\omega_\mathcal{F}$ on M is called *transverse Kähler* with respect to $\mathcal{F}$ if the following two conditions are satisfied:

(a) $\omega_\mathcal{F}$ is closed, i.e. $d\omega_\mathcal{F} = 0$;
(b) $\omega_\mathcal{F}$ is positive and the zero space of $\omega_\mathcal{F}$ is the tangent space of $\mathcal{F}$. (That is, $\omega(V, JV) \geqslant 0$ for any real tangent vector V, and $\omega_\mathcal{F}(V, JV) = 0$ if and only if V is tangent to $\mathcal{F}$; here J is the operator of complex structure.)

One way to define a transverse Kähler form on $\mathcal{Z}_\mathcal{K}$ is to use a modification of an argument of Loeb and Nicolau [30]; it works only for normal fans:

Proposition 4.4 *Assume that $\Sigma = \Sigma_P$ is the normal fan of a simple polytope P. Then the foliation $\mathcal{F}$ described in Construction 4.1 admits a transverse Kähler form $\omega_\mathcal{F}$.*

Proof Since Σ is a normal fan of P, we have a $\mathbb{T}^m$-equivariant diffeomorphism $\varphi\colon U(\mathcal{K})/C \xrightarrow{\cong} \mathcal{Z}_P$ between $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/C$ and the intersection of quadrics $\mathcal{Z}_P \subset \mathbb{C}^m$ (Theorem 3.4). Let $\omega = \frac{i}{2}\sum dz_k \wedge d\bar{z}_k$ be the standard form on $\mathbb{C}^m$, and $\omega_\mathcal{Z} = i^*_\mathcal{Z}\omega$ its restriction to the intersection of quadrics $\mathcal{Z}_P$. Define $\omega_\mathcal{F} = \varphi^*\omega_\mathcal{Z}$. Using the same argument as [30, Proposition 2] one verifies that the zero foliation of the form $\omega_\mathcal{Z}$ coincides with $\varphi(\mathcal{F})$, and therefore $\omega_\mathcal{F}$ is transverse Kähler. $\square$

The condition that Σ is a normal fan in Proposition 4.4 is important. Indeed, it was shown in [14] that general *LVMB-manifolds* do not admit transverse Kähler forms. A similar argument proves that there are no transverse Kähler forms on the quotients $U(\mathcal{K})/C$ for general complete simplicial fans.

We can relax the condition on the fan slightly at the cost of weakening the conditions on the form $\omega_\mathcal{F}$:

Definition 4.5 A complete simplicial fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$ is called *weakly normal* if there exists a (not necessarily simple) n-dimensional polytope P given by (2.6) such that Σ is a simplicial subdivision of the normal fan Σ_P.

Remark Let Σ be a complete simplicial fan with chosen generators $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ of its edges. Then Σ is normal if one can find constants $b_1, \dots, b_m$ such the polytope P defined by (2.6) is simple and the simplicial complex $\mathcal{K}_P$ (2.7) coincides with the underlying complex $\mathcal{K}$ of the fan. A fan Σ is weakly normal if one can find $b_1, \dots, b_m$ so that $\mathcal{K}$ is contained in $\mathcal{K}_P$; equivalently, $F_{i_1} \cap \dots \cap F_{i_k} \neq \varnothing$ in P if $\boldsymbol{a}_{i_1}, \dots, \boldsymbol{a}_{i_k}$ span a cone of Σ.

In the geometry of toric varieties, a lattice polytope (2.6) (in which $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ are primitive lattice vectors, and $b_1, \dots, b_m$ are integers) gives rise to an ample divisor on the toric variety

V_Σ corresponding to the normal fan $\Sigma = \Sigma_P$. In particular, toric varieties corresponding to (rational) normal fans are projective. Weakly normal rational fans Σ give rise to *semiample divisors* on their corresponding toric varieties (see [2, §II.4.2]); such a divisor defines a map from V_Σ to a projective space, which is not necessarily an embedding.

Given a cone $\sigma_I \in \Sigma$, the *quotient fan* Σ/σ_I is defined. Its associated simplicial complex is the link $\mathrm{lk}_\mathcal{K} I$. If Σ is weakly normal (with the corresponding polytope P), then Σ/σ_I is also weakly normal (with the corresponding polytope being a face of P). This fact will be used below in the inductive arguments for Theorems 4.11 and 4.18.

Theorem 4.6 *Assume that Σ is a weakly normal fan defined by $\{\mathcal{K}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$. Then there exists an exact $(1,1)$-form $\omega_\mathcal{F}$ on $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/C$ which is transverse Kähler for the foliation $\mathcal{F}$ on the dense open subset $(\mathbb{C}^\times)^m/C \subset U(\mathcal{K})/C$.*

Proof The plan of the proof is as follows. We construct a smooth function $f\colon U(\mathcal{K}) \to \mathbb{R}$ which is plurisubharmonic, so that $\omega = dd^c f$ is a positive $(1,1)$-form (here $d = \partial + \bar\partial$ and $d^c = -i(\partial - \bar\partial)$). We check that the kernel of ω consists of tangents to the orbits of G. Then we show that ω descends to a form $\omega_\mathcal{F}$ on $U(\mathcal{K})/C$ with the required properties.

We consider the short exact sequence

$$0 \to \mathrm{Ker}\,A \longrightarrow \mathbb{R}^m \xrightarrow{A} N_\mathbb{R} \to 0$$

where A is given by $\boldsymbol{e}_i \mapsto \boldsymbol{a}_i$. Since Σ is weakly normal, there is the corresponding polytope (2.6). We think of $(b_1, \dots, b_m)$ as a linear function $\boldsymbol{b}\colon \mathbb{R}^m \to \mathbb{R}$, and denote by $\chi_{\boldsymbol{b}}\colon \mathrm{Ker}\,A \to \mathbb{R}$ the restriction of $\boldsymbol{b}$ to $\mathrm{Ker}\,A$.

Let $I \in \mathcal{K}$ be a maximal simplex (i.e. the vectors $\boldsymbol{a}_i$, $i \in I$, span a maximal cone of Σ), and let $\boldsymbol{u}_I = \bigcap_{i\in I} F_i$ be the vertex of P corresponding to I. Define the linear function $\beta_I \in (\mathbb{R}^m)^*$ whose coordinates in the standard basis are $\langle \boldsymbol{a}_i, \boldsymbol{u}_I\rangle + b_i$, $i = 1, \dots, m$. It follows that all coordinates of β_I are nonnegative, and the coordinates corresponding to $i \in I$ are zero. (Some other coordinates of β_I may also vanish, since the polytope P is not necessarily simple and different I may give the same vertex.) Also, the restriction of β_I to $\mathrm{Ker}\,A$ is $\chi_{\boldsymbol{b}}$ because $A^*\colon N_\mathbb{R}^* \to (\mathbb{R}^m)^*$ is given by $\boldsymbol{u} \mapsto (\langle \boldsymbol{a}_1, \boldsymbol{u}\rangle, \dots, \langle \boldsymbol{a}_m, \boldsymbol{u}\rangle)$. Finally, by multiplying all b_i simultaneously by a positive factor we can obtain that all coordinates of all β_I are either zero or $\geqslant 2$.

We define the function $f\colon U(\mathcal{K}) \to \mathbb{R}$ as follows:

$$f(\boldsymbol{z}) = \log\Bigl(\sum_{\text{maximal } I\in\mathcal{K}} |\boldsymbol{z}|^{\beta_I}\Bigr)$$

where $|\boldsymbol{z}|^\alpha = |z_1|^{\alpha(\boldsymbol{e}_1)} \cdots |z_m|^{\alpha(\boldsymbol{e}_m)}$ is the monomial corresponding to a linear function $\alpha \in (\mathbb{R}^m)^*$, and we set $0^0 = 1$. By definition of $U(\mathcal{K})$, the set of zero coordinates of any point $\boldsymbol{z} \in U(\mathcal{K})$ is contained in a maximal simplex $I \in \mathcal{K}$, hence $|\boldsymbol{z}|^{\beta_I} > 0$. Therefore, the function f is smooth on $U(\mathcal{K})$.

Now define the real $(1,1)$-form $\omega = dd^c f$ on $U(\mathcal{K})$. By [15, Theorem I.5.6], the function f is plurisubharmonic, so that ω is positive.

Lemma 4.7 *The kernel of $\omega|_{(\mathbb{C}^\times)^m}$ consists of tangent spaces to the orbits of the action of G, see* (4.1).

Proof Let J be the operator of complex structure. Since ω is positive, its kernel coincides with the kernel of the symmetric 2-form $\omega(\,\cdot\,, J\cdot)$.

Take $z \in (\mathbb{C}^\times)^m$. By writing z in polar coordinates, $z_k = \rho_k e^{i\varphi_k}$, we decompose the real tangent space to $(\mathbb{C}^\times)^m$ at z as $T_z = T_\rho \oplus T_\varphi$, where T_ρ and T_φ consist of tangents to radial and angular directions respectively. Since f does not depend on the φ_i's, the matrix of $\omega(\,\cdot\,,\ J\cdot)$ is block-diagonal with respect to the decomposition $T_z = T_\rho \oplus T_\varphi$. The diagonal blocks are identical since ω is J-invariant. It follows that $\mathrm{Ker}\omega = (\mathrm{Ker}\omega \cap T_\rho) \oplus J(\mathrm{Ker}\omega \cap T_\rho)$. It remains to describe $\mathrm{Ker}\omega \cap T_\rho$. To do this, we identify T_ρ with the Lie algebra $\mathbb{R}^m$ of the group $\mathbb{R}^m_>$ acting on $(\mathbb{C}^\times)^m$ by coordinatewise multiplications.

Take a radial vector field $V \in T_\rho$ corresponding to a 1-parameter subgroup $t \mapsto (e^{\lambda_1 t}z_1, \dots, e^{\lambda_m t}z_m)$ where $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$. Then

$$\begin{aligned}\omega(V, JV) &= (dd^c f)(V, JV) = L_V\langle d^c f, JV\rangle - L_{JV}\langle d^c f, V\rangle\\ &= L_{JV}\langle df, JV\rangle - L_V\langle df, J^2V\rangle = \frac{d^2 f(e^{\lambda_1 t}z_1, \dots, e^{\lambda_m t}z_m)}{dt^2}\bigg|_{t=0},\end{aligned}$$

where L_V denotes the Lie derivative along V, the second equality holds because V and JV commute, and $\langle df, JV\rangle = 0$ because f does not depend on angular coordinates. It remains to calculate the second derivative. Note that for any linear function $\beta_I \in (\mathbb{R}^m)^*$, we have $|z(t)|^{\beta_I} = e^{\langle \beta_I, \lambda\rangle t}|z|^{\beta_I}$ along the curve $t \mapsto z(t) = (e^{\lambda_1 t}z_1, \dots, e^{\lambda_m t}z_m)$. Therefore,

$$\begin{aligned}&\frac{d^2}{dt^2} f(e^{\lambda_1 t}z_1, \dots, e^{\lambda_m t}z_m)\bigg|_{t=0} = \frac{d}{dt}\left(\frac{\sum_I \langle \beta_I, \lambda\rangle e^{\langle \beta_I, \lambda\rangle t}|z|^{\beta_I}}{\sum_I e^{\langle \beta_I, \lambda\rangle t}|z|^{\beta_I}}\right)\bigg|_{t=0}\\ &= \frac{1}{\left(\sum_I |z|^{\beta_I}\right)^2}\left(\sum_I \langle \beta_I, \lambda\rangle^2 |z|^{\beta_I} \cdot \sum_J |z|^{\beta_J} - \Big(\sum_I \langle \beta_I, \lambda\rangle |z|^{\beta_I}\Big)^2\right)\\ &= \frac{1}{\left(\sum_I |z|^{\beta_I}\right)^2}\left(\sum_{I,J} |z|^{\beta_I}|z|^{\beta_J}\Big(\langle \beta_I, \lambda\rangle - \langle \beta_J, \lambda\rangle\Big)^2\right).\end{aligned}$$

We claim that this vanishes precisely when $\lambda \in \mathrm{Ker}A$.

If $\lambda \in \mathrm{Ker}A$ then $\langle \beta_I, \lambda\rangle = \sum_{i=1}^m \lambda_i b_i$ by the definition of β_I, and this is independent of I. Therefore, the last sum in the displayed formula above vanishes.

Conversely, if the sum above is zero for a given $\lambda \in \mathbb{R}^m$, then $\langle \beta_I - \beta_J, \lambda\rangle = 0$ for any pair of maximal simplices $I, J \in \mathcal{K}$ (here we use the fact that $z \in (\mathbb{C}^\times)^m$). We have $\beta_I - \beta_J = A^*(\boldsymbol{u}_I - \boldsymbol{u}_J)$, where $\boldsymbol{u}_I - \boldsymbol{u}_J$ is the vector connecting the vertices $\boldsymbol{u}_I$ and $\boldsymbol{u}_J$ of P. Since P is n-dimensional, the linear span of all vectors $\beta_I - \beta_J$ is the whole $A^*(N^*_{\mathbb{R}})$. Thus, $\lambda \in \mathrm{Ker}A$.

We have therefore identified $\mathrm{Ker}\omega \cap T_\rho$ with $\mathrm{Ker}A \subset \mathbb{R}^m$. On the other hand, $\mathrm{Ker}A$ is the tangent space to the orbits of $R \subset G$, see (2.5). Since $\mathrm{Ker}\omega = (\mathrm{Ker}\omega \cap T_\rho) \oplus J(\mathrm{Ker}\omega \cap T_\rho)$, we can identify $\mathrm{Ker}\omega$ with $\mathrm{Ker}A \oplus J\mathrm{Ker}A$, which is exactly the tangent space to an orbit of G. □

Now we can finish the proof of Theorem 4.6. We need to show that the form $\omega = dd^c f$ descends to a form on $U(\mathcal{K})/C$. In other words, we need to show that ω is *basic* with respect to the foliation defined by the orbits of C, i.e. $\omega(V) = 0$ and $L_V\omega = 0$ for any vector field V tangent to C-orbits. Since $C \subset G$, the previous lemma implies that $\mathrm{Ker}\omega$ contains V, so $\omega(V) = 0$. By the Cartan formula, $L_V\omega = (di_V + i_V d)\omega = d(i_V\omega) = 0$, since $d\omega = 0$ and $i_V\omega = \omega(V) = 0$.

Let $\omega_{\mathcal{F}}$ be the form obtained by descending ω to $U(\mathcal{K})/C$. Then $\omega_{\mathcal{F}}$ is positive since ω is positive, and Lemma 4.7 implies that $\mathrm{Ker}\omega_{\mathcal{F}}$ consists exactly of the tangents to the orbits of G/C. Thus, $\omega_{\mathcal{F}}$ is transverse Kähler for $\mathcal{F}$ on $(\mathbb{C}^\times)^m/C$.

To see that the form $\omega_{\mathcal{F}}$ is exact we note that the fibres of the projection $p\colon U(\mathcal{K}) \to U(\mathcal{K})/C$ are contractible (as $C \cong \mathbb{C}^\ell$), so p is a homotopy equivalence. Hence p induces an isomorphism of the cohomology groups $p^*\colon H^*(U(\mathcal{K})/C;\mathbb{R}) \xrightarrow{\cong} H^*(U(\mathcal{K});\mathbb{R})$. Since $p^*\omega_{\mathcal{F}} = \omega$ and $[\omega] = 0$ in $H^2(U(\mathcal{K});\mathbb{R})$, the form $\omega_{\mathcal{F}}$ is also exact. One can actually construct a 1-form $\eta_{\mathcal{F}}$ satisfying $d\eta_{\mathcal{F}} = \omega_{\mathcal{F}}$ more explicitly as follows. Choose a smooth section $s\colon U(\mathcal{K})/C \to U(\mathcal{K})$. For any $x \in U(\mathcal{K})/C$ the restriction of ω to the fibre $p^{-1}(x)$ is zero, hence $d^c f|_{p^{-1}(x)}$ is closed. Since $p^{-1}(x)$ is contractible, $d^c f|_{p^{-1}(x)}$ is exact. Hence, $d^c f|_{p^{-1}(x)} = dg_x$ for some function $g_x\colon p^{-1}(x) \to \mathbb{C}$ defined up to a constant. We choose g_x such that $g_x(s(x)) = 0$. The collection of functions $\{g_x\}_x$ defines a smooth function $g\colon U(\mathcal{K}) \to \mathbb{C}$ such that for any $x \in U(\mathcal{K})/C$ we have $dg|_{p^{-1}(x)} = d^c f|_{p^{-1}(x)}$. Then the 1-form $\eta = d^c f - dg$ is basic with respect to the foliation generated by the fibers of p, since $L_V\eta = (di_V\eta + i_V d\eta) = i_V\omega = 0$ for any vector field V tangent to the fibre at p. Therefore η descends to a form $\eta_{\mathcal{F}}$ on $U(\mathcal{K})/C$ such that $d\eta_{\mathcal{F}} = \omega_{\mathcal{F}}$.

4.4 Kähler and Fujiki class $\mathcal{C}$ subvarieties

Now we can use the transverse Kähler form $\omega_{\mathcal{F}}$ to describe complex submanifolds and analytic subsets in $\mathcal{Z}_{\mathcal{K}}$.

Definition 4.8 A compact complex variety M is said to be of *Fujiki class* $\mathcal{C}$ if it is bimeromorphic to a Kähler manifold.

Remark Since a blow-up of a Kähler manifold is again Kähler, for each Fujiki class $\mathcal{C}$ variety X there exists a birational holomorphic map $\widetilde{X} \to X$, where $\widetilde{X}$ is a compact Kähler manifold.

Definition 4.9 Let M be a compact complex manifold, and Θ a closed (1, 1)-current on M (see Sect. 1.2). Assume that M admits a positive (1, 1)-form η such that the current $\Theta - \eta$ is positive. Then Θ is called a *Kähler current*.

Theorem 4.10 [16, Theorem 0.6] *A compact complex manifold admits a Kähler current if and only if it is of Fujiki class* $\mathcal{C}$.

We will use a modification of this statement for Fujiki class $\mathcal{C}$ *varieties* to prove the following:

Theorem 4.11 *Under the assumptions of Theorem 4.6, any Fujiki class $\mathcal{C}$ subvariety of $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/C$ is contained in a leaf of the ℓ-dimensional foliation $\mathcal{F}$.*

Proof Let $i\colon Y \hookrightarrow U(\mathcal{K})/C$ be a Fujiki class $\mathcal{C}$ subvariety. We may assume that Y is not contained in a coordinate submanifold of $\mathcal{Z}_{\mathcal{K}}$, i.e., Y contains a point from $(\mathbb{C}^\times)^m/C$. (Otherwise choose the smallest by inclusion coordinate submanifold $\mathcal{Z}_{\mathcal{K}_J} = U(\mathcal{K}_J)/C \subset U(\mathcal{K})/C$ such that $Y \subset \mathcal{Z}_{\mathcal{K}_J}$, and consider $\mathcal{Z}_{\mathcal{K}_J}$ instead of $\mathcal{Z}_{\mathcal{K}}$.)

We claim that Y is contained in a D-orbit (a leaf of $\mathcal{F}$). Suppose this is not the case. Then

$$k = \min_{y\in Y_{\mathrm{reg}}} \dim(T_yY \cap T_y\mathcal{F}) < \dim Y,$$

and the minimum is achieved at a generic point $y \in Y$.

Consider a smooth Kähler birational modification $(\widetilde{Y}, \omega_{\widetilde{Y}})$ of Y with a birational map $p\colon \widetilde{Y} \to Y$. Let $\omega_{\mathcal{F}}$ be the transverse Kähler form constructed in Theorem 4.6, and let $\widetilde{\omega}_{\mathcal{F}} = p^*i^*\omega_{\mathcal{F}}$ be its pullback to $\widetilde{Y}$. Consider the top-degree differential form

$$\nu = \widetilde{\omega}_{\mathcal{F}}^{\dim Y - k} \wedge \omega_{\widetilde{Y}}^k$$

on $\widetilde{Y}$. Then ν is a positive measure, since $\omega_{\mathcal{F}}$, $\widetilde{\omega}_{\mathcal{F}}$ and $\omega_{\widetilde{Y}}$ are all positive (1, 1)-forms. We claim that ν is nontrivial. Take a generic $y \in Y_{\mathrm{reg}}$, such that p is an isomorphism in a neighbourhood of $p^{-1}(y)$ and $\dim(T_yY \cap T_y\mathcal{F}) = k$. We claim that ν is nonzero (strictly positive) at $p^{-1}(y)$. Indeed, $i^*\omega_{\mathcal{F}}$ is a restriction of the transverse Kähler form $\omega_{\mathcal{F}}$, so it is strictly positive on a $(\dim Y - k)$-dimensional subspace transverse to its k-dimensional null space $T_yY \cap T_y\mathcal{F} \subset T_yY$. Hence, $\widetilde{\omega}_{\mathcal{F}}$ is also positive on the corresponding subspace of $T_{p^{-1}(y)}\widetilde{Y}$. Since $\omega_{\widetilde{Y}}$ is Kähler and positive in all directions, ν is strictly positive at $p^{-1}(y)$.

Now recall that $\widetilde{\omega}_{\mathcal{F}} = d\alpha$ is exact, so by the Stokes formula,

$$\int_{\widetilde{Y}} \widetilde{\omega}_{\mathcal{F}}^{\dim Y - k} \wedge \omega_{\widetilde{Y}}^k = \int_{\widetilde{Y}} d(\alpha \wedge \widetilde{\omega}_{\mathcal{F}}^{\dim Y - k - 1} \wedge \omega_{\widetilde{Y}}^k) = 0.$$

On the other hand, the measure ν is positive and nonzero at a point $p^{-1}(y)$, leading to a contradiction. Hence $k = \dim Y$ and Y is contained in a leaf of $\mathcal{F}$. □

Corollary 4.12 *Assume that the subspace* $\operatorname{Ker} A \subset \mathbb{R}^m$ *does not contain rational vectors. Then there are no positive dimensional Fujiki class $\mathcal{C}$ subvarieties through a generic point of $\mathcal{Z}_{\mathcal{K}}$. More precisely, there are no such subvarieties intersecting nontrivially the open subset $(\mathbb{C}^\times)^m/C \subset U(\mathcal{K})/C = \mathcal{Z}_{\mathcal{K}}$.*

Proof If $\operatorname{Ker} A \subset \mathbb{R}^m$ does not contain rational vectors, then, in the notation of Proposition 4.2, the group Γ is trivial. Therefore, any leaf in the open part $(\mathbb{C}^\times)^m/C \subset U(\mathcal{K})/C$ of the foliation $\mathcal{F}$ is isomorphic to $\mathbb{C}^\ell$, and therefore cannot contain Fujiki class $\mathcal{C}$ subvarieties. □

4.5 The case of 1-dimensional foliation

This case was studied by Loeb and Nicolau [30]. Here is how their results translate into our setting:

Theorem 4.13 *Assume that the foliation $\mathcal{F}$ is 1-dimensional, i.e. $\ell = 1$.*

(a) If $\operatorname{Ker} A \subset \mathbb{R}^m$ *is a rational subspace, then any analytic subset of $\mathcal{Z}_{\mathcal{K}}$ is either a point, or has the form $\pi^{-1}(X)$, where $\pi: U(\mathcal{K})/C \to V$ is the principal Seifert bundle over the variety $V = U(\mathcal{K})/G$ and $X \subset V$ is a subvariety;*

(b) If no rational linear function on $\mathbb{R}^m$ vanishes on $\operatorname{Ker} A$*, then any irreducible analytic subset of $\mathcal{Z}_{\mathcal{K}}$ is either a coordinate submanifold or a point.*

Proof First observe that if $m - n = 2\ell = 2$, then the fan Σ is normal. (Indeed, the corresponding polytope P can be obtained by truncating an n-simplex at a vertex.) Therefore, by Proposition 4.4, there exists a transverse Kähler form $\omega_{\mathcal{F}}$. Let Y be an analytic subset of positive dimension in $\mathcal{Z}_{\mathcal{K}}$. Then Y consists of leaves of the foliation $\mathcal{F}$, as otherwise the integral $\int_Y \omega_{\mathcal{F}}^{\dim Y}$ of the exact form $\omega_{\mathcal{F}}^{\dim Y}$ over Y is positive. In other words, Y is D-invariant.

Under assumption (a), both $G \subset (\mathbb{C}^\times)^m$ and $D = G/C \subset (\mathbb{C}^\times)^m/C$ are closed subgroups and Y has the form $\pi^{-1}(Y/D)$, where $Y/D \subset U(\mathcal{K})/G = V$.

When each vector $\boldsymbol{a}_i$ is primitive in the lattice $\mathbb{Z}\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \rangle$, the variety V is the toric variety V_Σ corresponding to the rational fan Σ. In general, V is a finite branched covering over V_Σ, see [32, Example 2.8].

Under assumption (b), we claim that the minimal closed complex subgroup $\overline{D}$ containing D is the whole $(\mathbb{C}^\times)^m/C$. This claim is equivalent to that $\overline{G} = (\mathbb{C}^\times)^m$. Indeed, since $\overline{G}$ is closed, the intersection $\operatorname{Lie} \overline{G} \cap i\mathbb{R}^m$ is a rational subspace. Now if $\operatorname{Lie} \overline{G} \cap i\mathbb{R}^m \neq i\mathbb{R}^m$, then there exists a rational linear function vanishing on $\operatorname{Lie} \overline{G}$ and therefore on $\operatorname{Lie} G = \operatorname{Ker} A_{\mathbb{C}}$, leading to a contradiction. Hence $\operatorname{Lie} \overline{G} \cap i\mathbb{R}^m = i\mathbb{R}^m$, i.e. $\operatorname{Lie} \overline{G} = \mathbb{C}^m$, and the claim is proved. It follows that the subset Y is $(\mathbb{C}^\times)^m/C$-invariant. An irreducible $(\mathbb{C}^\times)^m/C$-invariant analytic subset of $U(\mathcal{K})/C$ is a coordinate submanifold.

Example 4.14 (Hopf manifold) Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}$ be a set of vectors which span $N_{\mathbb{R}} \cong \mathbb{R}^n$ and satisfy a linear relation $\lambda_1\boldsymbol{a}_1 + \dots + \lambda_{n+1}\boldsymbol{a}_{n+1} = \boldsymbol{0}$ with all $\lambda_k > 0$. Let Σ be the complete simplicial fan in $N_{\mathbb{R}}$ whose cones are generated by all proper subsets of $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1}$. To make $m - n$ even we add one ghost vector $\boldsymbol{a}_{n+2}$. Hence $m = n + 2$, $\ell = 1$, and we have one more linear relation $\mu_1\boldsymbol{a}_1 + \dots + \mu_{n+1}\boldsymbol{a}_{n+1} + \boldsymbol{a}_{n+2} = \boldsymbol{0}$, this time the μ_k's are arbitrary reals.

The subspace $\operatorname{Ker} A \subset \mathbb{R}^{n+2}$ is spanned by $(\lambda_1, \dots, \lambda_{n+1}, 0)$ and $(\mu_1, \dots, \mu_{n+1}, 1)$.

Then $\mathcal{K} = \mathcal{K}_\Sigma$ is the boundary of an n-dimensional simplex with $n + 1$ vertices and one ghost vertex, $\mathcal{Z}_{\mathcal{K}} \cong S^{2n+1} \times S^1$, and $U(\mathcal{K}) = (\mathbb{C}^{n+1} \setminus \{\boldsymbol{0}\}) \times \mathbb{C}^\times$.

Conditions (a) and (b) of Construction 3.1 imply that C is a 1-dimensional subgroup in $(\mathbb{C}^\times)^m$ given in appropriate coordinates by

$$C = \left\{(e^{\zeta_1 w}, \dots, e^{\zeta_{n+1} w}, e^w) \colon w \in \mathbb{C}\right\} \subset (\mathbb{C}^\times)^m,$$

where $\zeta_k = \mu_k + \alpha\lambda_k$ for some $\alpha \in \mathbb{C} \setminus \mathbb{R}$. By changing the basis of $\operatorname{Ker} A$ if necessary, we may assume that $\alpha = i$. The moment-angle manifold $\mathcal{Z}_{\mathcal{K}} \cong S^{2n+1} \times S^1$ acquires a complex structure as the quotient $U(\mathcal{K})/C$:

$$\begin{aligned}
&\left(\mathbb{C}^{n+1} \setminus \{\boldsymbol{0}\}\right) \times \mathbb{C}^\times \big/ \left\{(z_1, \dots, z_{n+1}, t) \sim (e^{\zeta_1 w} z_1, \dots, e^{\zeta_{n+1} w} z_{n+1}, e^w t)\right\}\\
&\cong \left(\mathbb{C}^{n+1} \setminus \{\boldsymbol{0}\}\right) \big/ \left\{(z_1, \dots, z_{n+1}) \sim (e^{2\pi i \zeta_1} z_1, \dots, e^{2\pi i \zeta_{n+1}} z_{n+1})\right\},
\end{aligned}$$

where $\boldsymbol{z} \in \mathbb{C}^{n+1} \setminus \{\boldsymbol{0}\}$, $t \in \mathbb{C}^\times$. The latter is the quotient of $\mathbb{C}^{n+1} \setminus \{\boldsymbol{0}\}$ by a diagonalisable action of $\mathbb{Z}$. It is known as a *Hopf manifold*. For $n = 0$ we obtain a complex 1-dimensional torus (elliptic curve) of Example 3.2.

Suppose we are in the situation of Theorem 4.13 (a). Then all λ_k are commensurable (the ratio of each pair is rational). We can assume that all λ_k are integer (multiplying them by a common factor if necessary). Then Σ is a rational fan and $V_\Sigma = \mathbb{C}P^n(\lambda_1, \dots, \lambda_{n+1})$ is a weighted projective space, whose orbifold structure may have singularities in codimension one. This gives rise to a holomorphic principal Seifert bundle $\pi \colon \mathcal{Z}_{\mathcal{K}} \to \mathbb{C}P^n(\lambda_1, \dots, \lambda_{n+1})$ with fibre an elliptic curve.

Now suppose we are in the situation of Theorem 4.13 (b). For example, this is the case when $\lambda_1, \dots, \lambda_{m+1}$ are linearly independent over $\mathbb{Q}$. Then any submanifold of $\mathcal{Z}_{\mathcal{K}}$ is a Hopf manifold of lesser dimension (including elliptic curves and points).

4.6 Divisors and meromorphic functions

For simplicity, by a *divisor* on a complex manifold we mean an analytic subset of codimension one. (The description of divisors in $\mathcal{Z}_{\mathcal{K}}$ obtained below can be easily modified to match the more standard definition of divisors as linear combinations of analytic subsets of codimension one.)

For generic initial data, there are only finitely many divisors on the complex moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$, and they are of a very special type. This holds without any geometric restrictions on the fan:

Theorem 4.15 *Assume that the data $\{\mathcal{K}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ define a complete fan Σ in $N_{\mathbb{R}} \cong \mathbb{R}^n$, and $m - n = 2\ell$. Assume further that*

(a) there is at most one ghost vertex in $\mathcal{K}$;
(b) no rational linear function on $\mathbb{R}^m$ vanishes identically on $\operatorname{Ker} A$.

Then any divisor of $\mathcal{Z}_{\mathcal{K}}$ is a union of coordinate divisors.

Proof Let $\mathcal{D} \subset \mathcal{Z}_\mathcal{K}$ be a divisor. Consider the divisor $q^{-1}(\mathcal{D})$ in $U(\mathcal{K})$, where $q\colon U(\mathcal{K}) \to \mathcal{Z}_\mathcal{K} \cong U(\mathcal{K})/C$ is the quotient projection.

First assume that there are no ghost vertices in $\mathcal{K}$, so that all $\boldsymbol{a}_k$ are nonzero. Then $\mathbb{C}^m \backslash U(\mathcal{K})$ has codimension $\geqslant 2$. Hence the closure of $q^{-1}(\mathcal{D})$ in $\mathbb{C}^m$ is a C-invariant divisor in $\mathbb{C}^m$. Choose an element $\boldsymbol{u} = (u_1, \dots, u_m)$ in C such that $|u_k| > 1, k = 1, \dots, m$. Such an element exists since there is a relation $\sum_{k=1}^m \lambda_k \boldsymbol{a}_k = 0$ with all $\lambda_k > 0$ (this follows from the fact that Σ is a complete fan). Denote by L the discrete subgroup of C consisting of integral powers of $\boldsymbol{u}$; then $L \cong \mathbb{Z}$. Being a subgroup of C, the group L is diagonalisable and acts freely and properly on $\mathbb{C}^m \backslash \{\boldsymbol{0}\}$, so the quotient is a Hopf manifold. By Example 4.14, any Hopf manifold is a complex moment-angle manifold with $\ell = 1$. Then if follows from Theorem 4.13 (b) that any analytic subset of $(\mathbb{C}^m \backslash \{\boldsymbol{0}\})/L$ is a union of coordinate submanifolds. Hence the closure of $q^{-1}(\mathcal{D})/L$ in $(\mathbb{C}^m \backslash \{\boldsymbol{0}\})/L$ is a union of coordinate divisors, and the same holds for $\mathcal{D} \subset \mathcal{Z}_\mathcal{K}$.

Now assume that there is one ghost vertex in $\mathcal{K}$, say the first one. Then $U(\mathcal{K}) = \mathbb{C}^\times \times U(\widetilde{\mathcal{K}})$, where $\widetilde{\mathcal{K}}$ does not have ghost vertices. Since the divisor $q^{-1}(\mathcal{D}) \subset \mathbb{C}^\times \times U(\widetilde{\mathcal{K}})$ is C-invariant, its projection to the first factor $\mathbb{C}^\times$ is onto. Therefore, for any $z_1 \in \mathbb{C}^\times$, the intersection $\bigl(\{z_1\} \times U(\widetilde{\mathcal{K}})\bigr) \cap q^{-1}(\mathcal{D})$ is a divisor in $\{z_1\} \times U(\widetilde{\mathcal{K}})$. This divisor is invariant with respect to the subgroup $\widetilde{C} = \{(u_1, u_2, \dots, u_m) \in C\colon u_1 = 1\}$. Choose an element $\boldsymbol{u} = (1, u_2, \dots, u_m)$ in $\widetilde{C}$ such that $|u_i| > 1$ for $i \geqslant 2$. Such an element exist since now we have a relation $\sum_{i \geqslant 2} \mu_i \boldsymbol{a}_i = 0$ with all $\mu_i > 0$, by the completeness of the fan. Now we proceed as in the case when there are no ghost vertices, and conclude that each $\bigl(\{z_1\} \times U(\widetilde{\mathcal{K}})\bigr) \cap q^{-1}(\mathcal{D})$ is a union of coordinate divisors in $\{z_1\} \times U(\widetilde{\mathcal{K}})$. Since the number of coordinate divisors is finite, $\bigl(\{z_1\} \times U(\widetilde{\mathcal{K}})\bigr) \cap q^{-1}(\mathcal{D}) = \{z_1\} \times \mathcal{E}$, where $\mathcal{E} \subset U(\widetilde{\mathcal{K}})$ is a union of coordinate divisors. Thus, $q^{-1}(\mathcal{D}) = \mathbb{C}^\times \times \mathcal{E}$, and $\mathcal{D}$ also has the required form. □

Corollary 4.16 *Under the assumptions of Theorem 4.15, there are no non-constant meromorphic functions on $\mathcal{Z}_\mathcal{K}$.*

Proof Let f be a non-constant meromorphic function on $\mathcal{Z}_\mathcal{K}$. Choose a point $z_0 \in \mathcal{Z}_\mathcal{K}$ in the dense $(\mathbb{C}^\times)^m/C$-orbit outside of the pole set of f. Then the support of the zero divisor of the function $f(z) - f(z_0)$ contains a point z_0 in the dense $(\mathbb{C}^\times)^m/C$-orbit, so it does not lie in the union of coordinate divisors. This contradicts Theorem 4.15. □

4.7 General subvarieties

As we can see from Theorem 4.13, the geometry of $\mathcal{Z}_\mathcal{K}$ depends essentially on the geometric data, namely on a choice of maps A and Ψ. In the situation of Theorem 4.13 (i.e. $\ell = 1$), the case when no rational function vanishes on $\operatorname{Ker} A$ is generic; so that $\mathcal{Z}_\mathcal{K}$ has only coordinate submanifolds for generic geometric data. As we shall see, the situation is similar in the case of higher-dimensional foliations ($\ell > 1$), although the generic condition on the initial data will be more subtle.

Lemma 4.17 *Assume that data $\{\mathcal{K}; \boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ define a complete fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$ such that no rational linear function on $\mathbb{R}^m$ vanishes on* $\operatorname{Ker} A$*. Then for almost all subspaces $\mathfrak{c} \subset \mathbb{C}^m$ satisfying the conditions of Construction 3.1, the only complex subspace $L \subset \mathbb{C}^m$ such that*

(a) $\mathfrak{c} \subset L$,
(b) $\bar{\mathfrak{c}} \cap L \neq \{\boldsymbol{0}\}$,
(c) $L \cap i\mathbb{R}^m$ *is a rational subspace in* $i\mathbb{R}^m$,

is the whole $L = \mathbb{C}^m$.

Proof The subspace $\mathfrak{c} = \Psi(\mathbb{C}^\ell)$ is a point in the Grassmannian $\mathrm{Gr}_{\mathbb{C}}(\mathbb{C}^\ell, \mathrm{Ker} A_{\mathbb{C}})$, which has complex dimension ℓ^2. The conditions of Construction 3.1 specify an open subset in $\mathrm{Gr}_{\mathbb{C}}(\mathbb{C}^\ell, \mathrm{Ker} A_{\mathbb{C}})$; we refer to a map $\Psi\colon \mathbb{C}^\ell \to \mathbb{C}^m$ satisfying these conditions and the corresponding subspace $\mathfrak{c} = \Psi(\mathbb{C}^\ell)$ as *admissible*. We shall prove that the set of admissible $\mathfrak{c}$ for which there exist $L \subsetneq \mathbb{C}^m$ satisfying properties (a)–(c) is contained in a countable union of manifolds of dimension $<\ell^2$ and therefore has zero Lebesgue measure.

Let $L \subsetneq \mathbb{C}^m$ be a complex subspace satisfying conditions (a)–(c). Set $Q = \mathrm{Ker} A \cap L$, and let $\dim_{\mathbb{R}} Q = q$. Conditions (a) and (b) imply $q > 0$. Also, since no rational linear function vanishes on $\mathrm{Ker} A$ and $L \cap i\mathbb{R}^m$ is a proper rational subspace, L cannot contain the whole $\mathrm{Ker} A$. Hence, $0 < q < 2\ell$. Let $\pi_{\mathrm{Re}}, \pi_{\mathrm{Im}}\colon \mathfrak{c} \to \mathrm{Ker} A$ denote the projections onto the real and imaginary parts (which are both isomorphisms of real spaces). For a given $v \in Q$ there exists a unique $w \in \mathrm{Ker} A$ such that $v + iw \in \mathfrak{c} \subset L$. Since $v, v + iw \in L$, the vector w lies in L. Therefore,

$$\pi_{\mathrm{Im}} \circ \pi_{\mathrm{Re}}^{-1}(Q) = Q, \tag{4.2}$$

as the operator on the left hand side sends v to w. In particular, the operator $\pi_{\mathrm{Im}} \circ \pi_{\mathrm{Re}}^{-1}$ defines a complex structure on the space Q. Hence $\dim_{\mathbb{C}} \mathfrak{c} \cap (Q \otimes \mathbb{C}) = q/2$. Now, for a fixed subspace $Q \subsetneq \mathrm{Ker} A$, the set of complex subspaces $\mathfrak{c} \subset \mathrm{Ker} A_{\mathbb{C}}$ satisfying condition (4.2) is identified with an open subset in

$$\mathrm{Gr}_{\mathbb{C}}\big(\mathbb{C}^{q/2}, Q \otimes \mathbb{C}\big) \times \mathrm{Gr}_{\mathbb{C}}\big(\mathbb{C}^{\ell - q/2}, (\mathrm{Ker} A_{\mathbb{C}})/(Q \otimes \mathbb{C})\big)$$

and has complex dimension $(q/2)^2 + (\ell - q/2)^2 < \ell^2$. Since there are only countably many rational subspaces in $i\mathbb{R}^m$, there are countably many $Q \subset \mathrm{Ker} A$. Hence the set of spaces $\mathfrak{c}$ for which there exist $L \subsetneq \mathbb{C}^m$ satisfying properties (a)–(c) is contained in the union of countably many manifolds of dimension $<\ell^2$. Thus its Lebesgue measure is zero. □

Our final results describe analytic subsets in a complex moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$, under a generic assumption on the complex structure and a geometric assumption on the fan Σ:

Theorem 4.18 *Let $\mathcal{Z}_{\mathcal{K}}$ be a complex moment-angle manifold, with linear maps $A\colon \mathbb{R}^m \to N_{\mathbb{R}}$ and $\Psi\colon \mathbb{C}^\ell \to \mathbb{C}^m$ described in Construction 3.1. Assume that*

(a) no rational linear function on $\mathbb{R}^m$ vanishes on $\mathrm{Ker} A$*;*
(b) $\mathrm{Ker} A$ *does not contain rational vectors of $\mathbb{R}^m$;*
(c) the map Ψ satisfies the generic condition of Lemma 4.17;
(d) the fan Σ is weakly normal.

Then any irreducible analytic subset $Y \subsetneq \mathcal{Z}_{\mathcal{K}}$ of positive dimension is contained in a coordinate submanifold $\mathcal{Z}_{\mathcal{K}_J} \subsetneq \mathcal{Z}_{\mathcal{K}}$.

Proof Assume that $Y \subset \mathcal{Z}_{\mathcal{K}}$ is an irreducible analytic subset of the smallest positive dimension. Let $\mathcal{F}_Y$ be the foliation on Y associated with $\mathcal{F}$, i.e. $T_y\mathcal{F}_Y = T_yY \cap T_y\mathcal{F}$ (here T_yY denotes the Zariski tangent space). Assuming that Y contains a generic point (i.e. a point from $(\mathbb{C}^\times)^m/C \subset \mathcal{Z}_{\mathcal{K}}$), we need to show that Y is the whole $\mathcal{Z}_{\mathcal{K}}$. As we assume (d), Theorem 4.6 applies, giving a transverse Kähler form $\omega_{\mathcal{F}}$. Since $\omega_{\mathcal{F}}$ is exact, the integral $\int_Y \omega_{\mathcal{F}}^{\dim Y}$ vanishes, hence the foliation $\mathcal{F}_Y$ is nontrivial. For any $z \in \mathcal{Z}_{\mathcal{K}}$, the tangent space $T_y\mathcal{F}$ is naturally identified with the vector space $\mathrm{Ker} A_{\mathbb{C}}/\mathfrak{c}$. Let k be the complex dimension of $\mathcal{F}_Y$ at a generic point of Y, and let $\widetilde{Y} \subset Y \times \mathrm{Gr}_{\mathbb{C}}(\mathbb{C}^k, \mathrm{Ker} A_{\mathbb{C}}/\mathfrak{c})$ be the space of all k-dimensional planes $V \subset T_yY \cap T_y\mathcal{F}$. Denote by π_Y and π_G the projections of $\widetilde{Y}$ to Y and $\mathrm{Gr}_{\mathbb{C}}(\mathbb{C}^k, \mathrm{Ker} A_{\mathbb{C}}/\mathfrak{c})$, respectively. For any k-dimensional plane $V \subset \operatorname{Ker} A_{\mathbb{C}}/\mathfrak{c}$, the analytic subset $\pi_Y(\pi_G^{-1}(V))$ is identified with the closure of the set of all points $y \in Y$ such that $T_y Y = V$. Since $Y \subset \mathcal{Z}_{\mathcal{K}}$ is an analytic subset of the smallest dimension, $\pi_Y(\pi_G^{-1}(V))$ either is 0-dimensional for all V or coincides with Y.

Assume that $\dim \pi_Y(\pi_G^{-1}(V)) = 0$ for all $V \subset \operatorname{Ker} A_{\mathbb{C}}/\mathfrak{c}$. Then $\widetilde{Y}$ admits a meromorphic map to $\operatorname{Gr}(\mathbb{C}^k, \operatorname{Ker} A_{\mathbb{C}}/\mathfrak{c})$ which is finite at a generic point, hence $\widetilde{Y}$ is Moishezon. The map $\widetilde{Y} \to Y$ is surjective, so Y is also Moishezon. Then Y is of Fujiki class $\mathcal{C}$ by the classical result of [33]. By the assumption (b), Corollary 4.12 applies, leading to a contradiction.

Now assume that $\pi_Y(\pi_G^{-1}(V)) = Y$ for some k-dimensional plane $V \subset \operatorname{Ker} A_{\mathbb{C}}/\mathfrak{c}$. In other words, $T_y Y \cap T_y \mathcal{F} = V$ for a generic point $y \in Y$. Let $H \subset (\mathbb{C}^\times)^m/C$ be the largest closed complex subgroup preserving $Y \subset U(\mathcal{K})/C$, and let $\mathfrak{h} \subset \mathbb{C}^m/\mathfrak{c}$ be the Lie algebra of H. Let $L \subset \mathbb{C}^m$ be the preimage of $\mathfrak{h}$. Then

- $\mathfrak{c} \subset L$;
- $L \cap i\mathbb{R}^m$ is a rational subspace, since H is closed;
- $\bar{\mathfrak{c}} \cap L \neq \{\mathbf{0}\}$, since $\mathfrak{h} \supset V \subset \operatorname{Ker} A_{\mathbb{C}}/\mathfrak{c}$.

As we assume (a) and (c), Lemma 4.17 applies, implying that $L = \mathbb{C}^m$. Hence, $H = (\mathbb{C}^\times)^m/C$ and $Y = \mathcal{Z}_{\mathcal{K}}$.

Corollary 4.19 *Assume that every coordinate submanifold $\mathcal{Z}_{\mathcal{K}_J} \subseteq \mathcal{Z}_{\mathcal{K}}$ either satisfies the assumptions* (a)–(d) *of Theorem* 4.18*, or is a compact complex torus with no analytic subsets of positive dimension. Then any irreducible analytic subset $Y \subsetneq \mathcal{Z}_{\mathcal{K}}$ of positive dimension is a coordinate submanifold $\mathcal{Z}_{\mathcal{K}_J} \subsetneq \mathcal{Z}_{\mathcal{K}}$.*

Proof Let $Y \subsetneq \mathcal{Z}_{\mathcal{K}}$ be an irreducible analytic subset. Applying Theorem 4.18 to $\mathcal{Z}_{\mathcal{K}}$ we conclude that Y is contained in a coordinate submanifold $\mathcal{Z}_{\mathcal{K}_J} \subsetneq \mathcal{Z}_{\mathcal{K}}$. If $Y = \mathcal{Z}_{\mathcal{K}_J}$, we are done. Otherwise we have two options: either $\mathcal{Z}_{\mathcal{K}_J}$ itself satisfies the assumptions of Theorem 4.18 and we can proceed by induction, or $\mathcal{Z}_{\mathcal{K}_J}$ is a compact complex torus. In the latter case $Y = \mathcal{Z}_{\mathcal{K}_J}$, by the assumption. □

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