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Complexes of Modules over Exceptional Lie Superalgebras $E(3,8)$ and $E(5,10)$

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1 Introduction

In [3, 4, 5], we constructed all degenerate irreducible modules over the exceptional Lie superalgebra $E(3,6)$. In the present paper, we apply the same method to the exceptional Lie superalgebras $E(3,8)$ and $E(5,10)$.

The Lie superalgebra $E(3,8)$ is strikingly similar to $E(3,6)$. In particular, as in the case of $E(3,6)$, the maximal compact subgroup of the group of automorphisms of $E(3,8)$ is isomorphic to the group of symmetries of the Standard Model. However, as the computer calculations by Joris van Jeugt show, the fundamental particle contents in the $E(3,8)$ case is completely different from that in the $E(3,6)$ case [3]. All the nice features of the latter case, like the CPT symmetry, completely disappear in the former case. We believe that the main reason behind this is that, unlike $E(3,6)$, $E(3,8)$ cannot be embedded in $E(5,10)$, which, we believe, is the algebra of symmetries of the SU_5 Grand Unified Model (the maximal compact subgroup of the automorphism group of $E(5,10)$ is SU_5).

However, similarity with $E(3,6)$ allows us to apply to $E(3,8)$ all the arguments from [3] almost verbatim, and Figure 4.1 of the present paper, that depicts all degenerate $E(3,8)$-modules, is almost the same as [3, Figure 3] for $E(3,6)$.

The picture in the $E(5,10)$ case is quite different (see Figure 5.1). We believe that it depicts all degenerate irreducible $E(5,10)$-modules, but we still do not have a proof.

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2 Morphisms between generalized Verma modules

Let $L = \oplus_{j\in\mathbb{Z}} G_j$ be a $\mathbb{Z}$-graded Lie superalgebra by finite-dimensional vector spaces. Let

$$L_- = \oplus_{j<0} G_j, \qquad L_+ = \oplus_{j>0} G_j, \qquad L_0 = G_0 + L_+. \tag{2.1}$$

Given a G_0-module V, we extend it to a L_0-module by letting L_+ act trivially, and define the induced L-module

$$M(V) = U(L) \otimes_{U(L_0)} V. \tag{2.2}$$

If V is a finite-dimensional irreducible G_0-module, the L-module $M(V)$ is called a *generalized Verma module* (associated to V), and it is called *degenerate* if it is not irreducible.

Let A and B be two G_0-modules and let $\mathrm{Hom}(A, B)$ be the G_0-module of linear maps from A to B. The following proposition will be extensively used to construct morphisms between the L-modules $M(A)$ and $M(B)$.

Proposition 2.1. Let $\Phi \in M(\mathrm{Hom}(A, B))$ be such that

$$\nu \cdot \Phi = 0 \quad \forall \nu \in L_0. \tag{2.3}$$

Then one can construct a well-defined morphism of L-modules

$$\varphi : M(A) \longrightarrow M(B) \tag{2.4}$$

by the rule $\varphi(u \otimes a) = u\Phi(a)$. Explicitly, write $\Phi = \sum_m u_m \otimes \ell_m$, where $u_m \in U(L)$, $\ell_m \in \mathrm{Hom}(A, B)$. Then

$$\varphi(u \otimes a) = \sum_m (uu_m) \otimes \ell_m(a). \tag{2.5}$$

□

Proof. We have to prove that for $\nu \in U(L_0)$,

$$\varphi(u\nu \otimes a) = \varphi(u \otimes \nu a), \tag{2.6}$$

in order to conclude that φ is well defined. Notice that condition (2.3) means

$$\sum_m [\nu, u_m] \otimes \ell_m + \sum_m u_m \otimes \nu\ell_m = 0. \tag{2.7}$$

Therefore, we have

$$
\begin{aligned}
\varphi(uv \otimes a) &= \sum_m uvu_m \otimes \ell_m(a) \\
&= \sum_m u[v, u_m] \otimes \ell_m(a) + \sum_m uu_m v \otimes \ell_m(a) \\
&= \sum_m u[v, u_m] \otimes \ell_m(a) + \sum_m uu_m \otimes v(\ell_m(a)) \\
&= \sum_m u[v, u_m] \otimes \ell_m(a) + \sum_m uu_m \otimes (v\ell_m)(a) \\
&\quad + \sum_m uu_m \otimes \ell_m(va) \\
&= \sum uu_m \otimes \ell_m(va) = \varphi(u \otimes va) \quad \text{(by (2.7) and (2.5)).}
\end{aligned}
\tag{2.8}
$$

The fact that φ defines a morphism of L-modules is immediate from the definition. ■

Remark 2.2. If L_0 is generated by G_0 and a subset $T \subset L_+$, then condition (2.3) is equivalent to

$$G_0 \cdot \Phi = 0, \tag{2.9a}$$

$$a \cdot \Phi = 0 \quad \forall a \in T. \tag{2.9b}$$

Condition (2.9a) usually gives a hint to a possible shape of Φ and is checked by general invariant-theoretical considerations. After that, (2.9b) is usually checked by a direct calculation.

Remark 2.3. We can view $M(V)$ also as the induced $(L_- \oplus G_0)$-module: $U(L_- \oplus G_0) \otimes_{U(G_0)} V$. Then condition (2.9a) on $\Phi = \sum_m u_m \otimes \ell_m$, where $u_m \in U(L_- \oplus G_0)$ and $\ell_m \in \mathrm{Hom}(A, B)$, suffices in order for (2.5) to give a well-defined morphism of $(L_- \oplus G_0)$-modules. We can also replace G_0 by any of its subalgebras.

3 Lie superalgebras $E(3, 6)$, $E(3, 8)$, and $E(5, 10)$

Recall some standard notation:

$$W_n = \left\{ \sum_{j=1}^{n} P_i(x)\partial_i \mid P_i \in \mathbb{C}[[x_1, \ldots, x_n]], \partial_i \equiv \frac{\partial}{\partial x_i} \right\} \tag{3.1}$$

denotes the Lie algebra of formal vector fields in n indeterminates;

$$S_n = \left\{ D = \sum P_i \partial_i \mid \operatorname{div} D \equiv \sum_i \partial_i P_i = 0 \right\} \tag{3.2}$$

denotes the Lie subalgebra of divergenceless formal vector fields; $\Omega^k(n)$ denotes the space of formal differential forms of degree k in n indeterminates; $\Omega^k_{c\ell}(n)$ denotes the subspace of closed forms.

The Lie algebra W_n acts on $\Omega^k(n)$ via the Lie derivative $D \to L_D$. Given $\lambda \in \mathbb{C}$, we can twist this action

$$D\omega = L_D\omega + \lambda(\operatorname{div} D)\omega. \tag{3.3}$$

The W_n-module thus obtained is denoted by $\Omega^k(n)^\lambda$. Recall the following obvious isomorphism of W_n-modules:

$$\Omega^0(n) \simeq \Omega^n(n)^{-1}, \tag{3.4}$$

and the following slightly less obvious isomorphism of W_n-modules:

$$W_n \simeq \Omega^{n-1}(n)^{-1}. \tag{3.5}$$

The latter is obtained by mapping a vector field $D \in W_n$ to the $(n-1)$-form $\iota_D(dx_1 \wedge \cdots \wedge dx_n)$. Note that (3.5) induces an isomorphism of S_n-modules

$$S_n \simeq \Omega^{n-1}_{c\ell}(n). \tag{3.6}$$

Recall that the Lie superalgebra $E(5,10) = E(5,10)_{\bar{0}} + E(5,10)_{\bar{1}}$ is constructed as follows (see [1, 2]):

$$E(5,10)_{\bar{0}} = S_5, \qquad E(5,10)_{\bar{1}} = \Omega^2_{c\ell}(5), \tag{3.7}$$

$E(5,10)_{\bar{0}}$ acts on $E(5,10)_{\bar{1}}$ via the Lie derivative, and $[\omega_2, \omega_2'] = \omega_2 \wedge \omega_2' \in \Omega^4_{c\ell}(5) = S_5$ (see (3.6)) for $\omega_2, \omega_2' \in E(S,10)_{\bar{1}}$.

Next, recall the construction of the Lie superalgebras $E^\flat := E(3,6)$ and $E^\sharp := E(3,8)$ (see [1])

$$\begin{aligned} &E^\flat_{\bar{0}} = E^\sharp_{\bar{0}} = W_3 + \mathfrak{sl}_2(\Omega^0(3)) \text{ (the natural semidirect sum);} \\ &E^\flat_{\bar{1}} = \Omega^1(3)^{-1/2} \otimes \mathbb{C}^2, \qquad E^\sharp_{\bar{1}} = \left(\Omega^0(3)^{-1/2} \otimes \mathbb{C}^2\right) + \left(\Omega^2(3)^{-1/2} \otimes \mathbb{C}^2\right). \end{aligned} \tag{3.8}$$

The action of the even on the odd parts is defined via the Lie derivative and the multiplication of a function and a differential form. The bracket of two odd elements is defined by using the identifications (3.4) and (3.5) as follows. For $\omega_i, \omega_i' \in \Omega^i(3)$ and $\nu, \nu' \in \mathbb{C}^2$, we define the following bracket of two elements from $E^\flat_{\bar{1}}$:

$$[\omega_1 \otimes \nu, \omega_1' \otimes \nu'] = -(\omega_1 \wedge \omega_1') \otimes (\nu \wedge \nu') \\ - (d\omega_1 \wedge \omega_1' + \omega_1 \wedge d\omega_1') \otimes (\nu \cdot \nu'), \tag{3.9}$$

and the following bracket of two elements from $E^\sharp_{\bar{1}}$:

$$[\omega_2 \otimes \nu, \omega_2' \otimes \nu'] = 0, \qquad [\omega_0 \otimes \nu, \omega_0' \otimes \nu'] = -(d\omega_0 \wedge d\omega_0') \otimes (\nu \wedge \nu'), \tag{3.10}$$

$$[\omega_0 \otimes \nu, \omega_2 \otimes \nu'] = -(\omega_0 \wedge \omega_2) \otimes (\nu \wedge \nu') \\ - (d\omega_0 \wedge \omega_2 - \omega_0 \wedge \omega_2) \otimes (\nu \cdot \nu'). \tag{3.11}$$

Recall also an embedding of $E^\flat$ in $E(5, 10)$ [1, 3]. For that let $z_+ = x_4$, $z_- = x_5$, $\partial_+ = \partial_4$, $\partial_- = \partial_5$, and let ϵ^+, ϵ^- denote the standard basis of $\mathbb{C}^2$. Then $E^\flat_{\bar{0}}$ is embedded in $E(5, 10)_{\bar{0}} = S_5$ by ($D \in W_3$, $a, b, c \in \Omega^0(3)$):

$$D \longmapsto D - \frac{1}{2}(\operatorname{div} D)(z_+\partial_+ + z_-\partial_-), \\ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \longmapsto a(z_+\partial_+ - z_-\partial_-) + bz_+\partial_- + cz_-\partial_+, \tag{3.12}$$

and $E^\flat_{\bar{1}}$ is embedded in $E(5, 10)_{\bar{1}} = \Omega^2_{c\ell}(5)$ by ($f \in \Omega^0(3)$):

$$f dx_i \otimes \epsilon^\pm \longmapsto z_\pm dx_i \wedge df + f dx_i \wedge dz_\pm. \tag{3.13}$$

Introduce the following subalgebras $S^\flat \subset E^\flat$ and $S^\sharp \subset E^\sharp$:

$$S^\flat_{\bar{0}} = S^\sharp_{\bar{0}} = W_3 + \mathbb{C} \otimes \mathfrak{sl}_2(\mathbb{C}), \\ S^\flat_{\bar{1}} = \Omega^1_{c\ell}(3)^{-1/2} \otimes \mathbb{C}^2, \qquad S^\sharp_{\bar{1}} = \Omega^0(3)^{-1/2} \otimes \mathbb{C}^2. \tag{3.14}$$

Proposition 3.4. The map $S^\sharp \to S^\flat$, which is identical on $S^\sharp_{\bar{0}}$ and sends $f \otimes \nu \in S^\sharp_{\bar{1}}$ to $df \otimes \nu \in S^\flat_{\bar{1}}$ is a surjective homomorphism of Lie superalgebras with 2-dimensional central kernel $\mathbb{C} \otimes \mathbb{C}^2 \subset S^\sharp_{\bar{1}}$. □

Proof. The proof is straightforward using (3.9), (3.10), and (3.11). ■

This proposition is probably the main reason for a remarkable similarity between representation theories of $E^\sharp$ and $E^\flat$. We stress this similarity in our notation and develop representation theory of $E^\sharp$ along the same lines as that of $E^\flat$ done in [3, 4]. Sometimes we drop the superscript $\flat$ or $\sharp$ when the situation is the same.

Recall that $E^\flat$ carries a unique irreducible consistent $\mathbb{Z}$-gradation. It has depth 2, and it is defined by

$$\deg x_i = -\deg \partial_i = 2, \qquad \deg \partial_i = -1, \qquad \deg \epsilon^\pm = 0, \qquad \deg \mathfrak{sl}_2(\mathbb{C}) = 0. \tag{3.15}$$

The "nonpositive" part of this $\mathbb{Z}$-gradation is as follows:

$$\begin{aligned}
G_{-2} &= \langle \partial_i \mid i = 1,2,3 \rangle, \\
G_{-1} &= \langle d_i^a := \epsilon^a \otimes dx_i = dx_i \wedge dz_a \mid i = 1,2,3,\ a = +,- \rangle, \\
G_0 &= \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}Y,
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
\mathfrak{sl}_3(\mathbb{C}) &= \langle h_1 = x_1\partial_1 - x_2\partial_2,\ h_2 = x_2\partial_2 - x_3\partial_3,\ e_1 = x_1\partial_2, \\
&\qquad e_2 = x_2\partial_3,\ e_{12} = x_1\partial_3,\ f_1 = x_2\partial_1,\ f_2 = x_3\partial_2,\ f_{12} = x_3\partial_1 \rangle, \\
\mathfrak{sl}_2(\mathbb{C}) &= \langle h_3 = z_+\partial_+ - z_-\partial_-,\ e_3 = z_+\partial_-,\ f_3 = z_-\partial_+ \rangle, \\
Y &= \frac{2}{3}\sum x_i\partial_i - (z_+\partial_+ + z_-\partial_-).
\end{aligned} \tag{3.17}$$

The eigenspace decomposition of $\mathrm{ad}(3Y)$ coincides with the consistent $\mathbb{Z}$-grading of $E^\flat$. We fix the Cartan subalgebra $\mathcal{H} = \langle h_1, h_2, h_3, Y \rangle$ and the Borel subalgebra $\mathcal{B} = \mathcal{H} + \langle e_i (i = 1,2,3), e_{12} \rangle$ of G_0. Then $f_0 := d_1^+$ is the highest weight vector of the (irreducible) G_0-module G_{-1}, the vectors

$$e_0' := x_3 d_3^-, \qquad e_0^\flat := x_3 d_2^- - x_2 d_3^- + 2z_- dx_2 \wedge dx_3 \tag{3.18}$$

are all the lowest weight vectors of the G_0-module G_1, and we have

$$\begin{aligned}
[e_0', f_0] &= f_2, \\
[e_0^\flat, f_0] &= \frac{2}{3}h_1 + \frac{1}{3}h_2 - h_3 - Y =: h_0^\flat,
\end{aligned} \tag{3.19}$$

so that

$$h_0^\flat = -x_2\partial_2 - x_3\partial_3 + 2z_-\partial_-. \tag{3.20}$$

The following relations are also important to keep in mind:

$$\begin{aligned}
&[e_0', d_1^+] = f_2, \qquad [e_0', d_2^+] = -f_{12}, \qquad [e_0', d_3^+] = 0, \qquad [e_0', d_i^-] = 0, \\
&[d_i^\pm, d_j^\pm] = 0, \qquad [d_i^+, d_j^-] + [d_j^+, d_i^-] = 0.
\end{aligned} \tag{3.21}$$

Recall that G_0 along with the elements f_0, $e_0^\flat$, e_0' generate the Lie superalgebra $E^\flat$ [1].

The Lie superalgebra $E^\sharp$ carries a unique consistent irreducible $\mathbb{Z}$-gradation of depth 3:

$$E^\sharp = \oplus_{j\geq -3} G_j. \tag{3.22}$$

It is defined by

$$\deg x_i = -\deg \partial_i = \deg dx_i = 2, \qquad \deg \epsilon^{\pm} = -3, \qquad \deg \mathfrak{sl}_2(\mathbb{C}) = 0. \tag{3.23}$$

In view of Proposition 3.4 and the above $E^\flat$-notation, we introduce the following $E^\sharp$-notation

$$\begin{aligned} &d^{\pm} := 1 \otimes \epsilon^{\pm}, \qquad d_i^{\pm} := x_i \otimes \epsilon^{\pm}, \qquad e_0' := \frac{1}{2}x_3^2 \otimes \epsilon^{\pm}, \qquad f_0 := d_1^+, \\ &e_0^\sharp := -(dx_2 \wedge dx_3) \otimes \epsilon^-, \qquad h_0^\sharp := \frac{2}{3}h_1 + \frac{1}{3}h_2 - \frac{1}{2}h_3 + \frac{1}{2}Y. \end{aligned} \tag{3.24}$$

If, in analogy with $E^\flat$, we denote $\epsilon^a = dz_a$, then $G_{-3} = \langle dz_+, dz_- \rangle$, G_{-2}, G_{-1} and G_0 are the same as for $E^\flat$ (except that now $[G_{-1}, G_{-2}] \neq 0$), and the relations (3.19) and (3.21) still hold, but the formula for $h_0^\sharp$ is different:

$$h_0^\sharp = \frac{2}{3}h_1 + \frac{1}{3}h_2 - \frac{1}{2}h_3 + \frac{1}{2}Y = x_1\partial_1 - z_+\partial_+. \tag{3.25}$$

As in the $E^\flat$ case, the elements e_i, f_i, h_i for $i = 0, 1, 2, 3$ along with e_0' generate $E^\sharp$, the elements $e_0^\sharp$ and e_0' are all lowest weight vectors of the G_0-module G_1, and G_0 along with e_0' and $e_0^\sharp$ generate the subalgebra $\oplus_{j\geq 0} G_j$ [1]. Thus, by Remark 2.2, condition (2.9b) is equivalent to

$$e_0' \cdot \Phi = 0, \qquad e_0^\sharp \cdot \Phi = 0. \tag{3.26}$$

4 Complexes of degenerate Verma modules over $E(3,6)$

Let W be a finite-dimensional symplectic vector space and let H be the corresponding Heisenberg algebra

$$H = T(W)/\big(v \cdot w - w \cdot v - (v, w) \cdot 1\big), \tag{4.1}$$

where $T(W)$ denotes the tensor algebra over W and $(\,,)$ is the nondegenerate symplectic form on W. Given two transversal Lagrangian subspaces $L, L' \subset W$, we have a canonical isomorphism of symplectic spaces: $W = L + L' \simeq L \oplus L^*$, and we can canonically identify the symmetric algebra $S(L)$ with the factor of H by the left ideal generated by L'

$$V_L := H/(L') \simeq S(L)\mathbf{1}_L, \text{ where } \mathbf{1}_L = 1 + L'. \tag{4.2}$$

We thus acquire an H-module structure on $S(L)$.

We construct a symplectic space W by taking $x_1, x_2, x_3, z_+, z_-, \partial_1, \partial_2, \partial_3, \partial_+, \partial_-$ as a basis with the first half being dual to the second half:

$$(\partial_i, x_j) = -(x_j, \partial_i) = \delta_{ij}, \qquad (\partial_a, z_b) = -(z_b, \partial_a) = \delta_{a,b}, \quad \text{all other pairings zero.} \tag{4.3}$$

In general, the decomposition $W = L + L' \simeq L \oplus L^*$ provides the canonical maps

$$\mathrm{End}(L) \xrightarrow{\sim} L \otimes L^* \xrightarrow{\sim} L \cdot L' \hookrightarrow H, \tag{4.4}$$

which induce a Lie algebra homomorphism: $\mathfrak{gl}(L) \to H_{\mathrm{Lie}}$, where the Lie algebra structure is defined by the usual commutator.

We consider the following subspaces of W:

$$\begin{aligned} L_A &= L'_D = \langle x_i, z_a \rangle, \qquad & L_B &= L'_C = \langle x_i, \partial_a \rangle, \\ L_C &= L'_B = \langle \partial_c, z_a \rangle, \qquad & L_D &= L'_A = \langle \partial_i, \partial_a \rangle, \end{aligned} \tag{4.5}$$

where $i = 1, 2, 3$; $a = +, -$. Note that these are the only G_0-invariant Lagrangian subspaces of W.

As formulae (3.17) determine the inclusion $G_0 \hookrightarrow \mathfrak{gl}(V_X)$, where $X = A$, B, C, or D, we get a Lie algebra monomorphism:

$$G_0 \hookrightarrow \mathfrak{gl}(V_X) \hookrightarrow H_{\mathrm{Lie}}. \tag{4.6}$$

Thus we get a G_0-action on V_X. Notice that by (4.6)

$$\begin{aligned} &Y \longrightarrow Y^\flat = \frac{2}{3}\left(\sum_i x_i \partial_i\right) - \left(\sum_a z_a \partial_a\right), \\ &Y^\flat \mathbf{1}_A = 0, \qquad Y^\flat \mathbf{1}_B = 2\mathbf{1}_B, \qquad Y^\flat \mathbf{1}_C = -2\mathbf{1}_C, \qquad Y^\flat \mathbf{1}_D = 0, \end{aligned} \tag{4.7}$$

as it should be for $E^\flat = E(3,6)$ (see [3]).

In the $E(3,8)$ case we modify the G_0-action on V_X leaving it the same for $\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \subset G_0$, but letting

$$Y \longmapsto Y^\sharp = -\frac{4}{3}\left(\sum_i x_i\partial_i\right) + \left(\sum_a z_a\partial_a\right) = Y^\flat + 2T, \tag{4.8}$$

where, $T = -\sum_i x_i\partial_i + \sum_a z_a\partial_a$. We have

$$\begin{aligned} Yx_i^p z_a^r \mathbf{1}_A &= \left(-\frac{4}{3}p + r\right)x_i^p z_a^r \mathbf{1}_A, \\ Yx_i^p \partial_a^r \mathbf{1}_B &= \left(-\frac{4}{3}p - r - 2\right)x_i^p \partial_a^r \mathbf{1}_B, \\ Y\partial_i^q z_a^r \mathbf{1}_C &= \left(\frac{4}{3}q + r + 4\right)\partial_i^q z_a^r \mathbf{1}_C, \\ Y\partial_i^q \partial_a^r \mathbf{1}_D &= \left(\frac{4}{3}q - r + 2\right)\partial_i^q \partial_a^r \mathbf{1}_D. \end{aligned} \tag{4.9}$$

Let $F(p,q;r;y)$ denote the finite-dimensional irreducible G_0-module with highest weight $(p,q;r;y)$, where $p,q,r \in \mathbb{Z}_+$, $y \in \mathbb{C}$, and let $M(p,q;r;y) = M(F(p,q;r;y))$ be the corresponding generalized Verma module over $E(3,8)$. This module has a unique irreducible quotient denoted by $I(p,q;r;y)$. The latter module is called degenerate if the former is.

We announce below a classification of all degenerate irreducible $E(3,8)$-modules.

Theorem 4.5. All irreducible degenerate $E(3,8)$-modules $I(p,q;r,y)$ are as follows $(p,q,r \in \mathbb{Z}_+)$:

$$\begin{aligned} &\text{type A}: \ I(p,0;r;y_A), \quad y_A = -\frac{4}{3}p + r; \\ &\text{type B}: \ I(p,0;r;y_B), \quad y_B = -\frac{4}{3}p - r - 2; \\ &\text{type C}: \ I(0,q;r;y_C), \quad y_C = \frac{4}{3}q + r + 4; \\ &\text{type D}: \ I(0,q;r;y_D), \quad y_D = \frac{4}{3}q - r + 2, \quad \text{and } (q,r) \neq (0,0). \end{aligned} \tag{4.10}$$

□

We construct below certain $E(3,8)$-morphisms between the modules $M(p,q;r;y_X)$. This will imply that all modules $I(p,q;r;y)$ on the list are degenerate. The proof of the fact that the list is complete will be published elsewhere.

The theorem means that all the degenerate generalized Verma modules over $E(3,8)$ are in fact the direct summands of induced modules $M(V_X)$, $X = A,B,C,D$:

$$M(V_X) = \bigoplus_{m,n\in\mathbb{Z}} M(V_X^{m,n}), \tag{4.11}$$

where

$$V_X^{m,n} = \Big\{ f\mathbf{1}_X \mid \Big(\sum x_i \partial_i\Big) f = mf, \Big(\sum z_a \partial_a\Big) f = nf \Big\}, \tag{4.12}$$

(we normalize degree of $\mathbf{1}_X$ as $(0,0)$).

We construct morphisms between these modules with the help of Proposition 2.1. As in the $E^\flat$ case [3], introduce the following operators on $M(\mathrm{Hom}(V_X, V_X))$:

$$\nabla = \Delta^+\delta_+ + \Delta^-\delta_- = \delta_1\partial_1 + \delta_2\partial_2 + \delta_3\partial_3, \tag{4.13}$$

where

$$\Delta^\pm = \sum_{i=1}^{3} d_i^\pm \otimes \partial_i, \qquad \delta_i = \sum_{\alpha=\pm} d_i^a \otimes \partial_a. \tag{4.14}$$

Proposition 4.6. (a) The element ∇ gives a well-defined morphism $M(V_X) \to M(V_X)$, $X = A, B, C, D$, by formula (2.5).

(b) $\nabla^2 = 0$. □

Proof. The proof of (b) is the same as in [3]. In order to prove (a), we have to check conditions (2.9). It is obvious that $\Delta^\pm$ (resp. δ_i) are $\mathfrak{sl}_3(\mathbb{C})$-(resp., $\mathfrak{sl}_2(\mathbb{C})$-) invariant. Using both formulas for ∇, we conclude that it is G_0-invariant, proving (2.9a). In order to check (2.9b), first note that

$$e_0'\nabla = \big(f_2\partial_1\partial_+ - f_{12}\partial_2\partial_+\big) = \big(x_3\partial_2\partial_1 - x_3\partial_1\partial_2\big)\partial_+ = 0. \tag{4.15}$$

Now

$$\begin{aligned} e_0^\sharp\nabla &= h_0^\sharp\partial_1\partial_+ + f_1\partial_2\partial_+ + f_{12}\partial_3\partial_1 - f_3\partial_1\partial_- \\ &= \big(x_1\partial_1 - z_+\partial_+ + T\big)\partial_1\partial_+ + x_2\partial_1\partial_2\partial_+ + x_3\partial_1\partial_3\partial_+ - z_-\partial_+\partial_1\partial_- \\ &= \big(x_1\partial_1 - z_+\partial_+ + T + x_2\partial_2 + x_3\partial_3 - z_-\partial_-\big)\partial_1\partial_+ = 0, \end{aligned} \tag{4.16}$$

where we use (4.8) to check that

$$h_0^\sharp = \frac{2}{3}h_1 + \frac{1}{3}h_2 - \frac{1}{2}h_3 + \frac{1}{2}Y^\sharp = x_1\partial_1 - z_+\partial_+ + T. \tag{4.17}$$

■

Let $M_{X'} = M(V_{X'})$, where

$$\begin{aligned}
V_{A'} &= \mathbb{C}[x_1, x_2, x_3]\mathbf{1}_A = \oplus_{p\geq 0} V_A^{p,0},\\
V_{B'} &= \mathbb{C}[x_1, x_2, x_3]\mathbf{1}_B = \oplus_{p\geq 0} V_B^{p,0},\\
V_{C'} &= \mathbb{C}[\partial_1, \partial_2, \partial_3]\mathbf{1}_C = \oplus_{q\geq 0} V_C^{-q,0},\\
V_{D'} &= \mathbb{C}[\partial_1, \partial_2, \partial_3]\mathbf{1}_D = \oplus_{q\geq 0} V_D^{-q,0},
\end{aligned} \tag{4.18}$$

and let

$$\nabla_2 = \Delta^- \Delta^+ = d_1^- \Delta^+ \partial_1 + d_2^- \Delta^+ \partial_2 + d_3^- \Delta^+ \partial_3. \tag{4.19}$$

We define the morphism $\nabla_2 : M_{A'} \to M_{B'}$ by extending the map defined by ∇_2 as follows:

$$\begin{aligned}
&V_{A'} \xrightarrow{\sim} \mathbb{C}[x_1, x_2, x_3] \longrightarrow U(L_-) \otimes \mathbb{C}[x_1, x_2, x_3]\mathbf{1}_B \simeq U(L_-) \otimes V_{B'},\\
&f\mathbf{1}_A \longmapsto f \longmapsto \nabla_2 f \mathbf{1}_B.
\end{aligned} \tag{4.20}$$

In order to apply Proposition 2.1, we have to check conditions (2.9) for ∇_2. As before, condition (2.9a) obviously holds. Now

$$e_0' \nabla_2 \mathbf{1}_B = -\sum_i d_i^- (f_2 \partial_1 \partial_- - f_{12} \partial_2) \partial_i f \mathbf{1}_B = 0 \tag{4.21}$$

because $f_2 \partial_1 - f_{12} \partial_2 = x_3 \partial_2 \partial_1 - x_3 \partial_1 \partial_2 = 0$. Furthermore,

$$\begin{aligned}
e_0^\sharp \nabla_2 f \mathbf{1}_B &= \left(-\sum_i d_i^- (h_0^\sharp \partial_1 + f_1 \partial_2 + f_{12} \partial_3) \partial_i - f_3 \Delta^+ \partial_1 \right) f \mathbf{1}_B\\
&= -\left(\sum_i d_i^- (x_1 \partial_1 - z_+ \partial_+ + T + x_2 \partial_2 + x_3 \partial_3) \partial_1 \partial_2 - \Delta^- \partial_1 - \Delta^+ \partial_1 f_3 \right) f \mathbf{1}_B.
\end{aligned} \tag{4.22}$$

As $z_\pm \partial_\pm f \mathbf{1}_B = f z_\pm \partial_\pm \mathbf{1}_B = f(-1 + \partial_\pm z_\pm) \mathbf{1}_B = -f \mathbf{1}_B$, and $f_3 f \mathbf{1}_B = 0$, we conclude that

$$\begin{aligned}
-e_0^\sharp \nabla_2 f \mathbf{1}_B &= \sum_i d_i^- (x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 - z_+ \partial_+ + T + 1) f \mathbf{1}_B\\
&= \sum_i d_i^- (z_- \partial_- + 1) f \mathbf{1}_B = 0.
\end{aligned} \tag{4.23}$$

Thus Proposition 2.1 applies and we get the morphism $\nabla_2 : M_{A'} \to M_{B'}$.

In exactly the same fashion we construct the morphism $\nabla_2 : M_{C'} \to M_{D'}$ by taking $f \in \mathbb{C}[\partial_1, \partial_2, \partial_3]$.

Thus we have proved part (a) of the following proposition; part (b) was checked in [3].

Proposition 4.7. (a) Formulae (4.19) and (4.20) define the morphisms $\nabla_2 : M_{A'} \to M_{B'}$, $\nabla_2 : M_{C'} \to M_{D'}$ of $E(3,8)$-modules.

(b) $\nabla\nabla_2 = 0$, $\nabla_2\nabla = 0$. □

Similarly, we can construct morphisms with the help of the element

$$\nabla_3 = \delta_1\delta_2\delta_3 = \sum_{a,b,c} d_1^a d_2^b d_3^c \otimes \partial_a\partial_b\partial_c, \quad (a,b,c = +,-). \tag{4.24}$$

Consider modules $M_{X''} = M(V_{X''})$, where

$$\begin{aligned} V_{A''} &= \mathbb{C}[z_+, z_-]\mathbf{1}_A = \oplus_{r\geq 0} V_A^{0,r}, \qquad & V_{B''} &= \mathbb{C}[\partial_+, \partial_-]\mathbf{1}_B = \oplus_{r\geq 0} V_B^{0,-r}, \\ V_{C''} &= \mathbb{C}[z_+, z_-]\mathbf{1}_C = \oplus_{r\geq 0} V_B^{0,r}, \qquad & V_{D''} &= \mathbb{C}[\partial_+, \partial_-]\mathbf{1}_D = \oplus_{r\geq 0} V_D^{0,-r}. \end{aligned} \tag{4.25}$$

We construct morphisms $\nabla_3 : M_{A''} \to M_{C''}$ and $\nabla_3 : M_{B''} \to M_{D''}$ by extending the maps $\nabla_3' : V_{A''} \to U(L_-) \otimes V_{C''}$, $\nabla_3'' : V_{B''} \to U(L_-) \otimes V_{D''}$ given by the left and right diagrams below:

$$\begin{tikzcd} \mathbb{C}[z_a]\cdot\mathbf{1}_A \arrow[r, "\sim"] & \mathbb{C}[z_+, z_-] \arrow[d, "\nabla_3"] \\ & U(L_-)\otimes(\mathbb{C}[z_+, z_-]\mathbf{1}_C) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}[\partial_a]\cdot\mathbf{1}_B \arrow[r, "\sim"] & \mathbb{C}[\partial_+, \partial_-] \arrow[d, "\nabla_3"] \\ & U(L_-)\otimes(\mathbb{C}[\partial_+, \partial_-]\mathbf{1}_D). \end{tikzcd} \tag{4.26}$$

Here the horizontal maps are naturally $\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$-isomorphisms, but we have to define the action of Y on the target demanding the map to be a G_0-isomorphism. With this in mind, we have to check that Y commutes with ∇_3.

We use the Einstein summation convention argument for the vertical maps ∇_3 given by (4.26). Then for $f \in \mathbb{C}[z_+, z_-]$ we have $Y(f\mathbf{1}_A) = (\deg f)f\mathbf{1}_A$ and

$$\begin{aligned} Y\big(d_1^a d_2^b d_3^c \otimes \partial_a\partial_b\partial_c f\mathbf{1}_C\big) &= d_1^a d_2^b d_3^c \otimes \big(-1 + Y^\sharp\big)\partial_1\partial_b\partial_c f\mathbf{1}_C \\ &= d_1^a d_2^b d_3^c \otimes \partial_a\partial_b\partial_c f\big(-4 + \deg f + Y^\sharp\big)\mathbf{1}_C \\ &= (\deg f)\cdot d_1^a d_2^b d_3^c \otimes \partial_a\partial_b\partial_c f\mathbf{1}_C. \end{aligned} \tag{4.27}$$

Similarly for $f \in \mathbb{C}[\partial_+, \partial_-]$, we have $Y(f\mathbf{1}_B) = (-\deg f - 2)f\mathbf{1}_B$ and

$$\begin{aligned} Y\big(d_1^a d_2^b d_3^c \otimes \partial_a \partial_b \partial_c f \mathbf{1}_D\big) &= d_1^a d_2^b d_3^c \otimes \partial_a \partial_b \partial_c f\big(-4 - \deg f + Y^\sharp\big)\mathbf{1}_D \\ &= (-4 - \deg f + 2) d_1^a d_2^b d_3^c \otimes \partial_a \partial_b \partial_c f \mathbf{1}_D, \end{aligned} \tag{4.28}$$

where $Y^\sharp$ defined by (4.8). Thus we get the commutativity.

We meet no problem checking $e_0' \nabla_3 = 0$, but we consider calculations for $e_0^\sharp \nabla_3$ in more detail. If $f \in \mathbb{C}[z_+, z_-]$, then

$$e_0\big(\nabla_3 f \mathbf{1}_C\big) = h_0^\sharp d_2^b d_3^c \otimes \partial_+ \partial_b \partial_c f \mathbf{1}_C - f_3 d_2^b d_3^c \otimes \partial_- \partial_b \partial_c f \mathbf{1}_C, \tag{4.29}$$

because $d_1^a (x_2 \partial_1) d_3^c \otimes \partial_a \partial_+ \partial_c f \mathbf{1}_C = d_1^a d_2^b (x_3 \partial_1) \otimes \partial_a \partial_b \partial_+ f \mathbf{1}_C = 0$. Now $f_3(d_2^b d_3^c \otimes \partial_b \partial_c) = (d_2^b d_3^c \otimes \partial_b \partial_c) f_3$ and $h_0^\sharp (d_2^b d_3^c \otimes \partial_b \partial_c) = (d_2^b d_3^c \otimes \partial_b \partial_c)(h_0^\sharp - 2)$, where $h_0^\sharp = (2/3)h_1 + (1/3)h_2 - (1/2)h_3 + (1/2)Y^\sharp$, and again $Y^\sharp$ is defined by (4.8). Therefore,

$$\begin{aligned} e_0^\sharp\big(\nabla_3 f \mathbf{1}_C\big) &= \big(d_2^b d_3^c \otimes \partial_b \partial_c\big)\big(h_0^\sharp \partial_+ - 2\partial_+ - z_+ \partial_+ \partial_-\big) f \mathbf{1}_C \\ &= \big(d_2^b d_3^c \otimes \partial_b \partial_c\big)\big(h_0^\sharp - z_- \partial_- - 2\big)\partial_+ f \mathbf{1}_C \\ &= \big(d_2^b d_3^c \otimes \partial_b \partial_c\big)\big(-x_2 \partial_2 - x_3 \partial_3 - 2\big)\partial_+ f \mathbf{1}_C \\ &= \big(d_2^b d_3^c \otimes \partial_b \partial_c\big)\big(-\partial_2 x_2 - \partial_3 x_3\big)\partial_+ f \mathbf{1}_C = 0. \end{aligned} \tag{4.30}$$

The calculations in the case $f \in \mathbb{C}[\partial_+, \partial_-]$ are very much the same. So we have proved part (a) of the following proposition; part (b) was checked in [3].

Proposition 4.8. (a) Formulae (4.24) and (4.26) define the morphisms $\nabla_3 : M_{A''} \to M_{D''}$ and $M_{B''} \to M_{D''}$ of $E(3,8)$-modules.

(b) $\nabla \cdot \nabla_3 = 0$, $\nabla_3 \cdot \nabla = 0$. □

Furthermore, there are $E(3,8)$-module morphisms

$$\begin{aligned} &\nabla_4' : M\big(00;2;y_A\big) \longrightarrow M\big(01;1;y_D\big), \\ &\nabla_4'' : M\big(10;0;y_A\big) \longrightarrow M\big(00;2;y_D\big), \end{aligned} \tag{4.31}$$

defined by formulae (2.14) and (2.17) from [3], applied to $E(3,8)$. Arguments similar to those in [3] show that these are indeed well-defined morphisms.

Thus far we have constructed $E(3,8)$-homomorphisms ∇, ∇_2, ∇_3, ∇_4', and ∇_4'' between generalized Verma modules. Note that these maps have degree 1, 2, 3, and 4, respectively, with respect to the $\mathbb{Z}$-gradation of $U(L_-)$ induced by that of $E(3,8)$.

As in the case of $E(3,6)$ [3], all these maps are illustrated in Figure 4.1. The nodes in the quadrants A, B, C, D represent generalized Verma modules $M(p,0;r;y_X)$ if $X = A$

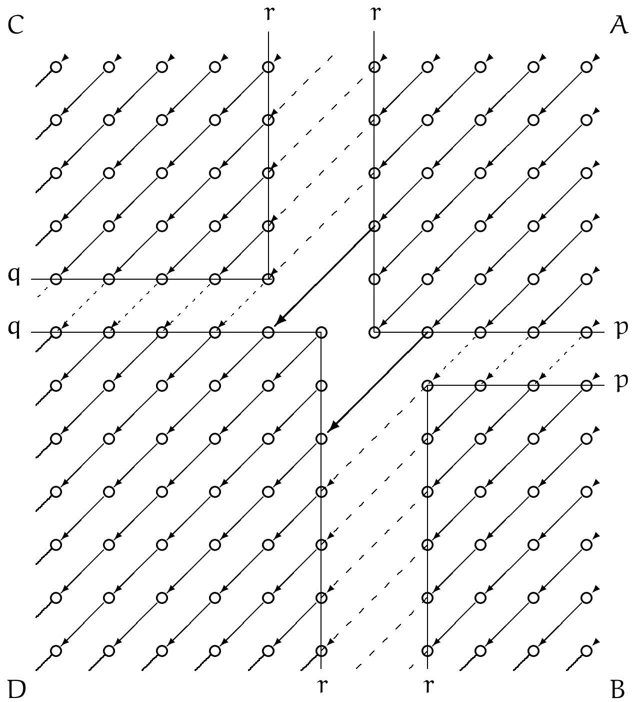

Figure 4.1

or B, and $M(0, q; r; y_X)$ if $X = C$ or D. The plain arrows represent ∇, the dotted arrows represent ∇_2, the interrupted arrows represent ∇_3, and the bold arrows represent ∇_4' and ∇_4''.

Note that the generalized Verma modules $M(00; 1; y_A)$ and $M(00; 1; y_D)$ are isomorphic since $y_A = y_D = 1$. We identify them. This allows us to construct the $E(3, 8)$-module homomorphism

$$\widetilde{\nabla} : M(00; 1; y_A) \longrightarrow M(01; 2; y_D), \tag{4.32}$$

which is *not* represented in Figure 4.1.

Note that $I(00; 1; y_A) = I(00; 1; y_D)$ is the coadjoint $E(3, 8)$-module. It follows from the above propositions that if we remove the module $M(00; 1; y_D)$ from Figure 4.1 and draw $\widetilde{\nabla}$, then all sequences in the modified Figure 4.1 become complexes. We denote by $H_A^{p,r}$, $H_B^{p,-r}$, $H_C^{-q,r}$, and $H_D^{-q,-r}$ the homology of these complexes at the position of $M(pq; r; y_X)$, $X = A, B, C, D$.

Theorem 4.9. (a) The kernels of all maps ∇, ∇_2, ∇_3, ∇_4', ∇_4'', $\widetilde{\nabla}$ are maximal submodules.

(b) The homology $H_X^{m,n}$ is zero except for six cases listed (as $E(3,8)$-modules) below

$$\begin{aligned} &H_A^{0,0} = \mathbb{C}, \qquad H_A^{1,1} = I\left(10;0;-\frac{4}{3}\right), \\ &H_A^{1,0} = H_D^{0,-2} = I(00;0;-2), \\ &H_D^{-1,-1} = H_D^{-1,-2} = I(00;1;1) \oplus \mathbb{C}. \end{aligned} \tag{4.33}$$

□

The proof is similar to that of the analogous $E(3,6)$-result in [3]. Note that this theorem gives the following explicit construction of all degenerate irreducible $E(3,8)$-modules:

$$I(pq;r;y_X) = M(pq;r;y_X)/\operatorname{Ker}\nabla, \tag{4.34}$$

where ∇ is the corresponding map in the modified Figure 4.1.

5 Three series of degenerate Verma modules over $E(5,10)$

As in [3] and in Section 3, we use for the odd elements of $E(5,10)$ the notation $d_{ij} = dx_i \wedge dx_j$ $(i,j = 1,2,\ldots,5)$; recall that we have the following commutation relation ($f,g \in \mathbb{C}[[x_1,\ldots,x_5]]$):

$$[fd_{jk}, gd_{\ell m}] = \epsilon_{ijk\ell m}\partial_i, \tag{5.1}$$

where $\epsilon_{ijk\ell m}$ is the sign of the permutation $ijk\ell m$ if all indices are distinct and 0 otherwise.

Recall that the Lie superalgebra $E(5,10)$ carries a unique consistent irreducible $\mathbb{Z}$-gradation $E(5,10) = \oplus_{j\geq -2}\mathfrak{p}_j$. It is defined by

$$\deg x_i = 2 = -\deg \partial_i, \qquad \deg d_{ij} = -1. \tag{5.2}$$

We have that $\mathfrak{p}_0 \simeq \mathfrak{sl}_5(\mathbb{C})$ and the $\mathfrak{p}_0$-modules occurring in the L_- part are

$$\begin{aligned} &\mathfrak{p}_{-1} = \langle d_{ij} \mid i,j = 1,\ldots,5\rangle \simeq \Lambda^2\mathbb{C}^5, \\ &\mathfrak{p}_{-2} = \langle \partial_i \mid i = 1,\ldots,5\rangle \simeq \mathbb{C}^{5*}. \end{aligned} \tag{5.3}$$

Recall also that $\mathfrak{p}_1$ consists of closed 2-forms with linear coefficients, that $\mathfrak{p}_1$ is an irreducible $\mathfrak{p}_0$-module and $\mathfrak{p}_j = \mathfrak{p}_1^j$ for $j \geq 1$.

We take for the Borel subalgebra of $\mathfrak{p}_0$ the subalgebra of the vector fields $\langle \sum_{i\le j} a_{ij}x_i\partial_j \,|\, a_{ij}\in\mathbb{C}, \mathrm{tr}(a_{ij})=0\rangle$, and denote by $F(m_1,m_2,m_3,m_4)$ the finite-dimensional irreducible $\mathfrak{p}_0$-module with the highest weight (m_1,m_2,m_3,m_4). Let

$$M(m_1,m_2,m_3,m_4)=M\big(F(m_1,m_2,m_3,m_4)\big) \tag{5.4}$$

denote the corresponding generalized Verma module over $E(5,10)$.

Conjecture 5.10. The following is a complete list of generalized Verma modules over $E(5,10)$ $(m,n\in\mathbb{Z}_+)$:

$$M(m,n,0,0), \qquad M(0,0,m,n), \qquad M(m,0,0,n). \tag{5.5}$$

□

In this section, we construct three complexes of generalized $E(5,10)$ Verma modules which shows, in particular, that all modules from the list given by Conjecture 5.10 are degenerate. Let

$$S_A=S\big(\mathbb{C}^5+\Lambda^2\mathbb{C}^5\big), \qquad S_B=S\big(\mathbb{C}^{5*}+\Lambda^2\mathbb{C}^{5*}\big), \qquad S_C=S\big(\mathbb{C}^5+\mathbb{C}^{5*}\big). \tag{5.6}$$

Denote by x_i $(i=1,\dots,5)$ the standard basis of $\mathbb{C}^5$, and by $x_{ij}=-x_{ji}$ $(i,j=1,\dots,5)$ the standard basis of $\Lambda^2\mathbb{C}^5$. Let x_i^* and $x_{ij}^*=-x_{ji}^*$ be the dual bases of $\mathbb{C}^{5*}$ and $\Lambda^2\mathbb{C}^{5*}$, respectively. Then S_A is the polynomial algebra in 15 indeterminates x_i, and x_{ij}, S_B is the polynomial algebra in 15 indeterminates x_i^* and x_{ij}^*, and S_C is the polynomial algebra in 10 indeterminates x_i and x_i^*.

Given two irreducible $\mathfrak{p}_0$-modules E and F, we denote by $(E\otimes F)_{\mathrm{high}}$ the highest irreducible component of the $\mathfrak{p}_0$-module $E\otimes F$. If $E=\oplus_i E_i$ and $F=\oplus_j F_j$ are direct sums of irreducible $\mathfrak{p}_0$-modules, we let $(E\otimes F)_{\mathrm{high}}=\oplus_{i,j}(E_i\otimes F_j)_{\mathrm{high}}$. If E and F are again irreducible $\mathfrak{p}_0$-modules, then $S(E\oplus F)=\oplus_{m,n\in\mathbb{Z}_+}S^mE\otimes S^nF$, and we let $S_{\mathrm{high}}(E\oplus F)=\oplus_{m,n\in\mathbb{Z}_+}(S^mE\otimes S^nF)_{\mathrm{high}}$. We also denote by $S_{\mathrm{low}}(E\oplus F)$ the complement to $S_{\mathrm{high}}(E\oplus F)$.

It is easy to see that we have as $\mathfrak{p}_0$-modules:

$$\begin{aligned} S_{A,\mathrm{high}} &\simeq \oplus_{m,n\in\mathbb{Z}_+}F(m,n,0,0),\\ S_{B,\mathrm{high}} &\simeq \oplus_{m,n\in\mathbb{Z}_+}F(0,0,m,n),\\ S_{C,\mathrm{high}} &\simeq \oplus_{m,n\in\mathbb{Z}_+}F(m,0,0,n). \end{aligned} \tag{5.7}$$

Introduce the following operators on the spaces $M(\mathrm{Hom}(S_X,S_X))$, $X=A$, B, or C:

$$\nabla_X=\sum_{i,j=1}^{5} d_{ij}\otimes\theta_{ij}^X, \tag{5.8}$$

where

$$\theta^A_{ij} = \frac{d}{dx_{ij}}, \qquad \theta^B_{ij} = x^*_{ij}, \qquad \theta^C_{ij} = x^*_i \frac{d}{dx_j} - x^*_j \frac{d}{dx_i}. \tag{5.9}$$

It is immediate to see that $p_0 \cdot \nabla_X = 0$. In order to apply Proposition 2.1, we need to check that

$$p_1 \cdot \nabla_X = 0. \tag{5.10}$$

This is indeed true in the case $X = C$, but it is not true in the cases $X = A$ and B. In fact (5.10) applied to $f \in S_X$, $X = A$ or B, is equivalent to the following equations, respectively $(a, b, c, d = 1, \dots, 5)$:

$$\left(\frac{d}{dx_{ab}}\frac{d}{dx_{cd}} - \frac{d}{dx_{ac}}\frac{d}{dx_{bd}} + \frac{d}{dx_{ad}}\frac{d}{dx_{bc}}\right) f = 0, \tag{5.11}$$

$$\left(x^*_{ab}x^*_{cd} - x^*_{ac}x^*_{bd} + x^*_{ad}x^*_{bc}\right) f = 0. \tag{5.12}$$

It is not difficult to check the following lemma.

Lemma 5.11. (a) The subspace of S_A defined by (5.11) is $S_{A,\text{high}}$.

(b) Equations (5.12) hold in $S_B / S_{B,\text{low}}$.

(c) Equation $\nabla_X^2 = 0$ is equivalent to the system of equations $(a, b, c, d = 1, \dots, 5)$:

$$\theta_{ab}\theta_{cd} - \theta_{ac}\theta_{bd} + \theta_{ad}\theta_{bc} = 0. \tag{5.13}$$

□

Let

$$V_A = S_{A,\text{high}}, \qquad V_{B(\text{resp. } C)} = S_{B(\text{resp. } C)} / S_{B(\text{resp. } C),\text{low}} \tag{5.14}$$

The above discussion implies the following proposition.

Proposition 5.12. (a) The operators ∇_X define $E(5,10)$-morphisms $M(V_X) \to M(V_X)$ $(X = A, B$ or $C)$.

(b) $\nabla_X^2 = 0$ $(X = A, B$ or $C)$.

(c) $\nabla_X = 0$ if and only if $X = A$ and $n = 0$, or $X = C$ and $m = 0$. □

The nonzero maps ∇_X are illustrated in Figure 5.1. The nodes in the quadrants A, B, and C represent generalized Verma modules $M(m, n, 0, 0)$, $M(0, 0, m, n)$, and $M(m, 0, 0, n)$, respectively. The arrows represent the $E(5,10)$-morphisms ∇_X, $X = A$, B, or C in the respective quadrants.

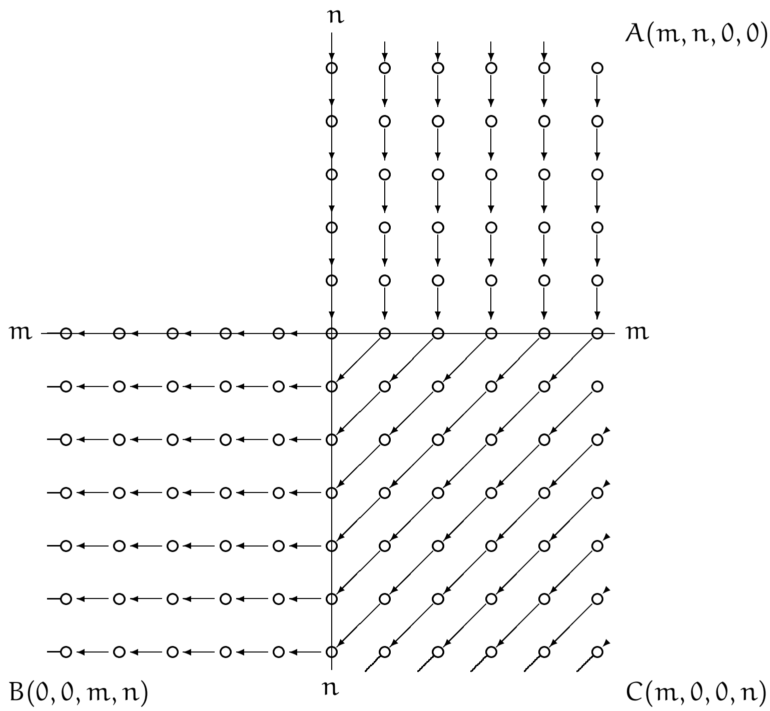

Figure 5.1

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