Linear Algebra and its Applications 466 (2015) 64–82

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Matrix-tree theorems and discrete path integration

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ARTICLE INFO

Article history:
Received 24 May 2013
Accepted 26 September 2014
Available online xxxx
Submitted by R. Brualdi

MSC:
15A15

Keywords:
Matrix-tree theorem
Graph Laplacian

ABSTRACT

We calculate characteristic polynomials of operators explicitly presented as polynomials of rank 1 operators. Corollaries of the main result (Theorem 2.3) include a generalization of the Forman's formula for the determinant of the graph Laplacian [6,8], the celebrated Matrix-tree theorem by G. Kirchhoff [9], and some its extensions and analogs, both known (e.g. the Matrix-hypertree theorem by G. Masbaum and A. Vaintrob [10]) and new.

1. Introduction

A celebrated Matrix-tree theorem (MTT) proved by G. Kirchhoff in 1847 [9] has been attracting a constant attention of specialists since then. It was given several new proofs (see e.g. [4] and the bibliography therein), was used in many contexts, sometimes quite

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http://dx.doi.org/10.1016/j.laa.2014.09.049
0024-3795/

unexpected ([3], for just an example); there are also many generalizations of the MTT ([1,2,5,6,8,10], to name just a few).

In its classical form, the MTT expresses the principal minor of some $n \times n$-matrix via summation over the set of trees on n numbered vertices. The matrix involved is a weighted sum of the operators $I - s$ where s runs through the set of all reflections in the Coxeter group A_{n-1}. In this article we generalize the MTT allowing any rank 1 operators instead of $I - s$, and any non-commutative polynomial instead of a weighted sum.

The article is organized as follows. In Section 2.1 we formulate and prove the main result, Theorem 2.3. It expresses the characteristic polynomial of an operator M given as a function of rank 1 operators $M_1, \dots, M_N$ by a sort of "discrete path integration". Then we consider two special cases of the theorem: linear polynomials (Section 2.2) and skew-symmetric ones (Section 2.3). In the first case the discrete path integration is reduced to summation over subsets (Corollary 2.4). In the second case we express a Pfaffian of the operator as a sum over the set of pair matchings (a.k.a. dimer structures; Theorem 2.8).

Section 3 contains some applications of the main theorem. In Section 3.1 we use Theorem 2.3 to prove a formula for the characteristic polynomial of the Laplacian of a line bundle on a graph (this formula was first obtained by R. Forman in [6] using a different method). In Section 3.2 we obtain two corollaries of the Forman's formula: the MTT (in [6] and [8] it was derived from Forman's formula as well) and the D-analog of the MTT (Corollary 3.5). In Section 3.3 we consider a discrete Scroedinger operator, which is a generalization of the graph Laplacian, and prove an expression for its characteristic polynomial.

In Section 3.4 we prove two results in a skew-symmetric case: Theorem 3.7, which is a generalization of the Matrix-hypertree theorem of [10], and its D-analog, Theorem 3.9.

2. General results

2.1. The main theorem

Let V be a vector space of dimension n with a scalar product $\langle \cdot, \cdot \rangle$. For $e, \alpha \in V$ denote by $M[e, \alpha]$ the operator

$$M[e,\alpha](v) \stackrel{\text{def}}{=} \langle \alpha, v \rangle e.$$

$M[e, \alpha] : V \to V$ has rank 1 or is zero.

Choose an integer N and consider two sequences of vectors, $e_1, \dots, e_N \in V$ and $\alpha_1, \dots, \alpha_N \in V$. Define then a linear operator $M : V \to V$ as

$$M = P\bigl(M[e_1, \alpha_1], \dots, M[e_N, \alpha_N]\bigr) \tag{1}$$

where

$$P(x_1, \dots, x_N) = \sum_{s=1}^{m} \sum_{1 \le i_1, \dots, i_s \le N} c_{i_1, \dots, i_s} x_{i_1} \dots x_{i_s}$$

is a noncommutative polynomial of degree m.

For $a, b \in \{1, \dots, N\}$ define

$$W_P(a, b) = \sum_{s=1}^{m} \sum_{\substack{i_1, i_2, \dots, i_s \\ i_1 = a, i_s = b}} c_{i_1, \dots, i_s} \langle \alpha_{i_2}, e_{i_1} \rangle \langle \alpha_{i_3}, e_{i_2} \rangle \dots \langle \alpha_{i_s}, e_{i_{s-1}} \rangle. \tag{2}$$

We can consider $\{1, \dots, N\}$ as the set of vertices in a complete directed graph K_N; then the internal summation in (2) is taken over the set of paths $i_1, \dots, i_s$ of length $s - 1$ joining the vertices $a = i_1$ and $b = i_s$. Let G be a subgraph of K_N, i.e. a set $\{d_1, \dots, d_k\}$ where every d_i, $i = 1, \dots, k$, is a directed edge joining vertices d_i^- and d_i^+. Define the *weight* of the graph G by

$$W_P(G) = \prod_{i=1}^{k} W_P\left(d_i^-, d_i^+\right) \cdot \det\left(\langle \alpha_{d_p^-}, e_{d_q^+} \rangle\right)_{p,q=1}^{k}. \tag{3}$$

Remark 2.1. To write down (3) one has to number the edges of the graph; it is clear, though, that $W_P(G)$ does not depend on the numbering. On the contrary, the *direction* of edges in G is essential: the polynomial P is noncommutative, so in general the weights $W_P(a, b)$ and $W_P(b, a)$ are unrelated.

Remark 2.2. Obviously, $W_P(G_1 \sqcup G_2) = W_P(G_1) W_P(G_2)$ where $G_1 \sqcup G_2$ is a union of *vertex-disjoint* graphs G_1 and G_2. For a usual union of graphs (which are edge-disjoint but may have common vertices) the weight W_P is not multiplicative.

Theorem 2.3. $\operatorname{char}_M(t) = \sum_{k=0}^{n} (-1)^k \mu_k t^{n-k}$ *where*

$$\mu_k = \sum_{G \in \mathcal{D}_k} W_P(G). \tag{4}$$

Here $\mathcal{D}_k$ is the set of directed graphs G with k edges, the vertices $1, \dots, N$, such that every connected component of G is either an oriented chain or an oriented cycle.

An oriented chain has vertices $i_1, \dots, i_s$ and edges $(i_1, i_2), (i_2, i_3), \dots, (i_{s-1}, i_s)$; an oriented cycle has the same vertices and the same edges plus (i_s, i_1). We will call a graph $G \in \mathcal{D}_k$ a *discrete oriented* 1*-manifold with boundary* (abbreviated as DOOMB), by an apparent graphical analogy.

Proof. Consider an orthonormal basis $u_1, \dots, u_n \in V$ and fix a sequence $j_1, \dots, j_k$, $1 \le j_1 < \dots < j_k \le N$. Then

$$M^{\wedge k}(u_{j_1} \wedge \cdots \wedge u_{j_k}) = \sum_{s_1,\dots,s_k=1}^{m} \sum_{\substack{1\le q\le k\\ 1\le t\le s_q\\ 1\le i_t^{(q)}\le N}} \prod_{q=1}^{k} \left(c_{i_1^{(q)}\dots i_{s_q}^{(q)}} \prod_{r=2}^{s_q} \langle \alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}} \rangle \right)$$

$$\times \prod_{q=1}^{k} \langle \alpha_{i_1^{(q)}}, u_{j_q} \rangle \times e_{i_{s_1}^{(1)}} \wedge \cdots \wedge e_{i_{s_k}^{(k)}}$$

$$= \sum_{s_1,\dots,s_k=1}^{m} \sum_{\substack{1\le q\le k\\ 1\le t\le s_q\\ 1\le i_t^{(q)}\le N\\ i_{s_1}^{(1)}<\dots<i_{s_k}^{(k)}}} \prod_{q=1}^{k} \left(c_{i_1^{(q)}\dots i_{s_q}^{(q)}} \prod_{r=2}^{s_q} \langle \alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}} \rangle \right)$$

$$\times \sum_{\sigma\in S_k} (-1)^{\operatorname{sgn}(\sigma)} \prod_{q=1}^{k} \langle \alpha_{i_1^{(q)}}, u_{j_{\sigma(q)}} \rangle \times e_{i_{s_1}^{(1)}} \wedge \cdots \wedge e_{i_{s_k}^{(k)}}$$

(where $\operatorname{sgn}(\sigma)$ means the parity of the permutation σ). So, the coefficient at $u_{j_1} \wedge \cdots \wedge u_{j_k}$ in $M^{\wedge k}(u_{j_1} \wedge \cdots \wedge u_{j_k})$ is equal to

$$\sum_{s_1,\dots,s_k=1}^{m} \sum_{\substack{1\le q\le k\\ 1\le t\le s_q\\ 1\le i_t^{(q)}\le N\\ i_{s_1}^{(1)}<\dots<i_{s_k}^{(k)}}} \prod_{q=1}^{k} \left(c_{i_1^{(q)}\dots i_{s_q}^{(q)}} \prod_{r=2}^{s_q} \langle \alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}} \rangle \right)$$

$$\times \det\left(\langle \alpha_{i_1^{(q)}}, u_{j_r} \rangle\right)_{q,r=1}^{k} \times \det\left(\langle u_{j_r}, e_{i_{s_q}^{(q)}} \rangle\right)_{q,r=1}^{k}.$$

The inequality $i_{s_1}^{(1)} < \cdots < i_{s_k}^{(k)}$ here and above means actually that the summation should be taken over the set of *unordered* k-tuples $\{i^{(1)}, \dots, i^{(k)}\}$ of multi-indices. Therefore

$$\mu_k = \operatorname{Tr} M^{\wedge k} = \sum_{s_1,\dots,s_k=1}^{m} \sum_{\substack{\{i^{(1)},\dots,i^{(k)}\}:\\ i^{(q)}=(i_1^{(q)},\dots,i_{s_q}^{(q)})\\ 1\le i_t^{(q)}\le N,\\ 1\le q\le k, 1\le i\le s_q}} \prod_{q=1}^{k} \left(c_{i_1^{(q)}\dots i_{s_q}^{(q)}} \prod_{r=2}^{s_q} \langle \alpha_{i_r^{(q)}}, e_{i_{r-1}^{(q)}} \rangle \right)$$

$$\times \det\left(\langle \alpha_{i_1^{(p)}}, e_{i_{s_q}^{(q)}} \rangle\right)_{p,q=1}^{k} \tag{5}$$

For every multi-index $i_1^{(q)}, \dots, i_{s_q}^{(q)}$, $1 \le q \le k$, denote by G a directed graph with the vertices $1, \dots, N$ and the edges d_q, $1 \le q \le k$, where d_q joins vertices $d_q^- \overset{\text{def}}{=} i_1^{(q)}$ and $d_q^+ \overset{\text{def}}{=} i_{s_q}^{(q)}$. Then $i_1^{(q)}, \dots, i_{s_q}^{(q)}$ is a path joining d_q^- with d_q^+, and (5) becomes

$$\mu_k = \sum_{G \text{ is a graph with } k \text{ directed edges}} W_P(G) \tag{6}$$

The last factor in (5) is zero if $i_1^{(q_1)} = i_1^{(q_2)}$ or $i_{s_{q_1}}^{(q_1)} = i_{s_{q_2}}^{(q_2)}$ for some $q_1 \neq q_2$. So, every vertex of G has at most one outgoing edge and at most one incoming edge. Therefore, G is a DOOMB, and the summation in (6) is actually performed over $G \in \mathcal{D}_k$. □

2.2. A special case: P is linear

Suppose that the polynomial P of Theorem 2.3 is linear: $P = \sum_{i=1}^N c_i x_i$. In this case all the paths in Eq. (2) have length 1. Consequently, the edges of G in (4) are loops attached to the vertices $1 \le i_1 < \dots < i_k \le N$; one has $W_P(i,i) = c_i$ for such edge. The vertices $i_1, \dots, i_k$ should be chosen so that the vectors $e_{i_1}, \dots, e_{i_k}$ are linearly independent, and the same is true for $\alpha_{i_1}, \dots, \alpha_{i_k}$ — else the determinant in (4) is zero.

In other words, the following corollary of Theorem 2.3 takes place:

Corollary 2.4. *Let* $M = \sum_{i=1}^N c_i M[e_i, \alpha_i]$. *Then* $\operatorname{char}_M(t) = \sum_{k=0}^n (-1)^k \mu_k t^{n-k}$ *where*

$$\mu_k = \sum_{\{i_1,\dots,i_k\} \subset \{1,\dots,N\}} c_{i_1} \dots c_{i_k} \det\bigl(\langle e_{i_p}, \alpha_{i_q}\rangle\bigr)_{p,q=1}^k. \tag{7}$$

Example 2.5. Let $V = \mathbb{R}^n$. Consider sequences of vectors e and α numbered by *pairs* of indices $1 \le i,j \le n$: $e_{ij} \stackrel{\text{def}}{=} u_i$ and $\alpha_{ij} \stackrel{\text{def}}{=} u_j$ where $u_1, \dots, u_n$ is an orthonormal basis in V (so, $N = n^2$). Let

$$P(x_{11}, \dots, x_{nn}) \stackrel{\text{def}}{=} \sum_{i,j=1}^n c_{ij} x_{ij};$$

then the matrix of the operator $M = P(M[e_{11}, \alpha_{11}], \dots, M[e_{nn}, \alpha_{nn}])$ in the basis $u_1, \dots, u_n$ is (c_{ij}). By Corollary 2.4 one has $\operatorname{char}_M(t) = \sum_{k=0}^n (-1)^k \mu_k t^{n-k}$ where $\mu_k = \sum c_{i_1 j_1} \dots c_{i_k j_k} (\langle e_{i_p j_p}, \alpha_{i_q j_q}\rangle)_{p,q=1}^k$; the summation is taken over the set of unordered k-tuples $(i_1, j_1), \dots, (i_k, j_k)$ with $1 \le i_s, j_s \le N$, $s = 1, \dots, k$. In other words, the summation is over the set of graphs F with the vertices $1, 2, \dots, n$ and k unnumbered directed edges (loops are allowed).

One has $\langle \alpha_{ij}, e_{kl}\rangle = \delta_{jk}$, so the contribution of a graph F into (7) is equal to $c_{i_1 j_1} \dots c_{i_k j_k} \det(\delta_{i_p j_q})_{p,q=1}^k$. It is easy to see that the determinant is nonzero only if all the i_p and all the j_q are distinct (else the matrix has identical rows or columns), and for every q there is a unique $p \stackrel{\text{def}}{=} \sigma(q)$ such that $j_q = i_p$ (else a matrix has a zero row). If these conditions are satisfied, the determinant is equal to $(-1)^{\operatorname{sgn}(\sigma)}$ where $\operatorname{sgn}(\sigma)$ is the parity of the permutation σ. Hence, Theorem 2.3 in this case is reduced to the usual formula expressing coefficients of the characteristic polynomial of the operator via its matrix elements.

2.3. A special case: $\alpha = e$ and P is skew-symmetric

Suppose now that $\alpha_i = e_i$ for all $i = 1, \dots, N$, and the polynomial P is skew-symmetric: $c_{i_s \dots i_1} = -c_{i_1 \dots i_s}$ for all $s = 1, \dots, m$ and $1 \leq i_1, \dots, i_s \leq N$. Then the operators $M[e_i, e_i]$ are symmetric: $\langle M[e_i, e_i]v_1, v_2\rangle = \langle e_i, v_1\rangle\langle e_i, v_2\rangle = \langle v_1, M[e_i, e_i]v_2\rangle$, and the operator $M = P(M[e_1, e_1], \dots, M[e_N, e_N])$ is skew-symmetric. Fix now an orthonormal basis $u_1, \dots, u_n$ in V, and define a *half-weight* $U_P^{i_1,\dots,i_{2k}}(F)$ of a directed graph F with the edges $d_1, \dots, d_k$ as

$$U_P^{i_1,\dots,i_{2k}}(F) = \prod_{p=1}^{k} W_P\big(d_p^-, d_p^+\big) \times \det\begin{pmatrix} \langle u_{i_1}, e_{d_1^-}\rangle & \langle u_{i_1}, e_{d_1^+}\rangle & \dots & \langle u_{i_1}, e_{d_k^-}\rangle & \langle u_{i_1}, e_{d_k^+}\rangle \\ \langle u_{i_{2k}}, e_{d_1^-}\rangle & \langle u_{i_{2k}}, e_{d_1^+}\rangle & \dots & \langle u_{i_{2k}}, e_{d_k^-}\rangle & \langle u_{i_{2k}}, e_{d_k^+}\rangle \end{pmatrix}. \quad (8)$$

Remark 2.6. Similarly to Remark 2.1, to write down the expression for the half-weight U one has to number the edges of F, but the result is independent of the numbering.

Remark 2.7. The skew symmetry of P implies $W_P(b, a) = -W_P(a, b)$; therefore reversing the direction of an edge in G will not change $U_P^{i_1\dots i_{2k}}(G)$. Thus we can speak about a half-weight of an *undirected* graph.

Theorem 2.8. $\mathrm{char}_M(t) = \sum_{k=0}^{n} (-1)^k \mu_{2k} t^{n-2k}$ *where*

$$\mu_{2k} = \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} \bigg(\sum_{F \in \mathcal{P}_k} U_P^{i_1,\dots,i_{2k}}(F)\bigg)^2. \quad (9)$$

Here $\mathcal{P}_k$ is the set of partial pair matchings (i.e. undirected graphs where different edges have no common vertices) with k edges. In particular, if n is even then the Pfaffian of the operator M is equal to $\sum_{F \in \mathcal{P}_{n/2}} U_P^{12\dots n}(F)$.

Proof. Note first that in (4) one has $\mu_{2k+1} = 0$ because M is skew-symmetric. Let now G be a DOOMB with $2k$ edges and a cycle $i_1 \dots i_{2d+1}$ of odd length, and let $\tilde{G}$ be a DOOMB obtained from G by reversal of all edges in this cycle. The skew symmetry of the polynomial P implies that $W_P(\tilde{G}) = -W_P(G)$, so contributions of G and $\tilde{G}$ into (4) cancel. Therefore, the summation in (4) is performed over the set of DOOMBs with cycles of even length only. Such DOOMBs can be represented as unions of two (directed) partial pair matchings.

To continue the proof we will need the following identity involving minors of a $2k \times 4k$ matrix Δ. Denote by $\mathcal{S}_d$ a set of all d-element subsets in $\{1, \dots, 2d\}$. For $I \in \mathcal{S}_{2k}$ denote by Δ_I a determinant of the $2k \times 2k$-submatrix of Δ formed by all the rows and the columns $i \in I$. Denote also $\Xi_I \overset{\text{def}}{=} \Delta_I \cdot \Delta_{I'}$ where $I' \overset{\text{def}}{=} \{1, 2, \dots, 4k\} \setminus I$.

Lemma 2.9. *Denote by $E_k \subset \mathcal{S}_{2k}$ a set of all $I \in \mathcal{S}_{2k}$ such that for every $s = 1, \dots, 2k$ the intersection $I \cap \{2s-1, 2s\}$ is empty or contains 2 elements, and by $O_k \subset \mathcal{S}_{2k}$ a set of all $I \in \mathcal{S}_{2k}$ such that for every $s = 1, \dots, 2k$ the intersection $I \cap \{2s-1, 2s\}$ contains 1 element. Then*

$$\sum_{I \in O_k} (-1)^{\Sigma I} \Xi_I = \sum_{I \in E_k} \Xi_I$$

where ΣI is the sum of elements in I.

Proof. For every $(2k-1)$-element set $J \subset \{1, \dots, 4k\}$ there holds a classical Plücker relation (see e.g. [7, p. 110]):

$$\sum_{a \in J'} (-1)^{m(a)} \Xi_{J \cup \{a\}} = 0 \tag{10}$$

where $m(a) = \#\{x \in J \mid x > a\} + \#\{x \in J' \mid x < a\}$. Let now

$$\rho(J) = (-1)^{\nu(J)} \binom{k-1}{2k - \nu(J) - 1}^{-1}$$

where $\nu(J) = \#\{s \mid J \cap \{2s-1, 2s\} \neq \emptyset\}$. Multiply the relation (10) by $\rho(J)$ and sum over all J. The resulting relation is linear in Ξ_I (that is, quadratic in Δ_I) and skew invariant under the permutations of $1, 2, \dots, 4k$ that respect the splitting $\{1, 2, \dots, 4k\} = \bigsqcup_{s=1}^{2k} \{2s-1, 2s\}$. So it suffices to calculate the coefficient at Ξ_{I_d} where $I_d = \{1, 2, \dots, 2d, 2d+1, 2d+3, \dots, 4k-2d-1\}$, $0 \le d \le k$.

Ξ_{I_d} enters relations (10) for $J = J_{d,s} \stackrel{\text{def}}{=} I_d \setminus \{s\}$, $1 \le s \le 2d$, and for $J = J'_{d,s} \stackrel{\text{def}}{=} I_d \setminus \{2d+2s-1\}$, $1 \le s \le 2(k-d)$. The coefficient at Ξ_{I_d} in (10) for $J = J_{d,s}$ is $-\binom{k-1}{d-1}^{-1}$ and for $J = J'_{d,s}$ it is $\binom{k-1}{d}^{-1}$ (in both cases it is independent of s). So for $0 < d < k$ the total coefficient is $-(2d)\binom{k-1}{d-1}^{-1} + 2(k-d)\binom{k-1}{d}^{-1} = 0$. For $d = 0$ it is $2k$, and for $d = k$, it is $-2k$. Since $I_0 \in O_k$ and $I_k \in E_k$, the lemma is proved. □

Finish now the proof of Theorem 2.8. Consider an undirected graph H that consists of p cycles containing $2n_1, \dots, 2n_p$ edges, q "even" chains containing $2\ell_1, \dots, 2\ell_q$ edges, and $2r$ "odd" chains containing $2m_1 + 1, \dots, 2m_{2r} + 1$ edges, respectively. The total number of edges in H is $2(n_1 + \dots + n_p) + 2(\ell_1 + \dots + \ell_q) + 2(m_1 + \dots + m_{2r}) + 2r \stackrel{\text{def}}{=} 2k$; the total number of vertices is $2r + q + 2k$. Without loss of generality one can suppose that the endpoints of the odd chains are numbered $1, 2, \dots, (4r-1), 4r$, the endpoints of the even chains are $4r+1, \dots, 4r+2q$, and the other vertices are $4r+2q+1, \dots, 2r+q+2k$.

Denote by $\mathcal{D}(H)$ the set of DOOMBs that become H after erasing orientations of all the edges. The set $\mathcal{D}(H)$ contains 2^{p+q+2r} elements; a graph $G \in \mathcal{D}(H)$ is determined by orientation of all the components. Denote also by $\mathcal{P}(H)$ the set of all pairs (F_1, F_2)

where F_1 and F_2 are partial pair matchings containing k edges each and such that the union of F_1 and F_2 is H. Edges of F_1 and F_2 in every component of H alternate, so the component can be split into edges of F_1 and F_2 in exactly two ways. The total number of edges of F_1 and F_2 in odd chains should be equal, which gives 1/2 of all possible splittings (in the even chains and in the cycles the balance is maintained automatically). Therefore, $\mathcal{P}(H)$ contains 2^{p+q+r} elements.

Fix a sequence of indices $1 \le i_1 < \cdots < i_{2k} \le N$. For a graph $G \in \mathcal{D}(H)$ let $I_-(G)$ be the set of initial vertices of its chains, $I_+(G)$, the set of final vertices of its chains, and $V(H)$, the set of all vertices (it depends on H only). Also denote by $E(H)$ the set of edges in H. Number the edges $d_1, \dots, d_{2k}$ and consider the matrix $\Delta(H)$ with $2k$ rows, $4k$ columns, and the matrix elements given by

$$\Delta(H)^s_{2t-1} = \langle u_{i_s}, e_{d_t^-} \rangle, \qquad \Delta(H)^s_{2t} = \langle u_{i_s}, e_{d_t^+} \rangle$$

(as usual, the edge d_t joins vertices d_t^- and d_t^+). Note that some columns in $\Delta(H)$ are repeated twice, namely, the columns corresponding to the vertices of the cycles and to the internal vertices of the chains; they depend on H only. Apply Lemma 2.9 to the matrix $\Delta(H)$, and multiply the result to $\prod_{j \in E(H)} W_P(d_j^-, d_j^+)$:

$$\sum_{G \in \mathcal{D}(H)} \prod_{j \in E(H)} W_P\left(d_j^-, d_j^+\right) \det \Delta(H)^{i_1 \dots i_{2k}}_{V(H) \setminus I_-(G)} \cdot \det \Delta(H)^{i_1 \dots i_{2k}}_{V(H) \setminus I_+(G)}$$
$$= \sum_{(F_1, F_2) \in \mathcal{P}(H)} U_P^{i_1 \dots i_{2k}}(F_1) U_P^{i_1 \dots i_{2k}}(F_2)$$

Take a sum over $1 \le i_1 < \cdots < i_{2k} \le n$ and use the Cauchy–Binet formula [11, §10.5]. In the left-hand side one obtains $\sum_{G \in \mathcal{D}_{2k}} W_P(G) = \mu_{2k}$; in the right-hand side the summation over F_1 and F_2 becomes independent, and (9) follows. □

3. Applications

3.1. Line bundles on a graph

Let G be an undirected graph without loops; parallel edges are allowed.

Definition 3.1. (See [8].) A *line bundle with connection on G* is a function attaching a number $\phi_d \neq 0$ to every directed edge d of G, such that $\phi_{-d} = \phi_d^{-1}$ where $-d$ is the edge d with the direction reversed.

To explain the name, attach a one-dimensional space L_v (a fiber of the bundle) to every vertex v of G. If an edge d joins vertices d^- and d^+, then the number ϕ_d can be interpreted as the 1×1-matrix of an operator $L_{d^-} \to L_{d^+}$ of parallel transport along d.

For a path $\Lambda = (d_1, \dots, d_k)$ in G denote $\phi_\Lambda \stackrel{\text{def}}{=} \phi_{d_1} \dots \phi_{d_k}$ (the operator of parallel transport along Λ). If Λ is a cycle then ϕ_Λ is called its holonomy. The holonomy of the same cycle with the orientation reversed is ϕ_Λ^{-1}.

Attach now a weight c_d to every edge d of G; the weight c_d does not depend on the edge direction.

Let $1, 2, \dots, n$ be vertices of G and $d_1, \dots, d_k$, its edges. Choose an orientation for every edge d_i, so that it joins the vertex d_i^- with the vertex d_i^+. Fix then an orthonormal basis $u_1, \dots, u_n$ in $\mathbb{R}^n$ and define $e_d \stackrel{\text{def}}{=} u_{d^-} - \phi_d^{-1} u_{d^+}$ and $\alpha_d \stackrel{\text{def}}{=} u_{d^-} - \phi_d u_{d^+}$. Consider a polynomial

$$P(x_1, \dots, x_k) = \sum_d c_d x_d, \tag{11}$$

and define an operator M by (1). If $v = \sum_{i=1}^n v_i u_i$ then

$$\begin{aligned} M(v) &= \sum_d c_d \big(v_{d^-} - \phi_d^{-1} v_{d^+}\big)\big(u_{d^-} - \phi_d u_{d^+}\big) \\ &= \sum_{i=1}^n u_i \sum_{d:\{d^-,d^+\}=\{i,j\}} c_d \big(v_i - \phi_d^{-1} v_j\big). \end{aligned} \tag{12}$$

The operator M is called (see [8]) a Laplacian of the bundle. Apparently, it does not depend on the direction of the edges.

A *unicycle* is a connected graph having as many vertices as edges. A unicycle is a simple cycle with trees (possibly empty) attached to its vertices. Call a graph G a *uni-forest* if every its connected component is either a tree or a unicycle; a *based* uni-forest is a uni-forest with a base point chosen in every tree component.

The following result generalizes the Matrix-CRSF theorem of [6] and [8]:

Theorem 3.2. *The characteristic polynomial of the Laplacian (12) of a line bundle on a graph is equal to $\sum_{k=0}^n (-1)^k \mu_k t^{n-k}$ where*

$$\mu_k = \sum_{F \in \mathcal{U}_{n,k}} \prod_{\substack{d \text{ is} \\ \text{an edge of } F}} c_d \prod_{\substack{\Lambda \text{ is} \\ \text{a cycle in } F}} (1 - \phi_\Lambda)\big(1 - \phi_\Lambda^{-1}\big). \tag{13}$$

Here $\mathcal{U}_{n,k}$ is the set of based uni-forests containing n vertices and k edges.

Note that the summand does not depend on the base vertices, so (13) contains groups of $\prod_{i=1}^{n-k}(m_i+1)$ identical terms; here m_i is the number of edges in the i-th tree component of F.

Theorem 3.2 follows from Corollary 2.4 and Lemma 3.3 below. Denote by Q_F the matrix $(\langle \alpha_{d_p}, e_{d_q} \rangle)_{p,q=1}^k$ (recall that $d_1, \dots, d_k$ are edges of the graph F).

Lemma 3.3.

1. *If F is a tree with k edges (and $k+1$ vertices) then* $\det Q_F = k+1$.
2. *If F is a unicycle with a cycle Λ then* $\det Q_F = (1-\phi_\Lambda)(1-\phi_\Lambda^{-1})$.
3. *If F is a connected graph with more than one cycle then* $\det Q_F = 0$.

Proof. Elementary transformations ($v_i \mapsto v_i + tv_j$) applied to vectors e and α will not change $\det Q_F$. If $I = (i_1, \dots, i_s)$ is a path in F (a sequence of vertices joined by successive edges $d_1 = (i_1 i_2), \dots, d_{s-1} = (i_{s-1} i_s)$) then elementary transformations applied to vectors $e_{d_1}, \dots, e_{d_{s-1}}$ allow to replace the last vector in this system by $u_{i_1} - \phi_I^{-1} u_{i_s}$; for α it is $u_{i_1} - \phi_I u_{i_s}$.

Choose a base vertex in F; in case F is a unicycle, it must belong to the cycle. Without loss of generality, the base vertex has number 1, and the other vertices in the cycle (in the unicycle case) are $2, \dots, \ell$. If F is a tree then for every vertex j there is a unique shortest path Λ_j joining 1 with j; applying the transformations described above one can replace the vectors $e_{d_1}, \dots, e_{d_k}$ and $\alpha_{d_1}, \dots, \alpha_{d_k}$ (where $d_1, \dots, d_k$ are the edges of F) by $\varepsilon_j \stackrel{\text{def}}{=} u_1 - \phi_{\Lambda_j}^{-1} u_j$ and $a_j \stackrel{\text{def}}{=} u_1 - \phi_{\Lambda_j} u_j$ where $j = 2, \dots, k+1$. If F is a unicycle then deleting an edge 1ℓ makes it a tree; take for Λ_j the shortest path joining 1 and j in this tree. Similar to the tree case, one replaces e_{d_s} and α_{d_s} by ε_j and a_j for $2 \le j \le k$, and additionally $\varepsilon_{k+1} \stackrel{\text{def}}{=} e_{\ell 1}$, $a_{k+1} \stackrel{\text{def}}{=} \alpha_{\ell 1}$.

If F is a tree then the matrix $(\langle a_i, \varepsilon_j \rangle)$, $2 \le i, j \le k+1$ has 2 on the main diagonal and 1 in all the other positions. Its determinant is $k+1$ by easy induction.

If F is a unicycle then an elementary transformation applied to vectors $\varepsilon_\ell = u_1 - \phi_{\Lambda_\ell}^{-1} u_\ell$ and $\varepsilon_{k+1} = e_{\ell 1} = u_m - \phi_{1\ell} u_1$ allows to replace the last one by $E_1 \stackrel{\text{def}}{=} u_1(1 - \phi_{\Lambda_\ell}^{-1} \phi_{1\ell}) = u_1(1 - \phi_\Lambda^{-1})$. Then an elementary transformation applied to E_1 and ε_j allows to replace ε_j by $E_j \stackrel{\text{def}}{=} -\Lambda_j^{-1} u_j$; here $j = 2, \dots, k$. Similar operations for α give the set of vectors $A_1 = u_1(1 - \phi_\Lambda)$ and $A_j \stackrel{\text{def}}{=} -\Lambda_j u_j$ for $j = 2, \dots, k$. The matrix $(\langle E_i, A_j \rangle)$, $1 \le i, j \le k$, is a diagonal matrix with $(1-\phi_\Lambda)(1-\phi_\Lambda^{-1})$ in the upper left corner and 1 in the other positions.

To prove the last statement of the lemma just note that there are k vectors e_d, and they all belong to the space $\mathbb{R}^n$ where n is the number of vertices in F. If F is connected and contains more than one cycle then $n < k$, so the rows of Q_F are linearly dependent. □

Proof of Theorem 3.2. Apply Corollary 2.4 to the vectors e_d, α_d defined above. Summation in (7) is done over the set of all unordered k-tuples $d_1, \dots, d_k$, that is, over the set of directed graphs F with k edges.

Let $F_1, \dots, F_\ell$ be connected components of F. If the edges d_i and d_j belong to different components then $\langle \alpha_{d_i}, e_{d_j} \rangle = 0$. So the matrix Q_F is block diagonal, and $\det Q_F = \det Q_{F_1} \dots \det Q_{F_\ell}$. It follows now from statement 3 of Lemma 3.3 that $\det Q_F = 0$ unless F is a uni-forest.

Let F be a uni-forest; by statement 1 of the lemma if F_i is a tree component then $\det Q_{F_i}$ is equal to the number of ways to choose a base vertex in it. By statement 2 of the lemma, if F_i is the cycle component then $\det Q_{F_i}$ is $(1-\phi_{\Lambda_i})(1-\phi_{\Lambda_i}^{-1})$ where Λ_i is the only cycle in F_i. Every cycle in F is Λ_i for some i, which finishes the proof. □

3.2. Matrix-tree theorems

Fix a space $\mathbb{R}^n$ with an orthonormal basis $u_1, \ldots, u_n$. For a permutation $\sigma \in \Sigma_n$ denote by $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ a linear operator permuting the basic vectors: $\sigma(u_i) \stackrel{\text{def}}{=} u_{\sigma(i)}$. This defines an action of the group Σ_n on $\mathbb{R}^n$. Transpositions (pq), $1 \le p < q \le n$, act by reflections; they generate the Coxeter group A_{n-1}.

Corollary 3.4 *(of Theorem 3.2). The characteristic polynomial of the operator* $M = \sum_{1\le p<q\le n} c_{pq}(1-(pq))$ *is equal to* $\sum_{k=0}^{n}(-1)^k \mu_k t^k$ *where*

$$\mu_k = \sum_{F\in\mathcal{F}_{n,k}} \prod_{\substack{(pq) \text{ is} \\ \text{an edge of } F}} c_{pq}.$$

Here $\mathcal{F}_{n,k}$ is the set of based forests with n vertices and k edges.

Proof. (Cf. [6].) This is a special case of Theorem 3.2 arising when $\phi_{ij} = 1$ for all i, j (a "trivial connection"). Since all the holonomies for such connection are equal to 1, only forests make nonzero contribution into (13). □

Apparently, $\det M = 0$ (there are no forests with n vertices and n edges), so the summation is indeed up to $k = n - 1$.

The corollary follows also from the classical Principal Minors Matrix-Tree Theorem, see e.g. [4] for proofs and related results.

Denote now by $(pq)' : \mathbb{R}^n \to \mathbb{R}^n$ a reflection in the hyperplane normal to the vector $u_p + u_q$:

$$(pq)'(u_p) = -u_q, \qquad (pq)'(u_q) = -u_p, \quad \text{and} \quad (pq)'(u_i) = u_i \quad \text{for } i \ne p, q. \tag{14}$$

Reflections (pq) and $(pq)'$ together, $1 \le p < q \le n$, generate a Coxeter group D_n. The D-version of the Matrix-tree theorem is:

Corollary 3.5. *The characteristic polynomial of the operator*

$$M = \sum_{1\le p<q\le n} c_{pq}^{-}\bigl(1-(pq)\bigr) + c_{pq}^{+}\bigl(1-(pq)'\bigr)$$

is equal to $\sum_{k=0}^{n}(-1)^k \mu_k t^k$ *where*

$$\mu_k = \sum_{0\le\ell\le k/2} 4^\ell \sum_{F\in\mathcal{UO}_{n,k,\ell}} \prod_{\substack{(pq)_s \text{ is} \\ \text{an edge of } F}} c^s_{pq}.$$

Here $\mathcal{UO}_{n,k,\ell}$ is the set of based uni-forests with n vertices, ℓ unicycle components, and k edges marked either $+$ or $-$ so that the number of $+$-edges in every cycle is odd.

Proof. Join every pair of vertices (i,j) with two edges: $(i,j)_-$ with $\phi^-_{ij} = 1$ and $(i,j)_+$ with $\phi^+_{ij} = -1$; the weights are c^+_{ij} and c^-_{ij}, respectively. By (12), $M = \sum_{1\le p<q\le n} c^-_{pq}(1-(pq)) + c^+_{pq}(1-(pq)')$. The holonomy of a cycle is $w = (-1)^j$ where j is the number of $+$-edges in the cycle. □

3.3. Discrete Schroedinger operator

Definition. A discrete Schroedinger operator is the operator $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}^n$ given by the formula $\mathcal{H} = \Delta + \Lambda$ where $\Delta \stackrel{\text{def}}{=} \sum_{1\le p<q\le n} 1-(pq)$ and $\Lambda(u_i) \stackrel{\text{def}}{=} \lambda_i u_i$ for every element of an orthonormal basis $u_1, \dots, u_n$.

In terms of (12) Δ is the Laplacian of the line bundle on a complete graph with trivial connection and edges of unit weight.

Theorem 3.6. *The characteristic polynomial of $\mathcal{H}$ is equal to*

$$\sum_{k=0}^{n} \sum_{F\in\mathcal{F}_{n,k}} \prod_{i=1}^{n-k} (t + \lambda_{p_i(F)}). \tag{15}$$

Here $\mathcal{F}_{n,k}$ is, like in Corollary 3.4, the set of based forests with n vertices and k edges; $p_i(F)$ is the base point of the i-th component of F.

Proof. Take $e_{ij} \stackrel{\text{def}}{=} u_i - u_j$, $1 \le i < j \le n$ (like in Section 3.1), and $e_i \stackrel{\text{def}}{=} u_i$, $1 \le i \le n$. If

$$P(x_{12}, \dots, x_{n-1,n}, y_1, \dots, y_n) = \sum_{1\le i<j\le n} x_{ij} + \sum_{i=1}^{n} \lambda_i y_i$$

then, apparently,

$$\mathcal{H} = P\big(M[e_{12}, e_{12}], \dots, M[e_{n-1,n}, e_{n-1,n}], M[e_1, e_1], \dots, M[e_n, e_n]\big).$$

Use Corollary 2.4. A k-element subset $\{i_1, \dots, i_k\} \subset \{(12), \dots, (n-1,n), 1, \dots, n\}$ in (7) can be interpreted as a graph F containing $k' \le k$ edges (corresponding to pairs (pq) chosen) and $k-k'$ marked vertices (corresponding to elements p). Denote, as usual, by Q_F the matrix mentioned in the term of (7) corresponding to F.

Like in the proof of Corollary 3.4, if the graph F contains a cycle $j_1, \dots, j_s$ then the corresponding vectors e_{pq} are linearly dependent: $e_{j_1 j_2} + \dots + e_{j_s j_1} = 0$, and therefore $\det Q_F = 0$. Similarly, if j_1 and j_s are marked vertices of F joined by a path $j_1, j_2, \dots, j_s$ then $e_{j_1} + e_{j_1 j_2} + \dots + e_{j_{s-1} j_s} + e_{j_s} = 0$, and $\det Q_F = 0$, too. Thus, summation in (7) is over the set of forests F with every component containing at most one marked vertex. If $F_1, \dots, F_\ell$ are connected components of F then $\det Q_F = \det Q_{F_1} \dots \det Q_{F_\ell}$.

By Lemma 3.3.1, if F_i is a component of F containing no marked vertices then $\det F_i = m_i + 1$ where m_i is the number of edges in F_i. Suppose now F_i that contains a marked vertex a_i. Let p be a dangling vertex of F_i; let q be the parent of p and r, the parent of q (a path joining p with the root starts with the vertices q and r). Replace F_i by a tree F_i' containing an edge rp instead of qp. Then the vectors $e_{pq} = \alpha_{pq}$ are replaced by $e_{pr} = \alpha_{pr} = e_{pq} + e_{qr}$. The matrix $Q_{F_i'}$ differs from Q_{F_i} by an elementary transformation, so $\det Q_{F_i'} = \det Q_{F_i}$. Applying this transformation several times, we convert F_i into a "bush" tree F_i^* where every vertex is joined by an edge with a root a_i, and conclude that

$$\det Q_{F_i} = \det Q_{F_i^*} = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ & & \dots & \\ 1 & 1 & \dots & 2 \end{pmatrix} = 1$$

(one assumes that a_i is in the first row and column). So, the coefficient at t^{n-k} in the characteristic polynomial of $\mathcal{H}$ is a sum over forests with $k' \le k$ edges (hence $n - k'$ components) and $k - k'$ marked vertices, at most one per component. The summand is $\prod_{i=1}^{k-k'} \lambda_{p_i(F)} \times \prod_{j=k-k'}^{n-k'} (m_j + 1)$; here we assume that a component numbered $i = 1, 2, \dots, k - k'$ contains a marked vertex $p_i(F)$ and a component numbered $j = k - k' + 1, \dots, n - k'$ contains m_j edges and no marked vertices. Obviously, this is equivalent to (15). □

3.4. Pfaffian-tree theorems

In this section we prove two corollaries of Theorem 2.8.

A *finite* 3-*graph* is a topological space obtained by gluing several triangles by their vertices (not sides!). Triangles (homeomorphic to disks) are called 3-edges of the 3-graph; their sides are called 2-edges.

A contractible 3-graph is called a 3-tree; a 3-forest is a 3-graph such that every its connected component is a 3-tree. If a root (a base vertex) is chosen in a 3-tree then every its 3-edge has one "inner" vertex (closer to the root) and two "outer" ones.

The following theorem is a generalization of the Matrix-hypertree theorem of [10]:

Theorem 3.7. *The Pfaffian of the principal* $(i_1, \dots, i_{2k})$*-minor of the skew-symmetric operator*

$$M = \sum_{1 \le p < q < r \le n} c_{pqr} \big((pqr) - (prq)\big) \tag{16}$$

is equal to

$$\mu_{i_1,\ldots,i_{2k}} = \sum_{G\in\mathcal{TF}_{i_1,\ldots,i_{2k}}} (-1)^{s(G)} \prod_{\substack{(pqr) \text{ is} \\ \text{a 3-edge of } G}} c_{pqr} \tag{17}$$

Here $\mathcal{TF}_{i_1,\ldots,i_{2k}}$ is the set of 3-forests with k 3-edges and the vertices $1,2,\ldots,n$ such that every their component contains exactly one vertex (a root) not in the set $\{i_1,\ldots,i_{2k}\}$; $s(G)$ is the number of inversions in a permutation σ of $1,\ldots,2k$ defined so that $i_{\sigma(2t-1)} < i_{\sigma(2t)}$ are outer vertices of the t-th edge of G, $1 \le t \le k$.

Remark 3.8. To define $s(G)$, the 3-edges of G should be numbered and the vertices of every 3-edge should be ordered. Nevertheless, the sign $(-1)^{s(G)}$ does not depend on the numbering because every 3-edge has two outer vertices, and therefore different numberings give rise to permutations of the same parity.

Proof of Theorem 3.7. Extend the definition of c_{pqr} to all $p,q,r = 1,\ldots,n$ to make it skew-symmetric: $c_{qpr} = -c_{pqr} = c_{prq}$. Also assume $c_{pqr} = 0$ if any two indices coincide. Take $e_{pq} = u_p - u_q$ and consider the polynomial

$$P = \frac{1}{6}\sum_{p,q,r=1}^{n} c_{pqr}x_{pq}x_{qr}, \tag{18}$$

where $x_{pq} = x_{qp}$ for all $1 \le p,q \le n$ by definition. A simple calculation shows that $M = P(M[e_{12},e_{12}],\ldots,M[e_{n-1,n},e_{n-1,n}])$ is the operator (16). (Note that $M[e_{pq},e_{pq}] = M[e_{qp},e_{qp}]$, so the relation $x_{pq} = x_{qp}$ is satisfied.)

Apply now Theorem 2.8. A partial pair matching $F \in \mathcal{P}_k$ is the set of ordered pairs $((p_s,q_s),(q_s,r_s))$, $s = 1,\ldots,k$, or, equivalently, a 3-graph with the 3-edges (p_s,q_s,r_s), where the vertices of every 3-edge are ordered.

Fix the set of indices $I = \{i_1,\ldots,i_{2k}\}$, $i_1 < \cdots < i_{2k}$, and denote by $Q_F = Q_F^I$ the matrix in the right-hand side of (8): its matrix elements are given by

$$(Q_F)_{s,2t-1} = \langle u_{i_s}, e_{q_tp_t}\rangle, \qquad (Q_F)_{s,2t} = \langle u_{i_s}, e_{q_tr_t}\rangle. \tag{19}$$

If $F_1,\ldots,F_\ell$ are connected components of the 3-graph F, then, as usual, $\det Q_F = \det Q_{F_1}\ldots\det Q_{F_\ell}$. Notice also that the term in (8) corresponding to F does not depend on the ordering of vertices in edges; so, taking into account the factor $1/6$ in (18), one can say that the summation in (8) is done over the set of 3-graphs with the vertices $1,2,\ldots,n$ and with k unoriented 3-edges.

Suppose now that the 3-edges $(p_1,q_1,r_1),\ldots,(p_s,q_s,r_s)$ of F form a cycle. Ordering of vertices in every 3-edge is not important, so one can suppose that the 2-edges (sides of the 3-edges) $(p_1,q_1),\ldots,(p_s.q_s)$ also form a cycle. Then $e_{p_1q_1}+\cdots+e_{p_sq_s} = 0$; therefore, the rows of Q_F are linearly dependent, and $\det Q_F = 0$. Thus, only 3-forests enter the sum.

Suppose now that vertices $p, q \notin I$ lie in the same connected component of F. Without loss of generality there exists a sequence of 3-edges $(p_1, q_1, r_1), \dots, (p_s, q_s, r_s)$ such that $p_1 = p$ and $q_s = q$. Then $e_{p_1 q_1} + \dots + e_{p_s q_s} = u_q - u_p$, so that $(u_{i_t}, e_{p_1 q_1}) + \dots + (u_{i_t}, e_{p_s q_s}) = 0$ for all t; so, the rows of Q_F are linearly dependent, and $\det Q_F = 0$. Hence, if a 3-forest F enters the sum then every its connected component contains *at most one* vertex $p \notin I$ (cf. with the proof of Theorem 3.6). On the other hand, the total number of vertices in F is equal to $2k+\ell$ (twice the number of 3-edges plus the number of connected components). Therefore, every connected component F_j of F contains *exactly one* vertex $a_j \notin I$; call it a root.

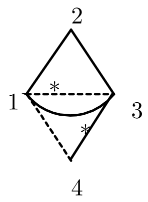
Let now σ be the permutation of $\{1, \dots, 2k\}$ defined in the formulation of the theorem. Let (pqr) be a dangling 3-edge of a component F_j of F; we assume that the vertices p and r are outer (belong to no other 3-edge), and $q \neq a_j$ belongs to a 3-edge $(p'q'r')$: $q = p'$. Replace the 3-edge (pqr) by a 3-edge $(pq'r)$ obtaining a new 3-forest F'. Equivalently, the vectors e_{pq} and e_{qr} are replaced by $e_{pq'} = e_{pq} + e_{qq'} = e_{pq} + e_{p'q'}$ and $e_{q'r} = e_{qr} + e_{q'q} = e_{qr} - e_{p'q'}$. Since $p, q, r \in I$, the matrix $Q_{F'}$ is obtained from Q_F by an elementary transformation, and $\det Q'_F = \det Q_F$. At the same time, one may assume that the permutations σ' and σ are the same (remember that the parity of σ does not depend on the edge numbering), so that $(-1)^{s(F')} = (-1)^{s(F)}$.

Applying the transformation several times we make every component of F look like a “3-bush”: a set of 3-edges attached to a common vertex $a \notin I$. For such 3-forests the equality $\det Q_F = (-1)^{s(F)}$ is obvious. □

Formulate now a D-version of Theorem 3.7 (in the same sense as Corollary 3.5 is a D-version of the Matrix-tree theorem). Define a *semi-open triangle* as a triangle pqr with a side pr removed (the internal points, the sides pq and qr, and the vertices p, q, r are preserved). A semi-open 3-graph is a topological space obtained by gluing semi-open triangles by their vertices (not sides!). A skeleton of a semi-open 3-graph is a union of sides pq and qr of all its 3-edges pqr; it is a graph. A semi-open 3-graph is called a 3-tree or a 3-unicycle if its skeleton is a tree or a unicycle, respectively. A union of sides pq and qr is a homotopy retract of the semi-open triangle pqr; consequently, a semi-open 3-graph is homotopy equivalent to its skeleton. Therefore, a semi-open 3-graph is a 3-tree if it is contractible, and a 3-unicycle if it is homotopy equivalent to a circle. A 3*-uni-forest* is a semi-open 3-graph where every connected component is either a 3-tree or a 3-unicycle.

Fix a system of weights c_{pqr}, $1 \le p, q, r \le n$, such that $c_{pqr} = -c_{rqp}$ and $c_{ppr} = c_{pqp} = c_{pqq} = 0$ for all p, q, r. Consider an operator $A_{pqr} = (pq)'(qr)' - (qr)'(pq)'$ where $(pq)'$ and $(qr)'$ are given by (14).

Define now a *semi-weight* $W_I(G)$ for any 3-uni-forest G and a set $I \stackrel{\text{def}}{=} \{i_1, \dots, i_{2k}\}$ where $1 \le i_1 < \dots < i_{2k} \le n$. Start from the case when G is a rooted 3-tree. Let $1 \le i_1 < \dots < i_{2k} \le n$ be its vertices different from the root (for other I the semi-weight $W_I(G) = 0$ by definition). In every 3-edge of G mark the “inner” vertex (the one closest to the root) with a star; apparently, every non-root vertex of G is marked in all the 3-edges containing it except exactly one. Then number and orient the 3-edges. Let $i_{\sigma(2t-1)}, i_{\sigma(2t)}$



$\sigma(G) = [3241], s(G) = +1$

$U(G) = c_{321}c_{431} = c_{123}c_{134}$

Fig. 1. A half-weight of a 3-unicycle.

$(1 \leq t \leq k)$ be the unmarked ("outer") vertices of the t-th 3-edge in the order determined by the edge orientation. (The orientation determines the *cyclic* order of the vertices, and the position of the deleted edge fixes their linear order: if the oriented 3-edge is pqr with the side pr deleted then the order of vertices is p, then q, then r.) Then σ is a permutation of $1, \dots, 2k$; denote by $\operatorname{sign}_I(G) = \pm 1$ its parity.

The permutation σ depends on the numbering of the 3-edges of G, but its parity $\operatorname{sign}_I(G)$ is well-defined because every 3-edge contains two unmarked vertices. At the same time, $\operatorname{sign}_I(G)$ changes sign if an orientation of a 3-edge of G is changed. Recall that the same is true for the weights of the 3-edges: $c_{rqp} = -c_{pqr}$. Thus, the *half-weight*

$$U_I(G) \stackrel{\text{def}}{=} \operatorname{sign}_I(G) \prod_{\substack{(pqr) \text{ is} \\ \text{a 3-edge of } G}} c_{pqr} \tag{20}$$

is independent of the orientations.

Let now G be a 3-unicycle. Again, number and orient its 3-edges arbitrarily and also choose an orientation of the cycle. Then mark with a star one vertex of every 3-edge using the following rule: if some vertices of a 3-edge pqr enter the cycle then mark the last of them (according to the orientation of the cycle). If no vertex of pqr is in the cycle then mark the vertex closest to the cycle. An easy induction shows that every vertex of G would be marked in all the 3-edges containing it except exactly one. (See Fig. 1 for an example of the marking; dashed lines mean deleted sides.) Then define the permutation σ, its parity $\operatorname{sign}_I(G)$ and the half-weight $U_I(G)$ exactly as in the 3-tree case. Note that in the 3-unicycle case I must be the set of all the vertices in G.

As before, $\operatorname{sign}_I(G)$ is independent of the 3-edge orientation and numbering; still, $\operatorname{sign}_I(G)$ depends on the orientation of the cycle in G. An easy induction shows that

$$\operatorname{sign}_I(-G) = (-1)^{c+1} \operatorname{sign}_I(G) \tag{21}$$

where $-G$ is the 3-unicycle G with the cycle reversed, and c is the number of 2-edges (sides of the 3-edges) in the cycle.

Finally, if $G_1, \dots, G_\ell$ are the connected components of a 3-uni-forest G then by definition $U_I(G) = U_{I_1}(G_1) \dots U_{I_\ell}(G_\ell)$ where I_s is the intersection of I with the set of vertices of G_s, $1 \leq s \leq \ell$. Note that $U_I(G) \neq 0$ only if I is the set of all vertices of G except the roots of all 3-tree components.

Theorem 3.9. *Fix a set* $I \overset{\text{def}}{=} \{i_1, \ldots, i_{2k}\}$, $1 \leq i_1 < \cdots < i_{2k} \leq n$. *The Pfaffian of the principal* $(i_1, \ldots, i_{2k})$*-minor of the skew-symmetric operator*

$$M = \frac{1}{2} \sum_{\substack{1 \leq p,q,r \leq n \\ p<r}} c_{pqr} A_{pqr} \tag{22}$$

is equal to

$$\mu_{i_1,\ldots,i_{2k}} = \sum_{G \in \mathcal{TUF}_{i_1,\ldots,i_{2k}}} U_{i_1,\ldots,i_{2k}}(G).$$

Here $\mathcal{TUF}_{i_1,\ldots,i_{2k}}$ *is the set of* 3*-uni-forests such that:*

1. *All the vertices of the* 3*-unicycle components of* G *are in the set* I.
2. *Every* 3*-tree component of* G *contains exactly one vertex (a root) which is not in* I.
3. *A cycle in every* 3*-unicycle component of* G *contains an odd number of* 2*-edges.*

The total number of 3-edges in any $G \in \mathcal{TUF}_{i_1,\ldots,i_{2k}}$ is k. By (21), condition 3 allows us not to worry about orientation of the cycles in G.

Proof. Take $e_{pq} \overset{\text{def}}{=} u_p + u_q$ and consider the polynomial

$$P = \frac{1}{2} \sum_{\substack{1 \leq p,q,r \leq n \\ p<r}} c_{pqr}(x_{pq}x_{qr} - x_{qr}x_{pq}), \tag{23}$$

where $x_{pq} = x_{qp}$ for all $1 \leq p, q \leq n$ by definition. A simple calculation shows that $M = P(M[e_{12}, e_{12}], \ldots, M[e_{n-1,n}, e_{n-1,n}])$ is the operator (22). (Again, $M[e_{pq}, e_{pq}] = M[e_{qp}, e_{qp}]$, so the relation $x_{pq} = x_{qp}$ is satisfied.)

Apply now Theorem 2.8. A partial pair matching $F \in \mathcal{P}_k$ is the set of ordered pairs $((p_s, q_s), (q_s, r_s))$, $s = 1, \ldots, k$, or, equivalently, a semi-open 3-graph with k 3-edges (p_s, q_s, r_s) (the side $p_s r_s$ deleted). Consider the matrix Q_F^I from (19). Like in the proof of Theorem 3.7, one has $\det Q_F^I = \det Q_{F_1}^{I_1} \ldots \det Q_{F_\ell}^{I_\ell}$ where $F_1, \ldots, F_\ell$ are connected components of F, and I_s, $1 \leq s \leq \ell$, is the intersection of I with the set of vertices of F_s.

Let the 2-edges $(p_1, p_2), (p_2, p_3), \ldots, (p_c, p_1)$ form a cycle. Then one has

$$e_{p_1p_2} - e_{p_2p_3} + \cdots + (-1)^s e_{p_cp_1} = \begin{cases} 0, & c \text{ is even}, \\ 2u_{p_1}, & c \text{ is odd}. \end{cases}$$

Thus, if F contains a cycle of even length, then the rows of the matrix Q_F^I are linearly dependent, and $\det Q_F^I = 0$; so, only graphs with all the cycles of odd length make a contribution. As a corollary, no connected component of F can contain more than one cycle: if it contains two, then either one of them or their union has even length (cf. the proof of Theorem 3.2). So, the summation is over the set of 3-uni-forests.

Like in the proof of Theorem 3.7 one shows that every 3-tree component of the semi-open 3-graph F contains exactly one vertex $p \notin I$ (a root). The total number of vertices of a 3-uni-forest is equal to $2k + t$ where k is the number of 3-edges and t is the number of tree components; therefore all the vertices of the unicycle components of F are in I.

Let F_s be a 3-unicycle component of F, and $I_s \subset I$ be the set of its vertices. To prove the equality $\det Q_{F_s}^{I_s} = \operatorname{sign}_{I_s}(F_s)$ we apply a series of elementary transformations to the columns of the matrix, or, equivalently, to the corresponding vectors e_{ij} (like in the proof of Theorem 3.7; elementary transformations leave the determinant unchanged). First, let $p_1, \ldots, p_c$ be vertices in the cycle of F_s; as we proved already, c is odd. Then there exists a sequence of elementary transformations converting the system of vectors $e_{p_1 p_2}, \ldots, e_{p_c p_1}$ (where $e_{p_i p_{i+1}} = u_{p_i} + u_{p_{i+1}}$) into $2u_{p_1}, u_{p_2}, \ldots, u_{p_c}$ (cf. the proof of Lemma 3.3). Let now pqr be a 3-edge of F such that p and q are in the cycle and r is not. Let the deleted edge be pr; if it is qr then the reasoning is similar. Applying the elementary transformation to the vectors u_q and $e_{qr} = u_q + u_r$, we replace e_{qr} with u_r.

Consider now "tree" parts of the 3-unicycle F_s. Applying to them the transformations similar to those used in the proof of Theorem 3.7 we transform F_s into a 3-unicycle where every out-of-cycle 3-edge pqr is attached to the cycle by a vertex. Suppose the vertex is p and the deleted side is pr (the other cases are similar). Then an elementary transformation applied to the vectors u_p (already obtained), $e_{pq} = u_p + u_q$ and $e_{qr} = u_q + u_r$ allows to replace e_{pq} and e_{qr} with u_q and u_r, respectively. Thus, a series of elementary transformations converts $Q_{F_s}^{I_s}$ to the matrix $R' = \langle (u_{i_a}, u_{i_{\tau(b)}}) \rangle$ where $\tau \stackrel{\text{def}}{=} \sigma_{I_s}(F_s)$. Then $\det Q_{F_s}^{I_s} = \det R' = \operatorname{sign}_{I_s}(F_s)$.

If F_s is a 3-tree component of F then the equality $\det Q_{F_s} = \operatorname{sign}_{I_s}(F_s)$ is proved exactly as in the proof of Theorem 3.7. $\square$

Acknowledgements

The authors are grateful to M. Kazarian for many valuable discussions, in particular, for the idea of the proof of Lemma 2.9. We also wish to thank A. Abdessalam and A. Bufetov for stimulating exchange of ideas and for introducing us to great examples of research in the field.

During the work the first named author was generously supported by the RFBR grant NSh-5138.2014.1 "Singularities theory and its applications", by the Higher School of Economics (HSE) Scientific Foundation grant 12-01-0015 "Differential geometry on graphs and discrete path integration", by Dobrushin Professorship grant 2013 (the Independent University of Moscow) and by the Simons Foundation grant (autumn 2013 and autumn 2014).

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