OPERS WITH IRREGULAR SINGULARITY AND SPECTRA OF THE SHIFT OF ARGUMENT SUBALGEBRA

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Abstract. The universal enveloping algebra of any simple Lie algebra $\mathfrak{g}$ contains a family of commutative subalgebras, called the quantum shift of argument subalgebras [R1, FFT]. We prove that generically their action on finite-dimensional modules is diagonalizable and their joint spectra are in bijection with the set of monodromy-free ${}^LG$-opers on $\mathbb{P}^1$ with regular singularity at one point and irregular singularity of order two at another point. We also prove a multi-point generalization of this result, describing the spectra of commuting Hamiltonians in Gaudin models with irregular singularity. In addition, we show that the quantum shift of argument subalgebra corresponding to a regular nilpotent element of $\mathfrak{g}$ has a cyclic vector in any irreducible finite-dimensional $\mathfrak{g}$-module. As a byproduct, we obtain the structure of a Gorenstein ring on any such module. This fact may have geometric significance related to the intersection cohomology of Schubert varieties in the affine Grassmannian.

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. The symmetric algebra $S(\mathfrak{g})$ carries a natural Poisson structure. A Poisson commutative subalgebra $\overline{\mathcal{A}}_\mu$ of $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$, called the *classical shift of argument subalgebra*, was defined in [MF] (see also [Ma]). It is generated by the derivatives of all orders in the direction of $\mu \in \mathfrak{g}^*$ of all elements of the algebra of $\mathfrak{g}$-invariants in $\mathbb{C}[\mathfrak{g}^*]$. Recently, this algebra was quantized in [R1, FFT]. More precisely, a commutative subalgebra $\mathcal{A}_\mu$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ was constructed and it was proved that for all regular $\mu \in \mathfrak{g}^*$ the associated graded of $\mathcal{A}_\mu$, with respect to the order filtration on $U(\mathfrak{g})$ is $\overline{\mathcal{A}}_\mu$. The algebra $\mathcal{A}_\mu$ is called the *quantum shift of argument subalgebra* of $U(\mathfrak{g})$. This is a free polynomial algebra in $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g})$ generators for any regular $\mu \in \mathfrak{g}^*$ (see [R1, FFT]). We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using an invariant inner product and regard μ as an element of $\mathfrak{g}$.

For any regular μ the algebra $\mathcal{A}_\mu$ contains the centralizer of μ in $\mathfrak{g}$. In particular, if μ is regular semisimple, then $\mathcal{A}_\mu$ contains a Cartan subalgebra of $\mathfrak{g}$. Hence it acts on weight subspaces of $\mathfrak{g}$-modules. We note that, as shown in [R1], a certain limit of $\mathcal{A}_\mu$ in the case when $\mathfrak{g} = \mathfrak{sl}_n$ may be identified with the Gelfand–Zetlin algebra. Hence the algebra $\mathcal{A}_\mu$ may be thought of as a generalization of

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the Gelfand–Zetlin algebra to an arbitrary simple Lie algebra. It is an interesting question to describe the joint (generalized) eigenvalues of $\mathcal{A}_\mu$ on $\mathfrak{g}$-modules.

The first steps towards answering this question were taken in [FFT], where it was shown that $\mathcal{A}_\mu$ is isomorphic to the algebra of functions on a certain space of ${}^LG$-*opers* on the projective line $\mathbb{P}^1$. More precisely, this is the space of ${}^LG$-opers on $\mathbb{P}^1$ with regular singularity at the point 0 and irregular singularity of order 2 at the point ∞, with a fixed 2-residue determined by μ (see [FFT] for details). Here ${}^LG$ is the *Langlands dual group* of G (${}^LG$ is taken to be of adjoint type), and ${}^LG$-opers are connections on a principal ${}^LG$-bundle over $\mathbb{P}^1$ satisfying a certain transversality condition, as defined in [BD]. The appearance of the Langlands dual group is not accidental, but is closely related to the geometric Langlands correspondence, through a description of the center of the completed enveloping algebra of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the critical level in terms of ${}^LG$-opers on the punctured disc [FF1, Fr2, Fr4].

Thus, we obtain that the spectra of $\mathcal{A}_\mu$ on a $\mathfrak{g}$-module M are encoded by ${}^LG$-opers on $\mathbb{P}^1$ satisfying the above properties. Furthermore, in [FFT] it was shown that if $M = V_\lambda$, the irreducible finite-dimensional $\mathfrak{g}$-module with dominant integral highest weight λ, then these ${}^LG$-opers satisfy two additional properties: they have a fixed residue at the point 0 (where the oper has regular singularity), determined by λ (we denote the space of such opers by $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\mu)}$), and they have *trivial monodromy*.

It was conjectured in [FFT] that in fact there is a bijection between the spectra of $\mathcal{A}_\mu$ on V_λ and the set of monodromy-free opers from $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\mu)}$ for regular $\mu \in \mathfrak{g}^*$. In this paper we prove this conjecture (see Corollary 3). Furthermore, we prove the following statement (see Corollary 4):

Theorem A. *For generic regular semisimple $\mu \in \mathfrak{g}^*$ and any dominant integral λ, the quantum shift of argument subalgebra $\mathcal{A}_\mu \subset U(\mathfrak{g})$ is diagonalizable and has simple spectrum on the irreducible $\mathfrak{g}$-module V_λ. Moreover, its joint eigenvalues (and hence eigenvectors, up to a scalar) are in one-to-one correspondence with monodromy-free opers from $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_{\pi(-\mu)}$.*

In this paper we do not address the question of constructing the eigenvectors of $\mathcal{A}_\mu$. Conjecturally, for generic μ, they may be constructed by the Bethe Ansatz method (see [FFT, Section 6]), but we do not attempt to prove this conjecture here, nor do we use Bethe Ansatz in the proof of Theorem A. Instead, we rely on the isomorphism between the algebra $\mathcal{A}_\mu$ and the algebra of functions on opers and the study of opers with irregular singularity and trivial monodromy.

The crucial step in our proof is the analysis of the action of $\mathcal{A}_\mu$ in the case when $\mu = f$, a regular *nilpotent* element of $\mathfrak{g}^* \simeq \mathfrak{g}$. The algebra $\mathcal{A}_f$ contains the centralizer $\mathfrak{a}_f$ of f in $\mathfrak{g}$. Elements of $\mathfrak{a}_f$ and more general elements of $\mathcal{A}_f$ are not diagonalizable operators on V_λ, but nilpotent operators. Hence it is natural to view them as "creation operators" and ask whether they generate V_λ from its highest weight vector. Theorem 1 in the main body of the paper implies that the answer to this question is affirmative (see Corollary 2 for more details).

Theorem B. *The $\mathfrak{g}$-module V_λ is cyclic as an $\mathcal{A}_f$-module. The annihilator of V_λ in $\mathcal{A}_f$ is generated by the no-monodromy conditions on opers from $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)^\lambda_0$.*

This provides a natural structure of a Gorenstein ring on any finite-dimensional irreducible $\mathfrak{g}$-module V_λ. We hope that this fact has a geometric interpretation. Namely, due to the results of [Gi, MV], the space V_λ may be naturally identified with the global cohomology of the irreducible perverse sheaf on the Schubert variety $\overline{\mathrm{Gr}}_\lambda$ in the affine Grassmannian of the Langlands dual group ${}^LG$. On the other hand, the universal enveloping algebra $U(\mathfrak{a}_f)$ of the centralizer $\mathfrak{a}_f$ of the principal nilpotent element f is identified with the cohomology ring of the affine Grassmannian. Hence it acts on the cohomology of our perverse sheaf. This action is precisely the action of $U(\mathfrak{a}_f)$ on V_λ [Gi]. But $U(\mathfrak{a}_f)$ is a subalgebra of the commutative algebra $\mathcal{A}_f$. This leads us to a natural question: what is the geometric meaning of $\mathcal{A}_f$? Perhaps, if one could answer this question, one could derive the cyclicity of V_λ as a $\mathcal{A}_f$-module by geometric means.

The shift of argument subalgebra $\mathcal{A}_\mu$ has a multi-point generalization, denoted by $\mathcal{A}_\mu(z_1,\dots,z_N)$, where $z_1,\dots,z_N$ are distinct points on $\mathbb{P}^1\backslash\infty$. This is a commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$, which consists of the Hamiltonians of the *Gaudin model* with irregular singularity [R1, FFT]. We show that for any regular $\mu\in\mathfrak{g}^*$ this algebra is isomorphic to the algebra of functions on the space of ${}^LG$-opers on $\mathbb{P}^1$ with regular singularities at $z_1,\dots,z_N$ and irregular singularity of order 2 at ∞, with the 2-residue determined by μ (see Proposition 3). We also prove a conjecture of [FFT] that for any regular $\mu\in\mathfrak{g}^*$ the set of joint eigenvalues of this algebra on the tensor product $V_{\lambda_1}\otimes\dots\otimes V_{\lambda_N}$ of irreducible finite-dimensional $\mathfrak{g}$-modules is in bijection with the set of opers of this kind with fixed residues determined by the highest weights $\lambda_1,\dots,\lambda_N$ and the no-monodromy condition (see Corollary 5). For generic $z_1,\dots,z_N$ and generic regular semisimple μ, this gives a multi-point generalization of Theorem A (see Corollary 6). We also prove a multi-point analogue of Theorem B (see Corollary 5).

We note that for the ordinary Gaudin model (with regular singularity) a description of the spectrum of the Hamiltonians in terms of an appropriate set of monodromy-free opers analogous to our Corollary 5 was conjectured in [Fr3, Conjecture 1] (it was proved in [Fr3, Theorem 2.7,(3)] that the spectrum does embed into this set of monodromy-free opers). In the course of writing this paper we learned that a variant of this conjecture was proved in [MTV1] in the case when $\mathfrak{g}=\mathfrak{gl}_M$ by a detailed analysis of intersections of Schubert varieties in the Grassmannian.[1] This result should be related to our Corollary 3 in the case of $\mathfrak{g}=\mathfrak{gl}_N$ via the duality of [TL].

Finally, we expect that the results of this paper may be generalized to the affine Kac–Moody algebras. As explained in [FF2], the affine analogue of the shift of argument subalgebra corresponding to regular semisimple μ is the algebra of quantum integrals of motion of the AKNS hierarchy of soliton equations. Such an algebra may also be defined for a regular nilpotent μ. In this case, its action is not diagonalizable, but it gives rise to a commutative algebra of creation operators. We expect that these operators generate highest weight modules over affine Kac–Moody algebras; for example, the irreducible integrable representations. In the latter case, we expect that the generators of the corresponding annihilating ideal

[1] In the subsequent paper [MTV2] a statement closely related to our Corollary 5 has been proved for $\mathfrak{g}=\mathfrak{gl}_M$.

are given by the no-monodromy conditions on the corresponding affine opers, by analogy with Theorem B in the finite-dimensional case. We plan to discuss this in more detail in our next paper.

The present paper is organized as follows. In section 2 we collect basic facts on Gaudin models and opers. In section 3 we formulate and discuss the main result of the paper. The detailed proof is given in the last section 4.

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2. Preliminaries

2.1. **Gaudin algebras.** Gaudin model was introduced in [Ga1] as a spin model related to the Lie algebra $\mathfrak{sl}_2$, and generalized to the case of an arbitrary semisimple Lie algebra in [Ga2], Section 13.2.2. For any $x \in \mathfrak{g}$, set

$$x^{(i)} = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \in U(\mathfrak{g})^{\otimes N}$$

(x at the ith place). Let $\{x_a\}$, $a = 1, \ldots, \dim \mathfrak{g}$, be an orthonormal basis of $\mathfrak{g}$ with respect to Killing form, and let $z_1, \ldots, z_N$ be pairwise distinct complex numbers. The Hamiltonians of Gaudin model are the following mutually commuting elements of $U(\mathfrak{g})^{\otimes N}$:

$$H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}. \tag{1}$$

In [FFR], a large commutative subalgebra $\mathcal{A}(z_1, \ldots, z_N)$ containing the H_i's was constructed with the help of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$, which is the universal central extension of $\mathfrak{g}((t))$. Namely, according to [FF1, Fr2], the completed enveloping algebra $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ of $\widehat{\mathfrak{g}}$ at the critical level (in the notation of [FFT]) contains a large center $Z(\widehat{\mathfrak{g}})$. Set $\widehat{\mathfrak{g}}_+ = \mathfrak{g}[[t]] \subset \widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}_- = t^{-1}\mathfrak{g}[t^{-1}] \subset \widehat{\mathfrak{g}}$. The natural homomorphism $Z(\widehat{\mathfrak{g}}) \to (\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})/\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}}) \cdot \widehat{\mathfrak{g}}_+)^{\widehat{\mathfrak{g}}_+}$ is surjective [FF1, Fr2]. Every element of the latter quotient has a unique representative in $U(\widehat{\mathfrak{g}}_-)$. Thus we obtain the following natural embedding

$$(\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})/\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}}) \cdot \widehat{\mathfrak{g}}_+)^{\widehat{\mathfrak{g}}_+} \hookrightarrow U(\widehat{\mathfrak{g}}_-).$$

Let $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset U(\widehat{\mathfrak{g}}_-)$ be the image of this embedding. The commutative subalgebra $\mathcal{A}(z_1, \ldots, z_N) \subset U(\mathfrak{g})^{\otimes N}$ is then the image of $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset U(\widehat{\mathfrak{g}}_-)$ under the homomorphism

$$U(\widehat{\mathfrak{g}}_-) \to U(\mathfrak{g})^{\otimes N}, \ x_a t^m \mapsto \sum_{i=1}^{n} z_i^m x_a^{(i)} \tag{2}$$

of evaluation at the points $z_1, \ldots, z_N$ (see [FFR]).

This construction is generalized as follows (see [FFT, R1]. For different approach in the case $\mathfrak{g} = \mathfrak{gl}_N$ see [ChT]). One constructs a family of homomorphisms depending on a collection of pairwise distinct nonzero complex numbers $z_1, \ldots, z_N$

$$U(\widehat{\mathfrak{g}}_-) \to U(\mathfrak{g})^{\otimes N} \otimes S(\mathfrak{g}), \ x_a t^m \mapsto \sum_{i=1}^{n} z_i^m x_a^{(i)} \otimes 1 + \delta_{-1,m} \otimes x_a. \tag{3}$$

Composing this with the evaluation $S(\mathfrak{g}) \to \mathbb{C}$ at any point $\mu \in \mathfrak{g}^* = \operatorname{Spec} S(\mathfrak{g})$, we obtain a family of homomorphisms $U(\widehat{\mathfrak{g}}_-) \to U(\mathfrak{g})^{\otimes N}$ depending on $z_1, \dots, z_N$ and $\mu \in \mathfrak{g}^*$:

$$x_a t^m \mapsto \sum_{i=1}^{n} z_i^m x_a^{(i)} + \delta_{-1,m}\mu(x_a).$$

Now, for any collection $z_1, \dots, z_N$ and $\mu \in \mathfrak{g}^*$, the image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under the corresponding homomorphism from this family is a certain commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$ depending on $z_1, \dots, z_N$ and $\mu \in \mathfrak{g}^*$. We denote it by $\mathcal{A}_\mu(z_1, \dots, z_N)$. In particular, $\mathcal{A}(z_1, \dots, z_N) = \mathcal{A}_0(z_1, \dots, z_N)$ corresponding to $\mu = 0$.

These subalgebras contain the following "inhomogeneous" Gaudin Hamiltonians:

$$H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k} + \sum_{a=1}^{\dim \mathfrak{g}} \mu(x_a) x_a^{(i)}.$$

It is shown in [R1, Proposition 4] that $\mathcal{A}_\mu(z_1, \dots, z_N) = \mathcal{A}_\mu(z_1 + c, \dots, z_N + c)$ for any $c \in \mathbb{C}$. Hence one can assume $z_1, \dots, z_N$ to be an arbitrary collection of pairwise distinct complex numbers, not necessarily nonzero. In particular, for $N = 1$, we obtain a family of commutative subalgebras $\mathcal{A}_\mu(z_1) \subset U(\mathfrak{g})$ which does not depend on z_1. We will set $z_1 = 0$ in this case and denote this algebra simply by $\mathcal{A}_\mu$. It is proved in [FFT, R1] that the associated graded algebra of $\mathcal{A}_\mu$ (with respect to the PBW filtration) for regular μ is the (classical) *shift of argument subalgebra* $\overline{\mathcal{A}}_\mu \subset S(\mathfrak{g})$.

This Poisson commutative subalgebra was first constructed by Mishchenko and Fomenko in [MF], in the following way. Let

$$S(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[P_1, \dots, P_\ell], \tag{4}$$

where $\ell = \operatorname{rank}(\mathfrak{g})$ be the center of $S(\mathfrak{g})$ with respect to the Poisson bracket, and the P_i are chosen so that they are homogeneous with respect to the natural grading on $S(\mathfrak{g})$. Let $\mu \in \mathfrak{g}^*$ be a regular element. Then the subalgebra $\overline{\mathcal{A}}_\mu \subset S(\mathfrak{g})$ is generated by the elements $\partial_\mu^n P_k$, where $k = 1, \dots, \ell$, $n = 0, \dots, \deg P_k - 1$, (or, equivalently, generated by central elements of $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ shifted by $t\mu$ for all $t \in \mathbb{C}$). These elements are algebraically independent (for a uniform proof, see [FFT, Theorem 3.11]). Hence the subalgebra $\overline{\mathcal{A}}_\mu \subset S(\mathfrak{g})$ is a free polynomial algebra in $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g})$ generators (and therefore has maximal possible transcendence degree). Since $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$, we obtain the following

Lemma 1. *For regular μ the algebra $\mathcal{A}_\mu$ is a free polynomial algebra in $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g})$ generators.*

From now on we will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using the Killing form and regard μ as an element of $\mathfrak{g}$.

Denote by $\mathfrak{g}^{\mathbb{R}}$ the compact real form of $\mathfrak{g}$, and let $x \mapsto \bar{x}$ be the corresponding $\mathbb{R}$-linear involution on $\mathfrak{g}$. Any irreducible finite-dimensional $\mathfrak{g}$-module has a $\mathfrak{g}^{\mathbb{R}}$-invariant Hermitian form. We shall use the following

Lemma 2. *For $z_1, \dots, z_N \in \mathbb{R}$ and $\mu \in i\mathfrak{g}^{\mathbb{R}}$, there is a $\mathbb{C}$-basis in the algebra $\mathcal{A}_\mu(z_1, \dots, z_N) \subset U(\mathfrak{g})^{\otimes N}$ acting by Hermitian operators on any irreducible finite-dimensional $\mathfrak{g}$-module.*

Proof. [2] Note that the center at the critical level and the homomorphism (3) are defined over $\mathbb{R}$. For $x \in \mathfrak{g}$, let $x[-k] = x \otimes t^{-k} \in \widehat{\mathfrak{g}}_-$. According to [R2], the subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset U(\widehat{\mathfrak{g}}_-)$ is the centralizer of the following quadratic element

$$H[-1] := \sum_{a=1}^{\dim \mathfrak{g}} x_a[-1]x_a[-1].$$

Consider the anti-involution ω on the enveloping algebra $U(\widehat{\mathfrak{g}}_-)$, defined on the generators as $x[-k] \mapsto -x[-k]$. Note that $\omega(H[-1]) = H[-1]$ and $H[-1] \in U(\widehat{\mathfrak{g}}_-^{\mathbb{R}})$. Therefore, the subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset U(\widehat{\mathfrak{g}}_-)$ is stable under two commuting involutions: ω and the involution $x[-k] \mapsto \bar{x}[-k]$. Hence, we can choose a basis $\{b_m\}$ of $\mathfrak{z}(\widehat{\mathfrak{g}})$ which consists of elements for which $\overline{\omega(b_m)} = b_m$. The evaluation homomorphism at $z_1, \ldots, z_N$ and μ sends an element $x[-k] \in \widehat{\mathfrak{g}}_-$ to $\sum_{i=1}^{n} z_i^{-k} x^{(i)} + (\mu, x)\delta_{1,k}$. For $x[-k] \in \widehat{\mathfrak{g}}_-^{\mathbb{R}}$, $z_1, \ldots, z_N \in \mathbb{R}$ and $\mu \in i\mathfrak{g}^{\mathbb{R}}$, this acts as an anti-Hermitian operator. This means that, for each $b \in U(\widehat{\mathfrak{g}}_-)$, the operators representing b and $\overline{\omega(b)}$ are Hermitian-conjugate. Hence the image of each $b_m \in \mathfrak{z}(\widehat{\mathfrak{g}})$ in $\mathcal{A}_\mu$ acts as a Hermitian operator. □

The algebra $U(\mathfrak{g})^{\otimes N}$ has an increasing filtration by finite-dimensional subspaces,

$$U(\mathfrak{g})^{\otimes N} = \bigcup_{k=0}^{\infty} U(\mathfrak{g})^{\otimes N}_{(k)},$$

where $U(\mathfrak{g})^{\otimes N}_{(k)}$ is spanned by monomials of total order less than or equal to k. We define the limit $\lim\limits_{s \longrightarrow \infty} B(s)$ for any one-parameter family of subalgebras $B(s) \subset U(\mathfrak{g})^{\otimes N}$, depending on s rationally, as

$$\bigcup_{k=0}^{\infty} \lim_{s \longrightarrow \infty} B(s) \cap U(\mathfrak{g})^{\otimes N}_{(k)}.$$

This limit here is understood as the limit of families of points in the corresponding Grassmannian (this limit exist since every rational map from $\mathbb{P}^1$ to the Grassmannian is regular). It is clear that the limit of a family of *commutative* subalgebras is a commutative subalgebra. It is also clear that passage to the limit is compatible with homomorphisms of filtered algebras (in particular, with the projection onto any factor and a finite-dimensional representation on any factor) in the following sense: for any homomorphism of filtered algebras $\varphi : U(\mathfrak{g})^{\otimes N} \to F$ one has

$$\varphi\left(\lim_{s \longrightarrow \infty} B(s)\right) \subset \lim_{s \longrightarrow \infty} \varphi(B(s)).$$

This implies the following standard result.

Proposition 1. *Let $B(s) \subset U(\mathfrak{g})^{\otimes N}$ be a family of commutative subalgebras, depending on s rationally, and V a finite-dimensional representation of $U(\mathfrak{g})^{\otimes N}$. If the algebra $\lim\limits_{s \longrightarrow \infty} B(s)$ has a cyclic vector in V, then the same holds for $B(s)$*

[2] We thank V. Toledano Laredo for pointing out that an earlier version of this proof was incomplete.

with generic $s \in \mathbb{C}$. *If the joint spectrum of* $\lim\limits_{s\longrightarrow\infty} B(s)$ *on* V *is simple, then the same holds for* $B(s)$ *with generic* $s \in \mathbb{C}$.

We need the following facts on the limit points of the family $\mathcal{A}_\mu(z_1, \dots, z_N)$.

Proposition 2. [R1]

$$\lim_{s\longrightarrow\infty} \mathcal{A}_\mu(sz_1, \dots, sz_N) = \lim_{s\longrightarrow\infty} \mathcal{A}_{s\mu}(z_1, \dots, z_N) = \mathcal{A}_\mu^{(1)} \otimes \dots \otimes \mathcal{A}_\mu^{(N)} \subset U(\mathfrak{g})^{\otimes N}$$

for regular μ.

Remark. In [R1] this Proposition is stated for regular semisimple μ, but the proof works for any regular μ without any changes.

We have $\mathcal{A}_\xi = \mathcal{A}_{t\xi}$ for any $t \in \mathbb{C}^\times, \xi \in \mathfrak{g}$. This follows from Lemma 1 of [R1], which states that the algebra $\mathcal{A}_\xi$ is generated by the elements homogeneous in ξ. We also have $\mathcal{A}_{t\,\mathrm{Ad}(g)\xi} = \mathrm{Ad}(g)\mathcal{A}_\xi \subset U(\mathfrak{g})$ (this follows from $\mathfrak{g}$-equivariance of the evaluation homomorphism $U(\widehat{\mathfrak{g}}_-) \to U(\mathfrak{g}) \otimes S(\mathfrak{g})$). Thus, the subalgebras $\mathcal{A}_\xi \subset U(\mathfrak{g})$ and $\mathcal{A}_{t\,\mathrm{Ad}(g)\xi} \subset U(\mathfrak{g})$ are conjugate for any $g \in G, t \in \mathbb{C}^*, \xi \in \mathfrak{g}$.

Let

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

be a Cartan decomposition of $\mathfrak{g}$. We denote by Π be the set of simple roots of $\mathfrak{g}$. We will choose generators $\{e_{-\alpha}\}_{\alpha\in\Pi}$ of the lower nilpotent subalgebra $\mathfrak{n}_-$. Let

$$f = \sum_{\alpha\in\Pi} e_{-\alpha} \in \mathfrak{g}$$

be a principal nilpotent element and $\{e,\ h,\ f\}$ be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, where $h \in \mathfrak{h}$. Let $\mathfrak{z}_\mathfrak{g}(e)$ be the centralizer of e in $\mathfrak{g}$. Consider Kostant's slice

$$\mathfrak{g}_{\mathrm{can}} := f + \mathfrak{z}_\mathfrak{g}(e).$$

This is an affine subspace in $\mathfrak{g}$, which consists of regular elements. On the other hand, for any regular $\mu \in \mathfrak{g}$, the $\mathrm{Ad}\,G$-orbit of μ intersects $\mathfrak{g}_{\mathrm{can}}$ at one point (this is a classical result due to Kostant [K]).

Lemma 3. *For any regular* μ *the closure of the family* $\mathcal{A}_{t\,\mathrm{Ad}(g)\mu} \subset U(\mathfrak{g})$ *(*$g \in G, t \in \mathbb{C}^*, \xi \in \mathfrak{g}$*) contains the subalgebra* $\mathcal{A}_f \subset U(\mathfrak{g})$.

Proof. According to [K], there exists $x \in \mathfrak{z}_\mathfrak{g}(e)$ such that $\mathcal{A}_{f+x}$ belongs to the family $\mathcal{A}_{t\,\mathrm{Ad}(g)\mu}$. Since $x \in \mathfrak{n}_+$, we have

$$\lim_{t\longrightarrow-\infty} \exp(2t)\,\mathrm{Ad}\exp(th)(f + x) = f,$$

and therefore $\mathcal{A}_f$ is a limit point of the family $\mathcal{A}_{t\,\mathrm{Ad}(g)\mu}$. □

The "most degenerate" subalgebra $\mathcal{A}_\mu \subset U(\mathfrak{g})$ among those corresponding to regular μ is the subalgebra $\mathcal{A}_f$. It is a free commutative algebra with generators $\Pi_i^{(n)}$ such that $\mathrm{gr}\,\Pi_i^{(n)} = \partial_f^n P_i \in S(\mathfrak{g})$, where $i = 1, \dots, l,\ n = 0, 1, \dots, d_i = \deg P_i - 1$, P_i are the generators of $S(\mathfrak{g})^\mathfrak{g}$ (see formula (4)).

The element $\frac{1}{2}h$ defines the *principal gradation* $\deg_{\mathrm{pr}}$ on $U(\mathfrak{g})$ such that

$$\deg_{\mathrm{pr}} e_\alpha = -\deg_{\mathrm{pr}} e_{-\alpha} = 1, \quad \alpha \in \Pi, \qquad \deg_{\mathrm{pr}} h_\alpha = 0, \quad h_\alpha \in \mathfrak{h}.$$

The generators of $\mathcal{A}_f$ are homogeneous with respect to this gradation, with $\deg_{\mathrm{pr}} \Pi_i^{(n)} = -n$. Thus, the algebra $\mathcal{A}_f$ is graded by the principal gradation:

$\mathcal{A}_f = \bigoplus_{n\geq 0} \mathcal{A}_f^{(-n)}$. Note that the Poincaré series of $\mathcal{A}_f$ with respect to the principal gradation is equal to that of the algebra $U(\mathfrak{n}_-)$. Thus, it is natural to expect that irreducible highest weight $\mathfrak{g}$-modules are cyclic as $\mathcal{A}_f$-modules (having the highest weight vector as a cyclic vector).

2.2. **Opers on the projective line.** Now let us describe the spectra of Gaudin algebras following [FFT].

Consider the Langlands dual Lie algebra ${}^L\mathfrak{g}$ whose Cartan matrix is the transpose of the Cartan matrix of $\mathfrak{g}$. By ${}^LG$ we denote the group of inner automorphisms of ${}^L\mathfrak{g}$. We fix a Cartan decomposition

$$ {}^L\mathfrak{g} = {}^L\mathfrak{n}_+ \oplus {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_-.$$

The Cartan subalgebra ${}^L\mathfrak{h}$ is naturally identified with $\mathfrak{h}^*$. We denote by ${}^L\Delta$, ${}^L\Delta_+$, and ${}^L\Pi$ the root system of ${}^L\mathfrak{g}$, the set of positive roots, and the set of simple roots, respectively. We denote by ${}^L\mathfrak{b}_+$ the Borel subalgebra ${}^L\mathfrak{n}_+ \oplus {}^L\mathfrak{h}$.

Set

$$p_{-1} = \sum_{\alpha^\vee \in \Pi^\vee} e_{-\alpha^\vee} \in {}^L\mathfrak{g}.$$

Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \in \mathfrak{h}^* = {}^L\mathfrak{h}.$$

The operator $\operatorname{Ad}\rho$ defines the principal gradation on ${}^L\mathfrak{g}$, with respect to which we have a direct sum decomposition ${}^L\mathfrak{b}_+ = \bigoplus_{i\geq 0}^L \mathfrak{b}_i$. Let p_1 be the unique element of degree 1 in ${}^L\mathfrak{n}_+$ such that $\{p_{-1}, 2\rho, p_1\}$ is an $\mathfrak{sl}_2$-triple (note that p_{-1} has degree -1). Let

$$V_{\mathrm{can}} = \bigoplus_{i=1}^{\ell} V_{\mathrm{can},i}$$

be the space of $\operatorname{Ad} p_1$-invariants in ${}^L\mathfrak{n}_+$, decomposed according to the principal gradation. Here $V_{\mathrm{can},i}$ has degree d_i, the ith exponent of ${}^L\mathfrak{g}$ (and of $\mathfrak{g}$). In particular, $V_{\mathrm{can},1}$ is spanned by p_1. Now choose a linear generator p_j of $V_{\mathrm{can},j}$.

Consider the Kostant slice in ${}^L\mathfrak{g}$,

$${}^L\mathfrak{g}_{\mathrm{can}} = \left\{ p_{-1} + \sum_{j=1}^{\ell} y_j p_j, \quad y \in \mathbb{C} \right\}.$$

By [K], the adjoint orbit of any regular element in the Lie algebra ${}^L\mathfrak{g}$ contains a unique element which belongs to ${}^L\mathfrak{g}_{\mathrm{can}}$. Thus, we have canonical isomorphisms

$$ {}^L\mathfrak{g}_{\mathrm{can}} \tilde{\to} {}^L\mathfrak{g}/{}^LG = {}^L\mathfrak{h}/{}^LW = \mathfrak{h}^*/W = \mathfrak{g}^*/G. \tag{5}$$

The notion of opers on an arbitrary curve was introduced by Beilinson and Drinfeld in [BD]. We refer the reader to [FFT] for details. Here we will only say that for an affine curve $U = \operatorname{Spec} R$ and an étale coordinate t on U, the space $\operatorname{Op}_{{}^LG}(U)$ of ${}^LG$-opers on U is isomorphic to the quotient of the space of connections of the form

$$d + (p_{-1} + \mathbf{v}(t))dt, \qquad \mathbf{v}(t) \in {}^L\mathfrak{b}_+ \otimes R$$

by the free action of the group ${}^L N_+ \otimes R$ of regular algebraic maps $U \to {}^L N_+$, where ${}^L N_+$ is the maximal unipotent subgroup of ${}^L G$ such that $\mathrm{Lie}\, {}^L N_+ = {}^L\mathfrak{n}_+$. Each oper has a unique representative of the form

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} u_j(t) p_j \right) dt, \qquad u_j(t) \in R.$$

In particular, the space $\mathrm{Op}_{{}^L G}(D^\times)$ of ${}^L G$-opers on the formal punctured disc $D^\times = \mathrm{Spec}\, \mathbb{C}((t))$ is isomorphic to the space of connections of the form

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} u_j(t) p_j \right) dt, \qquad u_j(t) \in \mathbb{C}((t)).$$

We will consider opers on $\mathbb{P}^1$ with singularities at a finite number of points. The oper has singularity of order k at $z \in \mathbb{P}^1$ if has the following form in local coordinate t at z:

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} (u_j (t-z)^{-k(d_j+1)} + O((t-z)^{-n-1})) p_j \right) dt.$$

The k-residue of such oper at the point z, for $k > 1$, is $p_{-1} + \sum_{j=1}^{\ell} u_j p_j$ (see [FFT, Section 4.3] for details).

Following [FFT], we denote by $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)}$ the space of ${}^L G$-opers on $\mathbb{P}^1 \backslash \{z_1, \dots z_N, \infty\}$ with regular singularities at the points $z_i, i = 1, \dots, N$, and with irregular singularity of order 2 at the point ∞ with the 2-residue $\pi(-\mu) \in {}^L\mathfrak{g}/{}^L G = \mathfrak{g}^*/G$, where $\pi : \mathfrak{g}^* \to \mathfrak{g}^*/G$ is the projection. Each oper from this space may be uniquely represented in the following form:

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} \overline{\mu}_j p_j + \sum_{i=1}^{N} \sum_{j=1}^{\ell} \sum_{n=0}^{d_j} u_{j,n}^{(i)} (t - z_i)^{-n-1} p_j \right) dt,$$

where

$$p_{-1} + \sum_{j=1}^{\ell} \overline{\mu}_j p_j$$

is the (unique) element of ${}^L\mathfrak{g}_{\mathrm{can}}$ contained in the ${}^L G$-orbit corresponding to the G-orbit of $\mu \in \mathfrak{g}^*$ under the isomorphism (5). Thus, $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)}$ is an affine space of dimension $\frac{1}{2}(\dim \mathfrak{g} + \mathrm{rk}\, \mathfrak{g})N$.

We will be mainly interested in the special case $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)_{(0);\pi(-\mu)}$ of opers having regular singularity at 0 and irregular singularity of order 2 at the point ∞ with the 2-residue $\pi(-\mu)$. By a slight abuse of notation we will denote this space simply by $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)_{\pi(-\mu)}$.

Rewriting an oper on $\mathbb{P}^1$ in local coordinate at any point z, we obtain an inclusion $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)} \subset \mathrm{Op}_{{}^L G}(D_z^\times)$, where $D_z^\times$ is the punctured disc at the point $z \in \mathbb{P}^1$. In particular, on the punctured disc $D_\infty^\times$ at ∞ (with the

coordinate $s = t^{-1}$) each element of $\mathrm{Op}_{L_G}(\mathbb{P}^1)_{\pi(-\mu)}$ may be uniquely represented by a connection of the form (see [FFT, Section 5.4]):

$$d - \left(p_{-1} - \sum_{j=1}^{\ell}(s^{-2d_j}\overline{\mu}_j + s^{-2d_j-1}u_j(s))p_j\right)ds, \qquad u_j(s) = \sum_{n=0}^{d_j} u_{j,n}s^n.$$

On the punctured disc $D_0^\times$ at 0, each element of $\mathrm{Op}_{L_G}(\mathbb{P}^1)_{\pi(-\mu)}$ may be represented uniquely by a connection of the form

$$d + \left(p_{-1} + \sum_{j=1}^{\ell}(\overline{\mu}_j + \sum_{n=0}^{d_j} u_{j,n}t^{-n-1})p_j\right)dt.$$

The 1-residue at 0 of this oper is equal to

$$p_{-1} + \sum_{j=1}^{\ell}(u_{j,d_j} + \frac{1}{4}\delta_{j,1})p_j \in {}^L\mathfrak{g}_{\mathrm{can}} \simeq {}^L\mathfrak{g}/{}^LG = \mathfrak{g}^*/G.$$

2.3. **Gaudin algebra and opers.** In [FF1] the center $Z(\widehat{\mathfrak{g}})$ of the completed enveloping algebra $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ at the critical level is identified with the algebra $\mathrm{Fun}(\mathrm{Op}_{L_G}(D^\times))$ of polynomial functions on the space $\mathrm{Op}_{L_G}(D^\times)$ of ${}^LG$-opers on the formal punctured disc $D^\times = \mathrm{Spec}\,\mathbb{C}((t))$.

By [FFT, Theorem 5.7,(4)], the algebra $\mathcal{A}_\mu(z_1,\dots,z_N)$, being a quotient of $Z(\widehat{\mathfrak{g}}) = \mathrm{Fun}(\mathrm{Op}_{L_G}(D^\times))$, is actually a quotient of the algebra of polynomial functions on the space $\mathrm{Op}_{L_G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)} \subset \mathrm{Op}_{L_G}(D_0^\times)$ (here $D_0^\times$ is the punctured disc at the point $0 \in \mathbb{P}^1$). The algebra $\mathrm{Fun}(\mathrm{Op}_{L_G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)})$ is a polynomial algebra in $\frac{1}{2}(\dim\mathfrak{g} + \mathrm{rk}\,\mathfrak{g})N$ generators.

On the other hand, Proposition 2 and Lemma 1 imply that for regular μ the algebra $\mathcal{A}_\mu(z_1,\dots,z_N)$ is a free polynomial algebra in the same number of generators (see also [R1, Theorem 2 and Corollary 4]). Indeed, for the same reason as in Lemma 3, it suffices to prove this for the principal nilpotent $\mu = f$. Note that for $s \in \mathbb{C}$ we have $\mathcal{A}_{sf}(z_1,\dots,z_N) = \mathcal{A}_{\mathrm{Ad}(\exp(-\frac{s}{2}h))f}(z_1,\dots,z_N) = \mathrm{Ad}(\exp(-\frac{s}{2}h)(\mathcal{A}_f(z_1,\dots,z_N))$, due to the G-equivariance of the evaluation homomorphism, and $\lim_{s\longrightarrow\infty}\mathcal{A}_{sf}(z_1,\dots,z_N) = \mathcal{A}_f^{(1)}\otimes\dots\otimes\mathcal{A}_f^{(N)} \subset U(\mathfrak{g})^{\otimes N}$, by Proposition 2. Thus we have a one-parameter family of subalgebras in $U(\mathfrak{g})^{\otimes N}$, which are all conjugate to $\mathcal{A}_f(z_1,\dots,z_N)$, having $\mathcal{A}_f^{(1)}\otimes\dots\otimes\mathcal{A}_f^{(N)}$ as a limit point. Hence the transcendence degree $\mathrm{tr\,deg}\,\mathcal{A}_f(z_1,\dots,z_N) = \mathrm{tr\,deg}\,\mathcal{A}_f^{\otimes N} = \frac{1}{2}(\dim\mathfrak{g} + \mathrm{rk}\,\mathfrak{g})N$.

Thus, we obtain the following assertion which was conjectured in [FFT, Conjecture 3].

Proposition 3. *For regular μ there is an isomorphism*

$$\mathcal{A}_\mu(z_1,\dots,z_N) \simeq \mathbb{C}[\mathrm{Op}_{L_G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)}].$$

In particular, for regular μ the algebra $\mathcal{A}_\mu$ is identified with the algebra of polynomial functions on the space $\mathrm{Op}_{L_G}(\mathbb{P}^1)_{\pi(-\mu)}$ of ${}^LG$-opers on $\mathbb{P}^1$ with regular singularity at the point 0 and with singularity of order 2 at ∞, with 2-residue $\pi(-\mu)$ (we recall that $\pi(-\mu)$ denotes the image of $-\mu \in \mathfrak{g}^*$ in $\mathfrak{g}^*/G = {}^L\mathfrak{g}/{}^LG$).

In particular, the space of opers corresponding to the algebra $\operatorname{Spec} \mathcal{A}_f$ looks as follows (in this case $\pi(-\mu) = 0$):

$$\operatorname{Op}_{{}^L G}(\mathbb{P}^1)_0 = \left\{ d - \left(p_{-1} - \sum_{j=1}^{\ell} s^{-2d_j-1} u_j(s) p_j \right) ds, \;\middle|\; u_j(s) = \sum_{n=0}^{d_j} u_{j,n} s^n \right\}$$
$$= \left\{ d + \left(p_{-1} + \sum_{j=1}^{\ell} \sum_{n=0}^{d_j} u_{j,n} t^{-n-1} p_j \right) dt \right\}$$

(here, as before, we omit z_1, which is set to 0).

Next, it is proved in [FFT, Theorem, 5.7], that for any collection of $\mathfrak{g}$-modules M_i with highest weights λ_i, $i = 1, \ldots, N$, the natural homomorphism

$$\mathcal{A}_\mu(z_1, \ldots, z_N) \to \operatorname{End}(M_1 \otimes \cdots \otimes M_N)$$

factors through the algebra of functions on the subspace $\operatorname{Op}_{{}^L G}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\mu)} \subset \operatorname{Op}_{{}^L G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)}$ which consists of the opers with the 1-residue $\pi(-\lambda_i - \rho)$ at z_i. Moreover, for integral dominant λ_i the action of $\mathcal{A}_\mu(z_1, \ldots, z_N)$ on the tensor product of the finite-dimensional modules V_{λ_i} factors through the algebra of functions on *monodromy-free* opers from $\operatorname{Op}_{{}^L G}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\mu)}$.

For every integral dominant weight λ, the set of monodromy-free opers on the punctured disc $D_z^\times$ at z with the regular singularity with the residue $-\lambda - \rho$ at z is defined by finitely many polynomial relations. Namely, each element of the space $\operatorname{Op}(D_z)^\lambda$ of opers on $D_z^\times$ with regular singularity and residue $\pi(-\lambda - \rho)$ may be uniquely represented as

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} \sum_{n=-\infty}^{d_j} u_{j,n} t^{-n-1} p_j \right) dt,$$

with

$$p_{-1} + \sum_{j=1}^{\ell} u_{j,d_j} \in \operatorname{Ad}({}^L G)(-\lambda - \rho).$$

It is shown in [FG, Section 2.9] that by using gauge action of $B_+(\!(t)\!)$ one can bring this connection to the form

$$d + \left(\sum_{\alpha^\vee \in \Pi^\vee} t^{\langle \alpha^\vee, \lambda \rangle} e_{-\alpha^\vee} + \mathbf{v}(t) \right) dt,$$

where $\mathbf{v}(t) \in t^{-1} {}^L\mathfrak{n}_+ \oplus {}^L\mathfrak{b}_+[[t]]$. Furthermore, this oper is monodromy-free if and only if $\mathbf{v}(t) \in {}^L\mathfrak{b}_+[[t]]$ [FG, Section 2.9]. Thus, the set of monodromy-free opers is defined by $\dim {}^L\mathfrak{n}_+$ polynomial relations $P_\alpha(u_{j,n})$ enumerated by positive roots $\alpha \in \Delta_+$. These polynomial relations have the degrees $(\alpha^\vee, \lambda + \rho)$ with respect to the $\mathbb{Z}$-grading defined by the formula $\deg u_{j,n} = -n + j$ (see [FG, Section 2.9], for details).

Introduce a $\mathbb{Z}$-grading on the algebra

$$\mathcal{A}_\mu(z_1, \ldots, z_N)^{(\lambda_i)} = \mathbb{C}\left[\operatorname{Op}_{{}^L G}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\mu)} \right] = \mathbb{C}[u^{(i)}_{j,n}]_{j=1,\ldots,\ell;\ n=0,\ldots,d_j;\ i=1,\ldots,N}$$

by the formula $\deg u_{j,n}^{(i)} = -n + j$. By comparing the degrees of the generators of the polynomial algebras $\mathcal{A}_f^\lambda$ and $\mathbb{C}[\mathrm{Op}_{^L G}(\mathbb{P}^1)_0^\lambda]$, we obtain the following (note that we abbreviate the notation $\mathrm{Op}_{^L G}(\mathbb{P}^1)_{(0);\pi(-\mu)}^\lambda$ to $\mathrm{Op}_{^L G}(\mathbb{P}^1)_0^\lambda$ when $\mu = f$):

Lemma 4. *For $\mathcal{A}_f^\lambda = \mathbb{C}[\mathrm{Op}_{^L G}(\mathbb{P}^1)_0^\lambda]$ this grading coincides with the principal grading on $\mathcal{A}_f^\lambda$.*

For any collection of integral dominant weights $\lambda_1, \ldots, \lambda_N$ attached to the points $z_1, \ldots, z_N$, we denote by $P_{z_k;\alpha}^{(z_1,\ldots,z_N);\mu;(\lambda_i)}$ the polynomial in $u_{j,n}^{(i)}$ expressing the "no-monodromy" condition at z_k corresponding to the root $\alpha \in \Delta_+$.

Lemma 5. *$P_{z_k;\alpha}^{(z_1,\ldots,z_N);\mu;(\lambda_i)}$ is an inhomogeneous polynomial of highest degree $\deg P_{z_k;\alpha}^{(z_1,\ldots,z_N);\mu;(\lambda_i)} = (\alpha^\vee, \lambda + \rho)$. The leading term (the sum of monomials of highest degree) of $P_{z_k;\alpha}^{(z_1,\ldots,z_N);\mu;(\lambda_i)}$ is equal to $P_{z_k;\alpha}^{(z_k);f;(\lambda_k)}$.*

Proof. Written in terms of a local coordinate t at z_k, an oper from the set $\mathrm{Op}_{^L G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)}^{(\lambda_i)}$ has the form

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} \sum_{n=-\infty}^{d_j} v_{j,n} t^{-n-1} p_j \right) dt, \quad v_{j,n} = u_{j,n}^{(k)} + \text{terms of lower degree.}$$

More precisely, $v_{j,n} = u_{j,n}^{(k)}$ for $n \geq 0$, and for $n < 0$, the polynomial $v_{j,n}$ is a linear combination of scalars and $u_{j,m}^{(i)}$, $m \geq 0$. Therefore, for $n < 0$, we have $\deg v_{j,n} < -n + j$. Hence the assertion. □

According to Lemma 5, in order to show that the no-monodromy conditions define a finite set of opers, it suffices to prove this for the space $\mathrm{Op}_{^L G}(\mathbb{P}^1)_0^\lambda$. This will be done in the next section.

3. Main results

3.1. **Formulation of the Main Theorem.** In section 4, we shall prove the following result.

Theorem 1. *The set of monodromy-free opers from $\mathrm{Op}_{^L G}(\mathbb{P}^1)_0^\lambda$ is 0-dimensional (equivalently, the trivial monodromy conditions $P_{0;\alpha}^{(0);f;(\lambda)}(u_{j,n})$ form a regular sequence).*

3.2. **Corollaries.** First, let us discuss some corollaries of this Theorem.

Let $\lambda_1, \ldots, \lambda_N$ be, as before, a collection of dominant integral weights of ${}^L G$ attached to the points $z_1, \ldots, z_N$.

Corollary 1. *The set of monodromy-free opers from $\mathrm{Op}_{^L G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)}^{(\lambda_i)}$ is 0-dimensional.*

Proof. This follows directly from Lemma 5. Indeed, the set of monodromy-free opers from $\mathrm{Op}_{^L G}(\mathbb{P}^1)_{(z_i);\pi(-\mu)}^{(\lambda_i)}$ is the set of common zeros of the polynomials $P_{z_k;\alpha}^{(z_1,\ldots,z_N);\mu;(\lambda_i)}$. By Lemma 5, the leading terms of these polynomials

are $P_{z_k;\alpha}^{(z_k);f;(\lambda_k)}$. By Theorem 1, the set of common zeros of $P_{z_k;\alpha}^{(z_k);f;(\lambda_k)}$ is 0-dimensional. Hence the set of common zeros of the polynomials $P_{z_k;\alpha}^{(z_1,\dots,z_N);\mu;(\lambda_i)}$ is also 0-dimensional. □

Corollary 2. *The $\mathfrak{g}$-module V_λ is cyclic as an $\mathcal{A}_f$-module. The annihilator of V_λ in $\mathcal{A}_f$ is the ideal $I_\lambda \subset \mathcal{A}_f = \mathbb{C}[\mathrm{Op}_{L_G}(\mathbb{P}^1)_0^\lambda]$ generated by the no-monodromy conditions on opers from $\mathrm{Op}_{L_G}(\mathbb{P}^1)_0^\lambda$.*

Proof. Note first that this assertion agrees with the q-analog of the Weyl dimension formula. Namely, the Poincaré series of any irreducible finite-dimensional $\mathfrak{g}$-module V_λ with respect to the principal grading is

$$\chi_\lambda(q) = \prod_{\alpha>0} \frac{1-q^{(\alpha^\vee,\lambda+\rho)}}{1-q^{(\alpha^\vee,\rho)}}.$$

We note that the non-central generators of $\mathcal{A}_f$ have the degrees $(\alpha^\vee,\rho)$ with respect to the principal grading, and the no-monodromy relations have the degrees $(\alpha^\vee,\lambda+\rho)$. Theorem 1 implies that the algebra $\mathcal{A}_f/I_\lambda$ is Gorenstein (i.e. its socle is one-dimensional), and has the same Poincaré series with respect to the principal grading as V_λ. Therefore the module V_λ is free as an $\mathcal{A}_f/I_\lambda$-module if and only if each nonzero element of the socle of $\mathcal{A}_f/I_\lambda$ sends the highest vector to some nonzero vector (which is proportional to the lowest weight vector). Thus it remains to show that there exists an element $a \in \mathcal{A}_f$ such that $av_\lambda = v_{w_0\lambda}$, (where $w_0 \in W$ is the longest element of the Weyl group and $v_{w_0\lambda}$ is the lowest weight vector).

Let e, h, f be the principal $\mathfrak{sl}_2$-triple containing f. The module V_λ decomposes into the direct sum of irreducible $\mathfrak{sl}_2$-modules with respect to this $\mathfrak{sl}_2$-triple. Let U be the irreducible $\mathfrak{sl}_2$-submodule containing v_λ (this is an $\mathfrak{sl}_2$-submodule with the highest weight $\langle h,\lambda\rangle$). Since $v_{w_0\lambda}$ is the unique vector of the weight $-\langle h,\lambda\rangle$ with respect to the principal $\mathfrak{sl}_2$, U contains $v_{w_0\lambda}$ as well. This means that we can take $a = f^{\dim U-1} \in \mathcal{A}_f$. □

Corollary 3. *For any regular μ the subalgebra $\mathcal{A}_\mu$ has a cyclic vector in V_λ. The annihilator of V_λ in $\mathcal{A}_\mu$ is generated by the no-monodromy conditions. In particular, the joint eigenvalues of $\mathcal{A}_\mu$ in V_λ (without multiplicities) are in one-to-one correspondence with monodromy-free opers from $\mathrm{Op}_{L_G}(\mathbb{P}^1)_{\pi(-\mu)}^\lambda$.*

Proof. Consider the family of commutative subalgebras $\mathcal{A}_{t\,\mathrm{Ad}(g)\mu} \subset U(\mathfrak{g})$. By Lemma 3, the subalgebra $\mathcal{A}_f \subset U(\mathfrak{g})$ is contained in the closure of this family. Note that the condition that annihilator of V_λ in $\mathcal{A}_{t\,\mathrm{Ad}(g)\mu}$ is generated by the no-monodromy conditions, as well as the existence of a cyclic vector is an open condition on $\mathcal{A}_{t\,\mathrm{Ad}(g)\mu} \subset U(\mathfrak{g})$. By Corollary 2, the subalgebra $\mathcal{A}_f$ satisfies both of these conditions, therefore the conditions are satisfied for some $\mathcal{A}_{t\,\mathrm{Ad}(g)\mu}$ (due to Proposition 1). Since the subalgebras $\mathcal{A}_\mu$ and $\mathcal{A}_{t\,\mathrm{Ad}(g)\mu}$ are conjugate, the assertion is true for $\mathcal{A}_\mu$ as well. This implies that the image of $\mathcal{A}_\mu$ in $\mathrm{End}(V_\lambda)$ is isomorphic to $\mathbb{C}[\mathrm{Op}_{L_G}(\mathbb{P}^1)_{\pi(-\mu)}^\lambda]$. Hence the joint eigenvalues of $\mathcal{A}_\mu$ in V_λ (without multiplicities) are in one-to-one correspondence with points of $\mathrm{Op}_{L_G}(\mathbb{P}^1)_{\pi(-\mu)}^\lambda$. (Note that this statement was conjectured in [FFT, Conjecture 2].) □

Corollary 4. *For generic μ and any dominant integral λ, the quantum shift of argument subalgebra $\mathcal{A}_\mu \subset U(\mathfrak{g})$ is diagonalizable and has simple spectrum on*

the $\mathfrak{g}$-module V_λ. Moreover, its joint eigenvalues (and hence eigenvectors, up to a scalar) are in one-to-one correspondence with monodromy-free opers from $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)^\lambda_{\pi(-\mu)}$.

Proof. First of all, by Lemma 2, the algebra $\mathcal{A}_\mu$ with purely imaginary μ acts by Hermitian operators on V_λ, and hence is diagonalizable. By Corollary 3, for regular μ it has a cyclic vector. The two properties may only be realized if $\mathcal{A}_\mu$ has simple spectrum. Hence $\mathcal{A}_\mu$ has simple spectrum for regular purely imaginary μ. Since the simple spectrum condition is open, $\mathcal{A}_\mu$ has simple spectrum for generic μ. By Corollary 3, the joint eigenvalues of $\mathcal{A}_\mu$ in V_λ are in one-to-one correspondence with monodromy-free opers from $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)^\lambda_{\pi(-\mu)}$. □

Corollary 5. *For any N-tuple of pairwise distinct complex numbers $z_1, \dots, z_N \in \mathbb{C}$ and any regular μ the subalgebra $\mathcal{A}_\mu(z_1, \dots, z_N) \subset U(\mathfrak{g})^{\otimes N}$ has a cyclic vector in $V_{(\lambda_i)} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_N}$. The annihilator of $V_{(\lambda_i)}$ in $\mathcal{A}_\mu(z_1, \dots, z_N)$ is generated by the no-monodromy conditions. In particular, the joint eigenvalues of $\mathcal{A}_\mu(z_1, \dots, z_N)$ in V_λ (without multiplicities) are in one-to-one correspondence with the monodromy-free opers from* $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\mu)}$.

Proof. For the same reason as in Corollary 3, it suffices to prove this for the principal nilpotent $\mu = f$. Let e, h, f be the principal sl_2-triple containing f, and let $s \in \mathbb{C}$. Then

$$\exp(\operatorname{ad} sh)(\mathcal{A}_f(z_1, \dots, z_N)) = \mathcal{A}_{\exp(-2s)f}(z_1, \dots, z_N).$$

We have

$$\lim_{s \to -\infty} \mathcal{A}_{\exp(-2s)f}(z_1, \dots, z_N) = \mathcal{A}_f^{(1)} \otimes \dots \otimes \mathcal{A}_f^{(N)},$$

by Proposition 2). By Corollary 3, the algebra $\mathcal{A}_f^{(1)} \otimes \dots \otimes \mathcal{A}_f^{(N)}$ has a cyclic vector in $V_{(\lambda_i)} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$. Hence $\mathcal{A}_{\exp(-2s)f}(z_1, \dots, z_N)$ has a cyclic vector for some $s \in \mathbb{C}$. Since

$$\exp(\operatorname{ad} sh)(\mathcal{A}_f(z_1, \dots, z_N)) = \mathcal{A}_{\exp(-2s)f}(z_1, \dots, z_N),$$

the algebra $\mathcal{A}_f(z_1, \dots, z_N)$ has a cyclic vector as well.

The assertion on the annihilator of $V_{(\lambda_i)}$ is proved by the same reasoning as in Corollary 3 with the reference to Lemma 5. (Note that this implies Conjecture 4 of [FFT].) □

Corollary 6. *For generic $z_1, \dots, z_N \in \mathbb{C}$, μ the subalgebra $\mathcal{A}_\mu(z_1, \dots, z_N) \subset U(\mathfrak{g})^{\otimes N}$ has simple spectrum in $V_{(\lambda_i)} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_N}$. Hence the joint eigenvectors for higher Gaudin Hamiltonians in $V_{(\lambda_i)}$ are in one-to-one correspondence with monodromy-free opers from* $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)^{(\lambda_i)}_{(z_i);\pi(-\mu)}$.

Proof. The same reasoning as in the proof of Corollary 4. □

3.3. **Idea of the Proof of the Main Theorem.** The proof of Theorem 1 will be given in the next section. In the proof, we will treat the elements of $\mathrm{Op}_{{}^L G}(\mathbb{P}^1)^\lambda_0$ simply as connections on the trivial principal ${}^L G$-bundle (where ${}^L G$ is the adjoint group of the Lie algebra ${}^L\mathfrak{g}$) on the punctured neighborhood of ∞ (we consider both formal and analytic neighborhoods) with irregular singularity of order 2 (i.e.,

we will use gauge transformations of such connections which do not preserve the oper structure).

More precisely, the group ${}^L G[[t]]$ acts on connections

$$d + A(t)dt, \quad A(t) \in {}^L\mathfrak{g} \tag{6}$$

on the trivial principal ${}^L G$-bundle on the formal punctured disc $D = \operatorname{Spec} \mathbb{C}(\!(t)\!)$ by *formal gauge transformations*. Namely, the action of $H(t) \in {}^L G[[t]]$ on $d + A(t)dt$ is given by the formula

$$d + (\operatorname{Ad} H(t)(A(t)) - H'(t)H(t)^{-1})dt. \tag{7}$$

For a horizontal section $s(t)$ with respect to the connection (6) the section $H(t)s(t)$ is horizontal with respect to (7). Equivalently, for each ${}^L\mathfrak{g}$-valued solution $\varphi(t)$ of the ${}^L\mathfrak{g}$-valued differential equation

$$d\varphi(t) + [A(t), \varphi(t)]dt = 0,$$

the ${}^L\mathfrak{g}$-valued function $\psi(t) = \operatorname{Ad} H(t)(\varphi(t))$ is a solution of

$$d\psi(t) + [\operatorname{Ad} H(t)(A(t)) - H'(t)H(t)^{-1}, \psi(t)]dt = 0.$$

In the same way, the group of analytic maps from an open subset U in $\mathbb{C}$ to ${}^L G$ acts by gauge transformations on analytic connections on the trivial principal ${}^L G$-bundle on U.

The main tools we use are the following local normalization theorems for irregular singular connections.

Fact 1. (Hukuhara–Turritin–Levelt theorem, see [BV], and also [IYa, V] for the $\mathfrak{gl}_n$ case) *Any connection $d + A(t)dt$ on the punctured disc $D^\times = \operatorname{Spec} \mathbb{C}(\!(t)\!)$, where $A(t) = \sum_{k=-2}^{\infty} A_k t^k$ may be reduced by a suitable* formal *shearing gauge transformation $H(t^{\frac{1}{N}})$ to its* formal normal form

$$d + B(w)dw \quad B(w) = \sum_{k=2}^{m} B_k w^{-k} + Cw^{-1}, \tag{8}$$

where $w^N = t$, $m \leq N + 1$ and all B_k commute with C and belong to a fixed Cartan subalgebra. Moreover, if A_{-2} is nilpotent, then $m \leq N$.

Fact 2. (Sibuya sectorial normalization theorem, see [B], Appendix A. For the $\mathfrak{gl}_n$ case, see [S], [W], [IYa].) *For any sector S of opening $\frac{\pi}{m-1}$ on the w-plane, the formal gauge transformation $H(w)$ may be extended to an* analytic *gauge transformation $H_S(w)$ conjugating the connection $d + A(t)dt$ to its formal normal form.*

The main idea of the proof of Theorem 1 is as follows. The opers we are interested in are represented by connections of the form $d + A(s)ds$, where $A(s) = \sum_{k=-2}^{\infty} A_k s^k$ with *nilpotent* A_{-2} in the neighborhood of ∞. By Hukuhara–Turritin–Levelt Theorem, its formal normal form is (8) with $m \leq N$. Moreover, all B_k are zero if and only if the initial connection has regular singularity at ∞, i.e. $u_{j,n} = 0$ for $n \neq d_j$. The solutions of ${}^L\mathfrak{g}$-valued ordinary differential equation

$$d\varphi(w) + [B(w), \psi(w)]dw = 0 \tag{9}$$

have the form

$$\mathrm{Ad}\left(\exp\left(\sum_{k=2}^{m} B_k w^{-k+1}\right) w^C\right) x = \mathrm{Ad}\left(\exp\left(\sum_{k=2}^{m} B_k t^{\frac{-k+1}{N}}\right) s^{\frac{C}{N}}\right) x,$$

where $x \in {}^L\mathfrak{g}, m \leq N$ (exponentials of linear combinations of *fractional* powers of s). Each sector S on the w-plane with the origin at 0, which does not contain real multiples of the eigenvalues of the operators $\mathrm{Ad}\exp(\frac{2\pi ik}{m-1})B_m$ for $k = 1, \ldots, m-1$, distinguishes a subspace of formal solutions of the equation (9) which *decay exponentially* as $w \to 0$ along each ray in this sector S. Due to the Sibuya theorem, on such a sector we also have a subspace of exponentially decaying solutions of the equation

$$d\varphi(s) + [A(s), \varphi(s)]ds = 0. \tag{10}$$

We have assumed that the connection $d + A(s)ds$ has trivial monodromy representation. Hence we may pick a global solution φ to the ${}^L\mathfrak{g}$-valued ordinary differential equation (10) which decays exponentially as $w \to 0$ along each ray in some sector S_0 on the w-plane. Since solution φ is a single-valued function of $s = w^N$, we find that φ also decays on the sector $\exp(2\pi ik/N)S_0, k = 1, \ldots, N-1$. Consider the sector $S_j = \exp(2\pi i/N)S_0$. We will show in section 4, that there exists a sector S_r (actually $S_r = \exp(\frac{\pi i}{m-1})S_0$) such that

(1) there is a formal solution of (10) which decays simultaneously on S_0 and S_j;
(2) there is no formal solution of (10) which decays simultaneously on S_r and S_0;
(3) there is a sector S on the w-plane of the opening $\frac{\pi}{m-1}$ which has nonzero intersection with S_j and S_r;
(4) there is a sector S' on the w-plane of the opening $\frac{\pi}{m-1}$ which has nonzero intersection with S_0 and S_r.

Applying Sibuya's sectorial normalization theorem to the sectors S and S', we obtain that there exists a nonzero formal solution of (10) exponentially decaying on both S_0 and S_r, which contradicts the second condition.

Thus, for any monodromy-free oper from $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)_0^\lambda$, we have $u_{j,n} = 0$ for $n \neq d_j$ and u_{j,d_j} are fixed by the residue at 0. Hence the space of monodromy-free opers from $\mathrm{Op}_{{}^LG}(\mathbb{P}^1)_0^\lambda$ is 0-dimensional. This completes the proof.

4. Proof of the Main Theorem

In this section we prove Theorem 1.

4.1. **The $\mathfrak{sl}_2$ case.** Let $d + A(t)dt$ be a $\mathfrak{sl}_2$-connection, such that

$$A(t) = e_{21} + (\lambda(\lambda+1)t^{-2} + a^2 t^{-1})e_{12}, \tag{11}$$

where λ is a fixed integral weight, $a \in \mathbb{C}$.

Proposition 4. *The connection (11) has trivial monodromy representation if and only if $a = 0$.*

Proof. Let $s = t^{-1}$ be the local coordinate at the infinity. We rewrite our connection in s as follows.

$$A(s) = s^{-2}e_{21} + (\lambda(\lambda+1) + a^2 s^{-1})e_{12}.$$

Changing the variable as $s = w^2$, we obtain the following connection:[3]

$$d + A(w)dw,\ A(w) = 2(w^{-3}f_{21} + (\lambda(\lambda+1)w + a^2 w^{-1}))e_{12}.$$

The latter is conjugate to

$$d + ((2ae_{11} - 2ae_{22})w^{-2} + B(w))dw, \tag{12}$$

where $B(w) = O(w^{-1})$ as $w \to 0$. The connection (12) is *formally* conjugate to

$$d + ((2ae_{11} - 2ae_{22})w^{-2} + (b_{11}e_{11} + b_{22}e_{22})w^{-1})dw. \tag{13}$$

Moreover, by the Sibuya sectorial normalization theorem, for any sector $S = \{w|\operatorname{Arg} w \in (\alpha, \alpha+\pi)\}$, the formal gauge transformation $H(w)$ can be extended to an analytical gauge transformation $H_S(w)$ conjugating (12) to (13).

Consider two sectors, $S_0 = \{w|\operatorname{Arg} w \in (\operatorname{Arg} a - \frac{\pi}{2}, \operatorname{Arg} a + \frac{\pi}{2})\}$ and $S_1 = \{w|\operatorname{Arg} w \in (\operatorname{Arg} a + \frac{\pi}{2}, \operatorname{Arg} a + \frac{3\pi}{2})\}$. We have the following basis of solutions to the linear ordinary differential equation (13)

$$\psi_0(w) = (0, \exp(-2aw^{-1})w^{b_{22}}), \quad \psi_1(w) = (\exp(2aw^{-1})w^{b_{11}}, 0).$$

Note that the solution ψ_0 decays exponentially as $w \to 0$ along each ray in the sector S_0 and blows up exponentially as $w \to 0$ along each ray in the sector S_1. Respectively, ψ_1 decays on S_1 and blows up on S_0.

Assume that the connection (11) has trivial monodromy. Let φ be the global solution to the equation (12) such that $\varphi|_{S_0} = H_{S_0}\psi_0$. Since ψ_0 decays exponentially on S_0 and the gauge transformation $H_{S_0}(w)$ is bounded in some neighborhood of 0, the solution φ also decays on S_0. Since solution φ is a single-valued function of $s = w^2$, we have $\varphi(w) = \varphi(-w)$, and hence $\varphi|_{S_1}$ decays as well.

Consider the sector $S = \{w|\operatorname{Arg} w \in (\operatorname{Arg} a, \operatorname{Arg} a + \pi)\}$. On S, we have the following basis of solutions to the equation (12):

$$\varphi_0 = H_S\psi_0, \quad \varphi_1 = H_S\psi_1.$$

We have $\varphi = k_0\varphi_0 + k_1\varphi_1$ for some $k_0, k_1 \in \mathbb{C}$. Since φ decays on $S \cap S_0$ while ψ_1 blows up on $S \cap S_0$, we have $k_1 = 0$. Since φ decays on $S \cap S_1$ while ψ_0 blows up on $S \cap S_1$, we have $k_0 = 0$. Hence $\varphi = 0$ and we have a contradiction. □

4.2. **General case.** We replace ${}^L G$ by G to simplify notation. Let $d + A(t)dt$ be a connection on the trivial G-bundle on $\mathbb{P}^1$ such that

$$A(t) = p_{-1} + \sum_{j=1}^{\ell} \left(c_j t^{-d_j - 1} p_j + \sum_{n=0}^{d_j - 1} u_{j,n} t^{-n-1} p_j \right).$$

Proposition 5. *The connection $d + A(t)dt$ has trivial monodromy representation if and only if $u_{j,n} = 0$ for all j, n.*

[3]Note that if we were considering an oper with regular semisimple μ, then we would be able to bring it to a normal form without extracting the square root of s. For this reason the argument given below would not work in this case.

Proof. Suppose that the connection $d + A(t)dt$ has no monodromy while some of the $u_{j,n}$ are nonzero.

Let $s = t^{-1}$ be the local coordinate at the infinity. We rewrite our connection in s as follows.

$$d + A(s)ds, \quad A(s) = s^{-2}p_{-1} + \sum_{j=1}^{\ell} c_j s^{d_j - 1} p_j + \sum_{n=0}^{d_j j - 1} u_{j,n} s^{n-1} p_j. \tag{14}$$

Let us change the variable as $w^N = s$, where $N = \prod\limits_{j \in E} j$. The connection rewrites as

$$d + A(w)dw, \quad A(w) = w^{-N-1}p_{-1} + \sum_{j \in E} c_j w^{jN-1} p_j + \sum_{n=0}^{j-1} u_{j,n} w^{nN-1} p_j. \tag{15}$$

Due to the Hukuhara–Turritin–Levelt theorem, the connection (15) may be reduced by a suitable *formal* gauge transformation $H(w)$ to its *formal normal form*

$$d + B(w)dw \quad B(w) = \sum_{k=2}^{m} B_k w^{-k} + Cw^{-1}, \tag{16}$$

where $m \leq N$ and all B_k commute with C and belong to the fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The coefficients B_k are zero (i.e., the connection has regular singularity) if and only if all $u_{j,n}$ are zero.

Let Δ_+ be the set of positive roots with respect to a Borel subalgebra containing $\mathfrak{h}$. To any root $\alpha \in \Delta_+$ such that $\alpha(B_m) \neq 0$, we assign a collection of $2m - 2$ *separation rays* (also known as *Stokes directions*) defined by the condition

$$\operatorname{Re} \frac{\alpha(B_m)}{w^{m-1}} = 0. \tag{17}$$

Remark. The set of Stokes directions is invariant under the transformations $w \mapsto \exp(\frac{\pi i}{m-1})w$ (clear from the condition (17)) and $w \mapsto \exp(\frac{2\pi i}{N})w$ (since the connection (14) depends on $s = w^N$, not w).

Generic case. Assume that B_m is *generic*, i.e., B_m is a regular element of $\mathfrak{h}$. Then all separation rays are distinct. The proof under this assumption is slightly simpler, so we shall give the detailed proof in this case first, and then explain what should be changed for an arbitrary B_m.

In the generic case we have precisely $M = (2m-2)r$ separation rays, where $r = |\Delta_+|$. We label these separation rays $d_1, d_2, \dots d_M = d_0$ going counterclockwise, and choose the initial sector S_0 to be between d_0 and d_1.

Consider the $\mathfrak{g}$-valued ordinary differential equation

$$d\varphi(w) + [A(w), \varphi(w)]dw = 0, \tag{18}$$

and its formal normal form

$$d\varphi(w) + [B(w), \varphi(w)]dw = 0. \tag{19}$$

We have the following basis of solutions of equation (19):

$$\psi_\alpha(w) = \mathrm{Ad}\exp\left(\sum_{k=2}^{m}\frac{1}{k-1}B_k w^{-k+1}\right)w^{-C}e_\alpha,\quad \psi_{h_j}(w) = \mathrm{Ad}\, w^{-C}h_j, \tag{20}$$

where $\alpha\in\Delta$, and $h_j, j=1,\dots,\ell$, is a basis of $\mathfrak{h}$. On each sector S_i between the separation rays d_i and d_{i+1}, the formal solutions behave as follows:

(1) If $\mathrm{Re}\,\frac{\alpha(B_m)}{w^{m-1}}<0$ on S_i, then $\psi_\alpha(w)$ decays exponentially as $w\to 0$ along each ray in S_i,
(2) If $\mathrm{Re}\,\frac{\alpha(B_m)}{w^{m-1}}>0$ on S_i, then $\psi_\alpha(w)$ blows up exponentially as $w\to 0$ along each ray in S_i,
(3) $\psi_{h_j}(w)$ has polynomial growth/decay as $w\to 0$ along each ray in S_i.

We note that if $|i-j|<r$ then there is a root $\alpha\in\Delta$ such that $\mathrm{Re}\,\frac{\alpha(B_m)}{w^{m-1}}<0$ on both S_i and S_j. This means, that, for $|i-j|<r$, there is a formal solution $\psi^{(ij)}=\psi_\alpha$ which decays on both sectors S_i and S_j.

Note that the formal solution are periodic with respect to $w\mapsto\exp(\frac{2\pi i}{m-1})w$. We shall consider the behavior of solutions in the "fundamental" sector $\widehat{S_0}$ between $d_M=d_0$ and d_{2r}. For each sector $S_i\subset\widehat{S_0}$ except S_0, there is a formal solution which decays simultaneously on S_i and S_r. Moreover, for each sector $S_i\subset\widehat{S_0}$ there is a sector of opening $\frac{\pi}{m-1}$ (a half-period) which has nonzero intersection with S_i and S_r.

On the other hand, the global solutions of the equation (18) are periodic with respect to $w\mapsto\exp(\frac{2\pi i}{N})w$. Since the set of separation rays is invariant under this transformation, this transformation permutes the sectors S_i. Hence the global solutions have the same asymptotic properties on the sectors S_0 and $S_j=\exp(\frac{2\pi i}{N})S_0$. Since $m-1<N$, the both sectors S_0 and $S_j=\exp(\frac{2\pi i}{N})S_0$ belong to the fundamental sector $\widehat{S_0}$.

Consider the sector $S_j=\exp(\frac{2\pi i}{N})S_0$. Since $m-1<N$, we have $0<j<2r$, and therefore there is a formal solution $\psi^{(jr)}$ which decays on S_j and S_r. Moreover, since the opening of the sector between d_j and d_r is less than $\frac{\pi}{m-1}$, there is a sector S_{jr} of opening $\frac{\pi}{m-1}$ which contains S_j and S_r. By Sibuya's theorem, the formal gauge transformation $H(w)$ can be extended to an analytical gauge transformation $H_{S_{jr}}(w)$ conjugating (15) to (16) on S_{jr}. Thus, there is a solution $\varphi^{(jr)}:=H_{S_{jr}}\psi^{(jr)}$ of the equation (18), which decays on both sectors S_j and S_r. Due to the no-monodromy condition, there exists a global solution φ such that $\varphi|_{S_{jr}}=\varphi^{(jr)}$. Moreover, the no-monodromy condition means that each solution of the equation (18) is a single-valued function of $s=w^N$, hence we have $\varphi(w)=\varphi(\exp(\frac{2\pi i}{N})w)$. This means that φ decays exponentially on $S_0=\exp(\frac{-2\pi i}{N})S_j$.

According to the Sibuya theorem, since the opening of the sector between $d_M=d_0$ and d_r is $\frac{\pi}{m-1}$, the equation (18) can be conjugated to its formal normal form in some sector S intersecting S_0 as well as S_r. On S, we have the following basis of solutions to the equation (18)

$$H_S\psi_\alpha,\ H_S\psi_{h_j},\quad \alpha\in\Delta,\ j=1,\dots,\ell.$$

We have

$$\varphi = \sum_{\alpha\in\Delta} k_\alpha H_S\psi_\alpha + \sum_{j=1}^{\ell} k_{h_j} H_S\psi_{h_j}.$$

For any h_j, the solution $H_S\psi_{h_j}$ does not decay exponentially on S, hence $k_{h_j} = 0$ for all $j = 1,\dots,\ell$. Next, for any $\alpha \in \Delta$ we have the following possibilities:

(1) either $\operatorname{Re}\frac{\alpha(B_m)}{w^{m-1}} > 0$ on S_0, and hence $H_S\psi_\alpha$ blows up on $S \cap S_0$,

(2) or $\operatorname{Re}\frac{\alpha(B_m)}{w^{m-1}} > 0$ on S_r, and hence $H_S\psi_\alpha$ blows up on $S \cap S_r$.

In both cases we obtain that $k_\alpha = 0$ for all $\alpha \in \Delta$. Hence $\varphi = 0$, and we have a contradiction.

General case. For non-generic B_m, the asymptotic behavior of formal solutions is not determined only by B_m, but depends also on B_k with $k < m$. Thus it is difficult to figure out on which sectors the given formal solution decays exponentially. The crucial observation is that for our purposes it is sufficient to watch only for the *most rapid* decay of solutions (i.e. faster than $\exp(bw^{-m+1})$ for some b), which is determined by the leading term B_m.

We have $M = (2m-2)r$ separation rays, where $r = \#\{\alpha|\alpha(B_m) \neq 0\}$. We label the separation rays $d_1,\dots d_M$ going in the positive sense, and choose the initial sector S_0 between d_M and d_1. Note that, for non-generic B_m, some of the separation rays may coincide. We choose the initial sector S_0 to be non-empty.

Consider the $\mathfrak{g}$-valued ordinary differential equation (18), and its formal normal form (19). We have the basis (20) of solutions to the equation (19), which behaves on each sector S_i between the separation rays d_i and d_{i+1} as follows:

(1) If $\operatorname{Re}\frac{\alpha(B_m)}{w^{m-1}} < 0$ on S_i, then $\psi_\alpha(w)$ decays *most rapidly* (i.e. faster than $\exp(bw^{-m+1})$ for some b) as $w \to 0$ along each ray in S_i,

(2) If $\operatorname{Re}\frac{\alpha(B_m)}{w^{m-1}} > 0$, then $\psi_\alpha(w)$ blows up as $w \to 0$ along each ray in S_i,

(3) If $\operatorname{Re}\frac{\alpha(B_m)}{w^{m-1}} = 0$, then $\psi_\alpha(w)$ does not decay or decays not faster than $\exp(bw^{-m+1})$ for all $b \in \mathbb{C}$ as $w \to 0$ along each ray in S_i,

(4) $\psi_{h_j}(w)$ has polynomial growth/decay as $w \to 0$ along each ray in S_i.

As in the generic case, we note that if $|i-j| < r$ then there is a root $\alpha \in \Delta$ such that $\operatorname{Re}\frac{\alpha(B_m)}{w^{m-1}} < 0$ on both S_i and S_j, and hence, there is a formal solution $\psi^{(ij)} = \psi_\alpha$ which decays most rapidly on both sectors S_i and S_j.

In the same way as in the generic case, we get the global solution φ of the equation (18) which decays most rapidly on S_0 and S_r. We take a sector S intersecting both S_0 and S_r, and represent the solution φ on S as

$$\varphi = \sum_{\alpha\in\Delta} k_\alpha H_S\psi_\alpha + \sum_{j=1}^{\ell} k_{h_j} H_S\psi_{h_j}.$$

For any h_j, the solution $H_S\psi_{h_j}$ does not decay exponentially on S, hence k_{h_j} for all $j = 1,\dots,\ell$. For any $\alpha \in \Delta$, we have the following possibilities:

(1) $\operatorname{Re}\frac{\alpha(B_m)}{w^{m-1}} > 0$ on S_0, and hence $H_S\psi_\alpha$ blows up on $S \cap S_0$,

(2) $\operatorname{Re}\frac{\alpha(B_m)}{w^{m-1}} > 0$ on S_r, and hence $H_S\psi_\alpha$ blows up on $S \cap S_r$,

(3) $\mathrm{Re}\,\frac{\alpha(B_m)}{w^{m-1}} = 0$, and hence $H_S\psi_\alpha$ does not decay or decays not faster than $\exp(bw^{-m+1})$ for all $b \in \mathbb{C}$ on S.

This means that $k_\alpha = 0$ for all $\alpha \in \Delta$. Hence $\varphi = 0$, and we have a contradiction. □

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