# Duck factory on the two-torus: multiple canard cycles without geometric constraints

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**Abstract**

Slow-fast systems on the two-torus are studied. As it was shown before, canard cycles are generic in such systems, which is in drastic contrast with the planar case. It is known that if the rotation number of the Poincaré map is integer and the slow curve is connected, the number of canard limit cycles is bounded from above by the number of fold points of the slow curve. In the present paper it is proved that for non-integer rotation numbers or unconnected slow curve there are no such geometric constraints: it possible to construct generic system with "as simple as possible" slow curve and arbitrary many limit cycles.

## 1 Introduction

Consider a generic slow-fast system on the two-dimensional torus

$$\begin{cases} \dot{x} = f(x, y, \varepsilon) \\ \dot{y} = \varepsilon g(x, y, \varepsilon) \end{cases} \qquad (x, y) \in \mathbb{T}^2 \cong \mathbb{R}^2/(2\pi\mathbb{Z}^2), \quad \varepsilon \in (\mathbb{R}_+, 0) \tag{1}$$

Assume that $f$ and $g$ are smooth enough and $g > 0$. The dynamics of this system is guided by the *slow curve*:

$$M = \{(x, y) \mid f(x, y, 0) = 0\}.$$

It consists of equilibrium points of the fast motion (i.e. motion determined by system (1) for $\varepsilon = 0$). Particularly, one can consider two parts of the slow curve: one is stable (consists attracting hyperbolic equilibrium points) and the other unstable (consists of repelling hyperbolic equlibrium points). On the plane $\mathbb{R}^2$, there is rather simple description of the generic trajectory of (1): it consists of interchanging phases of slow motion along stable parts of the slow curve and fast jumps along straight lines $y = const$ near folds of the slow curve. On the two-torus, more complicated behaviour can be locally generic.

**Definition 1.** A solution (or trajectory) is called *canard* if it contains an arc of length bounded away from zero uniformly in $\varepsilon$ that keeps close to the unstable part of the slow curve.

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It is usually also demanded in the definition of canard solution that it spends bounded away from 0 time near stable part of the slow curve as well. We will not expose this additional requirement because we are interested mostly in *attracting* canard cycles which have to satisfy it automatically.

Canards are not generic on the plane: one have to introduce an additional parameter to get an attracting canard cycle. However, they are generic on the two-torus, which was explained in [1] and strictly proved in [2]. The explanation involves the rotation number of the Poincaré map.

Assume that there exists global cross-section $\Gamma = \{y = const\}$ transversal to the field. Then one can define the Poincare map $P_\varepsilon \colon \Gamma \to \Gamma$. It is a diffeomorphism of a circle. The rotation number $\rho(\varepsilon)$ of the map $P_\varepsilon$ continuously depends on $\varepsilon$. For generic system (1) function $\rho(\varepsilon)$ is a Cantor function (also known as devil's staircase) whose horizontal steps occur at rational values which (in general case) corresponds to hyperbolic periodic points of map $P_\varepsilon$. These in turn corresponds to limit cycles of the original vector field. More precisely, if the Poincaré map has a rotation number with a denominator $n$ then the initial vector field has a limit cycle which makes $n$ full passes along the slow direction of the torus $y$. In particular, fixed points of Poincare map correspond to limit cycles which make only one round along the slow direction of the torus. Here and below we will assume that fast coordinate $x$ is vertical and slow coordinate $y$ is horizontal.

While hyperbolic limit cycles present, the rotation number is preserved under small perturbations. So when the rotation number increases, the limit cycles have to bifurcate through saddle-node (parabolic) bifurcation. Near the critical value of the parameter, the derivative of the Poincare map for both colliding cycles have to be close to 1. This is possible only if the cycles spend comparable time near stable and unstable part of the slow curve, and thus they are canards.

The next natural question is to provide an estimate for the number of canard cycles that can born in a generic slow-fast system on the two-torus. The answer to this question for the case of the integer rotation number and a rather wide class of systems was given in [3].

**Theorem 1.** *For generic slow-fast system on the two-torus with retractable nondegenerate connected slow curve the number of limit cycles that make one pass along the axis of slow motion is bounded by the number of fold points of the slow curve. This estimate is sharp in some open set in the space of slow-fast systems on the two-torus.*

In the present paper we consider two related cases not covered by theorem 1: non-integer rotation numbers and unconnected slow curve. The latter case is of special interest because slow-fast systems with unconnected slow curve appear naturally in physical applications, e.g. in the modelling of circuits with Josephson junction [5].

Our main results state that in contrast with Theorem 1, in these cases there are *no geometric constraints* on the number of (canard) limit cycles.

First for any desired number of limit cycles $l$ we construct open set in the space of the slow-fast systems on the two-torus with *convex* slow curve (i.e. having only two fold points) with at least $l$ canard cycles that make two passes along the axis of slow motion. (I.e. the corresponding Poincaré map has half-integer rotation number.) See Theorem 2.

Then we construct topologically generic (residual) set of slow-fast systems on the two-torus with two connected components with at least $l$ canard cycles and integer rotation number of Poincaré map. See Theorem 4.

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## 2 Main results

In this section we state our main results. We are interested only in the phase curves of system (1), so one can divide it by $g$ and consider without loss of generality case of $g \equiv 1$.

**Definition 2.** Slow curve $M$ is called *nondegenerate* if it has finite number of nondegenerate fold points: for every fold point $G$ the following conditions hold:

$$\left.\frac{\partial^2 f(x,y,0)}{\partial x^2}\right|_G \neq 0, \qquad \left.\frac{\partial f(x,y,0)}{\partial y}\right|_G \neq 0,$$

We will call $M$ *convex* if it is contained in the fundamental domain of the two-torus

$$\{-\pi < x < \pi, -\pi < y < \pi\}$$

and is a convex curve.

**Definition 3.** Fix some vertical segment $(x_1, x_2) \times \{y = const\}$ intersecting unstable part of the slow curve (to be chosen later). Every trajectory that cross this segment will be called *canard*.

**Theorem 2.** *For every desired number of limit cycles $l \in \mathbb{N}$ there exists an open set in the space of slow-fast systems on the two-torus with the following properties.*

1. *Slow curve $M$ is a connected convex nondegenerate curve containing in the fundamental domain of a torus.*
2. *For every system from this set there exists a sequence of intervals $\{R_n\}_{n=0}^{\infty} \subset \{\varepsilon > 0\}$, accumulating at zero, such that for every $\varepsilon \in R_n$ there exist exactly $l$ canard limit cycles.*

**Remark 3.** Using appropriate change of coordinates, one can replace the condition of convexity to the condition that $M$ has only two nondegenerate fold points .

Theorem 2 is proved in section 3.

The following theorem deals with the case of unconnected slow curve.

**Theorem 4.** *For every desired number of limit cycles $l \in \mathbb{N}$ there exists a topologically generic (residual) set in the space of slow-fast systems on the two-torus with the following properties.*

1. *Slow curve $M$ is nondegenerate curve with two connected components and 4 fold points.*

2. *For every system from this set there exists a sequence of intervals* $\{R_n\}_{n=0}^{\infty} \subset \{\varepsilon > 0\}$*, accumulating at zero, such that for every* $\varepsilon \in R_n$ *there exist exactly* $l$ *canard limit cycles.*

Theorem 4 is proved in section 5.

# 3 The construction of limit cycles

In this section we show that near every *neutral singular trajectory* there exists canard limit cycle. Such singular trajectory is defined by an equation involving integrals of the derivative of $f$ over some arcs of the slow curve. In the following section we will show that for every desired number of limit cycles there exists an open set of systems such that the corresponding equation has the desired number of roots. It finishes the proof of Theorem 2.

## 3.1 Settings and notations

Let $M$ be a slow curve. It consists of stable ($M^-$) and unstable ($M^+$) parts and two jump points: the forward jump point $G^-$ and the backward jump point $G^+$, see fig. 1:

$$M = M^+ \sqcup \{G^+\} \sqcup M^- \sqcup \{G^-\}.$$

Assume without loss of generality that $y(G^{\pm}) = \mp 1$. Here and below every equation with $\pm$'s and $\mp$'s corresponds to a couple of equations: with all top and all bottom signs.

We will also use the notation $M^{\pm}(y)$ assuming that $M^{\pm}$ here are functions, whose graph $x = M^{\pm}(y)$ defines unstable and stable parts of the slow curve.

Call $\Pi = S^1 \times [-1, 1]$ a *basic strip.*

Fix vertical segment $J^+$ (resp., $J^-$) which intersect $M^+$ ($M^-$) close enough to the jump point $G^-$ ($G^+$), and does not intersect $M^-$ ($M^+$). Let

$$y(J^{\pm}) =: \alpha^{\pm} = \pm 1 \mp \delta^{\pm},$$

where $\delta^{\pm}$ are small. Note that the definition of $J^{\pm}$ differs from one in [1, 2]: instead of placing $J^+$ near $G^+$ we place it near $G^-$ (and do opposite with $J^-$).

We have to reproduce the notation on oriented arcs on a circle from [2] and Poincaré maps. Consider arbitrary points $a$ and $b$ on the oriented circle $S^1$. They split the circle into two arcs. Denote the arc from point $a$ to point $b$ (in the sense of the orientation of the circle) by $[a, b\rangle$. The orientation of this arc is induced by the orientation of the circle. Also denote the same arc with the reversed orientation by $\langle a, b]$. (See fig. 2.)

Denote also the Poincaré map along the phase curves of the main system (1) from the cross-section $y = a$ to the cross-section $y = b$ in the forward time by $P_\varepsilon^{[a,b\rangle}$. Also, let $P_\varepsilon^{\langle a,b]} = (P_\varepsilon^{[a,b\rangle})^{-1}$: this is the Poincaré map from the cross-section $y = b$ to the cross-section $y = a$ in the backward time. This fact is stressed by the notation: the direction of the angle bracket shows the time direction.

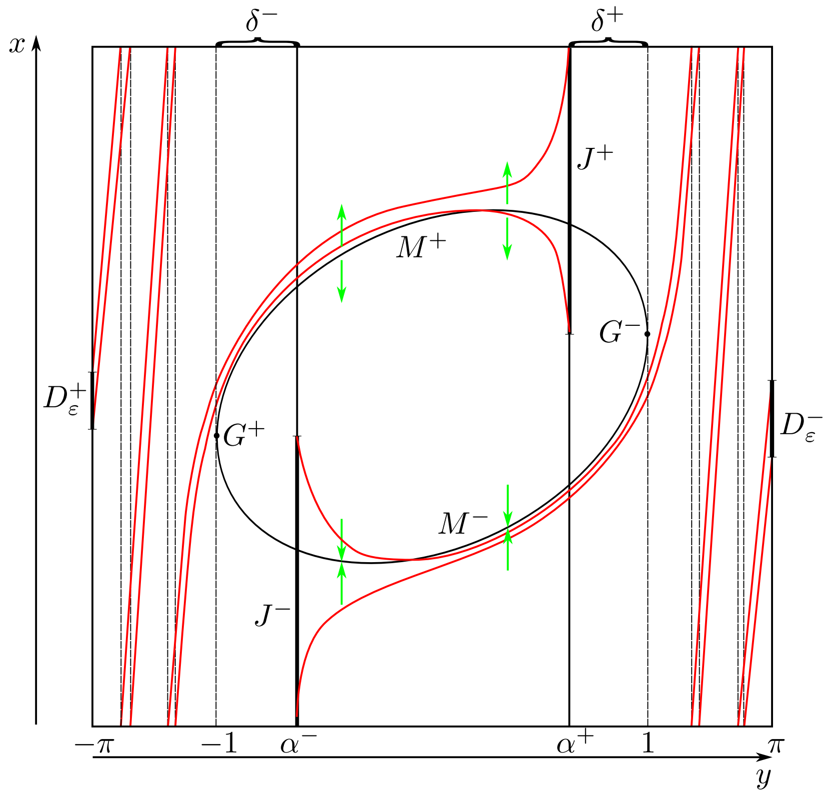


Figure 1: The slow curve and jump points. Note that horizontal axis is $y$ and vertical is $x$

## 3.2 Preliminary results

Denote

$$D_\varepsilon^+ := P_\varepsilon^{\langle -\pi, \alpha^+]}, \quad D_\varepsilon^- = P_\varepsilon^{[\alpha^-, \pi\rangle}$$

It is proved in [1, 2] (see Shape lemmas there) that $|D_\varepsilon^\pm| = O(e^{-C/\varepsilon})$. Note that as $\varepsilon$ decrease to 0, $D_\varepsilon^+$ moves downward and $D_\varepsilon^-$ moves upward, making infinitely many rotations (see Monotonicity lemmas in [1, 2]) and meet each other infinitely many times. The values of $\varepsilon$ for which $D_\varepsilon^+$ and $D_\varepsilon^-$ have nonempty intersection forms intervals $R_n$:

$$\{R_n\}_{n=0}^\infty = \{\varepsilon > 0\colon D_\varepsilon^+ \cap D_\varepsilon^- \neq \varnothing\}.$$

As it was shown in [1, 2], intervals $R_n$ has exponentially small length and accumulate at zero. If one pick any sequence $\varepsilon_n \in R_n$, $n = 1, 2, \ldots$, then

$$\varepsilon_n = O\left(\frac{1}{n}\right)$$

Let $\varepsilon \in R_n$ and pick some point $q \in D_\varepsilon^+ \cap D_\varepsilon^-$. Consider the trajectory which pass through $q$. In the forward time, this trajectory make several (about $O(1/\varepsilon)$) rotations, then perform backward jump, follow unstable part of the slow curve

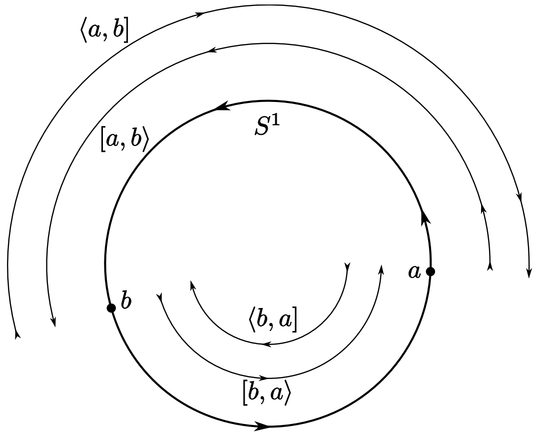


Figure 2: Orientation of the arcs

$M^+$ and finally intersect $J^+$. In the backward time, this trajectory (again after several rotations) will pass near the stable part of the slow curve $M^-$ and finally intersect $J^-$. We will call this trajectory *grand canard*.

Let $U$ be a segment of the stable or unstable part of the slow curve $M$. Introduce the notation:

$$\Phi(U) = \int_U f_x' dy \tag{2}$$

Denote also for brevity

$$\Phi^{\pm}[y_1, y_2\rangle := \Phi(U),$$

where $U$ is an arc of $M^{\pm}$ projected on $[y_1, y_2\rangle$.

Function $\Phi$ is approximately a logarithm of a derivative of a solution that pass near segment $U$ with respect to the initial condition. Formally, the following theorem holds:

**Theorem 5** (See [1])**.** *Let $U = [A, B] \subset M^{\pm}$ and $X = [x_1, x_2] \times \{y(A)\}$ be a segment that contains $A$ and do not cross $M$ in points different from $A$. Then*

$$\log \left( P_\varepsilon^{[y(A), y(B)\rangle}(x) \right)_x' \Big|_X = \frac{\Phi(U) + O(\varepsilon)}{\varepsilon}.$$

Moreover, similar (but a little weaker) estimate holds for trajectories extended through the jump point even after they make $O(1/\varepsilon)$ rotations along the $x$-axis after the jump. The exact statement follows.

**Theorem 6** (See [2, 3])**.** *Let $U = [A, G^-] \subset M^- \cup \{G^-\}$, $X$ as in previous Theorem and $y_1$ is a point outside of the projection of $M$ to $y$-axis, such that there is no other points of that projection on arc $[1, y_1\rangle$. Than the following holds:*

$$\log \left( P_\varepsilon^{[y(A), y_1\rangle}(x) \right)_x' \Big|_X = \frac{\Phi(U) + O(\varepsilon^\nu)}{\varepsilon},$$

*where $\nu \in (0, 1/4]$.*

Reverting the time, one can obtain a similar result for $M^+$ and $G^+$.

## 3.3 The first release point

Assume that $\varepsilon \in R_n$ for some $n$ and therefore there exists grand canard. Let $p$ be some point in the basic strip $\Pi$, such that $y(p)$ is strictly inside the arc $[\alpha^-, \alpha^+\rangle$ and $p$ is bounded away from $M$. Consider the trajectory that pass through $p$. Call it *p-trajectory.* In the forward time, it attracts to $M^-$ after time $O(\varepsilon)$ ("falls"). Assume without loss of generality that this fall is made in negative direction ("downwards"). After the fall, the trajectory follows $M^-$ exponentially close to the grand canard (being "above" the grand canard) until the jump point. Assume that the segment of the trajectory before the jump point is given as a graph of function $y = \psi^-(x)$.

After the jump, it will follow grand canard during the rotation phase, then perform backward jump and pass near some segment unstable part of the slow curve $M^+$. Assume that the segment of the trajectory after the backward jump point is given as a graph of function $y = \psi^+(x)$.

At some point it is possible that the trajectory will be released from the grand canard (and thus $M^+$) before intersection with $J^+$ and attracted to $M^-$ again. This release will be made in positive direction ("upward"), because $p$-trajectory is above the grand canard. If the release occur, denote the $y$-coordinate of the release point by $\widehat{\beta}(p)$. More exact, demand that

$$\psi^+(\widehat{\beta}(p)) - M^+(\widehat{\beta}(p)) = c,$$

where $c$ is some constant.

**Definition 4.** Assume that $|\Phi^-[y(p), 1\rangle| < \Phi^+[-1, 1\rangle$. Then there exist *singular release point* $\beta = \beta(p) \in [-1, 1\rangle$ such that

$$\Phi^-[y(p), 1\rangle + \Phi^+[-1, \beta(p)\rangle = 0 \tag{3}$$

**Proposition 7.** *The release point $\widehat{\beta}(p)$ is defined iff the singular release point $\beta(p)$ is defined. The following estimate holds:*

$$\widehat{\beta}(p) = \beta(p) + O(\varepsilon^\nu).$$

*Proof.* Assume that $\beta(p)$ is defined. Consider point $\beta_1 = \beta(p) + \delta_1$ for some small $\delta_1 > 0$, such that $\beta_1 < \alpha^+$. Then $\Phi^-[y(p), 1\rangle + \Phi^-[-1, \beta_1\rangle > 0$.

Let

$$R^1_\varepsilon = P_\varepsilon^{[-\pi, \beta_1\rangle} \circ P_\varepsilon^{[y(p), \pi\rangle}.$$

Theorem 6 and the chain rule imply that

$$\log(R^1_\varepsilon)'_x = \frac{\Phi^+[y(p), 1\rangle + \Phi^-[-1, \beta_1\rangle + O(\varepsilon^\nu)}{\varepsilon} \to \infty \text{ as } \varepsilon \to 0^+ \tag{4}$$

Let the segments of grand canard in basic strip given by functions $\varphi^+(y)$ (for part near $M^+$) and $\varphi^-(y)$ (for part near $M^-$). It follows from geometric singular perturbation theory [4] that outside of some neighborhood of the jump points,

$$\varphi^\pm(y) = M^\pm(y) + O(\varepsilon).$$

Therefore $x(p) - \varphi^-(y(p)) = x(p) - M^-(y(p)) + O(\varepsilon)$ is bounded away from 0. Mean value theorem implies that

$$R^1_\varepsilon(x(p)) - R^1_\varepsilon(\varphi^-(y(p))) = (x(p) - \varphi^-(y(p))) \cdot (R^1))'_x \to \infty$$

But by definition of grand canard, $R^1_\varepsilon(\varphi^-(y(p))) = \varphi^+(\beta_1)$ and the corresponding point belongs to $O(\varepsilon)$-neighborhood of $M^+$. Therefore, $p$-trajectory have to be far from $M^+$ when $y$-coordinate reaches $\beta_1$. Thus it is released earlier and $\widehat{\beta}(p)$ is defined.

Conversely, assume that $\widehat{\beta}(p)$ is defined. Let

$$R^2_\varepsilon = P_\varepsilon^{[-\pi,\hat{\beta}(p)\rangle} \circ P_\varepsilon^{[y(p),\pi\rangle}.$$

Then

$$\frac{R^2_\varepsilon(x(p)) - R^2_\varepsilon(\varphi^-(y(p)))}{x(p) - \varphi^-(y(p))} = \frac{c + O(\varepsilon)}{x(p) - M^-(y(p)) + O(\varepsilon)}.$$

Due to mean value theorem, it follows that the derivative of $R_\varepsilon$ in some point $x^*$ have to be of order constant. However Theorem 6 and the chain rule imply again that

$$\log(R^2_\varepsilon)'_x = \frac{\Phi^+[y(p),1\rangle + \Phi^-[-1,\widehat{\beta}(p)\rangle + O(\varepsilon^\nu)}{\varepsilon}. \tag{5}$$

If $\Phi^+[y(p),1\rangle + \Phi^-[-1,\widehat{\beta}(p)\rangle$ is not zero, the derivative have to be either exponentially small or exponentially big as $\varepsilon \to 0^+$. So the following estimate holds:

$$\Phi^+[y(p),1\rangle + \Phi^-[-1,\widehat{\beta}(p)\rangle = O(\varepsilon^\nu).$$

which differs from equation (3) only on $O(\varepsilon^\nu)$. Inverse function theorem finishes the proof. □

## 3.4 The second release and limit cycles

After the first release, $p$-trajectory attracts to $M^-$ and again follow it until the jump point, then jumps, rotates, perform backward jump and follows $M^+$ until possible second release. Note that the second release will be made in negative direction, because after the first release $p$-trajectory is below the grand canard. Denote the $y$-coordinate of the second release by $\widehat{\gamma}(p)$ (defined if the second release occurs).

Similarly to Proposition 7, let $\gamma(p)$ be the root of equation

$$\Phi^-[\beta(p),1\rangle + \Phi^+[-1,\gamma(p)\rangle = 0. \tag{6}$$

Then the following assertions hold:

$$\Phi^-[\beta(p),1\rangle + \Phi^+[-1,\widehat{\gamma}(p)\rangle = O(\varepsilon^\nu); \tag{7}$$

$$\widehat{\gamma}(p) = \gamma(p) + O(\varepsilon^\nu) \tag{8}$$

Consider function

$$S(p) = \Phi^-[\beta(p),1\rangle + \Phi^+[-1,y(p)\rangle$$

Note that $\beta(p)$ in fact depends only on $y(p)$, so one can write $S(y)$ instead of $S(p)$ assuming that $y(p) = y$.

**Proposition 8.** *Let $p$ be such point that $\gamma(p)$ is defined, $S(y(p)) = 0$ and $S'_y(y(p)) \neq 0$. Then there exists limit cycle that intersects $O(\varepsilon^\nu)$-neighborhood of point $p$.*

*Proof.* Consider points $p_1 = (x(p), y_1)$ and $p_2 = (x(p), y_2)$ such that $S(p_1) > 0$, $S(p_2) < 0$ and $p$ is close to both $p_1$ and $p_2$. Due to continuous dependence of the solution of differential equations on the initial condition, $p_1$- and $p_2$-trajectories performs two releases and $\gamma(p_1)$, $\gamma(p_2)$ are defined.

Denote the Poincaré map from the cross section $\{x = x(p)\}$ to itself by $Q(y)$. It is defined in some neighborhood of point $y(p)$ and

$$Q(y) = \widehat{\gamma}(y) + O(\varepsilon) = \gamma(y) + O(\varepsilon^{\nu}). \tag{9}$$

Due to (6),

$$S(p) = \Phi^{+}[-1, y(p)\rangle - \Phi^{+}[-1, \gamma(p)\rangle$$

Function $\Phi^{+}[-1, y\rangle$ is increasing in $y$. Therefore $\gamma(p_1) < c_1 < y_1$ and $\gamma(p_2) > c_2 > y_2$. Due to (9),

$$Q(y_1) < y_1, \quad Q(y_2) > y_2.$$

Intermediate value theorem imply that there exists point $y^*$ between $y_1$ and $y_2$ such that $Q(y^*) = y^*$ and the limit cycle present. □

# 4 Geometric construction

In this section we construct an open set of systems with arbitrary number of roots of equation $S(y) = 0$ defining the neutral contour. In the previous section it was shown that for every such root there exists a limit cycle of a system. Thus theorem 2 will be proved.

## 4.1 Problem statement

First of all state the problem in terms of integrals of the pair of functions. Let $N = ([\text{the desired number of limit cycles}] + 1)$. We have a sum of integrals

$$S(a) = \int_{-1}^{a} \lambda^{+} dy + \int_{a}^{1} \lambda^{-} dy + \int_{-1}^{\beta(a)} \lambda^{+} dy + \int_{\beta(a)}^{1} \lambda^{-} dy \tag{10}$$

where

$$\beta(a)\colon \int_{a}^{1} \lambda^{-} dy = -\int_{-1}^{\beta(a)} \lambda^{+} dy \tag{11}$$

and our goal is to construct such (smooth) functions

$$\lambda^{-}\colon [-1, 1] \to \mathbb{R}^{-} \text{ and } \lambda^{+}\colon [-1, 1] \to \mathbb{R}^{+} \tag{12}$$

that sum $S(a)$ changes sign at least $(N-1)$ times on the interval $(-1, 1)$. We also have to demand that the there exists such smooth function $f$ that $\lambda^{\pm} = f_x^{'}|_{M^{\pm}}$.

## 4.2 Algorithm

Slow curve $M$ is fixed. We choose sufficiently smooth function $f$ in system (1), which zero level set coincides with $M$. It defines functions $\lambda^{+}$ and $\lambda^{-}$ with range of values given by (12). Algorithm is intended to correct these functions to get sign changes for the sum $S$. We will construct such functions that sum of the first and fourth terms in sum $S$ will have sign changes, and the sum of the second and the third terms will be at the same moments equal to zero.

Algorithm consists of three steps. First of all we choose any $2N$ points $a_1, \dots, a_N$ and $b_N, \dots, b_1$ on the horizontal axis such that:

$$-1 = a_0 < a_1 < \dots < a_N < b_N < \dots < b_1 < b_0 = 1.$$

In other steps we will construct sign changes (release points) of the sum $S$ between each $a_i$ and $a_{i+1}$, $i = 1, \dots, N-1$.

On the second step we change function $\lambda^+$ on the interval $[-1, a_N]$ and function $\lambda^-$ on the interval $[b_N, 1]$ in such a way that sum of the first and the fourth terms in $S(a)$ change its sign $(N-1)$ times on the interval $(-1, a_N)$.

We take consequent $k = 0, 1, \dots, N$ and ask for next conditions according to its evennes: if $k$ is even, $0 \le k \le N$, we ask for

$$\left| \int_{-1}^{a_{k+1}} \lambda^+ dy \right| > \left| \int_{1}^{b_{k+1}} \lambda^- dy \right| \tag{13}$$

(Note that the second interval of integration is reversed.) We always can satisfy that condition choosing $\lambda^+$ big enough (by its absolute value) strictly inside the interval $[a_k, a_{k+1}]$. For odd values of $k$ we ask for reversed condition

$$\left| \int_{-1}^{a_{k+1}} \lambda^+ dy \right| < \left| \int_{1}^{b_{k+1}} \lambda^- dy \right| \tag{14}$$

and satisfy that condition choosing the function $\lambda^-$ big enough (by its absolute value) strictly inside the interval $[b_k, b_{k+1}]$ . It is important that changes on the current step doesn't affect the previously achieved equalities. The second step is over.

On the third (and last) step we will change $\lambda^+$ and $\lambda^-$ on the remaining intervals $[a_N, 1]$ and $[-1, b_N]$ resp. to satisfy

$$b_i = \beta(a_i), \ i = N, \dots, 1. \tag{15}$$

Rewrite in terms of integrals

$$\int_{a_i}^{1} \lambda^- dy = -\int_{-1}^{b_i} \lambda^+ dy, \ i = 1, \dots, N. \tag{16}$$

This condition states that sum of the second and the third terms in sum (10) equals to zero at the points $a_1, \dots, a_N$.

We do it by induction descending from the index $N$ to 1. Let $\beta(a_N)$ is not equal to $b_N$. If $\beta(a_N)$ is greater than $b_N$ then

$$\left| \int_{a_N}^{1} \lambda^- dy \right| > \left| \int_{-1}^{b_N} \lambda^+ dy \right|.$$

In this case we increase absolute value of $\lambda^+$ strictly inside the interval $[a_N, b_N]$, at that $\beta(a_N)$ is decreasing. We increase $\lambda^+$ until the equality becomes true. Otherwise if $\beta(a_N)$ is lower than $b_N$ we increase absolute value of $\lambda^-$ on the same interval. After these corrections $\beta(a_N) = b_N$.

Next we repeat the same for $b_{N-1}$ and $\beta(a_{N-1})$: now we can change $\lambda^+$ inside the interval $[b_N, b_{N-1}]$ and $\lambda^-$ inside the interval $[a_{N-1}, a_N]$. Note that these changes has no influence on the previously achieved equality $\beta(a_N) = b_N$.

Next we repeat the same for $k = N-2, \dots, 1$. Now all equalities in (16) becomes true, third step is over.

Now sum $S(a_i)$ is positive for odd indexes and negative for even.

# 5 One pass canards for unconnected slow curve

## 5.1 Problem statement for disjoint slow curve and two grand canards

Consider unconnected slow curve $M$ which consists of two connected components. Let the first one has projection on the $y$-axis $[0,1]$ and the second one has projection $[2,3]$. For simplicity consider curve with only 4 fold points, two on each component, with the following $y$-coordinates: $y(G_1^+) = 0$, $y(G_1^-) = 1$, $y(G_2^+) = 2$ and $y(G_2^-) = 3$. Fix four vertical segments $J_1^\pm$ and $J_2^\pm$ that intersect the slow curve near corresponding jump points $G_1^\mp$ and $G_2^\mp$ like in part 3.1. Let $y(J_1^\pm) = \alpha_1^\pm$, $y(J_2^\pm) = \alpha_2^\pm$. Denote

$$D_{1,\varepsilon}^+ = P_\varepsilon^{\langle -\pi, y(J_1^+)]}, \; D_{1,\varepsilon}^- = P_\varepsilon^{[y(J_1^-), \frac{3}{2}\rangle}, \tag{17}$$

$$D_{2,\varepsilon}^+ = P_\varepsilon^{\langle \frac{3}{2}, y(J_2^+)]}, \; D_{2,\varepsilon}^- = P_\varepsilon^{[y(J_2^-), -\pi\rangle}. \tag{18}$$

For the series of accumulating to zero exponentially small intervals $R^1 = \{R_n^1\}_{n=1}^\infty$ of parameter $\varepsilon$ interval $D_1^-$ intersects with interval $D_2^+$. For other series $R^2 = \{R_n^2\}_{n=1}^\infty$ intersection holds for two remaining intervals $D_2^-$ and $D_1^+$. For each of these series exists one of two possible grand canards. Series may intersect.

The question is how many (canard) limit cycles can be in described system. If series $R^1$ and $R^2$ intersects thus there is both grand canards and situation is close to the stated in part 4.1. But now instead of two passes there are one pass along two parts of the slow curve. We skip the description of the construction of limit cycles which is similar to section 3 and pass to geometric problem of the construction of neutral contours in the next section.

Case of the only one canard cycle is studied in part 5.4.

## 5.2 Problem statement in terms of integrals for two grand canards

Let $N = ([\text{the desired number of limit cycles}] + 1)$. We have a sum of integrals

$$S(a) = \int_0^a \lambda^+ dy + \int_a^1 \lambda^- dy + \int_2^{\beta(a)} \lambda^+ dy + \int_{\beta(a)}^3 \lambda^- dy \tag{19}$$

where

$$\beta(a)\colon \int_a^1 \lambda^- dy = -\int_2^{\beta(a)} \lambda^+ dy \tag{20}$$

and our goal is to construct smooth functions as in part 4.1 such that sum $S(a)$ changes sign at least $(N-1)$ times on the interval $(0,1)$. We still demand that there exists smooth function $f$ such that $\lambda^{\pm} = f'_x|_{M^{\pm}}$. Domain of functions $\lambda^-$ and $\lambda^+$ is disjointed and equal to the projection $[0,1]\bigcup[2,3]$ of the slow curve.

Every one-pass cycle will have two release points, one on each component of the slow curve.

### 5.3 Algorithm

Algorithm needs to be corrected respectively. Now we take points of the 'first' release $a_1, \dots, a_N$ at the projection $[0,1]$ of the first slow curve's component. Points $b_N, \dots, b_1$ are respectively in the interior of segment $[2,3]$.

$$0 < a_1 < a_2 < \dots < a_N < 1, \; 2 < b_N < b_{N-1} < \dots < b_1 < 3 \tag{21}$$

On the second step we do the same correction as before for the function $\lambda^+$ at the interior of segments $[0,a_1], \dots, [a_{N-1}, a_N]$ and respectively for the function $\lambda^-$ at the interior of segments $[b_N, b_{N-1}], \dots, [b_1, 3]$. It will guarantee the sign changes for the sum of the first and the last term in the sum (19).

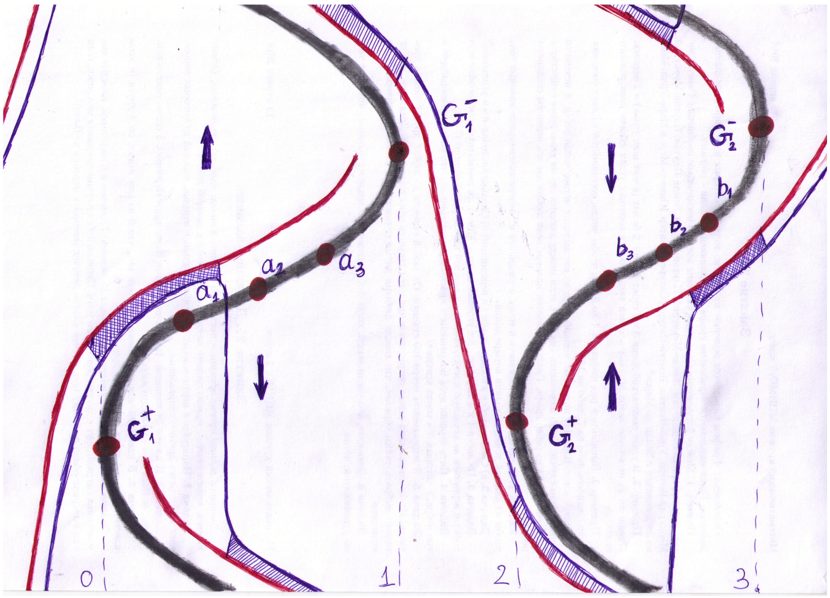


Figure 3: Canard for disjointed curve

On the third step we also should modify domain of correction. Now we achieve equalities (15) consequently correcting $\lambda^-$ at interior of segments $[a_N, 1]$, $[a_{N-1}, a_N]$, $\dots$, $[0, a_1]$ and respectively $\lambda^+$ at interior of segments $[2, b_N]$, $[b_N, b_{N-1}] \dots, [b_1, 3]$.

After that the sum (19) has (at least) $N$ zeroes at segment $[a_1, a_N]$. Each zero corresponds to canard limit cycle which releases two times: at zero of the sum on the first component and at the value which $\beta$ takes at this zero on the second component.

### 5.4 Case of only one grand canard

Consider the system with nondegenerate slow curve with only 4 fold points such that components's projections on the $y$-axis does not intersect. Let fold points and direction of fast motion for this system be as in 5.1.

Assume without loss of generality that

$$\left|\int_0^1 \lambda^- dy\right| < \left|\int_2^3 \lambda^+ dy\right|. \tag{22}$$

Thus strictly inside the segment $[2,3]$ exists a point $b$ such that

$$\left|\int_0^1 \lambda^- dy\right| = \left|\int_2^b \lambda^+ dy\right|. \tag{23}$$

Fix vertical segments $J_2^+$ and $\tilde{\Gamma}$, $y(\tilde{\Gamma}) = \tilde{b}$, that intersect only unstable part of the slow curve's second component and such that

$$b < \tilde{b} < y(J_2^+) < y(G_2^-).$$

On the stable part fix vertical segment $J_2^-$ such that $\tilde{b} < y(J_2^-)$.

On the first component's slow curve fix segment $J_1^-$ in neighbourhood of $G_1^+$ and $J_1^+$ in neighborhood of $G_1^-$.

For all these segments denote as before

$$D_{1,\varepsilon}^+ = P_\varepsilon^{\langle -\pi, y(J_1^+)]}, \qquad D_{1,\varepsilon}^- = P_\varepsilon^{[y(J_1^-), \frac{3}{2}\rangle}, \tag{24}$$

$$D_{2,\varepsilon}^+ = P_\varepsilon^{\langle \frac{3}{2}, y(J_2^+)]}, \qquad D_{2,\varepsilon}^- = P_\varepsilon^{[y(J_2^-), -\pi\rangle}. \tag{25}$$

**Proposition 9.** *Consider slow-fast system with following properties:*

1. *Slow curve $M$ is nondegenerate with two connected components and 4 fold points. Component's projections on the $y$-axis are nonintersecting.*
2. *Interval $D_1^-$ intersects with the interval $D_2^+$.*
3. *Interval $D_2^-$ has no intersection with intervals $D_1^-$ and $P_\varepsilon^{\langle -\pi, \tilde{b}]}\tilde{\Gamma}$.*

*Therefore there are only two limit cycles, both are canards.*

From the last two conditions follows existence of only one grand canard.

*Proof.* Take some trajectory that intersects $\Gamma$ in $P_\varepsilon^{\langle -\pi, \tilde{b}]}\tilde{\Gamma}$. It is contained in $\Gamma \setminus D_{2,\varepsilon}^-$. Thus in backward time it will be attracted by grand canard that passes nearby unstable (unstable in forward time, stable in backward) part of the second component of the $M$. So it will intersect cross-section $\tilde{\Gamma}$ and fall into the $P_\varepsilon^{\langle -\pi, \tilde{b}]}\tilde{\Gamma}$ again. That means that some limit cycle intersects $\tilde{\Gamma}$. Maybe it is not unique. But for any configuration of limit cycles stable and unstable ones are alternating, every stable has negative derivative of a Poincare map.

In backward time every trajectory from $\tilde{\Gamma}$ after single pass has positive (for arbitrary small values of $\varepsilon$) logarithm of derivative of order

$$\frac{1}{\varepsilon}\left|\int\limits_0^1 \lambda^- dy - \int\limits_2^3 \lambda^+ dy\right| = \frac{1}{\varepsilon}\left|\int\limits_{\tilde{b}}^3 \lambda^+ dy\right|. \tag{26}$$

Hence in forward time only one unstable canard limit cycle intersects $\tilde{\Gamma}$.

Now take any trajectory $\gamma$ that intersects $\Gamma \setminus (D^-_{2,\varepsilon} \bigcup P_\varepsilon^{\langle -\frac{1}{2},\tilde{b}]}\tilde{\Gamma})$. In forward time it will fall onto the second slow curve's stable component not later than at $\tilde{b}$, hence will jump from $G_2^-$ and intersect $\Gamma$ again at $D^-_{2,\varepsilon}$. Thus there are some limit cycles intersecting $D^-_{2,\varepsilon}$. These cycles can't be unstable because logarithm of their derivative is positive and has an order of (26). Hence because of alternating there is only one stable cycle. It is canard because it passes along grand canard from neighborhood of $G_1^+$ to some point that's $y$-coordinate is not larger than $\tilde{b}$. □

# 6 Topologically generic systems

We call slow-fast system (with four fold points, like in part 5) *good* if there exists $\varepsilon_0$ such that for any positive $\varepsilon \leqslant \varepsilon_0$ system doesn't have both grand canards:

$$D^+_{1,\varepsilon} \bigcap D^-_{2,\varepsilon} = \varnothing \text{ or } D^-_{1,\varepsilon} \bigcap D^+_{2,\varepsilon} = \varnothing. \tag{27}$$

**Theorem 10.** *Topologically generic slow-fast system is not good.*

*Proof.* Let $R_n^1$ be an interval on $\varepsilon$-axis described in theorem 2 that corresponds to intersection of $D^+_{1,\varepsilon}$ and $D^-_{2,\varepsilon}$ and $R_n^2$ corresponds to intersection of $D^-_{1,\varepsilon}$ and $D^+_{2,\varepsilon}$. For good system there exists $\varepsilon_0 > 0$ such that series $\{R_n^1\}_1^\infty$ and $\{R_n^2\}_1^\infty$ are nonintersecting for positive $\varepsilon < \varepsilon_0$.

Now we will perturb system to get intersection of the series. Take soFme segment $[y_1, y_2]$ that doesn't intersect projection of the slow curve. Add to $f$ some positive smooth function whose support is strictly inside the segment. Thus for arbitrarily small values of parameter $\varepsilon$ all trajectories that passes segment $[y_1, y_2]$ will made over this segment additional turn (or more) around fast direction on the torus.

In other terms: for some $n_0$ for any $n > n_0$ interval $R_n^1$ (or $R_n^2$) on $\varepsilon$-axis will move left and take the place at least of interval $R_{n+1}^1$ (or $R_{n+1}^2$).

Thus if we increase the value of $f$ continuously for some moment series $\{R_n^1\}_1^\infty$ and $\{R_n^2\}_1^\infty$ will intersect.

So we can get both intersections in (27) for some intervals $R_n^1$ and $R_n^2$ arbitrary close to any good system. Note that intersection in (27) holds for open set of systems. Thus we can consequently build countable number of intersections for $\varepsilon < \varepsilon_0$ in neighborhood of a good system. It corresponds to countable intersection of open and dense sets, that is topologically generic.

This finished proof of theorem 4 □

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