**Some quantitative methods and**

**models in economic theory**

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**Preface**

Currently the mathematical model of dynamic process embodies some dependence between quantitative characteristics of system which allows forecasting their behavior for the near or long-term periods. In economics as in science about production, distribution and consumption of goods the forecast and management of parameters of system leads to rational decision-making at all levels of society: government, region, firm, family or person. Therefore, the development of mathematical modeling is very actual. However, unlike in engineering sciences in economics the formal logic is still weak, there is no accurate system of assumptions (as, for example, axiomatic approach in mathematics), being accepted by the majority of experts or being meant by default. It leads to decreasing economic researches efficiency and narrowing applicability of results. Nevertheless, having assumed some idealization, having picked up the corresponding abstractions of economic system dynamics it can be approximately described by the mathematical equations reflecting cause-effect relationships.

In this book the following economic issues are anyway touched:

* forecast of exchange currency rate,
* forecast of product’s price at the goods exchange,
* management of investments (capital investment allocation),
* macro-economic dynamic models,
* models of rivalry and interaction of several firms on the common market or case of using common limited manufacturing resources
* oscillations of economic activity,
* models of international trade,
* management of level of customs import tariff,
* dynamic models of advertising activity and some others.

Some of the described above tasks have been completely carried out, that is the solution algorithm has been given (in some cases results of modeling have been also given), the other tasks have only been analyzed and ways of making models have been outlined. Here it is not the neglect of completed forms but the consequence of author’s another direction. It can be explained by striving for increasing instrumental possibilities of the theory of modeling dynamic processes. Conditionally one can split the theory into linear and non-linear parts. In the first part which is widely used at present there are some weak points. These weak points include uncertainty in choosing the length of identification interval, criteria diversity of model quality and as a result impossibility to consider one model better than another model.

Thus offering the model of dynamics to an economist a researcher says that this model is optimal for the given set of the observation of dynamics of an experimental parameter, for the linear model class and for the given (as a rule, a square criterion in one or another sense) criterion of model quality. An economist does not want to go into details, why say linear or square and which data are necessary for a researcher, he needs a reliable model with which he could make a decision properly, i.e. a criterion of optimality must be set by the ultimate applied purpose of modeling. These and other questions of the linear model theory are discussed in the book.

However, the main aim is connected with the author’s desire to include logistic equations with time delay in the system of instrumental means of the modeling theory. In biology such equations are known as Lotka-Volterra equations. They have been a focus of attention for a long time but without taking into account time lag and also without algorithms of coefficient identification their use one cannot consider worthwhile. It is shown in the book that the use of Lotka-Volterra equations is quite constructive and a dynamic diversity exceeds the behavior forms of trajectories of linear systems. This concerns more complicated structure of balanced identity, oscillation and asymptotic behavior.

In economics time lags (delays in system reaction on a situation change) have been used long ago in the description of dynamic problems but the complication of mathematical apparatus of the theory of equations with aftereffect does not allow using this theory widely. However, it would be a mistake to ignore time lags completely. So the author gives the analysis of dependence of economic conclusions on the results of mathematical modeling from taking into account time lags. Numerous examples of economic situations with estimation of time delay have been given, competitive activity of firms with delay in reaction has been considered and what is the most interesting, the algorithm of joint identification of system and delay parameters has been given. In the appendix many (but not all) facts of the theory of differential systems with aftereffect have been collected, but certainly they are mostly intended for mathematicians than economists. The most part of material has been published for the first time.

Structurally the book consists of four chapters and three appendices. The first chapter has an introductory character. It contains initial definitions of the modeling theory, the author’s point of view on compatibility of qualitative notions of mathematics and economic dynamics, historic palette of one of the divisions of economic dynamics i.e. the growth theory. The second chapter is devoted to the analysis of time series in wide sense. Two components of time series (determined and random) are normally distinguished in such tasks. In this book the focus is on the determined part of dependence, whereas the random component is estimated by any known method. The building of a time series trend as a process model is carried out by three ways: by direct approximation, i.e. from the given function class we choose the representative that draws nearer a time series in the best way; by building autoregressive models (as well as building models with delays and models having integral diversity); by building models of controlled dynamic processes, i.e. when a model is built on a known system response on a given input stimulus.

The third chapter contains the description, analysis and economic interpretation of Lotka-Volterra type models. Simple two-dimensional systems as well as multivariate systems have been investigated. Qualitative characteristics of solutions have been examined. In particular, the existence of boundary cycles has been proved, the asymptotical stability conditions of all balance positions have been determined. New algorithms of identification of Lotka-Volterra models without delay as well as with delay have been offered.

In the fourth chapter we investigate some specific models using real data: currency rate forecasting for long-term forward contracts; a model of interaction between a government, an importer and a smuggler in the international trade in order to determine an optimal import tariff, the competition on the USA car market and a model of sales response to the volume change of advertising for a certain new product. While the first of listed above tasks uses to the full extent the results and recommendations from the first chapter, then the construction of the second and the third models is based on economic peculiarities of the task which gives independent significance to the models.

The first appendix, as it has been already noted, demonstrates different aspects of the theory of differential equations with aftereffect. In full we list those theorems and describe those methods on which somehow or other the speculations of main chapters are based. The material is given with complete proofs as the author would like to create autonomous work so that a reader does not need any additional literature.

The second appendix contains the description of a known method used to determine the optimal size of a manufacturing enterprise from cost perspectives and existing prices for production factors.

In the third chapter the theory of production function is given, the overview of different forms of representing a production function is given and their peculiar features are distinguished. The task of influence of scientific-technical progress on the production process is considered.

Bibliography contains only factual references which show the origin of one or another result. The numbers of formulae, theorems, conclusions and figures are given in each chapter.

**Chapter 3. Lotka-Volterra models in economics**

The given chapter contains the description, the analysis and the ways of application of one remarkable model which is known as the Lotka-Volterra model in modern literature [146], [133]. Most often it is used for the description of processes in biology, medicine and ecology [108]. Recently economists have become interested in it [91]. We will develop this model further on supplementing it by new results and applications.

The structure of the chapter is organized in this way in order to encourage the economists who are not inimical to mathematics to apply the methods of nonlinear dynamics in economic processes as widely as possible. At the beginning of the chapter the formal-economic statements of dynamic situations, where it is impossible to avoid nonlinear models, are described. A special role in modelling is played by the time lag. The equations with the time lag have an adequate interpretation in economic terms. But it is not enough. It appears that the attempts to ignore the existence of time delay in the cause and effect relations result frequently in the erroneous conclusions.

In this chapter the nonlinear model of Lotka-Volterra with a lag argument is investigated from the point of view of stability and instability of the equilibrium states, an oscillation, an attraction domain, etc. The identification of model factors and lag values takes an important place in this model. All these results, incorporated in one book, give the universal analysis and constructive algorithms, which allow gradually to reduce the use of linear models in applications and to replace them with richer nonlinear models with a lag.

* 1. **The basic ideas, methods and results**

**3.1.1. The model description**

It is considered that the first formalization of dynamics of biological populations belongs to Maltus, who assumed that the change rate of species number *N* in population is proportional to the volume of population:

.

Naturally, this equation leads to the exponential growth of the population volume that is not observed (for a long period of time) in nature. The next model excluded the infinite growth and took into account an effect of saturation. It appeared in the work of Verhulst P. F. in 1838 under the name of a "logistic" model for the description of the biological population dynamics:

 

Here, the factor of increase *ε* is replaced by the linear function from the volume of population that is interpreted as follows: if the vital resources, such as food, life space, power resources of population are limited, the volume of population will be stabilized with time instead of growing indefinitely. In the last equation such level of volume of population is equal to  .

 In the most general form (without a lag) the logistic model was considered by A.N. Kolmogorov [69] for community of *n* kinds:



The functions  satisfy some special properties which allow a simple biological interpretation. We will not stop on this generalization because too large arbitrariness in choosing the specified functions hinders the creation of constructive methods of the model identification.

 V. Volterra used the linear functions:

 (1)

He investigated the cases *n*=2, 3, and also made some generalizations for the system with an arbitrary *n*.

The work of V. Volterra is of a special importance because being an expert in the field of the integral and integro differential equations, he introduced the lag into the consideration, i.e. he assumed that the interaction of biological species on common resources changes the rate of increase through a time interval. So in the task "predator - victim", the volume of population of "victims", being a food for "predators", is supposed to define the volume of population of the latter during a vegetative season. As a result of some logic steps V. Volterra received a system with the distributed lag:



Here *N1*, *N2* – the volumes of populations of the "victim" and the "predator" correspondingly, and *F1*, *F2* – the functions which characterize the distribution of "predators" and "victims" by ages.

If we consider, that there is the only lag, the Lotka-Volterra’s model with a lag without any restrictions on factors will have the form:

 

It is the system of the differential equations with a lag to which we will give the basic attention.

* + 1. **The research purpose and analysis methods.**

Since this book is supposed to be read by both mathematicians and economists, we mark those tasks, which can be solved by the offered methods. From the mathematician’s point of view the system of nonlinear differential equations with a lag is an independent object of research. In this case mathematicians usually consider the following questions:

* existence and uniqueness of solutions of the Cauchy’s task on the finite and infinite intervals of time;
* existence of stationary points (positions of the system’s balance);
* in more general sense, existence of invariant sets;
* the research of properties of stability and instability of the invariant sets;
* oscillation of solutions, i.e. the existence of periodic, almost periodic or recurrent solutions.

As the theory of differential equations with a lag is not yet an integral part of education of mathematicians, a large appendix, which is actually a contemporary textbook on the mentioned above problems, is given. All the necessary facts of the theory are given in it with the detailed proofs, which from our point of view are important for those mathematicians, who would like to apply their efforts in the theory development. Here, in the text, we will be limited by the unproved statements and will refer a reader to the corresponding place in the appendix.

For an economist the application of mathematical models with the purpose of forecasting the behavior of the system is connected with a number of conditions, which require the answers on very complex questions from the formally mathematical point of view:

* what economic assumptions is the model based on?
* how to identify all the parameters of the model and by what information?
* whether the qualitative pictures of behavior of the economic system, which are being forecast, are a trivial consequence of the assumptions?

In this sense, the conditions of adequacy and expediency of modeling frequently lead economists to two extremities: either to trust the model and its formal consequences undividedly, or to avoid mathematical forecasting of behavior of the model. We would like to see a substantial application of the model to the qualitative and quantitative forecasts of dynamics, but, at the same time, with a fair interpretation of economic assumptions.

One biologist, an expert in evolution of species, once asserted that "yeti" had disappeared as a biological species because of a competition with the human being for common life resources and this is in exact correspondence with the conclusions made by Volterra, which he received on the basis of the model of competition of two species for one life resource. Really, in [146] the model is considered:

 

where  such a positive function, which is permanently growing by each argument, that .  is a function biologically interpreted as an amount of food being eaten during a time unit. According to Volterra, the volume of population of one of the species is tending to zero as time goes on, whereas the volume of another one reaches some finite limit.

However, the human being and "yeti" eat different food; they use natural resources and energy in different way. And if the human being is constantly improving the technologies of life-support, "yeti" adapts himself only biologically. Therefore, the last model without a more detailed analysis of factors of reproduction in the square brackets in the equations cannot be applied to these two species.

It is possible that by virtue of the same Lotka-Volterra models with a lag, which will be discussed below, the dynamics of the "human being-yeti" system has fluctuations with a period of several millenniums, according to which the civilization of the human being will collapse, and "yeti", who is more adaptive biologically, will increase his population on the Earth for the next millenniums.

In any case, before we formulate the forecast by the mathematical model it is necessary to confirm its adequacy in some way. Unfortunately, an economist has not enough information to do it. A biologist can test the model even in the laboratory conditions, i.e. repeat all the dynamics with the same external impacts. An economist is deprived of this opportunity. He or she can only collect all the facts available in a history and separate those, which are close to the situation being considered. However, here the subjectivity of a researcher frequently plays the principal role.

* 1. **Economic interpretations**
		1. **General reasoning**

In this section we will discuss the mathematical formalization of dynamics of interaction between several firms, branches or states. Let us begin from the most simple: two competing companies in one economic niche, i.e. with common resources, consumers and identical goods or services. Assume that the price is constant when the volume of the product in the market is changing (i.e. the firms under consideration are not monopolists), and that the product is not a substitutable one. Examples of such rival firms can be two transport companies on transportation of cargoes or people in the given city. They can change organization of work, technology of service, introduce discounts, spend money on advertising, buy more perfect vehicles, reduce cost of repair and storage, choose new optimum routes to win the competition. It is possible to consider their interaction on the background of existing, powerful and not subjected to frequent changes the public transport facilities, i.e. to assume that a consumer will refuse to use the services of the considered companies when public transport becomes more preferable in its price or in quality.

The other examples are bakery goods production, apartment renovation service automobile repair service, tourist business etc.

It is clear that such competitive models can have large number of firms but this complexity will be possibly overcome by simple summation of goods volumes in the market through uniting the firms with close technologies of goods (or services) production.

A more complex example of economic interaction is the system of the global market. As the world resources of labour, capital, energy carriers, land, etc. are limited, the dynamic model of a competition between the countries under consideration may appear to be adequate. Certainly, it is impossible to construct the system covering all states and all goods, but it is reasonable to be limited by associations of countries with approximately equal GDP: USA + Canada + Mexico, Europe, Japan + S. Korea + Taiwan, Russia + Brazil + Arabian countries, etc. Or otherwise: the Northern America, the Southern America, Europe, Asia, and Africa.

For such model the economic interpretation will require much more complex assumptions: there are many companies which are international in manufacture and selling goods so it is difficult to ascribe them to a country or a continent. So "Ford" has branches in Europe, Southern America and Asia. Japanese and Korean firms of household electronics create their branches in America and Europe. German chemical enterprises have branches in Southern America and Asia. Mixing of capital and manufacture complicates the creation of a clear and absolutely "pure", in sense of logic, model. However, the support on such integrated highlights as GDP and life index could give quite an adequate model.

Note that economical agents not always struggle only for general resources or consumers. In each state there are branches whose interaction is more complex. Let us consider, for example, agriculture, industry, extractive sector of economy and the budgetary funds of the state. They compete and develop each other at the same time. As there are no absolutely isolated countries in the world, then there can be the states with one dominant sector from the named above. Let us conditionally agree that Burma, Thailand and Nicaragua are agricultural countries, Kuwait and Venezuela are extractive, England and Japan - industrial, Germany, Sweden and Luxembourg - budgetary. Nevertheless, in the majority of the countries all from the named above economic sectors function and they naturally cooperate. The governments and the parliaments try to control this interaction, for example, through introducing the special export-import taxes or supporting agriculture by special procurement prices.

Many economists tried to create the mathematical models of business cycles. Some of cycles have the names: long and short Kondratyev waves, seasonal cycles, etc. The others are told about: “ … there is an obviously expressed tendency to periodic fluctuations.” However, this phenomenon, variability, is not formalized up till now. Individual attempts (such as, for example, discussed below the Goodwin model) are unsuccessful from our point of view. The typical explanation of fluctuations based on introduction of new technologies does not seem to be convincing because in a wide spectrum of sciences and with regard for a chance of making a substantial discovery or invention the periodicity or even general law can hardly appear, except for increasing summation of all achievements. We will specially discuss this problem below.

* + 1. **Firms competition in the common market**

At first we consider the dynamic model of one enterprise. Let some firm, having a fixed capital *K*(*t*), attracting a labor force *L*(*t*), using the natural resources (raw material, water, energy, land, etc.) *R*(*t*), receive the volume of goods *x*(*t*). For the tasks with a complex structure of using the natural resources it is possible to consider conditionally, that *R*(*t*) is a working capital. Similarly, *K*(*t*) is a fixed capital. The volume of goods *x*(*t*) is expressed in the current prices. To simplify the consideration of economic assumptions at the first stage, we will fix the current prices. It means that the firm under consideration does not influence with its volume of goods on the market equilibrium price, i.e. it is not a monopolist. Designating the production function of the firm as *ϕ*, we receive the first mathematical expression: 

We will not consider in detail the theory of production functions, we will say only that *ϕ* and all its first derivatives are strictly positive (differentiability can be piecewise) [113]. Hereinafter, *t* designates the continuous or discrete time. If the time is continuous, we will designate the change rate of any variable as  or ; if the time is meant as discrete, the rate of change of any variable with time will be written as the first difference: *Δx*.

To expand the manufacture, the management of the firm should expand its fixed assets capital and use more labor and natural resources. Let us accept as the elementary variant of research, that the fixed assets requires amortization deductions on their maintenance, which are to be in proportion to the volume of the fixed assets, and on the development – the investments, *I*(*t*). Then, let the used resources be proportional to the fixed assets included in manufacturing:  The firm does not spend its profit for anything except the development (the tax component is considered as proportional to the output and is included in the production function): .

By adding to these assumptions the requirement of linearity of the function *ϕ*, we receive the equation of change of the fixed assets:

 

where *β* is a multiplier of depreciation of the fixed assets; . The last equation gives an exponential growth of the fixed assets, if . The output of production correspondingly will be growing or falling exponentially:

.

Thus, the model:

,

with the constant of growth rate *ε* characterizes the dynamics of one enterprise at the absence of any economic restrictions. But the restrictions always exist:

• the manufacture growth results in saturation of the market and in decrease of demand;

• in the time an obsolescence of goods or services takes place;

• the expansion of manufacture requires attraction of labor but if the labor market is limited, the expansion achieves the top limit;

• natural resources are always limited either in volume or in price when the demand on resources is growing (it concerns the labor as well).

To reach the Lotka-Volterra model, it is enough to present the multiplier *ε* as a decreasing linear function of growing value *x*(*t*) (a linear one as it is the simplest decreasing function) in the last equation: , with constant . Since the logic of these actions is clear, we will do it for a two-dimension case.

So, the first firm has the production function , and second - , where  and  - homogeneous linear functions. Let us assume that each firm spends their profit only on investment and the expenses for labor resources and working capital are proportional to the corresponding used fixed assets. The coefficients of proportionality, owing to the assumption of resources limitation (in the common market of labor and raw materials), is considered as linear decreasing functions *Kx, Ky* . Then the changes of the fixed assets of both firms consist of depreciation and investments:

 

Substituting the linear expressions as the factors for the labour and raw materials

 

we receive the system in relation to *Kx, Ky*:

$$\left\{\begin{matrix}\dot{K}\_{x}\left(t\right)=K\_{x}\left[-β\_{x}+φ\_{K}+\left(φ\_{L}-1\right)l\_{0x}+\left(φ\_{R}-1\right)r\_{0x}-\right.\\\left.-\left(\left(φ\_{L}-1\right)l\_{1x}+\left(φ\_{R}-1\right)r\_{1x}\right)K\_{x}-\left(\left(φ\_{L}-1\right)l\_{2x}+\left(φ\_{R}-1\right)r\_{2x}\right)K\_{y}\right],\\\begin{matrix}\dot{K}\_{y}\left(t\right)=K\_{y}\left[-β\_{y}+ψ\_{K}+\left(ψ\_{L}-1\right)l\_{0y}+\left(ψ\_{R}-1\right)r\_{0y}-\right.\\\left.-\left(\left(ψ\_{L}-1\right)l\_{1y}+\left(ψ\_{R}-1\right)r\_{1y}\right)K\_{x}-\left(\left(ψ\_{L}-1\right)l\_{2y}+\left(ψ\_{R}-1\right)r\_{2y}\right)K\_{y}\right].\end{matrix}\end{matrix}(3)\right.$$

To recognize the system (1), we redesignate the multipliers:







.

The economic sense of the multipliers *β* is clear. Let us interpret other parameters. The values *ϕK, ϕL, ϕR, ψL, ψK, ψR* characterize the production functions of both firms and, hence, we can consider them as given. Under the assumption of production functions these values are positive. The multipliers *ε* and *γ* should be identified according to the information on the volumes of fixed capital, *Kx, Ky*. The coefficients *l* and *r* are remained to be identified. But there are 12 of them in the six mentioned above equations. So they will receive a special consideration in the section on identification below.

The common resources limitation results in that with the growth of fixed assets *Kx, Ky* the multipliers of capital renewal become equal to zero and then become negative, i.e. the assets begin to decrease. The area in the plane of variables {*Kx, Ky*} when the fixed assets are not being withdrawn from manufacturing, is expressed analytically by inequalities:



Now it is clear, that the mutual competition *n* of the firms for common resources can be described by the system (1), however, the model parameters *ε* and *γ* are not of an abstract character, i.e. they represent some quite certain dependencies on economically interpretive parameters.

* + 1. **Dynamics of hierarchically connected branches**

The interaction of branches is very similar to the interaction of enterprises: branches also have fixed assets, involve labor and natural resources and, hence, compete with each other. However, here the dependence of branches is also possible.

Up till now we assumed that the parameters *ε* in the system (1) are positive, i.e. in absence of competition and limited resources all agents are developing exponentially (without any limits). But there exist branches whose dependence on other economic agents is much more complex. For example, the budget branches of the state (medicine, the armed forces, education, science, etc.) are not self-financing and suffer when the budget of the country is decreasing. Since the budget mainly consists of taxes, and they are proportional to the output of other branches, the negative factors *ε* can be expected to appear.

Let us consider the problem of subsidized agriculture. In economic history of our country (Russia) there were various relationships between agriculture and industry: the serfdom was replaced by patriarchy agriculture based on collective possession of the land. Stolypin’s attempt to make agriculture more productive through the initiative of individual peasants with private land ownership resulted in revolution. The socialist governing had no results in achieving the stability in agricultural production as well: the New Economic Policy (“NEP”), dispossession of small producers, the collectivization, the "price scissors", restrictive measures in manufacturing and fixed capital, outflow of the labor force from villages, unprepared and non-supported economically (and legally) farming – this is an incomplete list of strokes made to the agriculture of our country in the 20th century.

It is interesting that in the 80s of the 19th century, having considered the profitability of the grain growing in the Russian Empire, D.I. Mendeleev showed in detail with the analysis of the statistical data that in Russia with its unstable climate it is impossible to have reliable production of grain and that the huge export of rye to Europe at the end of the 19th century was a consequence of severe exploitation of peasants and malnutrition of the Russian citizens. His point of view was shared by a famous political and economic statesman of those years S.Yu. Vitte.

But in the industrialized countries, such as the USA, France or Germany, the agricultural production is a subject of permanent anxiety of governments, which are forced to make special state investments or to push the tax privileges through the parliaments or to do the budget purchases by overpricing. In economic history there are a lot of approaches in helping domestic farmers beginning from creating protectionism barriers on the way of a cheap import and up to price supports by the state for purchasing combines and tractors. All these approaches eventually result in the mathematical model of "predator - victim" which in the economic context means that the agriculture "parasitizes" on the industry. The model (1), to be exact the model (3), will have the following form:

  (4)

All the coefficients are positive here. Then we research this system and extend it for a multidimensional case.

* + 1. **International competition**

The economic interaction of states is defined by a set of various reasons of economic, political, geographical, historical content. There are no mathematical models which cover all manifestations of the interaction. As a rule, only its specific kinds are analyzed: for example, political group interests of states are investigated as the solution for a game problem, or a question on the trade flows is reduced to searching the extreme point of the function of many variables (the function of income). In view of complexity and multiplicity of interstate relations the models have mainly a static character.

Among all kinds of interactions between states we will concentrate only on the trade ones, ignoring military, political and other non-commercial relations. It is conditioned by a relatively comprehensive analysis of foreign trade which is based on the D. Ricardo's law of comparative advantages.

For almost two hundred years this law has been improved, changed and rejected but it is given in all textbooks on foreign trade. Moreover, the classical theory of comparative advantages in international trade was supplemented by the monetary theory, which is based on the payment balance in the export-import transactions of countries. Hence, this section of the economic theory can be considered as prepared enough for the mathematical modelling.

Here we formulate some known statements of the theory of international trade according to the book by P.H. Lindert "International Economics" [80].

The D. Ricardo's law says: *each country has a comparative advantage in manufacture of some product and receives a gain by trading with it in exchange for another one*.

The following basic statement bears the name of "the Heckscher - Ohlin theory" (both are Swedish economists, the former is a student of the latter): *the products, which require significant expenses (excessive factors of manufacture) for their manufacture and low expenses (scarce factors), are exported in exchange for the goods made using the factors in a reverse proportion. In such a way the excessive factors are exported in their latent form and the scarce factors of manufacture are imported.*

This conclusion does not always correspond to the economic statistics; therefore an American economist P. Samuelson limited the application of the last statement to several conditions:

* *we are talking about the case of two countries, two products and two production factors;*
* *the offer of the factors in each country is fixed and their transporting is allowed between the sectors inside the country, but not between the countries;*
* *the countries differ one from another only by provision of production factors;*
* *both countries have such a technology which provides a constant scale effect.*

Besides, in the distributive part of the theory P. Samuelson (together with W. Stolper) established that *at the same preconditions the establishment of trade relations and free trade will inevitably result in growth of compensation of that factor, which is mostly used in manufacture of the product and the price on which is growing; and in reduction of compensation of that factor, which is mostly used in manufacture of the products, the price on which is falling; without dependence on the structure of these products consumption by the owners of the manufacture factors.* It follows from this that with the beginning of foreign trade the strained relations will be growing in the society.

One more important fact is named "the theorem of leveling prices on the production factors" (it also belongs to P. Samuelson):

**if** *we add to the mentioned above preconditions the following ones:*

* *all markets are competitive;*
* *in both countries each factor is used with complete loading both in conditions of international trade and without it;*
* *there are no transport and information costs;*
* *the trade is free (without any tariffs);*
* *the production functions have no features of "convertibility of the factor intensity";*
* *after establishing of trade relations both countries continue to manufacture both products;*

**then** *foreign trade results in leveling the prices not only on products, but also on production factors without depending on the structure of demand or provision of factors in each country.*

The theory of international trade highlighted a problem of mobility of production factors both from its positive and negative sides. For example, the moving of the labor factor, i.e. the migration of people in searching for a job, which can satisfy them, is not essential, since the export of labor-consuming production is equal to moving of the labor factor. But, on the other hand, **the Rybchinsky** **theorem (**which is somewhat dual to the Stolper-Samuelson theorem) takes place: *the increasing offer of one of the factors results in a larger accelerated increase of output in that sector where this factor is used more intensively, while the absolute reduction of output rate is observed in the other sector.* Henceit follows that the other factors are attracted to manufacture with a hypertrophied (and lowered in price) factor leaving those manufactures where they dominated*.*

Thus, open foreign trade acquires new and difficult-to-formalize changes in the structure of manufacture and consumption when solving some state problems by the expansion of markets and the production factors base. But protectionism has also significant disadvantages of the general economic character. Though, in one way or another, governments constantly resort to different forms of protectionism, in its historical horizon, the protection of the domestic manufacturer frequently turns into preservation of disadvantages of development, creation of monopolistic groups, establishing of unreasonably high prices on goods, etc.

* + 1. **Kondratieff waves**

Nowadays a structural transformation of economy from planned methods of management in conditions of complete state ownership into market methods of manufacture, distribution and consumption of basic material wealth is occurring in Russia. It is natural that this change in the system of manufacture and proportions of consumption results in instability of conditions of macroeconomic parameters. They are sharply changed submitting to not only the economic laws but also to non-stationary political influences.

Therefore, the construction of mathematical models for unsteady, transient processes and their use in forecasting requires special substantiation. If we look at the change of parameters of the given country for a long period of time (hundreds of years), it is possible to consider some high-frequency fluctuations as integrally negligible. And then, three main components of the change can be distinguished:

а) more or less monotonous trend;

b) low-frequency fluctuations with a period of about 50 years (long Kondratieff waves);

c) high-frequency fluctuations with a period of about 12 years (short Kondratieff waves).

In 1928 our remarkable compatriot Nikolay Kondratieff gave a powerful impetus to study fluctuating processes of macroeconomic parameters in his work "The dynamics of prices of industrial and agricultural goods" [70]. After the discovery of long and short waves in fluctuations of prices, there appeared a lot of works which describe similar phenomena in business activity, the parameters of demand, etc. This question is very topical both in terms of researching the reasons of dynamics and in terms of forecasting the economic development.

Nowadays in Russia it is difficult to distinguish definitely one or several reasons which have a profound impact on the current processes, but in the scale of centuries it is quite possible to make some ideas and even the models of cause and effect connections in macro economy of the country. The majority of researchers are inclined to consider the technological innovations as the reason for the existence of long Kondratieff waves. We should learn to distinguish those inventions which are of the most importance for the considered macro parameters.

As this problem is unlikely to have a simple solution, the tools of the diffusion theory or of psychological conflict are involved for the description of fluctuations. The search of reasons for fluctuations is going on with great intensity: the scientific funds, associations, conferences publish several magazines. Apparently, the laws of fluctuations in economy will soon be established. In this book we would like to offer one of the approaches for the description of fluctuations which is based on the model Lotka-Volterra with a lag.

The basic cause, which has resulted in this model, is the limitation of economic space, to be more exact, the limitation of production factors used by the economy as well as a long reaction time of the economy to the change of environment. N.D. Kondratieff himself [70] also writes about it but without accents on the limitation of resources: "We can speak about the balance (and about the prices of balance) as applied to a longer period within which not only the demand, but the supply as well change: however the latter changes on the basis of the same fund of the basic capital wealth (the production factors). At last it is possible to speak about the balance (and about the prices of balance) as applied to a longer period, during which not only the demand and supply change, but also the amount of the mentioned basic capital wealth changes. This wealth (the largest building structures, melioration, skilled manpower, etc.) has an ability of long service. However, its creation requires a lot of time, which is longer than a usual commercial and industrial cycle.” The last paragraph of a quoted fragment pushes to the introduction of time lag in the mathematical model of fluctuations because without delay the Lotka-Volterra model does not suppose any fluctuations around the stationary point.

* 1. **Lag in economic models**

In this section some notes of methodical character are given. They appeared after acquaintance of the author with a plenty of mathematical dynamic models in macro- and microeconomics, to be more exact, in those dynamic models which use differential (and difference) equations with time lag (or delay of the argument). Many economists appeared to understand the significant importance of a time lag in cause and effect relationships of dynamic processes and are trying to reflect it in the appropriate models. However, quite complex mathematical technique of differential equations with a lag does not allow revealing all its advantages in full or at least up to a numerical result. In some cases the ignoring of a lag leads to inadequate results (further the first example is devoted to it). In other situations the attempts to achieve some special properties of the model are made. It is done through introducing the lag (or even an advancing argument) of the argument but in doing this inadequate transformations are assumed. The well-known model of Goodwin is considered by the author as an illustration of such approach.

The description of solutions behavior dependence on the value of the lag will complete the section. The form of periodic solution for a scalar case will be also given.

* + 1. **The model of the price dynamics in the neighbourhood**

**of the equilibrium point**

It is well-known [125] that in an ideal case the price for some product *p* is determined by the equilibrium of demand *D* and supply *S*. In fig. 9 this equilibrium corresponds to the point *E* on the plane (*q, p*), i.e. to the quantity of the product and its price.



Fig. 9. The equilibrium point *E* in crossing of the curves

of demand and supply

It is also known, that the state of equilibrium will be attracting (asymptotically stable in Lyapunov sense), if the angle *α* will be larger than the angle *β* (we mean the corresponding angles of tangents slopes to the curves of supply and demand). It is being proved by drawing the trajectory of price change in fig. 9 around the state of equilibrium [91].

If it is necessary to receive an analytical expression of the price as a function of time, then one of the following equations is usually used:



where  are the functions, which are inverse for the corresponding functions of supply and demand (in fig. 9 the inversion is reached by replacement of the price axis by the quantity axis and vice versa). In this case it is indifferent what equations are used and, therefore, we are stopping on the first. For the simplicity we consider that supply and demand are the linear functions of the price:



Here , therefore, the corresponding cotangents are necessarily positive. The equation for the change of the price will have the form:



Obviously the state of equilibrium of *pE* in the last equation is asymptotically stable in Lyapunov sense for any relationships of *α* and *β*, which contradicts the rule mentioned above. Taking into account, that the supply comes to the market with a delay *h* connected with the working cycle of manufacture of the given product (this fact is marked in [91]), we receive an adequate model:



Now the state of equilibrium *pE* is asymptotically stable at any delay, when  i.e. when 

For the last conclusion and the following reasoning to be understandable, it is expedient to give the analysis of the elementary equation with one concentrated lag [115] (also Appendix 1):



In the plane of parameters *a* and *b* the area of asymptotic stability has the form shown on fig. 10:

Fig. 10. The area of stability in the space of parameters (*a,b*)

So the introduction of the lag into the model allowed removing the obvious contradiction. Let us notice that sometimes the technological delay is absent or very small. The situation, when the product is available in sufficient quantity in the wholesale warehouses, corresponds to such case. But it is a section of cooperative games in the market.

* + 1. **Macroeconomic models with a lag**

Macroeconomic models connecting GDP, investments, fixed capital and consumption were considered by P. Samualson, J. Hicks, M. Kaletsky, R. Goodwin and others. The purpose of macroeconomic modeling is, as usual, a forecasting, the definition of tendencies and an attempt of the state management of the public economy. In the given work only one part of all dynamics is considered: the cyclic fluctuations produced by internal causes but not generated by some exogenous component for the chosen system. R. Goodwin constructed a mathematical model [45], emphasizing the nonlinearities and introduction of a lag for the received equations to have a stable limit cycle.

In the work [4], almost after 40 years D. Arrowsmith and K. Place repeated factually the process of modeling of R. Goodwin, having remained his logic and result. In our literature the similar models also repeatedly appeared, for example [52], and with a clearer logic of assumptions. However, in the large review [109] by A. Pomansky and G. Trofimov, the logic of Р.Goodwin is given again.

Generally speaking, the economic assumptions by R. Goodwin cannot be considered as adequate and the construction of the model causes the critical remarks, but here the main attention will be paid only on a mathematical side, namely, on an inaccuracy of mathematical transformations.

So, let at any moment of time *t* the national economy has the fixed capital *K*, the volume of manufacture (GDP) *Y* with a total consumption *C*. These values are connected between each other by the relationships:

 (5)

where the first equality means that the consumption depends linearly on the production volume and the second - that everything produced is spent either on consumption or on maintenance of the fixed capital.

Then R. Goodwin offered to operate the investments in such a way that the fixed capital oscillates around the value *γY* within the time. He suggested an impulse-relay function of *Y* as a control signal. It is natural that the volume of manufacture and the capital appear as piecewise-linear or piecewise-constant functions of the time. To smooth the jumps of the GDP, R. Goodwin forgot that the second equality in (5) means the distribution of the GDP and he said that it is the law of changing *Y* by the control  and he introduced the first lag *ε* :



As it is actually a nonlinear functional equation and its solution is problematic, *Y* is expanding into a series by *ε* and only the term of the first order is kept in the equation:



For the second time, striving to keep the variability but at the same time adding a gradual growth of economy, R. Goodwin believes, that the investments (control) depend on  nonlinearly and smoothly. To be exact, mathematically we receive the equation of an advanced type but its theory is far from being perfect. But expanding the corresponding functions in the Taylor series by degrees of the second delay *ϑ*, it is possible to receive the differential equation of the second order with smooth nonlinearity relatively to the speed *Y*:



It is the equation that has a stable limit cycle.

Thus, if the mathematical transformations were correct, the existence of stable fluctuations around the main trend of macro parameters of economy would follow from the clear economic preconditions. But are these transformations correct?

Below, the example of how the R. Goodwin’s operation made twice changes the qualitative properties of the model is given.

We consider a scalar equation with one lag

 .

It has an asymptotically stable zero solution at ; permits the periodic solution  at  (with any *A,B*) and is unstable in all the rest cases. It follows from the corresponding analysis in Appendix 1 (section 3).

The expansion of the right hand side of the last equation in a series by the degrees of *h* results in the ordinary differential equations of such order how many members of the expansion are left in the equation. In the first approximation the equation has the form:



At  the asymptotic stability takes place, at  the zero solution is unstable and there are no any periodic solutions. In the second approximation the initial equation turns into

,

which behaves itself as the equation of the first approximation, but at  it permits a periodic solution with the frequency . The approximation of the third order always results in instability. Thus, the qualitative picture in the solutions behavior of the precise model and the approximated one depends essentially on the value of the lag. If the equation of an advanced type is used in modeling, the stationary solution is always unstable there.

Thus, the ignoring of the time lags in forecast models can result in inadequate conclusions but also, the superficial manipulation of the instrument of equations with the argument lag can reduce the utility of mathematical modeling. It is impossible to consider the conclusions about the existence of a unique stable limit cycle as outgoing from the economic preconditions because the latter were strongly deformed by the subsequent rough transformations. The use of mathematics with an insufficiently advanced theory can result in inadequate predictions, though it is a very good stimulus for mathematicians in searching the new tools.

* + 1. **The change of qualitative behaviour of a model at**

**the increase of a lag.**

As has been just shown before, the vanish of stability is a characteristic feature for linear systems at increase of delay, and as the theorems of stability by the first approximation for the systems with a lag take place in general case, it will be always correct locally. But nonlinear systems at a big distance from the stationary points may reveal other properties. Let us consider qualitative change of a solution at increasing of delay on a scalar example of the Lotka-Volterra model.

At first we transform the scalar logistic equation, which was a prototype of the considered model, to the simplest kind. Namely, the equation



where *a,b>0* ; by replacement of the time *t=hs* and of the scale of measurement *x=y/b* reduce to the form:



Or in the integral form, which is more convenient for computing procedures, the last equation takes the form:

 (6)

We can assume *a=*1 without loss of generality and then the following conclusions are fair [13]:

• *y=*1 - the state of equilibrium of the equation (6);

• if the solution is below the state of equilibrium longer than a time unit, then *y* is growing;

• if the solution is above the unit longer than a time unit, then *y* is decreasing;

•as the solution cannot grow up more than  for a time unit, the value  limits the solution from above;

• locally, in some neighborhood of the unit, as has already been mentioned, the solution is asymptotically approaching the state of equilibrium at . If , *y=*1 will be unstable by Lyapunov.

The computing procedures are showing that at small *h* the asymptotic stability will be global in the positive area of the initial data, and at  around the non-trivial state of equilibrium there will appear periodic oscillations, which will be also globally attractive.

The type of these oscillations at increase of delay *h* is represented in figures below. Fig. 11 corresponds to *h=*1.6 and is similar to the sine-shaped oscillation but it reveals the tendency, which is shown with all its evidence in fig. 12 and 13:

Fig. 11. The solution of the scalar logistic equation with a lag,

which is a little higher than a critical one.

The bottom part of the oscillation is stretched horizontally and the top part -vertically.

Fig. 12. The lag is significantly bigger than the critical one.

Fig. 13. The delay is twice higher than the critical one.

The areas of the negative and positive parts in relation to the state of equilibrium remain equal. Indeed. If we want to talk about the periodic solutions, we may add the equality  for any *s* and the period *T* to (6). Then from (6)



or after transformations



that actually means the equality of the corresponding parts. Also, with the help of computer calculations it is possible to receive the dependence of the maximal amplitude and the period on the delay size *h*. In fig. 14 such dependence is given.

Fig. 14. Dependence of the period and the amplitude

of the stable oscillation at the delay increasing.

Here the abscissa axis corresponds to the delay size, a continuous line is a diagram of the period and the dashed line is the diagram of the amplitude.

* 1. **The complete analysis of the economic agents interaction**

In the given section we consider the situations with (and without) a lag in two-dimensional and many-dimensional cases. Let us begin with the discussion of possible states of equilibrium and the variants of behavior of economic agents. After having found out situations different from the mathematical point of view, we get down to their analysis moving from simple to complicated. The theoretical (i.e. absolute) facts will be alternated with partial (but evident) illustrations received by computation. The economic interpretation of results will be made in the conclusion of each subsection.

* + 1. **Stationary points of a model**

So, we consider the system of the differential equations:

  (7)

where *xi* characterizes one of *n* competing firms. It can be a fixed capital or an output of the product. The lag size will be especially supposed when the necessity appears.

By definition the stationary points of the system (7) can be calculated as a solution of the nonlinear equations:



where the coefficients  are positive or negative depending on the statement of the task. All the set of stationary points, as it is known from algebra [43], consists of several zero values and corresponding solutions of the remained linear system:



Here we left the square brackets in order to emphasize that it is not the whole system but only its "nontrivial" part.

For example, for the system (7) of the third order the set of the stationary points is written as follows:



If the corresponding linear system has no solutions, the state of equilibrium under consideration does not exist; if the system has an infinite set of solutions, the stationary points will compose a linear variety. Let, for example, the fourth point of the specified set is given by the system:



Obviously, it has no solutions. The system, which is changed a little



has the whole straight line of solutions: .

Let us note a very important fact about the system (7): *at any delay τ any coordinate plane and any crossing of the coordinate planes are invariant relatively to the system* (7). To see it, it is enough to rewrite the system (7) in the integral form:

 (8)

Obviously, for any initial functions (see Appendix 1) the sign of the solution is defined by the value . Thus, if the solution begins (at the moment of time ) in a positive orthant of the variables space, it will remain there further. At an infinite time of the solution existence it may adjoin to this or that face of the positive orthant.

This property of the model is very good for economic interpretations because an output of the firm or the fixed capital must not accept any negative values. If any *xi* approaches to zero with time, it means that the corresponding firm (or branch) ceases its activity. From the above-stated it follows that we are interested only in stationary points, laying inside and on the bounds of the positive orthant of the variables space of the system (7). Moreover, if the stationary point belongs to a coordinate plane, only asymptotically stable points are important for final conclusions. The research of the solutions behaviour on such a coordinate plane as on an invariant set is reduced to the same reasoning as in the whole space.

Thus we are interested only in nontrivial positive stationary points (and the existence of attracting points on the coordinate planes).

We will designate , then the linear algebraic system, which gives a nontrivial positive solution, is written in the form:

 (9)

where *X* is a vector of a required point. The following theorem about the positive solutions of system (9) exists [50]: *the alternative is fair: either the system (9) has a positive solution or the system of inequalities*

 (10)

*has a solution, where Y is an n-dimensional vector, (\*,\*) means a scalar product*.

It is geometrically equivalent to that the vector *E* belongs to a convex conic hull of columns of the matrix *Г*. This theorem is difficult to use and therefore, we will formulate the conditions of existence for the positive solution of the system (9) for each particular example specially.

* + 1. **The competitive activity without a lag**

At first let us consider a case of two firms’ activity competing for common resources. As follows from the above-stated, the dynamics of their activity without a lag (*τ=0*) is described by the equations:



where six coefficients  are certainly positive. By replacing the scale of measurement of variables:

,

we simplify the system to the type with only two parameters, which essentially influence on the solutions behavior:

  (11)

Here the parameters  are positive and are expressed through the initial

coefficients under the formulae:

.

The direct calculation of the non-trivial equilibrium point of the system (11) gives the result:

.

In fig. 15 this point is received in crossing the straight lines, the equations of which can be taken in square brackets of the system (11):



 Fig. 15. A non-trivial equilibrium point of the system (11).

From the last calculations it follows:

**Conclusion 1**. *The nontrivial positive stationary point of the system (11) exists and it is unique if:*

 ** (12)

*or*

 ** (13)

*If , the stationary solutions  compose a straight line with the equation ; and the condition  results in full absence of nontrivial stationary points*.

The state of equilibrium , which is being discussed, can be attracting (asymptotically stable) or repelling (unstable). To define the property of stability we will use the Lyapunov function:

 (14)

As we consider only positive *y*1, *y*2, then *V* is defined and continuous,  At all others positive *y*1, *y*2 the function *V* is strictly positive. It means that it is definitely positive for the state of equilibrium . As the functions of such type are seldom used by researchers, we will show an approximate diagram for a one-dimensional case in Fig.16:

Fig. 16. An approximate image of the Lyapunov’s function (14) for

a one-dimensional case.

We represent the curves of levels of the function in two-dimensional space:



Fig. 17. The approximated image of curves of levels of

the Lyapunov’s function (14) for a two-dimensional case

Its time derivative with respect to the system (11) has the form:

.

It is definitely negative, when. The last inequality is satisfied, when (12) is correct and is not satisfied at (13). Thus, conclusion 2 is fair.

**Conclusion 2.** *If the condition* (12) *is correct, then for any positive initial data the solution with time is approaching asymptotically to . If* (13) *takes place, the state of equilibrium is unstable (to be more exact, it is a special point of a "saddle" type). In this case, the points of the coordinate axis  are attracting on the variables plane y1,y2.*

The last statement of conclusion 2 is easy to prove by writing the system in deviations related to the specified points. The linear parts of these systems have the form:

 

where the prime mark shows a deviation of the variable from the corresponding equilibrium value. As it is known, the separatrix, which divides the positive quadrant of the plane into two parts, passes through the point  in this case: all trajectories approaching to the first coordinate axis are assembled in one part, those approaching to the second – in another one. The movement along the separatrix is performed to the point  . Hence,

**Conclusion 3.** *The trajectory of the system* (11) *with any non-negative initial data and any coefficients will gradually and asymptotically approach either to a nontrivial state of equilibrium or to one of the equilibrium points on the coordinate axis.*

The following diagrams show the field of directions in case of stability and

instability of a nontrivial state of equilibrium of the system (11):

 Fig.18.The nontrivial state of equilibrium is an attracting point



 Fig. 19. The nontrivial state of equilibrium is an unstable of

the "saddle" type.

**The economic interpretation**. *When two economic agents compete with each other for common resources the following results may occur as time goes on:*

* *at* *. the manufacture volumes of both firms is approaching to the magnitudes*  *and*  *correspondingly. Their initial state does not play any role; the volumes of manufacture can be reduced or increased, their total volume is also changing;*
* *at*  *one of the firms (except a very rare case of the separatrix), as time goes on, will stop manufacture, another one either will reduce or increase the output according to the available resources*. *The final output for the first firm is numerically equal to* *, and for second -* *. This case opens an opportunity for the management of the firms to control the process of competition. If the initial condition of production is changed in such a way that it appears in a preferable part of the plane, the chance to survive will appear. It is possible to achieve this, for example, by taking a large credit and expanding the production sharply. However, the expansion should not surpass the equilibrium value;*
* *in case of other relation between the model’s coefficients it is necessary to refuse from its use because the system will not have a positive equilibrium.*

Let's consider a case with an arbitrary number of economic agents which compete for common resources. The nontrivial positive solution of the system (9), as it was already noted, will exist if the positive vector *E* belongs to a convex conic hull built on the columns of the matrix *Г*. It is not a constructive criterion. As a rule, it is easier, in such cases, to solve the system (9) than to check general criteria. Assume that the solution of the system (9)  is received with available coefficients . The question about stability of the state of equilibrium in the point  is being solved with the help of the definitely positive function of Lyapunov:

 (15)

Note that this function is determined only for the strictly positive *xi* and equals to zero only in the point of equilibrium . Its derivative with the time, along the trajectories of the system (7), is written easier if to transform the system (7) with the use of solution of the algebraic system (9):

  (4\*)

So

 

Obviously, it is a definitely negative quadratic form if the matrix  is definitely positive.

**Conclusion 4.** *A nontrivial equilibrium state of the system* (7) *will be asymptotically stable, if the matrix with the positive elements is definitely positive; the trajectories of the system* (7) *with any strictly positive initial data approaching*  *as time goes on.*

 *If among the initial data there exists a zero one, i.e. the trajectory begins with some coordinate plane and due to the invariance, it will always remain on this Cartesian coordinate plane and the corresponding state of equilibrium will remain conditionally asymptotically stable. In relation to the whole space the coordinate states of equilibrium, (i.e. those which have zero coordinates), will be unstable*.

The situation, when the matrix  is not definitely positive, is not described for the general case because existence conditions of a nontrivial positive equilibrium point on the Cartesian coordinate planes are narrower than in the general case. Therefore, the conclusions about an attraction of the coordinate stationary points need to be checked every time.

So for example, we will analyse a three-dimensional model with two free parameters:



where  The system has a positive nontrivial stationary point if

, which is unstable, because the matrix



is not definitely positive. Thus, the general nontrivial equilibrium is unstable. Let us consider the Cartesian coordinate plane : the nontrivial equilibrium point on it at the already accepted restrictions always exists and is positive: . Having constructed the system in deviations in relation to the last state of equilibrium, it is easy to see that its linear part has negative characteristic numbers if, i.e. this stationary point will be always attracting.

On the Cartesian coordinate plane  the stationary points exist only at  It entails a strict negativeness of , which is not permitted. On the Cartesian coordinate plane  the positive stationary points can exist (at ) and in this case they are attracting but they may not exist.

From the asymptotic stability of the equilibrium point, the boundedness and extendibility on an infinite interval of time of solutions of the system (7) at *τ*=0 is followed. In case of instability such conclusion is not obvious. Therefore, we consider the system in the integral form (11). From the positiveness  the inequality follows:

 

and it entails the extendibility of solutions. But it is possible to show also the boundedness of all solutions at all positive coefficients and initial data (and the extendibility will also follow from it).

Assume that a variable *xi(s)* from its positive initial value is growing without restriction. Then there will be two quite closely related moments of time  that  and   and hence, taking into account the smoothness of solutions, we receive the contradiction: on the one hand, the monotonous increase, i.e , on the other hand - the right parts of the system (7) are negative at these values of variables.

It is necessary to note that beyond the limits of the positive orthant of the variable space the system (7) has non extendible solutions approaching the infinity at the bounded values of time. In this context the analysis of the right parts at various generalizations of the Lotka-Volterra systems is very important. So, for example, the generalized logistic model of A.N. Kolmogorov requires checking the extendibility for all variants of *fi*.

In case of instability of a nontrivial state of equilibrium (and, at the same time, the boundedness of solutions in the positive orthant), it would be interesting to investigate the question on the solution oscillations. In a plane case, the separatrix of unstable equilibrium state passes through the origin of coordinates and divides the positive quadrant into two parts, in each of which there exist attracting points on the coordinate axes and therefore there is no oscillation. It cannot be set for the case of more than two space dimension.

* + 1. **The competitive activity with a time lag**

Let us return to the equations (11) supposing that the lag *τ* is strictly positive:

  (16)

And the nontrivial state of equilibrium also remains as a previous one . At a small positive lag all conclusions about the qualitative behaviour of solutions of the system (16) continue to take place, but with increasing of the lag the picture is changing: the unstable stationary points remain unstable and the stable ones lose stability. In the latter case there appears an attracting limiting cycle (or stable fluctuation) around the nontrivial equilibrium centre . The exemplary behaviour of the trajectories is shown below:



 Fig. 20. The onset of an attracting boundary cycle at increasing of the lag.

Let us discuss the changes of properties of the trajectories with increase of a lag in detail.

**Statement.** *If the condition (12) is true, for any fixed lag τ there exists an area in a positive quadrant of the phase plane which contains triangles ABE and DEF and quadrangle CBED (see fig. 21), and this area is such one that all the trajectories from inside of this area remain in it and any trajectory outside enters inside it.*



Fig. 21. Exemplary portrait of a phase plane *(y1,y2)* for the systems (16)

under the condition (12).

For the proof we remark that the following inequalities result from (16):

• If the point , then 

• If the point , then 

• If the point , then 

1. • If the point  does not belong to the indicated areas, both derivatives are negative.

We consider only the continuous part of the trajectory (the initial function may have breaks. [See Appendix 1.]). Let the first variable has a value which can be as large as possible at an initial moment of time *t\** and the lag point that defines the sign of the derivative is located as close as possible to the origin of coordinates (point *C*), which corresponds to the largest value of the derivative. Then for the time *τ* the variable *y1*(*t\**)will increase by no more than  times and the lag point (*y1*(*t\*-τ*)*,* *y2*(*t\*-τ*)) will obligatory pass in (*y1*(*t\**)*,* *y2*(*t\**)) due to the trajectory continuity. As the last one under the assumption has a large value of the first variable, the derivative of the first variable will become negative. The sign of the indicated derivative will remain until the lag point does not return to the origin of coordinates.

The second variable has a similar behavior and therefore all distant points are moving to the origin of the coordinate system and then move aside at some distance. If we consider the second variable *y2* to be close to zero (it is the same to considering the trajectory which is almost on the abscissa axis), the first equation of the system (16) could be replaced by . Similarly, if we suppose the first variable to be small, we will receive an approximated second equation from the system (16): . In section 3.3.3. we discussed the behaviour of such equations solutions in detail. Thus, the area mentioned in the statement lies inside the rectangle .

The occurrence of the time lag is capable to change the whole qualitative picture of the trajectories on the phase plane: so with increasing the lag the attracting points become repulsing. Let us formulate all possible situations in the statement.

**Statement**. *The equilibrium state* (*1,0*) *of the systems (16) is asymptotically stable if , and unstable if either  or .*

*The equilibrium state position* (0,1/*b*) *is asymptotically stable if , and unstabl, if either  or .*

*The equilibrium state*  *is asymptotically stable if  and unstable if either  or  or .*

 *Here *

Let us discuss how the last conclusions are made. For each stationary point the set of equations in deviations is composed and its linear part is considered. For example, the first stationary point generates the linear approximation of the system in deviations of the kind:

.

It is obvious that the second equation gives solutions damping up to zero at ** and the first equation in this case becomes inhomogeneous linear with exponentially damping heterogeneity. Its homogeneous part will be asymptotically stable if the lag is not very high, namely, if **. The latter conclusion can be received from Appendix 1 where the precise solution of the area of the asymptotic stability for a scalar linear equation with one delay is given. Thus the conclusion of the statement is proved.

The solution of the problem with a nontrivial stationary point  is made with some difficulty. In this case the system in deviations (its linear part) has the form:

  (17)

Its characteristic quasi-polynomial is defined by the following equality



By changing the variables  the last equation is turned into the square one:



If ** it always has two real negative roots:

 ** (18)

Thus, the statement given above describes completely an asymptotical behavior of the trajectories in the positive quarter of the phase plane.

Now let us consider the problem of periodic solutions existence of the system (16). As we have already noted, at the zero time lag there cannot be any fluctuations (the fluctuating motion along a closed trajectory is meant). The interring of a lag changes the picture. It is not a point but a segment of the trajectory which generates the solution of a Cauchy problem for the system (16) and hence the trajectory can cross the separatrix (passing from the origin of the coordinate system to a nontrivial equilibrium point ). Let us show that if there is a closed trajectory, which is a carrier of a periodic motion, it satisfies the equalities



where *T* - is a corresponding period. Really from (16) the integral equalities follow:

  (19)

The assumed periodicity means that for any *t* the identities  and in particular,   are executed. Then from (16) we have the relationships:



which are a linear algebraic system. Its solution is written in the form:



Due to additivity of the integral and the periodicity the following equalities will be correct:



It means that the mean integral along the periodic motion coincides with the equilibrium state. Let us call such movements as  - central. Note that such coordinate-wise centrality does not guarantee us that the periodic trajectory will surround the center  from all sides, i.e. if we draw a ray from this center to any point of the periodic trajectory, it is not necessary that the ray will make a complete turn around the center at a complete going around by the contour point. The calculations demonstrate the existence of boundary cycles, which cannot be deformed in a unit circle with the center in  by continuous and homogeneous transformation in relation to this center.

In its arbitrarily small neighborhood, where the main role is played by the linear approximation of the system in deviations (17), the following property is correct in conditions of fulfillment of (12): *the linear system (17) has two sets of periodic solutions around the zero (that corresponds to the point  for the system (16)): the first corresponds to the lag  with a period , and the second – to the lag  with a period .* *The amplitude of these periodic solutions depends on initial functions. The negative values z1 and z2 are determined in (18).*

Here the calculation of the period is performed according to the scheme that has been already done: for the indicated fixed lag *τ* the characteristic quasi-polynomial  corresponds to the differential equation with a lag , which is transformed to the equation  by replacing the time scale . The latter has a solution of the form  with a period . Returning to the time *t* we receive a period which is equal to four lags.

The system (19) appears to have  - central periodic solution at any lag. Below we will justify this statement. For this purpose from the expressions (19) the iterative procedures are constructed in such a way that each iteration is periodic and  - central. The limit of the received iterations will be a required periodic solution.

At first, let us note that the arbitrary lag mentioned above can be replaced by **, because otherwise all solutions from the positive quadrant are attracted to the stationary point , i.e. it is a global attractor in our conditions. It is natural that the stationary solution satisfies the condition of periodicity and can serve as a limit mentioned above. But since there is not any practical utility from such an expanded interpretation of periodic solutions, we will consider that the time lag is more than the indicated value.

Thus, for the arbitrary lag *τ* and an arbitrary *T* we will select a pair of continuous,  -central, *T*-periodical functions . Let us substitute them in the integral representation of the system (16) and we will receive another pair of functions:

 

but they are not -central. To provide this property we will normalize each of the received functions with the help of an integral by a period:



Now the functions  are both continuous and -central, and *T*-periodic. Using the same way it is possible to receive any iterations  replacing the upper index zero by *k* and the unit by (*k*+1) in the last formulae. The last properties will be correct for all of them. Note that for the indicated iterations the succession of the initial points was created:

.

Since the each component of the pair  is continuous on the period, positive and has a fixed mean, there exists a majorant, which is common for all functions. If it were not so, i.e. for any as large as possible M such a number as *kM* and a moment of time  such as  could be found, it would be possible (from the centricity) to detect such a positive number as *w<τ*, that , and it would result in unboundedness of the right parts of the system (16).

From the existence of the general majorant for the functional succession, the existence of the limiting pair of the functions follows. It also should be -central and *T*-periodic. It is this limit which is a required periodic solution of the system (16).

Note. A procedure of iteration construction is hardly to be transformed into the operating numerical method of receiving periodical solutions. Here one of the problems is connected with the precise knowing of the period *T*.

*Let us consider two economic models to understand the parameters values which are included in the model. We will construct the system (7) for two firms. At first, we will suppose that the manufacturing companies work on the common market and use the same sources of production factors i.e. labour, raw materials and capital resources. It may be integrated house-building factory, the enterprises of light industry, etc.*

 *Recall that we discuss the system of equations:*



*Let us estimate the parameters of the model, which are responsible for their own enterprise capabilities disregarding the rival firm. Let the technological and administrative capabilities of the first enterprise be such that it is capable to double the output over a period of five years. Then, from the first equation of the model we receive at small x1*: * From this we have an approximated equality  Similarly, we suppose that the second enterprise at a small own output is capable to treble a volume of production over a period of five years. Hence, from the second equation of the model we receive *

*Each of the managers can evaluate the optimal size of the enterprise, using the well-known criterion through the prices on the production factors and the kind of production function (see Appendix 3). Let it follow from these estimations that the first enterprise is capable to produce 20 units and the second – 10 disregarding the competition. But we know that if there is no competition the limiting output at the first enterprise will be equal to , and on the second – to  and then the corresponding γ are calculated: γ11=0.007; γ22=0.022. The most difficult is to evaluate the mutual influence of two firms, i.e. γ12, γ21; or it is necessary to suppose that the managers of the firms have a possibility to agree the volume of the goods supply on the common market, or the regular marketing studies are being done from which it is possible to conclude about equality balance in the competitive relations. Suppose that the equal balance is reached at and then we calculate the rest parameters: γ12=*0.0044*; γ21=*0.003*.*

*Thus, the model of interaction of two manufacturing companies is constructed. To apply the conclusions received above we must know α, β, z1. According to the given above formulae α=*0.27*; β=*3.18 *and z1=* -0.19*. So, the nontrivial equilibrium state in the market of two manufacturing companies* (15,8) *will be asymptotically stable, i.e. whichever the initial conditions be, both companies will reach an equilibrium point and they might not worry about the time lags in their manufacturing processes or in the financial strategies, as the stability of a stationary point will be guaranteed till the lag, the duration of which is eight years. No fluctuating processes in the dynamics must not be observed.*

*The next example of competition in the common market of resources deals with trade. Let us consider the mutual influence of a small-sized retail trader (the first firm) and the trading company, which possesses the stationary premises, warehouses, showrooms, etc. (the second firm). As each of the firms is not capable to cover the whole market: the small retailer is closer to a buyer and more mobile but it cannot cover all the goods and provide the convenient and up-to-date methods of trade, we will conditionally accept that the retail trade can basically satisfy only* 30% *of the market, and the stationary one -* 80% *(further we will measure the trade activity in these units). Besides, let us assume that the first firm can increase the number of its shops up to ten times for one year if the stationary trade suddenly ceases its activity. The latter is capable only to double its turnover for a year because the construction, the lease, registration, sellers training, wholesale purchases requires a lot of time.*

*Thus, as in the first example we receive the values for the model coefficients:  γ11=*0.077*; γ22=*0.0086*. As the result of different kinds of rivalry (advertising, dumping prices, influence on the city’s authorities, etc.) the equilibrium  is formed from which it is possible to receive the missing parameters: γ12=*0.005*; γ21=*0.0018*. The model is ready and it is possible to learn what will result from it. Due to α=*0.078*; β=*5.73 *and z1=* -1.94 *the stationary point* (25%,75%) *will be asymptotically stable only in that case if the reaction of the market to the actions of each firm occur without a large delay, namely, the time lag should not exceed* 0.81 *of a year. Otherwise the fluctuations will begin around the stationary point: at times the small retail trade will seize more than* 25% *of the market, at other times the stationary trade will exceed its range near* 75%*. The longer is the lag, the longer is the period of fluctuations and their amplitude. When the delay exceeds its critical value of* 0.81 *a little, the fluctuation period will be a little longer than three years (four time lags).*

*If we use a computer, it is possible to observe the following picture. At the time lag of* 0.82 *(i.e. a little more than a critical one) the small retail trade starts to experience fluctuations with an amplitude of* 7*% of the whole market, i.e. approximately* 25*% of its volume. With the increase of the lag the amplitude is growing very fast and at τ=*0.9 *within each four years, the small retail trade is growing from the zero mark up to* 45*% of the whole market. But the fluctuations are generated in the stationary trade as well: if the lag is* 0.82, *then the amplitude is* 2% *of the whole market or* 3.2% *of its stationary volume. Such phenomena become evident on collection of taxes and on the city budget. With the growth of the lag the fluctuation amplitude and the period are growing nearly exponentially. It is necessary to note that the duration of the time lag is under the influence of the terms of credits in the banking system, the political preferences of the city power structures, remoteness from the world and domestic trade flows and etc.*

* + 1. **Interdependent activity with a lag**

In this section we consider such interaction of economic agents when one of them depends essentially on another. The elementary examples of such relations are the models of the budget or agrarian sector quadrants in the public economy. The biological analog of similar models is the «predator - prey» system. Without the time lag for two agents it is written in the form:

 (20)

where all coefficients are positive and  characterizes the dependent branch, for

example, agriculture and  characterizes the branch-donor, for example,

the whole industry or the mining sector.

 As in the previous section we will introduce more convenient coordinates (at

which the model has less parameters):

.

Then the system (20) has the form

  (21)

where .

The model has three stationary points (0,0), (0,1/*α*), (1-*α*,1). As before, when constructing a system in deviations, we will receive that the first point is always unstable «saddle» type, the second will be asymptotically stable at *α>*1 and unstable at 0<*α*<1. The third point, which corresponds to the normal operation of the economy (in the sense that none of branches terminates its activity), is asymptotically stable at 0<*α*<1. The last statement is easy justified with the help of the second method of Lyapunov. Let us consider a definitely positive function in relation to the third equilibrium point:

 

Its total derivative by time along solutions of the system (21) is a non-positive function:

**

According to the theorem of Barbashyn-Krasovskii [10], if the derivative is non-positive, it is enough for an asymptotic stability that the set (*y1,y2*), where it is equal to zero, does not contain the whole trajectories, except for a stationary point. This is executed in our case.

Thus, at 0<*α*<1 the nontrivial positive balance will be a global attractor, i.e. from any initial states the outputs of all related branches are approaching to the established values. As this takes place, the character of approaching can be different. If , the stationary point will be a special point of a «focus» type, or differently – of a «node» type. It means that in the first case the trajectories reach the equilibrium point by rotating along it in a spiral. In a limiting case, when *α=*0, there appear closed trajectories with the periodic motions along them around the equilibrium state. The equations of these closed curves can be received as the surfaces of levels of the Lyapunov definitely positive function, which has already been used by us:



Fig. 22 and 23 shows the exemplary trajectories in the positive quadrant of the coordinate plane in the cases described above.



Fig. 22. The nontrivial equilibrium state is an attracting «focus»



Fig. 23. The nontrivial equilibrium state is an attracting «node».

**Conclusion.** *If the time lag is absent, the behavior of interdependent branches, which allows their joint existence, is defined by the parameters of the model: if the inequality*  *is true, the «outputs» of the branches willfluctuate around the stationary point*  *being drawn into it in a spiral. If the inequality mentioned above is not fulfilled, the monotonic approaching to the equilibrium point is a characteristic of dynamics.*

*If γ22=*0, *which corresponds to the situation of unrestricted development of the industry, the interaction of the industry and the agriculture (as an example) is characterized by periodic fluctuations around the nontrivial stationary point with the period . Note that this fact was already established by V. Volterra.*

Taking into account the time lag in the models described above is also possible as we have done it before: in the systems (20) and (21) the variables in the square brackets are taken not in the current moment of time but in the previous (*t-τ*). We will consider only nontrivial equilibrium state (1-*α*, 1) and only under the condition of 0<*α*<1, that is when without any delay it will be asymptotically stable by Lyapunov. From the next system



we compose a system in deviations in relation to the equilibrium being considered and write out its linear part (as it has been done in the previous section):



Its characteristic quasi polynomial is defined by the following equality:



In relation to  the last equation turns into the square one:



Its solutions under the indicated conditions will always lie in the left-hand

half-plane of the complex plane:



If the received solution has been real (it will be, as it has already noted in the previous section, at ), then using an old method, we will establish at what time lags the stationary point will remain attracting: **. At the complex roots of the last equation it is necessary to use the note given at the end of the second paragraph in Appendix 1. From there it follows that the equilibrium state remains asymptotically stable if the point with coordinates  is in the part of the coordinate plane which is located above the abscissa axis but lower than the curve, which polar coordinates are connected by the equation: . The area of stability in the picture is shown as a shaded figure. The conditions look cumbersome but they are easy to be checked.

It is apparent that at small lags the mentioned point necessarily belongs to the figure and therefore, all trajectories are drawn into the equilibrium state as spirals. With the growth of the lag there appear stable fluctuations with the increasing amplitude and period after some critical value in dynamics of the economic agents’ behavior. Let us consider a numerical example.

*Let x1 be a gross product of the agriculture for one year, and x2 be the gross product of the industry of the country. Let us make a number of plausible assumptions to identify the coefficients of the model (20). If we propose that the industry is so dynamic that is capable to double the output for five years beginning from small volumes, we will receive  from an integral representation of the second from the equations (20). Having faced the overproduction or shortage of the production factors, the industry will stop in its development. We will designate this limit as 100%. Then, without any influence from the agriculture the second equation allows to calculate the following parameter  Now we will suppose that the government intends to help the agriculture by transmitting a part of industry output either through redistribution the investments, or influence on the prices, or paying the subsidies. The agriculture itself is such that, without some help from the outside, it will reduce the production to ten times for ten years, i.e. it will ultimately decline. The last assumption gives .*

*The presumable calculations in the government have shown that the planned help will result in stable operation of the economy near the point of equilibrium* (10%, 80%) *(the units of measurements correspond to the limit output of the industry). Thus, we can calculate:  In this case the remained parameter of the model α takes the value 0.8 and at small lags the equilibrium state will be asymptotically stable. To estimate the allowable lags it is necessary to use the criterion given above. Since it takes place for the first time, all operations will be done in detail. As* *, we cannot apply the inequality . The values of the found model parameters allow stating that if the point* (0.056*τ*, 0.0575*τ*) *is located below the line with the equation*  *in polar coordinates, the corresponding τ are allowable.*

*For the arbitrary τ the mentioned point lies on the central straight line with an angle of slope  Then, the length of a segment of this straight line, which lies inside the figure, is less than* 0.77*. Thus  It is ultimately stated that the equilibrium will remain asymptotically stable at delays almost up to* 10 *years. Though this period seems to be long, the economic relations of the industry and the agriculture can be adjusted on a long-term basis. For example, the investments in construction of agricultural equipment, long-term programs of the land- reclamation, electricity or gas supply lines for the rural locality are not to last more than 10 years according to this model.*

* + 1. **An arbitrary number of market participants**

In the tasks with many participants of economic activity, the mutual influence and competition are so interlaced that it is obviously not possible to use some definite model. Therefore, in the given section the general principles and methods of simulation of dynamics of the complex economic systems are formulated. Let us designate



Then the multidimensional model of Lotka-Volterra can be written in the form:

.

As it has already been noted at the beginning of this paragraph, the nontrivial state of equilibrium of the whole system is given by the solution of the equation (9). For the economic meaningfulness of the result it is necessary for this solution to be positive. We will designate it with the asterisk above:  Here, as before the character”+” shows a pseudo inverse matrix, since the inverse matrix may not exist in the general case. The system in deviations in relation to the considered positive equilibrium is written in the form:



and the characteristic equation of its linear approximation has the form:

.

Designating  as before, we can calculate  - the spectrum of the matrix .

**Statement.** *If among the complex numbers zi, there exists at least one with a positive real part, the equilibrium state is unstable according to Lyapunov. In this case there exists an attracting equilibrium state on the edges of the positive orthant. It corresponds to the destruction of one of the economic agents with the course of time.*

*If all numbers zi have strictly negative real parts, at small delays the equilibrium state is asymptotically stable by Lyapunov, i.e. all trajectories from any parts of the positive orthant are attracted to the equilibrium point with the course of time. The steady state fluctuations do not exist. All economic agents are developing in a smooth predicted way, approaching to the corresponding limiting values.*

*The increase of the lag up to some critical level results in disappearance of an asymptotic stability of the equilibrium state, but in its neighborhood there appear some fluctuating movements. At first they are periodical, and then at further increase of the lag they become almost periodical and recurrent.*

*The critical value of the lag τ\* is determined as follows: for any complex eigenvalue zi=-a+ib* (*a,b>*0) *of the matrix*  *the central straight line of the points* (*aτ,bτ*)*, which crosses the curve*  *at some τ\*, is being constructed, or in a more cumbersome analytical form:*

*.*

*The remained case is characterized by non-positive real parts of the complex numbers zi and by the existence of pairs of pure imaginary eigenvalues of the matrix* *. At the zero time lags for two equations, the cyclic motions are formed in the neighbourhood of the equilibrium. The lag makes the stationary point unstable in this case. But whether there will be a stable fluctuation or whether there will be a manifold (not necessarily linear) with stable fluctuations in a multidimensional case, is not known at present.*

* 1. **Identification of the system**

The real dynamic processes are under the influence of a reasons set and disturbing factors. No mathematical formalization of the processes can take them into account in full. Therefore, the stage of identifying the system, i.e. determining the parameters of equations, which are responsible for the particular process, is an optimization stage of modeling. All the previous reasoning is related to the choice of the class of functions, which is specified with an accuracy of some parameters. Now it is time to select the parameters in such a way so that some quality functional reaches the minimum. In this case the solutions of the system of logistics equations or otherwise of the Lotka-Volterra model is the functions' class of the model. Below we will discuss and give the related formulae and recommendations for the model without any lag, with a lag and with errors in the observed data.

* + 1. **The identification of the model without a time lag and without any information distortion**

At first we will consider a scalar logistics equation

  (22)

where *a, b* are the positive parameters. Suppose that we know the solution of this equation in discrete moments of time  *xi*. The identification of the equation (22) involves the search of the numbers *a, b* by the observations {*xi*}.

The given equation is easy to solve:

,

and therefore, if we have only three observations *x0, x1, x2*, it is possible to compose an algebraic system of two equations for searching the numbers *a, b*. But this approach will not result in the general constructive algorithm.

Let us go to the discrete time in the equation (22). Due to

,

we receive for the adjacent moments of time *tk, tk+1,* integrating the last equality:



The calculation of the integral is impossible to be carried out with accuracy because the values *x*(*t*) between the observations are unknown. Therefore, the integral has to be approximated. It can be done by different methods with a different accuracy of the final result. Keeping in mind the general algorithm we choose the approximations of the first order:



From this equality we deduce a functional, the minimum of which should be found to restore *a, b*:

 . (23)

Unfortunately, it is impossible to state that at precise value of the numbers *a, b* the functional will take a zero value, but taking into account the errors of measurement and modelling, the real processes would not give the exact zero. Therefore, we accept as a definition that the numbers *a, b* received at minimization of the functional *F(a,b)*, define the best model from all equations (22) for the given set of observations over the process {*xi*}.

We extend this result on the vector case. Let *X* be an *n*-dimensional vector with the components *xi* and the discrete time be designated by an integer argument:  From the form of equations (7) it follows (at zero delay) that

.

Here, the sign of the approximated equality underlines the availability of the same procedure of integral approximating as it has been done in a scalar case.

Let us select a functional, which characterizes the quality of parameters adjustment, as a sum of squares of deviations through all vectors of the observations:



It is easy to see that the functional is a squared one in respect of the parameters required *εi, γij*. As it has already been executed repeatedly (sections 2.3.3., 2.3.4., for example) by the replacement of variables



we reduce the systems to a well-known problem of minimization of the squared functional:



Then, as it follows from reasoning in section 2.3.4., if there exist (*n*+1) linearly independent vectors among the vectors {*Zk*}, *k*=0,…,*N*-1, then the minimum of the last functional is reached in the unique point:

 

**Note.** Here, it is very important to note that the observations  should be strictly positive, since, on the other hand, it will be impossible to use the function ln. This circumstance restricts the application of the models under consideration. For example, the GNP has fluctuations around the curve of the economy growth, therefore the modeling of changing of the GNP but not of the GNP itself is considered usual. But changing of the GNP can be negative in the period of political and economic shocks of a country. Thus, in the tasks of the macro level the Lotka-Volterra models are more natural for countries with stable economy.

* + 1. **The identification of the model without a lag and with errors**

**in information**

In the paragraphs of the second chapter we have discussed the time series with errors of measurement, on which the limitations of independence of distribution on time were imposed. Suppose again that

 (24)

where the symbol “∧” marks the deterministic components of observations and *μi* is a random error of the values being measured for each component of the vector *X*(*k*), this error has a zero expectation and quite small dispersion. Then, by expanding the function ln into the Taylor series, we receive an approximated equality:



Thus,  is determined and continuous by *μ*, if 

Let us consider a scalar case as a more simple, i.e. we check what conditions should be imposed on the sequences  for the optimal values *a, b* received at minimization of the functional (23) with the perturbed data (24) to be continuous in respect of the perturbations *μ*. Here,  is a sequence of realizations of a random disturbance *μ*.

Calculating a partial derivative from (23) by *a,* we receive:

.

Then, calculating a partial derivative from (23) by *b* and substituting the value *a* received, we detect *b*:

.

Here we have all sums by *k* from 0 up to *N*-1. In the denominator of the last formula there locates an estimation of a mean square deviation  from its mean value multiplied by the length of sampling. Therefore, the conditions of correctness of the last formulae consist of inequalities:

  (25)

where  is a mean of sampling .

The first of inequalities (25) is easy to check before the data processing and hence it can be not interpreted. The second should always be executed by virtue of the assumption (24) and of that the data  satisfy the Lotka-Volterra equation (and do not correspond to the equilibrium state), and at quite small dispersion of a random error *μ* the sequence  will also satisfy the second inequality in the (25).

Now it remains to note that substituting (24) in the expressions for *a* and *b*, we will receive continuous functions by *μ* (at quite small *μ* ). Therefore, if *μ→*0,  will be replaced everywhere by , and for the latter the inequalities (25) take place necessarily.

In a multidimensional case the condition of non-singularity of the matrix is added to the conditions (25) and this, in its turn, is transformed to existence among *N* vectors *Zk* of precisely (*n*+1) linearly independent, that is necessary to require from the multidimensional data for identification of the model.

* + 1. **The identification of the model with a lag**

Since the equations with a delay argument involved in the sphere of modeling of the dynamic processes play a significant role in the book, it is natural to extend the algorithm of section 3.3.4., in which the method of identification of the linear system with a lag is offered, onto the non-linear Lotka-Volterra models. An approach to the problem of simultaneous identification of the lag value and the system coefficients remains the same: for each discrete lag from a definite interval we receive the optimal coefficients, and then from the final set of the systems we select one system for which the value of the quality functional will be the least.

Let  be the observations over the states of the system (7) in the discrete moments of time *t=kΔ*. We consider the time lag to be multiple for the interval of observation  Then from the system (7) it follows that



If we take a logarithm of last equality, substitute the variable of integration and approximate the integral by the formula of a trapezoid, we will receive an approximated (with an accuracy which is in proportion to *Δ*) equality:



at 

Now it is possible to construct the functional, the minimization of which brings us to the solution of the problem of identification:



where 

**The** **algorithm** *of choice of the relevant system of the Lotka-Volterra equations* (7) *with delay according to a vector time series consists in the following: for everyone*  *we do replacement of variables:*



*and solve the problem of minimization of the functional:*

 

*As a result we have , and from this finite set of parameters we select the number l, for which the functional Fl(El,Γl) gets the least value.*

**3.6. Conclusion**

In this chapter the Lotka-Volterra multidimensional model with a time lag is considered. The economic problems class is chosen, in which dynamics can be described by such a model. In the given circle of problems all kinds of enterprises interaction on a common economic field are included: in production and distribution of goods, both in competition and in the case of dependence of one enterprise on operation of the other one. The same takes place in interaction between branches, regions and countries. Therefore, the model under consideration may appear to be useful when analysing the dynamics of foreign trade or cyclicity of economic development of countries.

The application of non-linear equations opens the way to formation of several areas of attraction (or repulsing) in the variables space, and the introduction of a lag is not only natural for the economic problems, but it also enriches essentially the dynamic properties of the model and makes both elementary periodic fluctuations and almost periodic and recurrent movements a subject of study. In the chapter the profound research of such dynamic properties for the multidimensional case has been given. It has been shown, partially analytically, partially with help of numerical methods, how such complex motions look like and how their characteristics are changed at increasing of the value of the time lag.

The offer to use so complex models much wider would hang in air if there were not propositions of the particular algorithms of identification of the model coefficients. The offered numerical method is quite transparent and gives quite an adequate mathematical model for the dynamic processes, whose reasons can be related to the competition for common life resources.

The introduction of the described above models into the economic practice would help to make the forecast more precise, to clear up the reasons of these or those phenomena, and also to undertake the corresponding managerial steps to change the situation.