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On Expansion of Zeta(3) in Continued Fraction[1]

Part 2

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Abstract

This is continuation of our article [4]. When F and G in [4] are constant sequences, we obtain continued fraction for zeta(3) parametrized by some family of points (F,G) on projective line. This family of points can be obtained if from full projective line would be removed some no more than countable nowhere dense exeptional set of finite points. A countable nowhere dense set, which contains the above exeptional set of finite points, is specified also.

To the thirty fifth anniversary of Apéry's discovery

Introduction

Let $\tau = \tau(\nu) = \nu + 1$, $\sigma = \sigma(\nu) = \tau(\tau - 1) = \nu(\nu + 1)$, where $\nu \in \mathbb{N}$. Let further u and v are variables,

(0.1)
$$c_{u,v,2}(\nu) = -\tau(\tau + 1)^2(2\tau - 1) \times$$

$$-3(4\sigma(\nu) + 1)u^2 - (10\sigma(\nu) + 3)uv + 2\sigma(3\sigma(\nu) + 1)v^2 \in \mathbb{N}[u, v],$$

(0.2)
$$c_{u,v,1}(\nu) = -12(68\tau^6 - 45\tau^4 + 12\tau^2 - 1)u^2 -$$

$$8(157\tau^6 - 106\tau^4 + 30\tau^2 - 3)uv+$$

[1] Short version

$$4(102\tau^8 - 170\tau^6 + 89\tau^4 - 24\tau^2 + 3)v^2 \in \mathbb{N}[u, v],$$

$$c_{u,v,0}(\nu) = \tau(\tau-1)^2(2\tau+1) \times \tag{0.3}$$
$$(3(4\sigma(\nu+1)+1)u^2 + (10\sigma(\nu+1)+3)uv - 2\sigma(\nu+1)(3\sigma(\nu+1)+1)v^2),$$

$$b_{u,v}(\nu+1) = -c_{u,v,1}(\nu) \in \mathbb{Q}[u, v] \text{ for } \nu \in \mathbb{N}, \tag{0.4}$$

$$a_{u,v}(\nu+1) = -c_{u,v,0}(\nu)c_{u,v),2}(\nu-1) \text{ for } \nu \geq 2,\ \nu \in \mathbb{N}, \tag{0.5}$$

$$P_{u,v}(0) = b_{u,v}(0) = 4(3u+2v),\ Q_{u,v}(0) = 1, \tag{0.6}$$

$$a_{u,v}(2) = -c_{u,v,0}(1),\ Q_{u,v}(1) = b_{u,v}(1) = (34u+52v)/(u+v) \tag{0.7}$$

$$P_{u,v}(1) = (327u+500v),\ a_{u,v}(1) = P_{u,v}(1) = -b_{u,v}(0)b_{u,v}(1). \tag{0.8}$$

Let $r_{u,v}(\nu)$ be the ν-th convergent of continuous fraction

$$b_{u,v}(0) + \frac{a_{u,v}(1)|}{|b_{u,v}(1)} + \frac{a_{u,v}(2)|}{|b_{u,v}(2)} + \frac{a_{u,v}(3)|}{|b_{u,v}(3)} + \dots . \tag{0.9}$$

over the field $\mathbb{Q}(u, v)$. Let $P_{u,v}(\nu)$ and $Q_{u,v}(\nu)$ be respectively nominator and denominator of $r_{u,v}(\nu)$, $\rho_k(x) = \left(5x + 3 + (-1)^k\sqrt{\Delta(x)}\right)/(x(3x+2))$, with $\Delta(x) = 18x^2(2x+1) + (7x+3)^2$, $x = 2\nu(\nu+1)$, $\nu \in \mathbb{N}$, $k = 1, 2$, and let $\mathfrak{A} = \{\rho_k(x) \colon x = 2\nu(\nu+1),\ \nu \in \mathbb{N},\ k = 1, 2\}$,

$$\mathfrak{B} = \left\{ -\beta_2^{*(2)}(1;\nu)/\beta_2^{*(1)}(1;\nu) \colon \nu \in \mathbb{N}_0 \right\},$$

where

$$\beta_2^{*(r)}(z;\nu) = \sum_{k=0}^{\nu+1} \left(\binom{\nu+1}{k} \binom{\nu+k}{k} \right)^2 k^r z^k. \tag{0.10}$$

Then we have

Theorem B. *Let $F + G \neq 0$, $(F+G)G \geq 0$, $G/F \notin \mathfrak{B} \cup \mathfrak{A}$, if $F \neq 0$. Then all convergents $r_{u,v}(\nu)$ are well defined for $u = F$, $v = G$ and*

$$8(F+G)\zeta(3) = b_{F,G}(0) + \frac{a_{F,G}(1)|}{|b_{F,G}(1)} + \frac{a_{F,G}(2)|}{|b_{F,G}(2)} + \frac{a_{F,G}(3)|}{|b_{F,G}(3)} + \dots . \tag{0.11}$$

Moreover $P_{u,v}(\nu)$ and $Q_{u,v}(\nu)$ are homogeneous polynomials in $\mathbb{Z}[u, v]$,

$$\max(2\nu, 1) = \deg_u(P_{u,v}(\nu)) = \deg_v(P_{u,v}(\nu)) = \deg(P_{u,v}(\nu)), \tag{0.12}$$

$$\max(2\nu-1, 0) = \deg_u(Q_{u,v}(\nu)) = \deg_v(Q_{u,v}(\nu)) = \deg(Q_{u,v}(\nu)), \tag{0.13}$$

where $\nu \in \mathbb{N}_0$.

Remark. The values $\rho_k(x)$ with $k = 1, 2$ are zeros of the following trinomial $a_0(x) + 2a_1(x)\rho + a_2(x)\rho^2$, where

$$a_0(x) = -6(2x+1),\ a_1(x) = -2(5x+3),\ a_2(x) = x(3x+2).$$

Since $-a_0(x)/a_2(x) = 3/x + 3/(3x+2), -a_1(x)/a_2(x) = 3/x + 1/(3x+2)$, decrease together with increasing of $x > 0$ it follows that

$$\Delta(x)/(a_2(x))^2 = (-a_1(x)/a_2(x))^2 - (-a_0(x)/a_2(x)),$$

and $\rho_2(x)$ decrease with increasing of x. Moreover, $0 < -\rho_1(x) < \rho_2(x)$ for any $x > 0$ and $\lim_{x\to\infty} r_2(x) = 0$. Consequently, for given $F > 0$ and $G > 0$ the condition of the Theorem B must be checked for finite family of ν; for example, if $G/F > r(4)$, then condition of the Theorem B is fulfilled. We note that $r(4) < 0,36$.

I give here sketch of proof of Theorem B. Full version of this work can be found in [5]. Initial variants of this article can be found in [3], [4]

§1. Auxiliary functions.

We use the same auxiliary functions as in [3] $-$ [5]. For example,

$$f_1^*(z;\nu) = \beta_2^{*(0)}(z;\nu), \delta^r f_1^*(z;\nu) = \beta_2^{*(r)}(z;\nu), \tag{1.1}$$

where $\delta = \frac{\partial}{\partial z}$ Expressions for my other auxiliary functions and connection of them with polylogarithms can be found in (12) $-$ (17) of [4]. Expressions for $\beta_k^{*(r)}(z;\nu)$ for $k = 1, 3, 4$ can be found in (4.37) $-$ (4.38), §4 of [5].

§2. Auxiliary difference equation.

As in [4], we put

$$a^{***}_{F,G,j}(z;\nu) = F a^{**}_{2,j}(z;\nu) + G a^{**}_{3,j}(z,\nu), \tag{2.1}$$

$$y^{***}_{F,G,k}(z;\nu) = F\delta f_k(z;\nu) + G\delta^2 f_k(z,\nu) \tag{2.2}$$

for $\nu \in M_1^* = ((\mathbb{R}\diagdown(-2,1)) \cap \mathbb{Z}$, $j = 1,\ 2,\ 3$, $k = 1, 3$ In view of (40) in [4],

$$\mu_1(\nu)^2\nu^5 y^{**}_{F,G,k}(1,\nu-1) = \sum_{j=1}^{2} a^{***}_{F,G,j+1}(1;\nu)(\delta^j f_{1,0,k})(1,\nu) \tag{2.3}$$

with $\nu \in M_1^* = ((-\infty,-2] \cup [1,+\infty)) \cap \mathbb{Z}$ and $k = 1,\ 3$. Replacing $\nu \in M_1^*$ by $\nu := -\nu - 2 \in M_1^{**} = ((\mathbb{R}\diagdown(-3,0)) \cap \mathbb{Z}$ in (2.3), we see that

$$-\mu_1(\nu)^2(\nu+2)^5 y^{**}_{F,G,k}(1,\nu+1) = \sum_{j=1}^{2} a^{***}_{F,G,j}(1;-\nu-2)(\delta^j f_k)(1,\nu) \tag{2.4}$$

with $k = 1,\ 3$ and $\nu \in M_1^{**} = ((-\infty, -3] \cup [0, +\infty)) \cap \mathbb{Z}$. Let

$$\vec{w}_{F,G,j}(\nu) = \begin{pmatrix} a^{***}_{F,G,j+1}(1; -\nu-2) \\ F(2-j) + G(j-1) \\ a^{***}_{F,G,j+1}(1;\nu) \end{pmatrix}, \tag{2.5}$$

where $j = 1, 2,\ \nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$,

$$W_{F,G}(\nu) = \begin{pmatrix} a^{***}_{F,G,2}(1; -\nu-2) & a^{***}_{F,G,3}(1; -\nu-2) \\ F & G) \\ a^{***}_{F,G,2}(1;\nu) & a^{***}_{F,G,3}(1;\nu) \end{pmatrix} = \tag{2.6}$$

$$\begin{pmatrix} \vec{w}_{F,G,1}(\nu) & \vec{w}_{F,G,2}(\nu) \end{pmatrix},\ Y_k^{***}(\nu) = \begin{pmatrix} (\delta f_k)(1,\nu) \\ (\delta^2 f_k)(1,\nu) \end{pmatrix},$$

$$Y^{****}_{F,G,k}(\nu) = \begin{pmatrix} \mu_1(-\nu-2)^2(-\nu-2)^5 y^{**}_{F,G}(z, -\nu-3) \\ y^{**}_{F,G}(z,\nu) \\ \mu_1(\nu)^2\nu^5 y^{**}_{F,G}(z, \nu-1) \end{pmatrix}, \tag{2.7}$$

where $k = 1,\ 3,\ \nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$. Let further

$$\vec{w}_{F,G,3}(\nu) = \begin{pmatrix} w_{F,G,3,1}(\nu) \\ w_{F,G,3,2}(\nu) \\ w_{F,G,3,3}(\nu) \end{pmatrix} = [\vec{w}_{F,G,1}(\nu), \vec{w}_{F,G,2}(\nu)]. \tag{2.8}$$

is vector product of $\vec{w}_{F,G,1}(\nu)$ and $\vec{w}_{F,G,2}(\nu)$. Let $\bar{w}_{F,G,3}(\nu) = (\vec{w}_{F,G,3}(\nu))^t$ is the row conjugate to the column $\vec{w}_{F,G,3}(\nu)$. Then for scalar products $(\vec{w}_{F,G,3}(\nu), \vec{w}_{F,G,j}(\nu))$ we have the equalities

$$\bar{w}_{F,G,3}(\nu)\vec{w}_{F,G,j}(\nu) = (\vec{w}_{F,G,3}(\nu), \vec{w}_{F,G,j}(\nu)) = 0,$$

where $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z},\ j = 1,\ 2$. Therefore

$$\bar{w}_{F,G,3}(\nu) W_{F,G}(\nu) = \begin{pmatrix} 0 & 0 \end{pmatrix}, \tag{2.9}$$

where $\nu \in M_1^{****} = (\mathbb{R}\diagdown(-3,\ 1)) \cap \mathbb{Z}$. In view of (2.3) (2.4) and (2.9),

$$\bar{w}_{F,G,3}(\nu) Y^{****}_{F,G,k}(\nu) = \bar{w}_{F,G,3}(\nu) W_{F,G,3}(\nu) Y_k^{***}(\nu) = 0, \text{ for } k = 1,\ 3 \tag{2.10}$$

and $\nu \in M_1^{****} = (\mathbb{R}\diagdown(-3, 1)) \cap \mathbb{Z}$. Since $\tau_{-\nu-2} = -\nu - 1 = -\tau_\nu = -\tau$, it follows from (31) in [4], (2.1), (2.6) $-$ (2.8), that

$$a^{***}_{F,G,2}(1;\nu) = F a^{**}_{1,0,2,2}(1,\nu) + G a^{**}_{3,2}(1,\nu) = \tau^5(\tau-1) \times \tag{2.11}$$

$$(F(\tau^3 + 2(2\tau-1)^3) - 2G(\tau-1)(2\tau-1)(\tau^3 - (\tau-1)^3)),$$

$$a^{***}_{F,G,2}(z;-\nu-2)=Fa^{**}_{2,2}(1,-\nu-2)+Ga^{**}_{3,2}(1,-\nu-2)= \tag{2.12}$$

$$\tau^5(\tau+1)(-F(\tau^3+2(2\tau+1)^3)-2G(\tau+1)(2\tau+1)((\tau+1)^3)-\tau^3),$$

$$a^{***}_{F,G,3}(1;\nu)=Fa^{**}_{2,3}(1,\nu)+Ga^{**}_{3,3}(1,\nu)=\tau^4(\tau-1)\times \tag{2.13}$$

$$(-3F(2\tau-1)^3+G(\tau-1)((\tau-1)^3+2(2\tau-1)^3)),$$

$$a^{***}_{F,G,3}(z;-\nu-2)=Fa^{**}_{2,3}(1,-\nu-2)+Ga^{**}_{3,3}(1,-\nu-2)= \tag{2.14}$$

$$\tau^4(\tau+1(-3F(2\tau+1)^3-G(\tau+1)((\tau+1)^3+2(2\tau+1)^3))),$$

In view of (2.5), (2.11) – (2.14)

$$\vec{w}_{1-i,i,j}(\nu)=\begin{pmatrix} a^{***}_{1-i,i,j+1}(1;-\nu-2)\\ (1-i)(2-j)+i(j-1)\\ a^{***}_{1,0,j+1}(1;\nu), \end{pmatrix},$$

where $i=0,\ 1,\ j=1,2,\ \nu\in M_1^{****}=((-\infty,-3]\cup[1,+\infty))\cap\mathbb{Z}$. Then

$$\vec{w}_{F,G,j}(\nu)=F\vec{w}_{1,0,j}(\nu)+G\vec{w}_{0,1,j}(\nu)$$

where $j=1,2,\ \nu\in M_1^{****}=((-\infty,-3]\cup[1,+\infty))\cap\mathbb{Z}$. In view of (2.8),

$$\vec{w}_{F,G,3}(\nu)=FG([\vec{w}_{1,0,1}(\nu),\vec{w}_{0,1,2}(\nu)]+[\vec{w}_{0,1,1}(\nu),\vec{w}_{1,0,2}(\nu)])+ \tag{2.15}$$

$$F^2[\vec{w}_{1,0,1}(\nu),\vec{w}_{1,0,2}(\nu)]+G^2[\vec{w}_{0,1,1}(\nu),\vec{w}_{0,1,2}(\nu)].$$

For any $\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ $i=1,\ 2,\ 3$ we put $(\vec{a})_i=a_i$. Let further

$$\vec{w}_{i,j,4}(\nu)=[\vec{w}_{i,1-i,1}(\nu),\vec{w}_{j,1-j,2}(\nu)] \text{ with } \ i=0,\ 1,\ j=0,\ 1. \tag{2.16}$$

In view of (2.16),(2.11) − (2.14), $\vec{w}_{1,1,4}(\nu)=[\vec{w}_{1,0,1}(\nu),\vec{w}_{1,0,2}(\nu)]=\vec{w}_{1,0,3}(\nu)]$,

$$(\vec{w}_{1,1,4}(\nu))_1=(\vec{w}_{1,0,3}(\nu))_1=\det\begin{pmatrix}1 & 0\\ a^{***}_{1,0,2}(1;\nu) & a^{***}_{1,0,3}(1;\nu)\end{pmatrix}=$$

$$a^{***}_{1,0,3}(1;\nu)=a^{**}_{2,3}(1,\nu)=-3\tau^4(\tau-1)(2\tau-1)^3,$$

$$(\vec{w}_{1,1,4}(\nu))_2=(\vec{w}_{1,0,3}(\nu))_2=-\det\begin{pmatrix}a^{***}_{1,0,2}(1;-\nu-2) & a^{***}_{1,0,3}(1;-\nu-2)\\ a^{***}_{1,0,2}(1;\nu) & a^{***}_{1,0,3}(1;\nu)\end{pmatrix}=$$

$$-\det\begin{pmatrix}a^{**}_{2,2}(1;-\nu-2) & a^{**}_{2,3}(1;-\nu-2)\\ a^{**}_{2,2}(1;\nu) & a^{**}_{2,3}(1;\nu)\end{pmatrix}=$$

$$a^{**}_{2,2}(1;\nu)a^{**}_{2,3}(1;-\nu-2)-a^{**}_{2,3}(1;\nu)a^{**}_{2,3}(1;-\nu-2)=$$
$$\tau^5(\tau-1)(\tau^3+2(2\tau-1)^3)(-3\tau^4(\tau+1)(2\tau+1)^3)-$$
$$(-3\tau^4(\tau-1)(2\tau-1)^3)(-\tau^5(\tau+1)(\tau^3+2(2\tau+1)^3))=$$
$$-3\tau^9(\tau^2-1)(\tau^3((2\tau-1)^3+(2\tau+1)^3)+4(4\tau^2-1)^3)=$$
$$-12\tau^9(\tau^2-1)(68\tau^6-45\tau^4+12\tau^2-1),$$
$$(\vec{w}_{1,1,4}(\nu))_3=(\vec{w}_{1,0,3}(\nu))_3=\det\begin{pmatrix}a^{***}_{1,0,2}(1;-\nu-2) & a^{***}_{1,0,3}(1;-\nu-2)\\ 1 & 0\end{pmatrix}=$$
$$-a^{***}_{1,0,3}(1;-\nu-2)=-a^{**}_{2,3}(1,-\nu-2)=3\tau^4(\tau+1)(2\tau+1)^3.$$

In view of (2.16),(2.11) $-$ (2.14), $\vec{w}_{0,0,4}(\nu)=[\vec{w}_{0,1,1}(\nu),\vec{w}_{0,1,2}(\nu)]=\vec{w}_{0,1,3}(\nu)$,

$$(\vec{w}_{0,0,4}(\nu))_1=(\vec{w}_{0,1,3}(\nu))_1=\det\begin{pmatrix}0 & 1\\ a^{***}_{0,1,2}(1;\nu) & a^{***}_{0,1,3}(1;\nu)\end{pmatrix}=$$
$$-a^{***}_{0,1,2}(1;\nu)=-a^{**}_{3,2}(1,\nu)=2\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3),$$
$$(\vec{w}_{0,0,4}(\nu))_2=(\vec{w}_{0,1,3}(\nu))_2=-\det\begin{pmatrix}a^{***}_{0,1,2}(1;-\nu-2) & a^{***}_{0,1,3}(1;-\nu-2)\\ a^{***}_{0,1,2}(1;\nu) & a^{**1*}_{0,1,3}(1;\nu)\end{pmatrix}=$$
$$-\det\begin{pmatrix}a^{**}_{3,2}(1;-\nu-2) & a^{**}_{3,3}(1;-\nu-2)\\ a^{**}_{3,2}(1;\nu) & a^{**}_{3,3}(1;\nu)\end{pmatrix}=$$
$$a^{**}_{3,2}(1;\nu)a^{**}_{3,3}(1;-\nu-2)-a^{**}_{3,3}(1;\nu)a^{**}_{3,2}(1;-\nu-2)=$$
$$-2\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3)(-\tau^4(\tau+1)^2((\tau+1)^3+2(2\tau+1)^3)-$$
$$(-2\tau^5(\tau+1)^2(2\tau+1)((\tau+1)^3-\tau^3))(\tau^4(\tau-1)^2((\tau-1)^3+2(2\tau-1)^3)=$$
$$4\tau^9(\tau^2-1)^2(102\tau^6-68\tau^4+21\tau^2-3),$$
$$(\vec{w}_{0,0,4}(\nu))_3=(\vec{w}_{0,1,3}(\nu))_3=\det\begin{pmatrix}a^{***}_{0,1,2}(1;-\nu-2) & a^{***}_{0,1,3}(1;-\nu-2)\\ 0 & 1\end{pmatrix}=$$
$$a^{**}_{3,2}(1;-\nu-2)=-2\tau^5(\tau+1)^2(2\tau+1)((\tau+1)^3-\tau)^3),$$

In view of (2.16),(2.11) $-$ (2.14), $\vec{w}_{0,1,4}(\nu)=[\vec{w}_{0,1,1}(\nu),\vec{w}_{1,0,2}(\nu)]$,

$$(\vec{w}_{0,1,4}(\nu))_1=([\vec{w}_{0,1,1}(\nu),\vec{w}_{1,0,2}(\nu)])_1=\det\begin{pmatrix}0 & 0\\ a^{***}_{0,1,2}(1;\nu) & a^{***}_{1,0,3}(1;\nu)\end{pmatrix}=0,$$
$$(\vec{w}_{0,1,4}(\nu))_2=([\vec{w}_{0,1,1}(\nu),\vec{w}_{1,0,2}(\nu)])_2=$$
$$-\det\begin{pmatrix}a^{***}_{0,1,2}(1;-\nu-2) & a^{***}_{1,0,3}(1;-\nu-2)\\ a^{***}_{0,1,2}(1;\nu) & a^{***}_{1,0,3}(1;\nu)\end{pmatrix}=$$
$$-a^{**}_{3,2}(1;-\nu-2)a^{**}_{2,3}(1;\nu)+a^{**}_{3,2}(1;\nu)a^{**}_{2,3}(1;-\nu-2)=$$
$$-12t^9(\tau^2-1)(4\tau^2-1)(12\tau^4-6\tau^2+1),$$

$$(\vec{w}_{0,1,4}(\nu))_3 = ([\vec{w}_{0,1,1}(\nu), \vec{w}_{1,0,2}(\nu)])_3 =$$
$$\det \begin{pmatrix} a^{***}_{0,1,2}(1; -\nu-2) & a^{***}_{1,0,3}(1; -\nu-2) \\ 0 & 0 \end{pmatrix} = 0,$$

In view of (2.16),(2.11) $-$ (2.14), $\vec{w}_{1,0,4}(\nu) = [\vec{w}_{1,0,1}(\nu), \vec{w}_{0,1,2}(\nu)]$,

$$(\vec{w}_{1,0,4}(\nu))_1 = ([\vec{w}_{1,0,1}(\nu), \vec{w}_{0,1,2}(\nu)])_1 = \det \begin{pmatrix} 1 & 1 \\ a^{***}_{1,0,2}(1; \nu) & a^{***}_{0,1,3}(1; \nu) \end{pmatrix} =$$
$$a^{**}_{3,3}(1; \nu) - a^{**}_{2,2}(1; \nu) = -t^4(t-1)(2t-1)(10t^2-10t+3),\ w_{1,0,4}(\nu))_2 =$$
$$(\vec{(}[w_{1,0,1}(\nu), w_{0,1,2}(\nu)])_2 = -\det \begin{pmatrix} a^{***}_{1,0,2}(1; -\nu-2) & a^{***}_{0,1,3}(1; -\nu-2) \\ a^{***}_{1,0,2}(1; \nu) & a^{***}_{0,1,3}(1; \nu) \end{pmatrix} =$$
$$-a^{**}_{2,2}(1; -\nu-2)a^{**}_{3,3}(1; \nu) + a^{**}_{2,2}(1; \nu)a^{**}_{3,3}(1; -\nu-2) =$$
$$-4t^9(t^2-1)(170t^6-104t^4+30t^2-3),\ (\vec{w}_{1,0,4}(\nu))_3 =$$
$$([\vec{w}_{1,0,1}(\nu), \vec{w}_{0,1,2}(\nu)])_3 = \det \begin{pmatrix} a^{***}_{1,0,2}(1; -\nu-2) & a^{***}_{0,1,3}(1; -\nu-2) \\ 1 & 1 \end{pmatrix} =$$
$$a^{**}_{2,2}(1; -\nu-2) - a^{**}_{3,3}(1; -\nu-2) = t^4(t+1)(2t+1)(10t^2+10t+3),$$
$$(\vec{w}_{0,1,4}(\nu))_2 + (\vec{w}_{1,0,4}(\nu))_2 = -8t^9(t^2-1)(157t^6-106t^4+30t^2-3).$$

Therefore,

(2.17)
$$(\vec{w}_{F,G,3}(\nu))_1 = -3t^4(\tau-1)(2\tau-1)^3)F^2 - \tau^4(\tau-1)(2\tau-1)\times$$
$$((10\tau^2-10\tau+3)FG - 2\tau(\tau-1)(\tau^3-(\tau-1)^3)G^2),$$

(2.18)
$$(\vec{w}_{F,G,3}(\nu))_2 = -12\tau^9(\tau^2-1)(68\tau^6-45\tau^4+12\tau^2-1)F^2 -$$
$$8t^9(t^2-1)(157t^6-106t^4+30t^2-3)FG +$$
$$4\tau^9(\tau^2-1)^2(102\tau^6-68\tau^4+21\tau^2-3)G^2,$$

(2.19)
$$(\vec{w}_{F,G,3}(\nu))_3 = 3t^4(t+1)(2\tau+1)^3F^2 + t^4(t+1)(2t+1)\times$$
$$((10t^2+10t+3)FG - 2\tau(\tau+1)((\tau+1)^3-\tau^3)G^2).$$

According to (2.2), (2.10), (2.7), (2.17), (2.18), (2.19),

(2.20)
$$-\tau^4(\tau+1)^5 w_{F,G,3,1}(\nu)y^{**}_{F,G,k}(\nu+1) +$$
$$w_{F,G,3,2}(\nu)y^{**}_{F,G,k}(\nu) + \tau^4(\tau-1)^5 w_{F,G,3,3}(\nu)y^{**}_{F,G,k}(\nu-1) = 0.$$

Since $f_{1,0,k}(1,\nu) = f^*_{1,0,k}(1,\nu)/(\nu+1)^2$, it follows from (2.20), (0.1) – (0.3) that

(2.21)
$$c_{F,G,2}(\nu)x(\nu+1) + c_{F,G,1}(\nu)x(\nu) + c_{F,G,0}(\nu)x(\nu-1) = 0$$

for $x(\nu) = x_{F,G,k}(\nu)$, where

$$x_{F,G,k}(\nu) = F\delta f_k^*(1,\nu) + G\delta^2 f_k^*(1,\nu)),\ k = 1,\ 3. \tag{2.22}$$

Let

$$\beta^{**}_{F,G,i}(z;\nu) := F\beta^{*(1)}_{2i}(z;\nu) + G\beta^{*(2)}_{2i}(z;\nu)$$

for $i = 1, 2$. In view of (1.1),

$$\beta^{**}_{F,G,1}(1;\nu) := F\beta^{*(1)}_{2}(1;\nu) + G\beta^{*(2)}_{2}(1;\nu) =$$

$$F\delta f_1^*(1,\nu) + G\delta^2 f_1^*(1,\nu) = x_{F,G,1}(\nu)$$

In view of (16) in [4] with $j = 1$ and (2.22),

$$x_{F,G,3}(\nu) = 2\zeta(3)\beta^{**}_{F,G,1}(1;\nu) - \beta^{**}_{F,G,2}(1;\nu). \tag{2.23}$$

The equality (2.21) have been checked for $\nu = 1$, $k = 3$. in [5], pages 30–32.

§3. Auxiliary continued fraction.

Let

$$c^*_{u,v,k}(\nu) = c_{u,v,k}(\nu)/(u+v)^2 \text{ for } k = 0,\ 1,\ 2, \tag{3.1}$$

$$b^*_{u,v}(\nu+1) = -c^*_{u,v,1}(\nu) \in \mathbb{Q}[u,v]/(u+v)^2 \text{ for } \nu \in \mathbb{N}, \tag{3.2}$$

$$a^*_{u,v}(\nu+1) = -c^*_{u,v,0}(\nu)c^*_{u,v,2}(\nu-1) \text{ for } \nu \in [2+\infty) \cap \mathbb{N}, \tag{3.3}$$

$$a^*_{u,v}(2) = -c^*_{u,v,0}(1), \tag{3.4}$$

$$P^*_{u,v}(0) = b^*_{u,v}(0) = (3u+2v)/(u+v),\ Q_{u,v}(0) = 1, \tag{3.5}$$

$$Q^*_{u,v}(1) = b^*_{u,v}(1) = (34u+52v)(u+v) \tag{3.6}$$

$$P_{u,v}(1) = (327u+500v)(4u+4v) \tag{3.7}$$

$$a^*_{u,v}(1) = P^*_{u,v}(1) = -b^*_{u,v}(0)b^*_{u,v}(1). \tag{3.8}$$

Let $r^*_{u,v}(\nu)$ be the ν-th convergent of the continued fraction

$$b^*_{u,v}(0) + \frac{a^*_{u,v}(1)|}{b^*_{u,v}(1)} + \frac{a^*_{u,v}(2)|}{b^*_{u,v}(2)} + \frac{a^*_{u,v}(3)|}{b^*_{u,v}(3)} + \frac{a^*_{u,v}(4)|}{|b^*_{F,G}(4)}\ldots. \tag{3.9}$$

Let $P^*_{u,v}(\nu)$ and $Q^*_{u,v}(\nu)$ be respectively nominator and denominator of $r^*_{u,v}(\nu)$. If $F+G\neq 0$, then, clearly, the equations

$$c_{F,G,2}(\nu)x_{\nu+1}+c_{F,G,1}(\nu)x_\nu+c_{F,G,0}(\nu)x_{\nu-1}=0, \tag{3.10}$$

$$c^*_{F,G,2}(\nu)x_{\nu+1}+c^*_{F,G,1}(\nu)x_\nu+c^*_{F,G,0}(\nu)x_{\nu-1}=0, \tag{3.11}$$

are equivalent. If $x_\nu=x_{F,G,k}(\nu)=F\delta f^*_{1,0,k}(1,\nu)+G\delta^2 f_{1,0,k}(1,\nu)$, for $\nu\in\mathbb{N}_0$ and fixed $k\in\{1,\,3\}$, then, in view of (2.21), the equality (3.10) holds. In view of (0.1) $-$ (0.3), $c_{F,G,2}(\nu)=-12\tau^8G^2(1+o(1))\,(\tau\to\infty)$,

$$c_{F,G,1}(\nu)=408\tau^8G^2(1+o(1)),\; c_{F,G,0}(\nu)=-12\tau^8G^2(1+o(1))\,(\tau\to\infty),$$

if $G\neq 0$, and, if $G=0$, then $c_{F,G,2}(\nu)=24\tau^6F^2(1+o(1))\,(\tau\to\infty)$,

$$c_{F,G,1}(\nu)=-24\tau^6 34F^2(1+o(1)),\; c_{F,G,0}(\nu)=24\tau^6F^2(1+o(1))\,(\tau\to\infty).$$

In any case the equation (3.10) is difference equation of Poincaré type with characteristic polynomial $\lambda^2-34\lambda+1$. Hence, if $\{x_\nu\}_{\nu=1}^{+\infty}$ is a non-zero solution of (3.10), $\varepsilon\in(0,1)$, then there are $C_1(\varepsilon)>0$ and $C_2(\varepsilon)>0$ such that either

$$C_1(\varepsilon)\left(1+\sqrt{2}\right)^{-4\nu(1+\varepsilon)}\leq|x_\nu|\leq C_2(\varepsilon)\left(1+\sqrt{2}\right)^{-4\nu(1-\varepsilon)} \tag{3.12}$$

for all $\nu\in\mathbb{N}$ or

$$C_1(\varepsilon)\left(1+\sqrt{2}\right)^{4\nu(1-\varepsilon)}\leq|x_\nu|\leq C_2(\varepsilon)\left(1+\sqrt{2}\right)^{4\nu(1+\varepsilon)}. \tag{3.13}$$

for all $\nu\in\mathbb{N}$. In view of (0.10), if $x_\nu=\beta^{*(r)}(1;\nu)=\delta^r f^*_1(1,\nu)$ with $r=0,1,2$, then (3.12) is impossible. Therefore

$$C_1(\varepsilon)\left(1+\sqrt{2}\right)^{4\nu(1-\varepsilon)}\leq\beta^{*(r)}(1;\nu)\leq C_2(\varepsilon)\left(1+\sqrt{2}\right)^{4\nu(1+\varepsilon)}. \tag{3.14}$$

for $r=0,1,2$, and all $\nu\in\mathbb{N}$.

Lemma 3.1. *The following equalities hold:*

$$\lim_{\nu\to\infty}\beta^{*(1)}_{1,0,2}(1;\nu))=+\infty,\; \lim_{\nu\to\infty}\beta^{*(2)}_2(1;\nu)/\beta^{*(1)}_2(1;\nu)=+\infty, \tag{3.15}$$

Proof. Proof can be found in [5], §6, Lemma 6.1. □

Let conditions Theorem B are fulfilled. Then, in view of (0.10),

$$\beta^{**}_{F,G,1}(1;\nu)=(F+G)\beta^{*(1)}_{1,0,2}(1;\nu)+G(\beta^{*(2)}_{1,0,2}(1;\nu)-\beta^{*(1)}_{1,0,2}(1;\nu))\neq 0 \tag{3.16}$$

for $\nu \in \mathbb{N}_0$, and therefore, in view of (3.15),

$$\beta^*_{F,G,1}(1;\nu) = \beta_2^{*(1)}(1;\nu)(F + G\beta_2^{*(2)}(1;\nu)/\beta_2^{*(1)}(1;\nu)) \to \infty, \tag{3.17}$$

when $\nu \to \infty$. Moreover, if $F \neq 0$ and $G/F \notin \mathfrak{B}$ then (3.16) and (3.17) hold, and, if in this case $x_\nu = x_{F,G,1}(\nu) = \beta^{**}_{F,G,1}(1;\nu)$, then (3.12) is impossible. In view of (11) $-$ (16) with $j = \alpha = 1$ in [4] and (4.3), (4.35) in [5], we have the equality $\delta^r f^*_{1,0,3}(1,\nu) = (\nu+1)^2 O(1)$ for $r = 0$, 1, 2; hence (3.13) is impossible for $x_\nu = x_{F,G,3}(\nu) = F\delta f^*_{1,0,3}(1,\nu) + G\delta^2 f^*_{1,0,3}(1,\nu)$. Therefore

$$\frac{C_1(\varepsilon)/C_2(\varepsilon)}{\left(1+\sqrt{2}\right)^{8\nu(1+\varepsilon)}} \leq \left|2\zeta(3) - \frac{\beta^{**}_{F,G,2}(1,\nu)}{\beta^{**}_{F,G,1}(1,\nu)}\right| \leq \frac{C_2(\varepsilon)/C_1(\varepsilon)}{\left(1+\sqrt{2}\right)^{8\nu(1-\varepsilon)}}. \tag{3.18}$$

Let

$$\delta^*_{u,v}(\nu) = \prod_{j=1}^{\nu-1} c^*_{u,v,2}(j) \text{ for } \nu \in \mathbb{N}_0. \tag{3.19}$$

So, $\delta^*_{F,G}(1) = \delta^*_{F,G}(0) = 1$. Let $c^{**}_{F,G,1}(\nu) = c^*_{F,G,1}(\nu)$ for all $\nu \in \mathbb{N}$, let

$$c^{**}_{F,G,0}(\nu) = c^*_{F,G,0}(\nu)c^*_{F,G,0}(\nu-1) \text{ for all } \nu \in [2,+\infty) \cap \mathbb{N},$$

and let $c^{**}_{F,G,0}(1) = c^*_{F,G,0}(1)$. If conditions of Theorem B are fulfilled,
then $\delta_{F,G}(\nu) \neq 0$ far all the $\nu \in \mathbb{N}$, and the equation (3.10) and the system

$$\begin{cases} y_{\nu+1} + c^*_{F,G,1}(\nu)y(\nu) + c^*_{F,G,0}(\nu)c^*_{F,G,2}(\nu-1)y(\nu-1) = 0 \\ y_\nu = \delta^*_{F,G}(\nu)x_\nu,\ \nu \in \mathbb{N} \end{cases}$$

are equivalent. Moreover, $P^*_{F,G}(\nu)$ and $\delta^*_{F,G}(\nu)\beta^{**}_{F,G,2}(1,\nu)/\beta^{**}_{F,G,1}(1,0)$ satisfy to the first of equations (3.10) and the same initial conditions. Therefore

$$P^*_{F,G}(\nu) = \delta_{F,G}(\nu)\beta^{**}_{F,G,2}(1,\nu)/\beta^{**}_{F,G,1}(1,0), \tag{3.20}$$

Analogously,

$$Q^*_{F,G}(\nu) = \delta_{F,G}(\nu)\beta^{**}_{F,G,1}(1,\nu)/.\beta^{**}_{F,G,1}(1,0), \tag{3.21}$$

Hence, $r^*_{F,G}(\nu) = \beta^{**}_{F,G,2}(1,\nu)/\beta^{**}_{F,G,1}(1,\nu)$ for all $\nu \in \mathbb{N}_0$. In view of (3.18)

$$\lim_{\nu\to\infty} r^*_{F,G}(\nu) = 2\zeta(3).$$

§4. End of the proof of Theorem B.

Lemma 4.1. *The following equalities hold:*

$$P^*_{u,v}(\nu) = P^*_{u/v,1}(\nu),\, Q^*_{u,v}(\nu) = Q^*_{u/v,1}(\nu), \delta_{u,v}(\nu) = \delta_{u/v,1}(\nu). \tag{4.1}$$

Proof. If $\nu = 0,\, 1$, then the equalities (4.1) directly follows from (3.5) – (3.7) and (3.19) In view of (3.1),

$$c^*_{u,v,k}(\nu) = c^*_{u/v,1,k}(\nu) \tag{4.2}$$

for $k = 0,\, 1,\, 2, \nu \in \mathbb{N}$. Therefore the last equality in (4.1) holds for all $\nu \in \mathbb{N}$. In view (3.8)), (3.5), (3.6), (3.3), (3.2) and (4.2),

$$a_{u,v}(\nu) = a_{u/v,1}(\nu),\, b_{u,v}(\nu) = b_{u/v,1}(\nu)$$

for $\nu \in [2, +\infty) \cap \mathbb{Z}$. Let $\nu \in [2, +\infty) \cap \mathbb{Z}$, and let (4.1) hold for all $\nu - \kappa$ with $\kappa \in [0, \nu - 1] \cap \mathbb{Z}$. Then we have

$$P^*_{u,v}(\nu) = b^*_{u,v}(\nu)P^*_{u,v}(\nu - 1) + a^*_{u,v}(\nu)P^{(}_{u,v}\nu - 2) =$$

$$b^*_{u/v,1}(\nu)P^*_{u/v,1}(\nu - 1) + a^*_{u/v,1}(\nu)P^*_{u/v,1}(\nu - 2) = P^*_{u/v,1}(\nu)$$

$$Q^*_{u,v}(\nu) = b^*_{u,v}(\nu)Q^*_{u,v}(\nu - 1) + a^*_{u,v}(\nu)Q^*_{u,v}(\nu - 2) =$$

$$b^*_{u/v,1}(\nu)Q^*_{u/v,1}(\nu - 1) + a^*_{u/u,1}(\nu)Q^*_{u/v,1}(\nu - 2) = Q_{u/v,1}(\nu).$$

□

Lemma 4.2. *The following equalities hold:*

$$\delta^*_{u,v}(\nu) = \delta_{u,v}(\nu)/(u + v)^{\max(2\nu - 2,0)} \tag{4.3}$$

where $\delta^*_{u,v}(\nu)$ is homogeneous polynomial in $\mathbb{Z}[u, v]$, and

$$\max(2\nu - 2, 0) = \deg_u(\delta_{u,v}(\nu)) = \deg_v(\delta_{u,v}(\nu)) = \deg(\delta_{u,v}(\nu)). \tag{4.4}$$

Proof. We have to prove the last equality in (4.4), because other assertions of the Lemma are obvious. In view of (0.3) and (0.1),

$$c_{F,G,0}(\nu) = -((\tau - 1)^2(2\tau + 1)/(\tau + 1)^2(2\tau - 1))\tau(\tau + 1)c_{F,G,2}(\nu + 1) \neq 0$$

for all $\nu \in \mathbb{N}_0$. Therefore the last equality in (4.4) holds also.
Since (see (5.61), (5.62) in §5 of [5]) $\beta^{**}_{F,G,1}(1, 1) = 34F + 52G$,

$$\beta^{**}_{F,G,1}(1, 0) = F + G,\, \beta^{**}_{F,G,2}(1, 2) = (14843F + 32845G)/6,$$

$$\beta^{**}_{F,G,2}(1, 1) = (327F + 500G)/4, \beta^{**}_{F,G,2}(1, 0) = 3F + 2G,$$

it follows from (3.20), (3.21) and (4.1) that

$$P_{u,v}(\nu) = 4P^*_{u,v}(\nu)(u+v)^{2\nu} = 16(u+v)\delta_{u,v}(\nu)\beta^{**}_{u,v,2}(1,\nu),$$

$$Q_{u,v}(\nu) = Q^*_{u,v}(\nu)(u+v)^{2\nu-1} = 4\delta_{u,v}(\nu)\beta^{**}_{u,v,1}(1,\nu),$$

$$\max(2\nu,1) = \deg_u(P_{u,v}(\nu)) = \deg_v(P_{u,v}(\nu)) = \deg(P_{u,v}(\nu)),$$

$$\max(2\nu-1,0) = \deg_u(Q_{u,v}(\nu)) = \deg_v(Q_{u,v}(\nu)) = \deg(Q_{u,v}(\nu)),$$

where $\nu \in \mathbb{N}_0$. $\square$

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