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Maps of several variables of finite total variation. I. Mixed differences and the total variation ☆

Vyacheslav V. Chistyakov *, Yuliya V. Tretyachenko

Department of Applied Mathematics and Informatics, State University Higher School of Economics, Bol'shaya Pechërskaya Street 25/12, Nizhny Novgorod 603155, Russian Federation

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ABSTRACT

Given two points $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$ from $\mathbb{R}^n$ with $a<b$ componentwise and a map f from the rectangle $I_a^b=[a_1,b_1]\times\dots\times[a_n,b_n]$ into a metric semigroup $M=(M,d,+)$, we study properties of the *total variation* $\mathrm{TV}(f,I_a^b)$ of f on I_a^b introduced by the first author in [V.V. Chistyakov, A selection principle for mappings of bounded variation of several variables, in: Real Analysis Exchange 27th Summer Symposium, Opava, Czech Republic, 2003, pp. 217–222] such as the additivity, generalized triangle inequality and sequential lower semicontinuity. This extends the classical properties of C. Jordan's total variation ($n=1$) and the corresponding properties of the total variation in the sense of Hildebrandt [T.H. Hildebrandt, Introduction to the Theory of Integration, Academic Press, 1963] ($n=2$) and Leonov [A.S. Leonov, On the total variation for functions of several variables and a multidimensional analog of Helly's selection principle, Math. Notes 63 (1998) 61–71] ($n\in\mathbb{N}$) for real-valued functions of n variables.

1. Introduction to part I

The notion of a *function of bounded variation* was introduced by Jordan in [32] for real-valued functions on the closed interval $I=[a,b]\subset\mathbb{R}$ with $a,b\in\mathbb{R}$, $a<b$. In the present time this notion is well studied and applied in many directions even for metric space valued maps (e.g., [3,6,7,9,11,15,18–21,26,35,39,40]) and can be formulated as follows. Let (M,d) be a metric space with metric d and $f:I\to M$ be a map. Then f is said to be *of bounded variation* on I (in symbols, $f\in\mathrm{BV}(I;M)$) if its (*Jordan*) *variation* defined by

$$V_a^b(f)=\sup_{\mathcal{P}}\sum_{i=1}^m d\big(f(x_{i-1}),f(x_i)\big)$$

is finite, the supremum being taken over all partitions $\mathcal{P}=\{x_i\}_{i=0}^m$ of the interval I, i.e., $m\in\mathbb{N}$ and $a=x_0<x_1<\dots<x_{m-1}<x_m=b$. Let us note that the value $V_a^b(f)$ depends only on the order relation on the domain I, which is a linear order, and the distance function d in the target space M, and these are minimal assumptions for the value $V_a^b(f)$ to be meaningful.

Given $x,y\in I$ with $x\leqslant y$ and $f:I\to M$, the value $V_x^y(f)$ is defined in a similar way, and it is well known (e.g., from the references above, or as almost straightforward consequences of the definition of $V_x^y(f)$) that it has the following three

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* Corresponding author. Fax: +7 831 4169650.
E-mail addresses: czeslaw@mail.ru, vchistyakov@hse.ru (V.V. Chistyakov), tretyachenko_y_v@mail.ru (Y.V. Tretyachenko).

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basic properties: (a) it is *additive* in the sense that if $a \leqslant x \leqslant y \leqslant z \leqslant b$, then $V_x^z(f) = V_x^y(f) + V_y^z(f)$; (b) the inequality $d(f(x), f(y)) \leqslant V_x^y(f)$ holds for all $x, y \in I$, $x \leqslant y$; and (c) it is *sequentially lower semicontinuous* in the sense that if $x, y \in I$, $x \leqslant y$, and a sequence of maps $f_j : I \to M$, $j \in \mathbb{N}$, *converges pointwise* on I to a map $f : I \to M$ (i.e., $d(f_j(x), f(x)) \to 0$ as $j \to \infty$ for all $x \in I$), then $V_x^y(f) \leqslant \liminf_{j\to\infty} V_x^y(f_j)$.

In particular, these properties are essential for the validity of the following Helly-type pointwise selection principle in the space $\mathrm{BV}(I; M)$ ([28] if $M = \mathbb{R}$, and [7,11,15,18] in the general case): *if a sequence* $\{f_j\} \equiv \{f_j\}_{j\in\mathbb{N}}$ *of maps from* $\mathrm{BV}(I; M)$ *is such that the closure in M of the set $\{f_j(x)\}_{j\in\mathbb{N}}$ is compact for each $x \in I$ and the sequence $\{V_a^b(f_j)\}_{j\in\mathbb{N}}$ is bounded, then there exists a subsequence of $\{f_j\}$, which converges pointwise on I to a map f from* $\mathrm{BV}(I; M)$. This compactness result effectively applies to the proof of the existence of selections of bounded variation for univariate multi-valued maps with compact images from M (cf. [7,10,11,15,22]).

At the same time the above Helly-type selection principle is a motivation for this paper (part I). In what follows we will be interested to what extent the three properties above of the Jordan variation carry over to maps of several real variables so that the Helly theorem still holds. These properties are also of independent importance in the study of (multi-valued) nonlinear superposition operators acting on BV maps of several variables (in a paper under preparation) as it was exposed in more particular cases in [16] and [17]. Basing on the results and technique of this part, two extensions of the Helly-type *pointwise* selection principle to metric semigroup-valued maps of several variables will be established in part II of this paper (we refer to part II in this issue for more details).

Properties (a), (b) and (c) above depend upon notions of bounded variation for functions of several variables, which are known to be quite numerous in the literature (e.g., [4,8,23,25,29,31,37,41–43]) and which generalize different aspects of the classical Jordan variation. Under some approaches to the multidimensional variation [2,8,36], which involve certain integration procedures, Helly-type theorems are rather concerned with the *almost everywhere* convergence of extracted subsequences, and no stronger convergence can be expected in this case. On the other hand, for *real-valued* functions of several variables there are definitions of the notion of variation [29,34], which go back to Vitali [41], Hardy [27] and Krause [1,23], such that a complete analogue of the Helly theorem holds with respect to the *pointwise* convergence of extracted subsequences. Moreover, it was shown recently [17,34] that certain counterparts of properties (a) and (b) hold for the variation in the sense of Vitali–Hardy–Krause.

The aim of part I is to study properties of the notion of *total variation* [29,34] for maps of several variables under minimal requirements on the target space M such that the approach of Vitali, Hardy and Krause is still applicable (in some generalized sense [14]). This is interesting in connection with the notion of a multifunction of several variables of bounded variation: it suffices to assume that the target space M is a metric semigroup (cf. Section 2). Note that in this case most approaches to the multidimensional variation lose sense (except the approach in [2]).

This paper (part I) is organized as follows. In Section 2 we present necessary definitions and our three main results, Theorems 1, 2 and 3, which extend the properties of additivity, generalized triangle inequality and sequential lower semicontinuity of the Jordan variation to metric semigroup-valued maps of several variables. Sections 3–5 are devoted to their proofs.

2. Definitions and main results

Throughout the paper we adopt and follow the Vitali–Hardy–Krause approach to the notion of variation for maps of several variables in the multiindex notation initiated in [12,14] and developed in detail in [17] (equivalent approaches in different notation for real functions can be found in [33,34]).

Let $\mathbb{N}$ and $\mathbb{N}_0$ stand for the sets of positive and nonnegative integers, respectively, and $n \in \mathbb{N}$. Given $x, y \in \mathbb{R}^n$, we write $x = (x_1, \dots, x_n) = (x_i\colon\ i \in \{1, \dots, n\})$ for the coordinate representation of x, and set $x + y = (x_1 + y_1, \dots, x_n + y_n)$, and $x - y$ is defined similarly. The inequality $x < y$ will be understood componentwise, i.e., $x_i < y_i$ for all $i \in \{1, \dots, n\}$, and a similar meaning applies to $x = y$, $x \leqslant y$, $y \geqslant x$ and $y > x$. If $x < y$ or $x \leqslant y$, we denote by I_x^y the rectangle $\prod_{i=1}^n [x_i, y_i] = [x_1, y_1] \times \cdots \times [x_n, y_n]$. Elements of the set $\mathbb{N}_0^n$ are as usual said to be *multiindices* and denoted by Greek letters and, given $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we set $|\theta| = \theta_1 + \cdots + \theta_n$ (the order of θ) and $\theta x = (\theta_1 x_1, \dots, \theta_n x_n)$. The n-dimensional multiindices $0_n = (0, \dots, 0)$ and $1_n = (1, \dots, 1)$ will be denoted simply by 0 and 1, respectively (actually, the dimension of 0 and 1 will be clear from the context). We also put $\mathcal{E}(n) = \{\theta \in \mathbb{N}_0^n\colon\ \theta \leqslant 1 \text{ and } |\theta| \text{ is even}\}$ (the set of 'even' multiindices) and $\mathcal{O}(n) = \{\theta \in \mathbb{N}_0^n\colon\ \theta \leqslant 1 \text{ and } |\theta| \text{ is odd}\}$ (the set of 'odd' multiindices). For elements from the set $\mathcal{A}(n) = \{\alpha \in \mathbb{N}_0^n\colon\ 0 \neq \alpha \leqslant 1\}$ we simply write $0 \neq \alpha \leqslant 1$.

The domain of (almost) all maps under consideration will be a rectangle I_a^b with fixed $a, b \in \mathbb{R}^n$, $a < b$, called the *basic rectangle*. The range of maps will be a *metric semigroup* $(M, d, +)$, i.e., (M, d) is a metric space, $(M, +)$ is an Abelian semigroup with the operation of addition $+$, and d is translation invariant: $d(u, v) = d(u + w, v + w)$ for all $u, v, w \in M$. A nontrivial example of a metric semigroup is as follows [24,38]: Let $(X, \|\cdot\|)$ be a real normed space and M be the family of all nonempty closed bounded convex subsets of X equipped with the Hausdorff metric d given by $d(U, V) = \max\{\mathrm{e}(U, V), \mathrm{e}(V, U)\}$, where $U, V \in M$ and $\mathrm{e}(U, V) = \sup_{u\in U} \inf_{v\in V} \|u - v\|$. Given $U, V \in M$, defining $U \oplus V$ as the closure in X of the Minkowski sum $\{u + v\colon\ u \in U,\ v \in V\}$ we find that the triple $(M, d, \oplus)$ is a metric semigroup (actually, M is an abstract convex cone but this is irrelevant for our purposes).

Given $f : I_a^b \to (M, d, +)$, we define the *Vitali-type n-th mixed 'difference'* of f on a subrectangle $I_x^y \subset I_a^b$, where $x, y \in I_a^b$ and $x < y$, by (cf. [14])

$$\mathrm{md}_n\big(f, I_x^y\big) = d\Bigg(\sum_{\theta\in\mathcal{E}(n)} f\big(x+\theta(y-x)\big), \sum_{\eta\in\mathcal{O}(n)} f\big(x+\eta(y-x)\big)\Bigg). \tag{2.1}$$

For example, for the first three dimensions we have: if $n = 1$, then $\mathcal{E}(1) = \{0\}$ and $\mathcal{O}(1) = \{1\}$, and so, $\mathrm{md}_1(f, I_x^y) = d(f(x), f(y))$; if $n = 2$, then $\mathcal{E}(2) = \{(0,0), (1,1)\}$ and $\mathcal{O}(2) = \{(0,1), (1,0)\}$, and so,

$$\mathrm{md}_2\big(f, I_x^y\big) = d\big(f(x_1,x_2) + f(y_1,y_2), f(x_1,y_2) + f(y_1,x_2)\big);$$

if $n = 3$, then $\mathcal{E}(3) = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$ and $\mathcal{O}(3) = \{(1,1,1), (1,0,0), (0,1,0), (0,0,1)\}$, and so,

$$\begin{aligned}\mathrm{md}_3\big(f, I_x^y\big) = d\big(&f(x_1,x_2,x_3) + f(y_1,y_2,x_3) + f(y_1,x_2,y_3) + f(x_1,y_2,y_3),\\ &f(y_1,y_2,y_3) + f(y_1,x_2,x_3) + f(x_1,y_2,x_3) + f(x_1,x_2,y_3)\big)\end{aligned}$$

(one may draw corresponding pictures to see the points where f is evaluated at the left- and right-hand places of d('left', 'right')).

Remark 2.1. Formally, the value $\mathrm{md}_n(f, I_x^y)$ from (2.1) is defined for $x < y$. Now if $x, y \in I_a^b$, $x \leqslant y$ and $x \not< y$, then the right-hand side in (2.1) is equal to zero for any map $f : I_a^b \to M$. In fact, if $x_i = y_i$ for some $i \in \{1, \dots, n\}$, then

$$\sum_{\theta\in\mathcal{E}(n)} f\big(x+\theta(y-x)\big) = \sum_{\bar\theta\in\mathcal{O}(n)} f\big(x+\bar\theta(y-x)\big).$$

In order to see this, given $\theta = (\theta_1, \dots, \theta_n) \in \mathcal{E}(n)$, we set $\bar\theta = (\bar\theta_1, \dots, \bar\theta_n) = (\theta_1, \dots, \theta_{i-1}, 1-\theta_i, \theta_{i+1}, \dots, \theta_n)$ and note that $\bar\theta \in \mathcal{O}(n)$ and, moreover, the map $\theta \mapsto \bar\theta$ is a bijection between $\mathcal{E}(n)$ and $\mathcal{O}(n)$. It remains to take into account that $x + \theta(y-x) = x + \bar\theta(y-x)$ for all $\theta \in \mathcal{E}(n)$, because

$$x_i + \theta_i(y_i - x_i) = x_i = x_i + (1-\theta_i)(y_i - x_i) = x_i + \bar\theta_i(y_i - x_i).$$

The *Vitali-type n-th variation* [17,34,41] of $f : I_a^b \to M$ is defined by

$$V_n\big(f, I_a^b\big) = \sup_{\mathcal{P}} \sum_{1\leqslant\sigma\leqslant\kappa} \mathrm{md}_n\big(f, I_{x[\sigma-1]}^{x[\sigma]}\big), \tag{2.2}$$

the supremum being taken over all multiindices $\kappa \in \mathbb{N}^n$ and all *net partitions* of I_a^b of the form $\mathcal{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$, where points $x[\sigma] = (x_1(\sigma_1), \dots, x_n(\sigma_n))$ from I_a^b are indexed by $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{N}_0^n$ with $\sigma \leqslant \kappa$ and satisfy the conditions: $x[0] = a$, $x[\kappa] = b$ and $x[\sigma-1] < x[\sigma]$ for all $1 \leqslant \sigma \leqslant \kappa$ (in other words, a net partition $\mathcal{P}$ is the Cartesian product of ordinary partitions of closed intervals $[a_i, b_i]$, $i = 1, \dots, n$). Note that all rectangles $I_{x[\sigma-1]}^{x[\sigma]}$ of a net partition are non-degenerated, non-overlapping and their union is I_a^b.

In order to define the notion of the total variation of a map $f : I_a^b \to M$ we need the notion of variation of f of order less than n. Following [17], we define the *truncation of a point* $x \in \mathbb{R}^n$ *by a multiindex* $0 \ne \alpha \leqslant 1$ by $x\lfloor\alpha = (x_i\colon i \in \{1, \dots, n\},\ \alpha_i = 1)$, and set $I_a^b\lfloor\alpha = I_{a\lfloor\alpha}^{b\lfloor\alpha}$. Clearly, $x\lfloor 1 = x$ and $I_a^b\lfloor 1 = I_a^b$, and if $x \in I_a^b$, then $x\lfloor\alpha \in I_a^b\lfloor\alpha$. For example, if $x = (x_1, x_2, x_3, x_4)$ and $\alpha = (1, 0, 0, 1)$, we have $x\lfloor\alpha = (x_1, x_4)$ and $I_a^b\lfloor\alpha = [a_1, b_1] \times [a_4, b_4]$. Given $f : I_a^b \to M$ and $z \in I_a^b$, we define the *truncated map* $f_\alpha^z : I_a^b\lfloor\alpha \to M$ *with the base at* z by $f_\alpha^z(x\lfloor\alpha) = f(z + \alpha(x - z))$ for all $x \in I_a^b$. It follows that f_α^z depends only on $|\alpha|$ variables $x_i \in [a_i, b_i]$, for which $\alpha_i = 1$, and the other variables remain fixed and equal to z_j when $\alpha_j = 0$. In the above example we get $f_\alpha^z(x_1, x_4) = f_\alpha^z(x\lfloor\alpha) = f(x_1, z_2, z_3, x_4)$ for $(x_1, x_4) \in [a_1, b_1] \times [a_4, b_4]$.

Now, given $f : I_a^b \to M$ and $0 \ne \alpha \leqslant 1$, the function $f_\alpha^a : I_a^b\lfloor\alpha \to M$ with the base at $z = a$ depends only on $|\alpha|$ variables, and so, making use of the definitions (2.2) and (2.1) with n replaced by $|\alpha|$, f replaced by f_α^a and I_a^b replaced by $I_a^b\lfloor\alpha$, we get the notion of the (*Hardy–Krause-type* [1,23,27]) $|\alpha|$*-th variation* of f, which is denoted by $V_{|\alpha|}(f_\alpha^a, I_a^b\lfloor\alpha)$.

The *total variation* of $f : I_a^b \to M$ in the sense of Hildebrandt ([13,16], [29, III.6.3], [30] if $n = 2$) and Leonov ([12,14,17,34] if $n \in \mathbb{N}$) is defined by

$$\mathrm{TV}\big(f, I_a^b\big) = \sum_{0\ne\alpha\leqslant 1} V_{|\alpha|}\big(f_\alpha^a, I_a^b\lfloor\alpha\big) \equiv \sum_{i=1}^{n} \sum_{\alpha\leqslant 1,\,|\alpha|=i} V_i\big(f_\alpha^a, I_a^b\lfloor\alpha\big), \tag{2.3}$$

the summations here and throughout the paper being taken over *n-dimensional* multiindices in the ranges specified under the summation sign.

For the first three dimensions $n = 1, 2, 3$ we have, respectively,

$$\mathrm{TV}\big(f, I_a^b\big) = V_a^b(f), \quad \text{the usual Jordan variation on the interval } [a,b],$$

$$\mathrm{TV}\big(f, I_a^b\big) = V_{a_1}^{b_1}\big(f(\cdot, a_2)\big) + V_{a_2}^{b_2}\big(f(a_1, \cdot)\big) + V_2\big(f, I_{a_1,a_2}^{b_1,b_2}\big),$$

$$\begin{aligned}\mathrm{TV}\big(f, I_a^b\big) = {} & V_{a_1}^{b_1}\big(f(\cdot, a_2, a_3)\big) + V_{a_2}^{b_2}\big(f(a_1, \cdot, a_3)\big) + V_{a_3}^{b_3}\big(f(a_1, a_2, \cdot)\big) \\ & + V_2\big(f(\cdot,\cdot, a_3), I_{a_1,a_2}^{b_1,b_2}\big) + V_2\big(f(\cdot, a_2, \cdot), I_{a_1,a_3}^{b_1,b_3}\big) \\ & + V_2\big(f(a_1, \cdot,\cdot), I_{a_2,a_3}^{b_2,b_3}\big) + V_3\big(f, I_{a_1,a_2,a_3}^{b_1,b_2,b_3}\big).\end{aligned}$$

We denote by $\mathrm{BV}(I_a^b; M)$ the space of all maps $f : I_a^b \to M$ of *finite* (or bounded) *total variation* (2.3).

Our first main result, a counterpart of property (a) from the Introduction to be proved in Section 3, is the *additivity* property of $|\alpha|$-th variation $V_{|\alpha|}$ for each $0 \neq \alpha \leqslant 1$ (which is well known when $M = \mathbb{R}$):

Theorem 1. *Given $f : I_a^b \to M$, $x, y \in I_a^b$ with $x < y$, $z \in I_a^b$ and $0 \neq \alpha \leqslant 1$, if $\{x[\sigma]\}_{\sigma=0}^{\kappa}$ is a net partition of I_x^y, then*

$$V_{|\alpha|}\big(f_\alpha^z, I_x^y \lfloor \alpha\big) = \sum_{1\lfloor\alpha \leqslant \sigma\lfloor\alpha \leqslant \kappa\lfloor\alpha} V_{|\alpha|}\big(f_\alpha^z, I_{x[\sigma-1]}^{x[\sigma]} \lfloor \alpha\big), \tag{2.4}$$

where the summation is taken only over those σ_i in the range $1 \leqslant \sigma_i \leqslant \kappa_i$ with $i \in \{1, \dots, n\}$, for which $\alpha_i = 1$.

In order to present our second main result (Theorem 2), we need a lemma concerning mixed differences and some short notations, which will be used throughout the paper. Given $0 \neq \alpha \leqslant 1$, the sum over '$\mathrm{ev}\,\theta \leqslant \alpha$' denotes the sum over '$\theta \in \mathcal{E}(n)$ s.t. $\theta \leqslant \alpha$', where 's.t.' is the usual abbreviation for 'such that', and a similar convention applies to the sum over '$\mathrm{od}\,\theta \leqslant \alpha$'.

Lemma 1. *If $f : I_a^b \to M$, $x, y \in I_a^b$, $x \leqslant y$, $z \in I_a^b$ and $0 \neq \alpha \leqslant 1$, then*

$$\mathrm{md}_{|\alpha|}\big(f_\alpha^z, I_x^y \lfloor \alpha\big) = d\bigg(\sum_{\mathrm{ev}\,\theta \leqslant \alpha} f\big(z + \alpha(x-z) + \theta(y-x)\big), \sum_{\mathrm{od}\,\theta \leqslant \alpha} f\big(z + \alpha(x-z) + \theta(y-x)\big)\bigg). \tag{2.5}$$

In particular, if $z = a$ or $z = x$, we have, respectively,

$$\begin{aligned}\mathrm{md}_{|\alpha|}\big(f_\alpha^a, I_x^y \lfloor \alpha\big) &= \mathrm{md}_{|\alpha|}\big(f_\alpha^{a+\alpha(x-a)}, I_{a+\alpha(x-a)}^y \lfloor \alpha\big), \\ \mathrm{md}_{|\alpha|}\big(f_\alpha^x, I_x^y \lfloor \alpha\big) &= d\bigg(\sum_{\mathrm{ev}\,\theta \leqslant \alpha} f\big(x + \theta(y-x)\big), \sum_{\mathrm{od}\,\theta \leqslant \alpha} f\big(x + \theta(y-x)\big)\bigg).\end{aligned} \tag{2.6}$$

The proof of Lemma 1 is the same as in [17, Part I, Lemma 5] (and so, details are omitted): we note only that $\theta' \in \mathbb{N}_0^{|\alpha|}$ and $|\theta'|$ is even (odd) if and only if there exists a unique $\theta \in \mathbb{N}_0^n$ s.t. $\theta \leqslant \alpha$, $|\theta|$ is even (odd, respectively) and $\theta' = \theta\lfloor\alpha$, and apply definition (2.1) where n is replaced by $|\alpha|$.

Theorem 2. *If $f \in \mathrm{BV}(I_a^b; M)$, $x, y \in I_a^b$ and $x \leqslant y$, then*

$$d\big(f(x), f(y)\big) \leqslant \sum_{0 \neq \alpha \leqslant 1} \mathrm{md}_{|\alpha|}\big(f_\alpha^x, I_x^y \lfloor \alpha\big) \leqslant \mathrm{TV}\big(f, I_x^y\big).$$

This theorem will be proved in Section 4. It is a generalization of property (b) from the Introduction and a counterpart of Leonov's (in)equalities established in [34, Theorem 2 and Corollary 5] for real-valued functions of n variables (cf. also [17, Part I, Lemma 6 and (3.5)]). The inequalities in Theorem 2 are also known for metric semigroup-valued maps of two variables [5,16]. However, in the general case Theorem 2 needs a different proof as compared to the cases of maps of one or two variable(s) or $M = \mathbb{R}$.

The last property, to be established in Section 5, is the *sequential lower semicontinuity* of the total variation $\mathrm{TV}(\cdot, I_a^b)$:

Theorem 3. *If a sequence of maps $\{f_j\}$ from I_a^b into M converges pointwise on I_a^b to a map $f : I_a^b \to M$, then* $\mathrm{TV}(f, I_a^b) \leqslant \liminf_{j\to\infty} \mathrm{TV}(f_j, I_a^b)$.

3. Proof of Theorem 1

In order to prove Theorem 1, we need several lemmas. The following equality will be needed in Lemma 2 (cf. [17, Part I, equality (3.4)]): given two multiindices $0 \leqslant \beta \leqslant \gamma \leqslant 1$, we have:

$$\bigl|\{\alpha\colon\ \beta \leqslant \alpha \leqslant \gamma \text{ and } |\alpha| = i\}\bigr| = C_{|\gamma|-|\beta|}^{i-|\beta|} \quad \text{for all } |\beta| \leqslant i \leqslant |\gamma|, \tag{3.1}$$

where $|A|$ denotes the number of elements in the set A and, given $0 \leqslant j \leqslant m$, $C_m^j = \binom{m}{j} = \frac{m!}{j!(m-j)!}$ is the usual binomial coefficient (with $0! = 1$). Also, recall (cf. Section 2) that a multiindex α is said to be even (odd) if $\alpha \in \mathcal{E}(n)$ ($\alpha \in \mathcal{O}(n)$, respectively).

Lemma 2. (a) *Given $m \in \mathbb{N}$ and integer $0 \leqslant k \leqslant m-1$, we have*:

$$\sum_{i \geqslant k/2}^{\leqslant m/2} C_{m-k}^{2i-k} = \sum_{i \geqslant (k+1)/2}^{\leqslant (m+1)/2} C_{m-k}^{2i-1-k} = 2^{m-k-1},$$

where the summations are taken over integer i in the ranges specified.

(b) *Given two multiindices $0 \leqslant \beta \leqslant \gamma \leqslant 1$ with $\beta \neq \gamma$, we have*:

$$\bigl|\{\text{even } \alpha\colon\ \beta \leqslant \alpha \leqslant \gamma\}\bigr| = \bigl|\{\text{odd } \alpha\colon\ \beta \leqslant \alpha \leqslant \gamma\}\bigr|.$$

Proof. (a) Since the case $m = 1$ is clear, let $m \geqslant 2$. By the binomial formula, we find $0 = (1-1)^{m-k} = \sum_{j=0}^{m-k} (-1)^j C_{m-k}^j$. Considering the possibilities when m and k are of different evenness or of the same evenness and changing the summation index j appropriately, we arrive at the desired equality.

(b) By virtue of (3.1), the left-hand side of the equality is equal to

$$\bigl|\{\alpha\colon\ \beta \leqslant \alpha \leqslant \gamma \text{ and } |\alpha| = 2i \text{ for all } i \text{ s.t. } |\beta| \leqslant 2i \leqslant |\gamma|\}\bigr| = \sum_{i \geqslant |\beta|/2}^{\leqslant |\gamma|/2} C_{|\gamma|-|\beta|}^{2i-|\beta|},$$

and the right-hand side of the equality is equal to

$$\bigl|\{\alpha\colon\ \beta \leqslant \alpha \leqslant \gamma \text{ and } |\alpha| = 2i-1 \text{ for all } i \text{ s.t. } |\beta| \leqslant 2i-1 \leqslant |\gamma|\}\bigr| = \sum_{i \geqslant (|\beta|+1)/2}^{\leqslant (|\gamma|+1)/2} C_{|\gamma|-|\beta|}^{2i-1-|\beta|}.$$

It remains to put $m = |\gamma|$ and $k = |\beta|$, note that $k < m$ and apply the equality from the previous assertion (a). □

Since $(M, d, +)$ is a metric semigroup, then, by virtue of the triangle inequality for d and the translation invariance of metric d on M, we have, for all $u, v, u', v' \in M$:

$$\begin{aligned} &d(u, v) \leqslant d(u', v') + d(u + u', v + v'),\\ &d(u + u', v + v') \leqslant d(u, v) + d(u', v'). \end{aligned} \tag{3.2}$$

Inequality (3.2) yields that the addition operation $(u, v) \mapsto u + v$ is a continuous map from $M \times M$ into M. More generally, if $u_j \to u$, $v_j \to v$, $u_j' \to u'$ and $v_j' \to v'$ as $j \to \infty$ (convergence of sequences in M), then $\lim_{j\to\infty} d(u_j + v_j, u_j' + v_j') = d(u + v, u' + v')$.

Before we turn to the proof of Theorem 1, a few remarks are in order. Note that if $\mathcal{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ is a net partition of I_a^b, then

$$I_a^b = \bigcup_{1 \leqslant \sigma \leqslant \kappa} I_{x[\sigma-1]}^{x[\sigma]} = \bigcup_{1 \leqslant \sigma \leqslant \kappa} \prod_{i=1}^{n} \bigl[x_i(\sigma_i - 1), x_i(\sigma_i)\bigr] = \prod_{i=1}^{n} \left(\bigcup_{l=1}^{\kappa_i} I_{x_i(l-1)}^{x_i(l)} \right) \tag{3.3}$$

is a union of non-overlapping non-degenerated rectangles $I_{x[\sigma-1]}^{x[\sigma]}$ with the sides parallel to the coordinate axes. In this section it will be convenient and brief to term the union as in (3.3) also a *partition* of I_a^b (by non-overlapping non-degenerated rectangles).

If $\mathcal{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ and $\mathcal{P}' = \{x'[\sigma']\}_{\sigma'=0}^{\kappa'}$ are two net partitions of I_a^b, we say that $\mathcal{P}'$ is a *refinement* of $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}'$. Also, for the sake of convenience we define the *n-th prevariation* of $f : I_a^b \to M$, corresponding to $\mathcal{P}$, by

$$\mathrm{v}_n(f; \mathcal{P}) = \sum_{1 \leqslant \sigma \leqslant \kappa} \mathrm{md}_n\bigl(f, I_{x[\sigma-1]}^{x[\sigma]}\bigr).$$

It follows that the Vitali-type n-th variation of f is given by $V_n(f, I_a^b) = \sup_{\mathcal{P}} \mathrm{v}_n(f; \mathcal{P})$, where the supremum is taken over all net partitions $\mathcal{P}$ of I_a^b.

The basic ingredient in the proof of Theorem 1 is the following

Lemma 3. *Given $f : I_a^b \to M$, if $\mathcal{P}$ and $\mathcal{P}'$ are two net partitions of I_a^b s.t. $\mathcal{P} \subset \mathcal{P}'$, then $\mathrm{v}_n(f; \mathcal{P}) \leqslant \mathrm{v}_n(f; \mathcal{P}')$.*

In order to prove this lemma we need three more Lemmas 4–6. In what follows we fix a map $f : I_a^b \to M$.

Lemma 4. *Given $x, y \in I_a^b$ with $x < y$ and $x' \in I_a^b$, we have the following partition of I_x^y, induced by the point x':*

$$I_x^y = \bigcup_{1-\xi \leqslant \alpha \leqslant 1} I_{x+\alpha\xi(x'-x)}^{x'+\alpha(y-x')}, \tag{3.4}$$

where the multiindex $\xi \equiv \xi(x, x', y) = (\xi_1, \ldots, \xi_n)$ is given by

$$\xi_i \equiv \xi_i\big(x, x', y\big) = \begin{cases} 1 & \text{if } x_i < x_i' < y_i, \\ 0 & \text{if } x_i' \leqslant x_i \text{ or } x_i' \geqslant y_i, \end{cases} \qquad i \in \{1, \ldots, n\}, \tag{3.5}$$

and

$$\mathrm{md}_n\big(f, I_x^y\big) \leqslant \sum_{1-\xi \leqslant \alpha \leqslant 1} \mathrm{md}_n\big(f, I_{x+\alpha\xi(x'-x)}^{x'+\alpha(y-x')}\big). \tag{3.6}$$

Before we prove Lemma 4, let us establish two of its particular variants as Lemmas 5 and 6 (note that in Lemma 5 the rectangles in the union may degenerate).

Lemma 5. *If $x, y \in I_a^b$ with $x < y$ and $x' \in I_x^y$, then we have the following union of non-overlapping (possibly, degenerated) rectangles*

$$I_x^y = \bigcup_{0 \leqslant \alpha \leqslant 1} I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')}, \tag{3.7}$$

and the following inequality holds:

$$\mathrm{md}_n\big(f, I_x^y\big) \leqslant \sum_{0 \leqslant \alpha \leqslant 1} \mathrm{md}_n\big(f, I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')}\big). \tag{3.8}$$

Proof. Since $x_i \leqslant x_i' \leqslant y_i$ for all $i \in \{1, \ldots, n\}$, we have:

$$I_{x_i}^{y_i} = [x_i, y_i] = \big[x_i, x_i'\big] \cup \big[x_i', y_i\big] = I_{x_i}^{x_i'} \cup I_{x_i'}^{y_i} = \bigcup_{\alpha_i=0}^{1} I_{x_i+\alpha_i(x_i'-x_i)}^{x_i'+\alpha_i(y_i-x_i')},$$

and so (cf. Eq. (2.5) in [17, Part II]),

$$I_x^y = \prod_{i=1}^{n} I_{x_i}^{y_i} = \prod_{i=1}^{n} \left(\bigcup_{\alpha_i=0}^{1} I_{x_i+\alpha_i(x_i'-x_i)}^{x_i'+\alpha_i(y_i-x_i')} \right) = \bigcup_{0 \leqslant \alpha \leqslant 1} I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')}.$$

The mixed difference at the left-hand side of (3.8) is given by (2.1), and again by virtue of (2.1), the mixed difference at the right-hand side of (3.8) is equal to

$$\mathrm{md}_n\big(f, I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')}\big) = d\left(\sum_{\mathrm{ev}\,\beta \leqslant 1} h(\alpha, \beta), \sum_{\mathrm{od}\,\beta \leqslant 1} h(\alpha, \beta) \right),$$

where $h(\alpha, \beta) = f(x + (\alpha \vee \beta)(x' - x) + \alpha\beta(y - x'))$ and $\alpha \vee \beta = \alpha + \beta - \alpha\beta$. Noting that if $\alpha = \beta$, then $\alpha \vee \beta = \beta$ and $\alpha\beta = \beta$, we find

$$\begin{aligned} \sum_{0 \leqslant \alpha \leqslant 1} \sum_{\mathrm{ev}\,\beta \leqslant 1} h(\alpha, \beta) &= \sum_{\mathrm{ev}\,\beta \leqslant 1} \sum_{\substack{0 \leqslant \alpha \leqslant 1, \\ \alpha \neq \beta}} h(\alpha, \beta) + \sum_{\mathrm{ev}\,\beta \leqslant 1} h(\beta, \beta) \\ &= \sum_{\mathrm{ev}\,\beta \leqslant 1} \sum_{\substack{0 \leqslant \alpha \leqslant 1, \\ \alpha \neq \beta}} h(\alpha, \beta) + \sum_{\mathrm{ev}\,\beta \leqslant 1} f\big(x + \beta(y - x)\big) \\ &\equiv U + u. \end{aligned}$$

Let us show that the double sum U can be represented as

$$U=\sum_{0\neq\gamma\leqslant 1}\ \sum_{\substack{0\leqslant\delta\leqslant\gamma,\\ \delta\neq\gamma}} c_{\gamma\delta} f\bigl(x+\gamma\bigl(x'-x\bigr)+\delta\bigl(y-x'\bigr)\bigr)$$

with certain integer factors $c_{\gamma\delta}$ to be evaluated below. In fact, given $0\neq\gamma\leqslant 1$ and $0\leqslant\delta\leqslant\gamma$ with $\delta\neq\gamma$, there exist even $\beta\leqslant 1$ and $0\leqslant\alpha\leqslant 1$, $\alpha\neq\beta$, s.t. $\alpha\vee\beta=\gamma$ and $\alpha\beta=\delta$. In order to see this, if γ is even or δ is even, we may set $\beta=\gamma$ and $\alpha=\delta$, or $\beta=\delta$ and $\alpha=\gamma$, respectively. Now, if γ and δ are odd, then since $\delta\neq\gamma$, we can find $i\in\{1,\dots,n\}$ s.t. $\delta_i=0$ and $\gamma_i=1$, and so, if we set $\beta=(\delta_1,\dots,\delta_{i-1},1,\delta_{i+1},\dots,\delta_n)$, then $\delta\leqslant\beta\leqslant\gamma$, $\delta\neq\beta\neq\gamma$ and $|\beta|=|\delta|+1$ is even, and it remains to put $\alpha=\gamma+\delta-\beta$.

Given γ and δ as above, let us evaluate $c_{\gamma\delta}$. Since $\delta=\alpha\beta\leqslant\beta\leqslant\alpha\vee\beta=\gamma$ and, given even β, the multiindex $0\leqslant\alpha\leqslant 1$, $\alpha\neq\beta$, s.t. $\alpha\vee\beta=\gamma$ and $\alpha\beta=\delta$, is determined uniquely by $\alpha=\gamma+\delta-\beta$, we have $c_{\gamma\delta}=|\{\text{even }\beta\colon\ \delta\leqslant\beta\leqslant\gamma\}|$.

In a similar manner, we find

$$\sum_{0\leqslant\alpha\leqslant 1}\ \sum_{\mathrm{od}\,\beta\leqslant 1} h(\alpha,\beta)=\sum_{\mathrm{od}\,\beta\leqslant 1}\ \sum_{\substack{0\leqslant\alpha\leqslant 1,\\ \alpha\neq\beta}} h(\alpha,\beta)+\sum_{\mathrm{od}\,\beta\leqslant 1} f\bigl(x+\beta(y-x)\bigr)\equiv V+v,$$

where

$$V=\sum_{0\neq\gamma\leqslant 1}\ \sum_{\substack{0\leqslant\delta\leqslant\gamma,\\ \delta\neq\gamma}} d_{\gamma\delta} f\bigl(x+\gamma\bigl(x'-x\bigr)+\delta\bigl(y-x'\bigr)\bigr)$$

with $d_{\gamma\delta}=|\{\text{odd }\beta\colon\ \delta\leqslant\beta\leqslant\gamma\}|$. By Lemma 2(b), $c_{\gamma\delta}=d_{\gamma\delta}$, and so, $U=V$. Applying the translation invariance of d and inequality (3.2), we obtain inequality (3.8):

$$\begin{aligned} d(u,v)=d(U+u,V+v)&=d\biggl(\sum_{0\leqslant\alpha\leqslant 1}\ \sum_{\mathrm{ev}\,\beta\leqslant 1} h(\alpha,\beta),\ \sum_{0\leqslant\alpha\leqslant 1}\ \sum_{\mathrm{od}\,\beta\leqslant 1} h(\alpha,\beta)\biggr)\\ &\leqslant\sum_{0\leqslant\alpha\leqslant 1} d\biggl(\sum_{\mathrm{ev}\,\beta\leqslant 1} h(\alpha,\beta),\ \sum_{\mathrm{od}\,\beta\leqslant 1} h(\alpha,\beta)\biggr).\qquad\square \end{aligned}$$

Lemma 6. *Given $x, y\in I_a^b$ with $x<y$ and $x'\in I_x^y$, we have the following partition of I_x^y, induced by x':*

$$I_x^y=\bigcup_{\lambda\leqslant\alpha\leqslant\mu} I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')},\tag{3.9}$$

where the multiindices $\lambda\equiv\lambda(x,x')=(\lambda_1,\dots,\lambda_n)$ and $\mu\equiv\mu(x',y)=(\mu_1,\dots,\mu_n)$ are defined for $i\in\{1,\dots,n\}$ by

$$\lambda_i\equiv\lambda_i\bigl(x,x'\bigr)=\begin{cases}1 & \text{if } x_i=x_i',\\ 0 & \text{if } x_i<x_i',\end{cases}\quad\text{and}\quad \mu_i\equiv\mu_i\bigl(x',y\bigr)=\begin{cases}0 & \text{if } x_i'=y_i,\\ 1 & \text{if } x_i'<y_i,\end{cases}$$

and the following inequality holds:

$$\mathrm{md}_n\bigl(f,I_x^y\bigr)\leqslant\sum_{\lambda\leqslant\alpha\leqslant\mu}\mathrm{md}_n\bigl(f,I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')}\bigr).\tag{3.10}$$

Proof. First, we note that, since $x_i<y_i$ for all $i\in\{1,\dots,n\}$, then $\lambda\leqslant\mu$. In particular, if $x<x'<y$, then $\lambda=0$ and $\mu=1$, and we get (3.7) as a consequence of (3.9).

In order to prove (3.9), given $i\in\{1,\dots,n\}$, consider the following possibilities: (i) $x_i'=x_i$ and $x_i'<y_i$; (ii) $x_i<x_i'$ and $x_i'=y_i$; and (iii) $x_i<x_i'$ and $x_i'<y_i$. We have, respectively:

(i) $\lambda_i=1$ and $\mu_i=1$, and so, if $\lambda_i\leqslant\alpha_i\leqslant\mu_i$, then $\alpha_i=1$ and

$$I_{x_i}^{y_i}=I_{x_i'}^{y_i}=\bigcup_{\alpha_i=1} I_{x_i+\alpha_i(x_i'-x_i)}^{x_i'+\alpha_i(y_i-x_i')};$$

(ii) $\lambda_i=0$ and $\mu_i=0$, and so, if $\lambda_i\leqslant\alpha_i\leqslant\mu_i$, then $\alpha_i=0$ and

$$I_{x_i}^{y_i}=I_{x_i}^{x_i'}=\bigcup_{\alpha_i=0} I_{x_i+\alpha_i(x_i'-x_i)}^{x_i'+\alpha_i(y_i-x_i')};$$

(iii) $\lambda_i = 0$ and $\mu_i = 1$, and so, if $\lambda_i \leqslant \alpha_i \leqslant \mu_i$, then $\alpha_i \in \{0, 1\}$ and

$$I_{x_i}^{y_i} = I_{x_i}^{x_i'} \cup I_{x_i'}^{y_i} = \bigcup_{\alpha_i=0}^{1} I_{x_i+\alpha_i(x_i'-x_i)}^{x_i'+\alpha_i(y_i-x_i')}.$$

Moreover, in all the cases (i)–(iii) the left endpoint $x_i + \alpha_i(x_i' - x_i)$ is less than the right endpoint $x_i' + \alpha_i(y_i - x_i')$, and so, all the closed intervals above are non-degenerated. It follows that

$$I_x^y = \prod_{i=1}^{n} I_{x_i}^{y_i} = \prod_{i=1}^{n}\left(\bigcup_{\lambda_i\leqslant\alpha_i\leqslant\mu_i} I_{x_i+\alpha_i(x_i'-x_i)}^{x_i'+\alpha_i(y_i-x_i')}\right) = \bigcup_{\lambda\leqslant\alpha\leqslant\mu} I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')}.$$

The point x' gives rise to a net partition $\{x[\sigma]\}_{\sigma=0}^{\kappa}$ of I_x^y as follows: we put $\kappa = \mu - \lambda + 1$ and, given $i \in \{1, \dots, n\}$, we set $x_i(0) = x_i$ and $x_i(1) = y_i$ if $\kappa_i = 1$, and $x_i(0) = x_i$, $x_i(1) = x_i'$ and $x_i(2) = y_i$ if $\kappa_i = 2$. We note that if $0 \leqslant \sigma \leqslant \mu - \lambda$, then $x[\sigma] = x + (\sigma + \lambda)(x' - x)$, and if $1 \leqslant \sigma \leqslant \kappa = \mu - \lambda + 1$, then $x[\sigma] = x' + (\sigma - 1 + \lambda)(y - x')$. Also, note that $x + \lambda(x' - x) = x$ and $x' + \mu(y - x') = y$. It follows that

$$I_x^y = \bigcup_{\lambda\leqslant\alpha\leqslant\mu} I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')} = \bigcup_{1\leqslant\sigma\leqslant\mu-\lambda+1} I_{x+(\sigma-1+\lambda)(x'-x)}^{x'+(\sigma-1+\lambda)(y-x')} = \bigcup_{1\leqslant\sigma\leqslant\kappa} I_{x[\sigma-1]}^{x[\sigma]}.$$

Now, we turn to the proof of (3.10). By Lemma 5, inequality (3.8) holds. Clearly, if $\lambda = 0$ and $\mu = 1$ (i.e., $x < x' < y$), then (3.8) implies (3.10). Assume that $\lambda \neq 0$ (i.e., $x \not< x'$) and suppose that $0 \leqslant \alpha \leqslant 1$ is s.t. $\lambda \not\leqslant \alpha$. Then there exists $i \in \{1, \dots, n\}$ s.t. $\lambda_i = 1$ and $\alpha_i = 0$, and so, $x_i = x_i'$, which implies $x_i + \alpha_i(x_i' - x_i) = x_i = x_i' = x_i' + \alpha_i(y_i - x_i')$. It follows from Remark 2.1 that $\mathrm{md}_n(f, I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')}) = 0$. Similarly, if we assume that $\mu \neq 1$ (i.e., $x' \not< y$) and suppose that $0 \leqslant \alpha \leqslant 1$ is s.t. $\alpha \not\leqslant \mu$, then there exists $i \in \{1, \dots, n\}$ s.t. $\alpha_i = 1$ and $\mu_i = 0$, and so, $x_i' = y_i$. Noting that $x_i + \alpha_i(x_i' - x_i) = x_i' = y_i = x_i' + \alpha_i(y_i - x_i')$, we find $\mathrm{md}_n(f, I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')}) = 0$. In this way inequality (3.10) follows. □

Proof of Lemma 4. Suppose that $x, y \in I_a^b$, $x < y$ and $x' \in I_a^b$. We set $x'' = x + \xi(x' - x)$, where ξ is defined in (3.5) (the point x'' will play the role of x' from (3.9)). We have $x \leqslant x'' < y$; in fact, given $i \in \{1, \dots, n\}$, we find: if $\xi_i = 1$, then $x_i < x_i' < y_i$ and $x_i'' = x_i'$ implying $x_i < x_i'' < y_i$, and if $\xi_i = 0$, then $x_i' \leqslant x_i$ or $x_i' \geqslant y_i$, and $x_i'' = x_i$ implying $x_i = x_i'' < y_i$. Applying (3.9) with x' replaced by x'', we get the following partition of I_x^y induced by x'' and, hence, by x':

$$I_x^y = \bigcup_{\lambda''\leqslant\alpha\leqslant\mu''} I_{x+\alpha(x''-x)}^{x''+\alpha(y-x'')}, \tag{3.11}$$

where $\lambda'' = \lambda(x, x'')$ and $\mu'' = \mu(x'', y)$ are defined in Lemma 6, i.e., given $i \in \{1, \dots, n\}$, we have:

$$\lambda_i'' = \begin{cases} 1 & \text{if } x_i = x_i'', \\ 0 & \text{if } x_i < x_i'', \end{cases} \quad \text{and} \quad \mu_i'' = \begin{cases} 0 & \text{if } x_i'' = y_i, \\ 1 & \text{if } x_i'' < y_i. \end{cases}$$

We assert that $\lambda'' = 1 - \xi$ and $\mu'' = 1$. In fact, since $x'' < y$, then $\mu'' = 1$. In order to see that $\lambda'' = 1 - \xi$, let $i \in \{1, \dots, n\}$. If $x_i < x_i' < y_i$, then $\xi_i = 1$, and so, $x_i'' = x_i + \xi_i(x_i' - x_i) = x_i'$, which implies $x_i < x_i''$ and $\lambda_i'' = 0 = 1 - \xi_i$. Now if $x_i' \leqslant x_i$ or $x_i' \geqslant y_i$, then $\xi_i = 0$, and so, $x_i'' = x_i$, which gives $\lambda_i'' = 1 = 1 - \xi_i$.

Now, let us calculate the lower and upper indices in (3.11). We have: $x + \alpha(x'' - x) = x + \alpha\xi(x' - x)$ and

$$x'' + \alpha\big(y - x''\big) = x + (1 - \alpha)\xi\big(x' - x\big) + \alpha(y - x).$$

Noting that the union in (3.11) is taken over $\alpha \leqslant 1$ s.t. $1 - \xi \leqslant \alpha$, we get $1 - \alpha \leqslant \xi$, and so, $(1 - \alpha)\xi = 1 - \alpha$ implying

$$x'' + \alpha\big(y - x''\big) = x + (1 - \alpha)\big(x' - x\big) + \alpha(y - x) = x' + \alpha\big(y - x'\big).$$

These calculations and observations above prove equality (3.4).

Let us show that partition (3.4) is actually induced by x'. Since $x' \in I_a^b$, by Lemma 6, the point x' induces a partition of I_a^b of the form (3.9):

$$I_a^b = \bigcup_{\lambda'\leqslant\beta\leqslant\mu'} I_{a+\beta(x'-a)}^{x'+\beta(b-x')},$$

where the multiindices $\lambda' = \lambda(a, x')$ and $\mu' = \mu(x', b)$ are defined in Lemma 6, i.e., given $i \in \{1, \dots, n\}$, we have:

$$\lambda_i' = \begin{cases} 1 & \text{if } a_i = x_i', \\ 0 & \text{if } a_i < x_i', \end{cases} \quad \text{and} \quad \mu_i' = \begin{cases} 0 & \text{if } x_i' = b_i, \\ 1 & \text{if } x_i' < b_i. \end{cases}$$

We assert that for each α with $1 - \xi \leqslant \alpha \leqslant 1$ there exists a unique $\beta \equiv \beta(\alpha)$ with $\lambda' \leqslant \beta \leqslant \mu'$ s.t.

$$I_{x+\alpha\xi(x'-x)}^{x'+\alpha(y-x')} = I_x^y \cap I_{a+\beta(x'-a)}^{x'+\beta(b-x')}. \tag{3.12}$$

In order to prove (3.12), we define $\beta = \beta(\alpha) = (\beta_1, \dots, \beta_n)$ by

$$\beta_i \equiv \beta_i(\alpha) = \begin{cases} \alpha_i & \text{if } x_i' < y_i, \\ 0 & \text{if } x_i' \geqslant y_i, \end{cases} \quad i \in \{1, \dots, n\},$$

and establish equality (3.12) componentwise. Given $i \in \{1, \dots, n\}$, we consider the following two cases: (a) $x_i' < y_i$, and (b) $x_i' \geqslant y_i$.

In case (a) we have $\beta_i = \alpha_i$. First, assume that $x_i < x_i'$, and so, $\xi_i = 1$. It follows that if $1 - \xi_i \leqslant \alpha_i \leqslant 1$, then $\alpha_i = 0$ or $\alpha_i = 1$. If $\alpha_i = 0$, then we find (for $\beta_i = \alpha_i = 0$)

$$I_{x_i}^{x_i'} = [x_i, x_i'] = [x_i, y_i] \cap [a_i, x_i'] = I_{x_i}^{y_i} \cap I_{a_i+\beta_i(x_i'-a_i)}^{x_i'+\beta_i(b_i-x_i')},$$

and if $\alpha_i = 1$, then we find (for $\beta_i = \alpha_i = 1$)

$$I_{x_i'}^{y_i} = [x_i', y_i] = [x_i, y_i] \cap [x_i', b_i] = I_{x_i}^{y_i} \cap I_{a_i+\beta_i(x_i'-a_i)}^{x_i'+\beta_i(b_i-x_i')}.$$

Now, assume that $x_i' \leqslant x_i$, and so, $\xi_i = 0$ and $x_i' \leqslant x_i < y_i \leqslant b_i$. It follows that if $1 - \xi_i \leqslant \alpha_i \leqslant 1$, then $\beta_i = \alpha_i = 1$ implying

$$I_{x_i}^{y_i} = [x_i, y_i] = [x_i, y_i] \cap [x_i', b_i] = I_{x_i}^{y_i} \cap I_{a_i+\beta_i(x_i'-a_i)}^{x_i'+\beta_i(b_i-x_i')}.$$

In case (b) we have $\xi_i = 0$, $\beta_i = 0$ and $a_i \leqslant x_i < y_i \leqslant x_i'$, and so, if $1 - \xi_i \leqslant \alpha_i \leqslant 1$, then $\alpha_i = 1$ and

$$I_{x_i}^{y_i} = [x_i, y_i] = [x_i, y_i] \cap [a_i, x_i'] = I_{x_i}^{y_i} \cap I_{a_i+\beta_i(x_i'-a_i)}^{x_i'+\beta_i(b_i-x_i')}.$$

Let us show that $\lambda' \leqslant \beta \leqslant \mu'$. Let $i \in \{1, \dots, n\}$. If $a_i = x_i'$, then $\lambda_i' = 1 = \mu_i'$ and, since $x_i' < y_i$, then $\beta_i = \alpha_i$. By (3.5), $\xi_i = 0$, and so, since $1 - \xi_i \leqslant \alpha_i \leqslant 1$, then $\alpha_i = 1$, which implies $\lambda_i' = \beta_i = \mu_i'$. Now, if $x_i' = b_i$, then $\lambda_i' = 0 = \mu_i'$ and, since $x_i' \geqslant y_i$, then $\beta_i = 0$ (and $\xi_i = 0$), and so, $\lambda_i' = \beta_i = \mu_i'$. Finally, if $a_i < x_i' < b_i$, then $\lambda_i' = 0$ and $\mu_i' = 1$, and so, since $\beta_i \in \{0, 1\}$, then $\lambda_i' \leqslant \beta_i \leqslant \mu_i'$.

The uniqueness of $\beta(\alpha)$, for each $1 - \xi \leqslant \alpha \leqslant 1$, is a consequence of the following: if $\lambda' \leqslant \beta \leqslant \mu'$ and $\beta \neq \beta(\alpha)$, then there exists $i \in \{1, \dots, n\}$ s.t. $\beta_i = 1 - \beta_i(\alpha)$. Arguing as in (a) and (b) above, we find that the equality (3.12) cannot hold for this β.

Now, inequality (3.6) readily follows from Lemma 6, (3.11) and (3.4). $\square$

Remark 3.1. (a) If $x' \in I_x^y$ in Lemma 4, then it is easily seen that $\xi = \mu - \lambda$, and so, equality (3.4) assumes the form:

$$I_x^y = \bigcup_{1-(\mu-\lambda)\leqslant\alpha\leqslant 1} I_{x+\alpha(\mu-\lambda)(x'-x)}^{x'+\alpha(y-x')}.$$

Although this equality looks different from (3.9), the two equalities are the same: this is verified as in (i)–(iii) of the proof of Lemma 6.

(b) If $x < x' < y$, then $\xi = 1$, $\lambda = 0$ and $\mu = 1$, and so, (3.4), (3.9) and (3.7) are identical.

(c) Here we consider a certain particular case of (3.12) and establish conditions on x', under which x' does not induce a (further) partition of I_x^y. In view of (3.12), we have:

$$I_{x+\alpha\xi(x'-x)}^{x'+\alpha(y-x')} = I_x^y \quad \text{if and only if} \quad \xi = 0 \text{ and } \alpha = 1, \tag{3.13}$$

which is also equivalent to

$$a + \beta(x' - a) \leqslant x \quad \text{and} \quad y \leqslant x' + \beta(b - x') \quad \text{with } \beta = \beta(1). \tag{3.14}$$

Clearly, if $\xi = 0$ and $\alpha = 1$, then the left-hand side equality in (3.13) holds. Conversely, if the left-hand side equality in (3.13) holds for some $1 - \xi \leqslant \alpha \leqslant 1$, then $x + \alpha\xi(x' - x) = x$ and $x' + \alpha(y - x') = y$, and so, if we suppose that $\xi_i = 1$ for some $i \in \{1, \dots, n\}$, then, by (3.5), $x_i < x_i' < y_i$, and so, $\alpha_i = 0$ and $x_i' = y_i$, which is a contradiction. Thus, $\xi = 0$ and $\alpha = 1$.

Now, if $\xi = 0$ and $\alpha = 1$, then, by (3.12) and (3.13),

$$I_x^y = I_x^y \cap I_{a+\beta(x'-a)}^{x'+\beta(b-x')} \quad \text{with } \beta = \beta(1), \tag{3.15}$$

which implies (3.14). Conversely, (3.14) implies (3.15), and so, the left-hand side equality in (3.13) holds, i.e., $\xi = 0$ and $\alpha = 1$.

This observation also shows that a point $x' \in I_a^b$ induces a 'true' partition of I_x^y provided that, for all β with $\lambda' \leqslant \beta \leqslant \mu'$, we have:

$$a + \beta(x' - a) \nleqslant x \quad \text{or} \quad y \nleqslant x' + \beta(b - x'),$$

which is also equivalent to $\xi \neq 0$.

Proof of Lemma 3. Let $\mathcal{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ for some $\kappa \in \mathbb{N}^n$ and $x' \in \mathcal{P}'$. Given $1 \leqslant \sigma \leqslant \kappa$, we set $x_\sigma = x[\sigma - 1]$, $y_\sigma = x[\sigma]$ and $\xi_\sigma(x') = \xi(x_\sigma, x', y_\sigma)$, where ξ is defined in (3.5), and note that $x_\sigma < y_\sigma$. The point x' induces a partition of $I_{x_\sigma}^{y_\sigma} = I_{x[\sigma-1]}^{x[\sigma]}$ of the form (3.4) with $x = x_\sigma$ and $y = y_\sigma$, and so, by virtue of (3.3), we get the following partition of I_a^b, induced by x':

$$I_a^b = \bigcup_{1 \leqslant \sigma \leqslant \kappa} \; \bigcup_{1-\xi_\sigma(x') \leqslant \alpha \leqslant 1} I_{x_\sigma + \alpha\xi_\sigma(x')(x'-x_\sigma)}^{x' + \alpha(y_\sigma - x')}. \tag{3.16}$$

We denote by $\mathcal{P}^1$ the net partition of I_a^b corresponding to (3.16). Moreover, by (3.6), for each $1 \leqslant \sigma \leqslant \kappa$ we have the inequality:

$$\mathrm{md}_n\big(f, I_{x_\sigma}^{y_\sigma}\big) \leqslant \sum_{1-\xi_\sigma(x') \leqslant \alpha \leqslant 1} \mathrm{md}_n\big(f, I_{x_\sigma + \alpha\xi_\sigma(x')(x'-x_\sigma)}^{x' + \alpha(y_\sigma - x')}\big). \tag{3.17}$$

With no loss of generality we may assume that $x' \notin \mathcal{P}$: if $x' \in \mathcal{P}$, i.e., $x' = x[\sigma']$ for some $1 \leqslant \sigma' \leqslant \kappa$, then x' does not affect the partition $\mathcal{P}$ of I_a^b in the sense that $\mathcal{P}^1 = \mathcal{P}$, and so, $\mathrm{v}_n(f; \mathcal{P}^1) = \mathrm{v}_n(f; \mathcal{P})$. In order to see this, we note that (3.5) implies $\xi_\sigma(x') = \xi(x[\sigma - 1], x[\sigma'], x[\sigma]) = 0$, and so, by Remark 3.1(c), conditions (3.13) and (3.14) hold with $\beta = \beta(1) = (\beta_1, \ldots, \beta_n)$ s.t.

$$\beta_i = \begin{cases} 1 & \text{if } x_i(\sigma_i') < x_i(\sigma_i), \\ 0 & \text{if } x_i(\sigma_i') \geqslant x_i(\sigma_i) \end{cases} = \begin{cases} 1 & \text{if } \sigma_i' < \sigma_i, \\ 0 & \text{if } \sigma_i' \geqslant \sigma_i. \end{cases}$$

Summing over $1 \leqslant \sigma \leqslant \kappa$ in (3.17) and taking into account (3.3) and (3.16), we obtain the inequality

$$\mathrm{v}_n(f; \mathcal{P}) \leqslant \mathrm{v}_n\big(f; \mathcal{P}^1\big).$$

Replacing $\mathcal{P}$ by $\mathcal{P}^1$ in the arguments above, taking $x' \in \mathcal{P}' \setminus \mathcal{P}^1$ and denoting by $\mathcal{P}^2$ the partition of I_a^b induced from $\mathcal{P}^1$ by x', we get $\mathrm{v}_n(f; \mathcal{P}^1) \leqslant \mathrm{v}_n(f; \mathcal{P}^2)$. Since $\mathcal{P}' \setminus \mathcal{P}$ is a finite set, we exhaust it by points x' in a finite number of steps, arrive at the partition $\mathcal{P}'$ of I_a^b and prove the desired inequality $\mathrm{v}_n(f; \mathcal{P}) \leqslant \mathrm{v}_n(f; \mathcal{P}')$. □

Proof of Theorem 1. 1. First, we establish (2.4) for $\alpha = 1 = 1_n$, i.e.,

$$V_n\big(f, I_x^y\big) = \sum_{1 \leqslant \sigma \leqslant \kappa} V_n\big(f, I_{x[\sigma-1]}^{x[\sigma]}\big). \tag{3.18}$$

Modulo the notation, there is no loss of generality if we assume that $x = a$ and $y = b$, so that $\{x[\sigma]\}_{\sigma=0}^{\kappa}$ is a net partition of I_a^b.

Let $\mathcal{P}$ be an arbitrary net partition of I_a^b. Denote by $\mathcal{P}'$ the net partition of I_a^b induced from $\mathcal{P}$ by points $\{x[\sigma]\}_{\sigma=0}^{\kappa}$, so that $\mathcal{P}'$ is a refinement of $\mathcal{P}$. Given $1 \leqslant \sigma \leqslant \kappa$, set $\mathcal{P}_\sigma = \mathcal{P}' \cap I_{x[\sigma-1]}^{x[\sigma]}$ and note that $\mathcal{P}_\sigma$ is a net partition of $I_{x[\sigma-1]}^{x[\sigma]}$, and $\mathcal{P}' = \bigcup_{1 \leqslant \sigma \leqslant \kappa} \mathcal{P}_\sigma$. Then by virtue of Lemma 3, we have:

$$\mathrm{v}_n(f; \mathcal{P}) \leqslant \mathrm{v}_n\big(f; \mathcal{P}'\big) = \sum_{1 \leqslant \sigma \leqslant \kappa} \mathrm{v}_n(f; \mathcal{P}_\sigma) \leqslant \sum_{1 \leqslant \sigma \leqslant \kappa} V_n\big(f, I_{x[\sigma-1]}^{x[\sigma]}\big).$$

Since $\mathcal{P}$ is arbitrary, the left-hand side in (3.18) is not greater than the right-hand side.

Let us prove the reverse inequality. If $V_n(f, I_{x[\sigma-1]}^{x[\sigma]})$ is infinite for some $1 \leqslant \sigma \leqslant \kappa$, then since $I_{x[\sigma-1]}^{x[\sigma]} \subset I_a^b = I_x^y$, the value $V_n(f, I_x^y)$ is infinite as well. Thus, we suppose that the right-hand side of (3.18) is finite. Let $\varepsilon > 0$ be arbitrary. Given $1 \leqslant \sigma \leqslant \kappa$, by the definition of $V_n(f, I_{x[\sigma-1]}^{x[\sigma]})$, there exists a net partition of $I_{x[\sigma-1]}^{x[\sigma]}$, denoted by $\mathcal{P}_\sigma(\varepsilon)$, s.t.

$$\mathrm{v}_n\big(f; \mathcal{P}_\sigma(\varepsilon)\big) \geqslant V_n\big(f, I_{x[\sigma-1]}^{x[\sigma]}\big) - (\varepsilon/c),$$

where $c = |\{\sigma\colon\ 1 \leqslant \sigma \leqslant \kappa\}|$. We denote by $\mathcal{P}(\varepsilon)$ the net partition of I_a^b induced from $\{x[\sigma]\}_{\sigma=0}^{\kappa}$ by points from $\bigcup_{1 \leqslant \sigma \leqslant \kappa} \mathcal{P}_\sigma(\varepsilon)$. Given $1 \leqslant \sigma \leqslant \kappa$, we set $\mathcal{P}'_\sigma(\varepsilon) = \mathcal{P}(\varepsilon) \cap I_{x[\sigma-1]}^{x[\sigma]}$ and note that $\mathcal{P}'_\sigma(\varepsilon)$ is a refinement of $\mathcal{P}_\sigma(\varepsilon)$, and $\mathcal{P}(\varepsilon) = \bigcup_{1 \leqslant \sigma \leqslant \kappa} \mathcal{P}'_\sigma(\varepsilon)$. By virtue of Lemma 3, we find

$$\begin{aligned} V_n\big(f, I_a^b\big) &\geqslant \mathrm{v}_n\big(f; \mathcal{P}(\varepsilon)\big) = \sum_{1 \leqslant \sigma \leqslant \kappa} \mathrm{v}_n\big(f; \mathcal{P}'_\sigma(\varepsilon)\big) \geqslant \sum_{1 \leqslant \sigma \leqslant \kappa} \mathrm{v}_n\big(f; \mathcal{P}_\sigma(\varepsilon)\big) \\ &\geqslant \sum_{1 \leqslant \sigma \leqslant \kappa} V_n\big(f, I_{x[\sigma-1]}^{x[\sigma]}\big) - \varepsilon\Bigg(\sum_{1 \leqslant \sigma \leqslant \kappa} 1\Bigg)\Big/c, \end{aligned}$$

where the factor by ε is, actually, equal to 1. The desired inequality follows if we take into account the arbitrariness of $\varepsilon > 0$.

2. Now, suppose that $0 \neq \alpha \leqslant 1$ and $\alpha \neq 1$. Note that $x\lfloor\alpha, y\lfloor\alpha \in I_{a\lfloor\alpha}^{b\lfloor\alpha}$ and $x\lfloor\alpha < y\lfloor\alpha$, and that $\{x[\sigma]\lfloor\alpha\}_{\sigma\lfloor\alpha=0}^{\kappa\lfloor\alpha}$ is a net partition of $I_{a\lfloor\alpha}^{b\lfloor\alpha}$. So, replacing $1 = 1_n$ by $1\lfloor\alpha$ (so that $|1\lfloor\alpha| = |\alpha|$) and f—by f_α^z in (3.18), we get:

$$V_{|\alpha|}\big(f^z_\alpha, I^y_x\lfloor\alpha\big) = V_{|1\lfloor\alpha|}\big(f^z_\alpha, I^{y\lfloor\alpha}_{x\lfloor\alpha}\big) = \sum_{1\lfloor\alpha\leqslant\sigma\lfloor\alpha\leqslant\kappa\lfloor\alpha} V_{|1\lfloor\alpha|}\big(f^z_\alpha, I^{x[\sigma]\lfloor\alpha}_{x[\sigma-1]\lfloor\alpha}\big),$$

which is equal to the right-hand side of (2.4).

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

In order to prove Theorem 2, we need two more lemmas (Lemmas 7 and 8).

Lemma 7. *If $m \in \mathbb{N}$, $u, v \in M$, $\{u_j\}_{j=1}^m, \{v_j\}_{j=1}^m \subset M$ and*

$$\sum_{i=1}^{\leqslant m/2} u_{2i} + u + \sum_{i=1}^{\leqslant (m+1)/2} v_{2i-1} = \sum_{i=1}^{\leqslant m/2} v_{2i} + v + \sum_{i=1}^{\leqslant (m+1)/2} u_{2i-1}, \tag{4.1}$$

then

$$d(u,v) \leqslant \sum_{j=1}^m d(u_j, v_j). \tag{4.2}$$

Proof. Observe that if $u + \ell_1 + \cdots + \ell_k = v + r_1 + \cdots + r_k$ for some $k \in \mathbb{N}$ and $\{\ell_i, r_i\}_{i=1}^k \subset M$, then $d(u,v) \leqslant \sum_{i=1}^k d(\ell_i, r_i)$. In fact, by the translation invariance of d and inequality (3.2), we have:

$$\begin{aligned} d(u,v) &= d\left(u + \sum_{i=1}^k \ell_i, v + \sum_{i=1}^k \ell_i\right) = d\left(v + \sum_{i=1}^k r_i, v + \sum_{i=1}^k \ell_i\right) \\ &= d\left(\sum_{i=1}^k r_i, \sum_{i=1}^k \ell_i\right) \leqslant \sum_{i=1}^k d(r_i, \ell_i). \end{aligned}$$

Applying this observation and equality (4.1), we get:

$$d(u,v) \leqslant \sum_{i=1}^{\leqslant m/2} d(u_{2i}, v_{2i}) + \sum_{i=1}^{\leqslant (m+1)/2} d(v_{2i-1}, u_{2i-1}) = \sum_{j=1}^m d(u_j, v_j). \qquad \square$$

Remark 4.1. In particular, (in)equalities (4.1) and (4.2) hold for *odd* m if

$$u + \sum_{i=1}^{(m-1)/2} u_{2i} = \sum_{i=1}^{(m+1)/2} u_{2i-1} \quad \text{and} \quad v + \sum_{i=1}^{(m-1)/2} v_{2i} = \sum_{i=1}^{(m+1)/2} v_{2i-1}, \tag{4.3}$$

and for *even* m if either

$$u + \sum_{i=1}^{m/2} u_{2i} = v + \sum_{i=1}^{m/2} u_{2i-1} \quad \text{and} \quad \sum_{i=1}^{m/2} v_{2i} = \sum_{i=1}^{m/2} v_{2i-1}, \tag{4.4}$$

or

$$\sum_{i=1}^{m/2} u_{2i} = v + \sum_{i=1}^{m/2} u_{2i-1} \quad \text{and} \quad \sum_{i=1}^{m/2} v_{2i} = u + \sum_{i=1}^{m/2} v_{2i-1}. \tag{4.5}$$

In the next lemma we set $\mathcal{A}_0 \equiv \mathcal{A}_0(n) = \{\theta \in \mathbb{N}_0^n\colon\ \theta \leqslant 1\}$. Also, we stick to the following conventions: '$u \doteq 0$' will mean that u is omitted in the formula under consideration (especially in a metric semigroup with no zero), and a sum over the empty set is also omitted in any context (i.e., $\sum_\emptyset \doteq 0$).

Lemma 8. *Given a map $h : \mathcal{A}_0 \to M$ and a multiindex $\gamma \in \mathcal{A}_0$, we have*:

$$\sum_{\mathrm{ev}\,\alpha\leqslant\gamma}\ \sum_{\mathrm{ev}\,\theta\leqslant\alpha} h(\theta) = c_\gamma + \sum_{\mathrm{od}\,\alpha\leqslant\gamma}\ \sum_{\mathrm{ev}\,\theta\leqslant\alpha} h(\theta), \tag{4.6}$$

where $c_\gamma \doteq 0$ if γ is odd, and $c_\gamma = h(\gamma)$ if γ is even, and

$$\sum_{\mathrm{od}\,\alpha\leqslant\gamma}\ \sum_{\mathrm{od}\,\theta\leqslant\alpha} h(\theta) = d_\gamma + \sum_{\mathrm{ev}\,\alpha\leqslant\gamma}\ \sum_{\mathrm{od}\,\theta\leqslant\alpha} h(\theta), \tag{4.7}$$

where $d_\gamma = h(\gamma)$ if γ is odd, and $d_\gamma \doteq 0$ if γ is even.

Proof. 0. Denote by $\mathcal{L}$ (by $\mathcal{R}$) the set of all 'admissible' θ's at the left- (right-)hand side of the equality under consideration and, given $\theta\in\mathcal{L}$ (and $\theta\in\mathcal{R}$), by $L(\theta)$ (and by $R(\theta)$)—the multiplicity of the term $h(\theta)$ at the left- (and right-)hand sum(s). Then the equality can be rewritten as

$$\sum_{\theta\in\mathcal{L}} L(\theta)h(\theta) = \sum_{\theta\in\mathcal{R}} R(\theta)h(\theta), \tag{4.8}$$

where $L(\theta)h(\theta)$ denotes the sum of terms of the form $h(\theta)$ taken $L(\theta)$ times (and likewise for $R(\theta)h(\theta)$). In what follows in order to prove (4.8), we show that $\mathcal{L}=\mathcal{R}$ and $L(\theta)=R(\theta)$ for all $\theta\in\mathcal{L}=\mathcal{R}$.

We divide the proof into four steps for clarity.

In the first two steps we let γ be odd (i.e., $0\leqslant\gamma\leqslant 1$ and $|\gamma|$ is odd).

1. Let us establish (4.6). We have $\mathcal{L}=\{\text{even } \theta\colon\ \exists \text{ even } \alpha\leqslant\gamma \text{ s.t. } \theta\leqslant\alpha\}$, i.e., $\mathcal{L}=\{\text{even } \theta\colon\ \theta\leqslant\gamma\}$, and $\mathcal{R}=\{\text{even } \theta\colon\ \exists \text{ odd } \alpha\leqslant\gamma \text{ s.t. } \theta\leqslant\alpha\}$. The sets $\mathcal{L}$ and $\mathcal{R}$ are nonempty ($0\in\mathcal{L}$ and $0\in\mathcal{R}$) and $\mathcal{L}=\mathcal{R}$. In fact, the inclusion $\mathcal{L}\supset\mathcal{R}$ is clear, and so, we let $\theta\in\mathcal{L}$. Since θ is even, γ is odd and $\theta\leqslant\gamma$, there exists $i\in\{1,\dots,n\}$ s.t. $\theta_i=0$ and $\gamma_i=1$. We set $\alpha=(\theta_1,\dots,\theta_{i-1},1,\theta_{i+1},\dots,\theta_n)$. It follows that $\alpha\leqslant\gamma$, $|\alpha|=|\theta|+1$ is odd and $\theta\leqslant\alpha$, and so, $\theta\in\mathcal{R}$.

Given $\theta\in\mathcal{L}=\mathcal{R}$, we find $\theta\neq\gamma$, $L(\theta)=|\{\text{even } \alpha\colon\ \theta\leqslant\alpha\leqslant\gamma\}|$ and $R(\theta)=|\{\text{odd } \alpha\colon\ \theta\leqslant\alpha\leqslant\gamma\}|$. By Lemma 2(b), $L(\theta)=R(\theta)$, and so, (4.8) holds implying (4.6) with $c_\gamma\doteq 0$.

2. Let us prove (4.7). If $|\gamma|=1$, then the equality is immediate: the left-hand side is equal to $h(\gamma)=d_\gamma$, while the double sum at the right is omitted (in fact, even $\alpha\leqslant\gamma$ implies $\alpha=0$, and so, no odd θ s.t. $\theta\leqslant 0$ exists). Now, if $|\gamma|>1$, then $\mathcal{L}=\{\text{odd } \theta\colon\ \theta\leqslant\gamma\}$ and $\mathcal{R}=\{\text{odd } \theta\colon\ \exists \text{ even } \alpha\leqslant\gamma \text{ s.t. } \theta\leqslant\alpha\}\cup\{\gamma\}$ (disjoint union), and $\mathcal{L}=\mathcal{R}$. Let $\theta\in\mathcal{L}=\mathcal{R}$. If $\theta\neq\gamma$, then $L(\theta)=|\{\text{odd } \alpha\colon\ \theta\leqslant\alpha\leqslant\gamma\}|$ and $R(\theta)=|\{\text{even } \alpha\colon\ \theta\leqslant\alpha\leqslant\gamma\}|$, and so, by Lemma 2(b), $L(\theta)=R(\theta)$. Now if $\theta=\gamma$, then $L(\gamma)=|\{\text{odd } \alpha\colon\ \gamma\leqslant\alpha\leqslant\gamma\}|=1$, and since $d_\gamma=h(\gamma)$, then $R(\gamma)=1$ as well. The conclusion follows as in Step 1.

Suppose that γ is even.

3. In order to prove (4.6), we first note that if $\gamma=0$, then the double sum at the right is omitted and the double sum at the left is equal to $h(0)=c_0$. Assume that $\gamma\neq 0$. Then $\mathcal{L}=\{\text{even } \theta\colon\ \theta\leqslant\gamma\}$ and $\mathcal{R}=\{\gamma\}\cup\{\text{even } \theta\colon\ \exists \text{ odd } \alpha\leqslant\gamma \text{ s.t. } \theta\leqslant\alpha\}$ (disjoint union), and $\mathcal{L}=\mathcal{R}$. Let $\theta\in\mathcal{L}=\mathcal{R}$. Then $L(\theta)=|\{\text{even } \alpha\colon\ \theta\leqslant\alpha\leqslant\gamma\}|$ and, in particular, $L(\gamma)=1$. If $\theta=\gamma$, then, since $c_\gamma=h(\gamma)$, we have $R(\gamma)=1$, and if $\theta\neq\gamma$, then $R(\theta)=|\{\text{odd } \alpha\colon\ \theta\leqslant\alpha\leqslant\gamma\}|$, and so, by Lemma 2(b), $L(\theta)=R(\theta)$.

4. Finally, we prove (4.7). Since the equality is clear for $\gamma=0$ (i.e., 'empty' equality), we assume that $|\gamma|>0$. We have $\mathcal{L}=\{\text{odd } \theta\colon\ \theta\leqslant\gamma\}$, $\mathcal{R}=\{\text{odd } \theta\colon\ \exists \text{ even } \alpha\leqslant\gamma \text{ s.t. } \theta\leqslant\alpha\}$ and $\mathcal{L}=\mathcal{R}$. Given $\theta\in\mathcal{L}=\mathcal{R}$, we find $\theta\neq\gamma$, $L(\theta)=|\{\text{odd } \alpha\colon\ \theta\leqslant\alpha\leqslant\gamma\}|$ and $R(\theta)=|\{\text{even } \alpha\colon\ \theta\leqslant\alpha\leqslant\gamma\}|$, and so, by Lemma 2(b), $L(\theta)=R(\theta)$. □

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. It suffices to prove only the first inequality: the second one follows from the first inequality, (2.2) and (2.3). Setting $u=f(x)$ and $v=f(y)$ and taking into account (2.6), the first inequality in Theorem 2 can be rewritten equivalently as

$$d(u,v)\leqslant\sum_{0\neq\alpha\leqslant 1} d\big(u(\alpha),v(\alpha)\big)=\sum_{j=1}^{n}\sum_{|\alpha|=j} d\big(u(\alpha),v(\alpha)\big) \tag{4.9}$$

(the sum over $|\alpha|=j$ designates the sum over $0\neq\alpha\leqslant 1$ s.t. $|\alpha|=j$), where, given $\alpha,\theta\in\mathcal{A}_0$, we set $h(\theta)=f(x+\theta(y-x))$,

$$u(\alpha)=\sum_{\mathrm{ev}\,\theta\leqslant\alpha} h(\theta),\quad\text{and}\quad v(\alpha)=\sum_{\mathrm{od}\,\theta\leqslant\alpha} h(\theta)\quad\text{if } \alpha\neq 0 \text{ and } v(0)\doteq 0.$$

In order to establish (4.9), given integer $0\leqslant j\leqslant n$, we also set

$$u_j=\sum_{|\alpha|=j} u(\alpha)\quad\text{and}\quad v_j=\sum_{|\alpha|=j} v(\alpha)$$

and note that

$$u_0=u(0)=h(0)=u,\qquad v_0=v(0)\doteq 0,\qquad v=h(1),\qquad u_n=u(1)\quad\text{and}\quad v_n=v(1).$$

Suppose that we have already verified equalities (4.3) if $m=n$ is odd and (4.4) if $m=n$ is even. Applying Lemma 7, we get inequality (4.2), where, by virtue of (3.2), we have:

$$d(u_j, v_j) = d\Bigl(\sum_{|\alpha|=j} u(\alpha), \sum_{|\alpha|=j} v(\alpha)\Bigr) \leqslant \sum_{|\alpha|=j} d\bigl(u(\alpha), v(\alpha)\bigr).$$

Now, (4.9) follows if we sum these inequalities over $j = 1, \dots, n$ and take into account (4.2).

It remains to verify equalities (4.3) and (4.4). For this, we apply Lemma 8 with $\gamma = 1$ and note that $m = n = |\gamma| = |1|$. Suppose that $n = |1|$ is odd. By virtue of (4.6), we have:

$$\begin{aligned} u + \sum_{i=1}^{(m-1)/2} u_{2i} &= \sum_{i=0}^{(n-1)/2} u_{2i} = \sum_{i=0}^{(n-1)/2} \sum_{|\alpha|=2i} u(\alpha) = \sum_{\mathrm{ev}\,\alpha \leqslant 1} u(\alpha) \\ &= \sum_{\mathrm{od}\,\alpha \leqslant 1} u(\alpha) = \sum_{i=1}^{(n+1)/2} \sum_{|\alpha|=2i-1} u(\alpha) = \sum_{i=1}^{(m+1)/2} u_{2i-1}, \end{aligned}$$

and by virtue of (4.7), we get:

$$\begin{aligned} v + \sum_{i=1}^{(m-1)/2} v_{2i} &= h(1) + \sum_{i=0}^{(n-1)/2} v_{2i} = h(1) + \sum_{i=0}^{(n-1)/2} \sum_{|\alpha|=2i} v(\alpha) = h(1) + \sum_{\mathrm{ev}\,\alpha \leqslant 1} v(\alpha) \\ &= \sum_{\mathrm{od}\,\alpha \leqslant 1} v(\alpha) = \sum_{i=1}^{(n+1)/2} \sum_{|\alpha|=2i-1} v(\alpha) = \sum_{i=1}^{(m+1)/2} v_{2i-1}, \end{aligned}$$

which establishes (4.3). Now suppose that $n = |1|$ is even. By (4.6), we get:

$$\begin{aligned} u + \sum_{i=1}^{m/2} u_{2i} &= \sum_{i=0}^{n/2} u_{2i} = \sum_{i=0}^{n/2} \sum_{|\alpha|=2i} u(\alpha) = \sum_{\mathrm{ev}\,\alpha \leqslant 1} u(\alpha) \\ &= h(1) + \sum_{\mathrm{od}\,\alpha \leqslant 1} u(\alpha) = v + \sum_{i=1}^{n/2} \sum_{|\alpha|=2i-1} u(\alpha) = v + \sum_{i=1}^{m/2} u_{2i-1}, \end{aligned}$$

and by virtue of (4.7), we have:

$$\begin{aligned} \sum_{i=1}^{m/2} v_{2i} &= \sum_{i=0}^{n/2} v_{2i} = \sum_{i=0}^{n/2} \sum_{|\alpha|=2i} v(\alpha) = \sum_{\mathrm{ev}\,\alpha \leqslant 1} v(\alpha) \\ &= \sum_{\mathrm{od}\,\alpha \leqslant 1} v(\alpha) = \sum_{i=1}^{n/2} \sum_{|\alpha|=2i-1} v(\alpha) = \sum_{i=1}^{m/2} v_{2i-1}, \end{aligned}$$

which establishes (4.4) and completes the proof of Theorem 2. □

Remark 4.2. The left-hand side inequality in Theorem 2 is of interest when $x < y$. However, if $x \leqslant y$ and $x \not< y$, it can be refined in the following way (cf. [17, Part I, Lemma 6]): given $x, y \in I_a^b$, $x < y$, and $0 \neq \gamma \leqslant 1$, we have:

$$d\bigl(f(x), f\bigl(x + \gamma(y - x)\bigr)\bigr) \leqslant \sum_{0 \neq \alpha \leqslant \gamma} \mathrm{md}_{|\alpha|}\bigl(f_\alpha^x, I_x^y \lfloor \alpha\bigr).$$

In fact, by Theorem 2, we find

$$d\bigl(f(x), f\bigl(x + \gamma(y - x)\bigr)\bigr) \leqslant \sum_{0 \neq \alpha \leqslant 1} \mathrm{md}_{|\alpha|}\bigl(f_\alpha^x, I_x^{x+\gamma(y-x)} \lfloor \alpha\bigr),$$

where, by virtue of (2.6), the mixed difference at the right is equal to

$$d\Bigl(\sum_{\mathrm{ev}\,\theta \leqslant \alpha} f\bigl(x + \theta\gamma(y - x)\bigr), \sum_{\mathrm{od}\,\bar\theta \leqslant \alpha} f\bigl(x + \bar\theta\gamma(y - x)\bigr)\Bigr). \tag{4.10}$$

If $\alpha \not\leqslant \gamma$, then $\alpha_i = 1$ and $\gamma_i = 0$ for some $i \in \{1, \dots, n\}$, and so, arguing as in Remark 2.1 we find $x + \theta\gamma(y - x) = x + \bar\theta\gamma(y - x)$ for all even θ with $\theta \leqslant \alpha$ implying that (4.10) is equal to zero. Now if $\alpha \leqslant \gamma$, then $\theta\gamma = \theta$ for any $\theta \leqslant \alpha$, and so, (4.10) coincides with the right-hand side of (2.6).

5. Proof of Theorem 3

Proof of Theorem 3. 1. First, we show that if $x, y \in I_a^b$, $x < y$, and $0 \neq \alpha \leqslant 1$, then

$$\mathrm{md}_{|\alpha|}\big(f_\alpha^a, I_x^y\lfloor\alpha\big) = \lim_{j\to\infty} \mathrm{md}_{|\alpha|}\big((f_j)_\alpha^a, I_x^y\lfloor\alpha\big). \tag{5.1}$$

By virtue of (2.5), we have:

$$\mathrm{md}_{|\alpha|}\big(f_\alpha^a, I_x^y\lfloor\alpha\big) = d\Bigg(\sum_{\mathrm{ev}\,\theta\leqslant\alpha} f\big(\underbrace{a+\alpha(x-a)+\theta(y-x)}_{(\cdots)}\big), \sum_{\mathrm{od}\,\theta\leqslant\alpha} f(\cdots)\Bigg),$$

and a similar equality holds for f_j in place of f. Applying the inequalities $|d(u,v) - d(u',v')| \leqslant d(u,u') + d(v,v')$, $u, v, u', v' \in M$, and (3.2) and taking into account the pointwise convergence of f_j to f, we find

$$\begin{aligned}
&\big|\mathrm{md}_{|\alpha|}\big((f_j)_\alpha^a, I_x^y\lfloor\alpha\big) - \mathrm{md}_{|\alpha|}\big(f_\alpha^a, I_x^y\lfloor\alpha\big)\big| \\
&\quad\leqslant d\Bigg(\sum_{\mathrm{ev}\,\theta\leqslant\alpha} f_j(\cdots), \sum_{\mathrm{ev}\,\theta\leqslant\alpha} f(\cdots)\Bigg) + d\Bigg(\sum_{\mathrm{od}\,\theta\leqslant\alpha} f_j(\cdots), \sum_{\mathrm{od}\,\theta\leqslant\alpha} f(\cdots)\Bigg) \\
&\quad\leqslant \sum_{\mathrm{ev}\,\theta\leqslant\alpha} d\big(f_j(\cdots), f(\cdots)\big) + \sum_{\mathrm{od}\,\theta\leqslant\alpha} d\big(f_j(\cdots), f(\cdots)\big) \\
&\quad= \sum_{0\leqslant\theta\leqslant\alpha} d\big(f_j(\cdots), f(\cdots)\big) \to 0 \quad \text{as } j\to\infty.
\end{aligned}$$

2. In the rest of this proof we need only the inequality

$$\mathrm{md}_{|\alpha|}\big(f_\alpha^a, I_x^y\lfloor\alpha\big) \leqslant \liminf_{j\to\infty} \mathrm{md}_{|\alpha|}\big((f_j)_\alpha^a, I_x^y\lfloor\alpha\big), \tag{5.2}$$

which readily follows from (5.1). If $\mathcal{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$ is a net partition of I_a^b, then $\mathcal{P}\lfloor\alpha = \{x[\sigma]\lfloor\alpha\}_{\sigma\lfloor\alpha}^{\kappa\lfloor\alpha}$ is net partition of $I_a^b\lfloor\alpha$, and so, given $1 \leqslant \sigma \leqslant \kappa$, setting $x = x[\sigma-1]$ and $y = x[\sigma]$ in (5.2), we find

$$\begin{aligned}
\sum_{1\lfloor\alpha\leqslant\sigma\lfloor\alpha\leqslant\kappa\lfloor\alpha} \mathrm{md}_{|\alpha|}\big(f_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]}\lfloor\alpha\big) &\leqslant \sum_{1\lfloor\alpha\leqslant\sigma\lfloor\alpha\leqslant\kappa\lfloor\alpha} \liminf_{j\to\infty} \mathrm{md}_{|\alpha|}\big((f_j)_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]}\lfloor\alpha\big) \\
&\leqslant \liminf_{j\to\infty} \sum_{1\lfloor\alpha\leqslant\sigma\lfloor\alpha\leqslant\kappa\lfloor\alpha} \mathrm{md}_{|\alpha|}\big((f_j)_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]}\lfloor\alpha\big) \\
&\leqslant \liminf_{j\to\infty} V_{|\alpha|}\big((f_j)_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]}\lfloor\alpha\big).
\end{aligned}$$

By the arbitrariness of $\mathcal{P}$, we infer that

$$V_{|\alpha|}\big(f_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]}\lfloor\alpha\big) \leqslant \liminf_{j\to\infty} V_{|\alpha|}\big((f_j)_\alpha^a, I_{x[\sigma-1]}^{x[\sigma]}\lfloor\alpha\big).$$

We conclude that

$$\begin{aligned}
\mathrm{TV}\big(f, I_a^b\big) &= \sum_{0\neq\alpha\leqslant1} V_{|\alpha|}\big(f_\alpha^a, I_a^b\lfloor\alpha\big) \leqslant \sum_{0\neq\alpha\leqslant1} \liminf_{j\to\infty} V_{|\alpha|}\big((f_j)_\alpha^a, I_a^b\lfloor\alpha\big) \\
&\leqslant \liminf_{j\to\infty} \sum_{0\neq\alpha\leqslant1} V_{|\alpha|}\big((f_j)_\alpha^a, I_a^b\lfloor\alpha\big) = \liminf_{j\to\infty} \mathrm{TV}\big(f_j, I_a^b\big). \qquad \square
\end{aligned}$$

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