



## Moduli via double pants decompositions <sup>☆</sup>

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### ABSTRACT

We consider (local) parameterizations of Teichmüller space  $\mathcal{T}_{g,n}$  (of genus  $g$  hyperbolic surfaces with  $n$  boundary components) by lengths of  $6g - 6 + 3n$  geodesics. We find a large family of suitable sets of  $6g - 6 + 3n$  geodesics, each set forming a special structure called “admissible double pants decomposition”. For admissible double pants decompositions containing no double curves we show that the lengths of curves contained in the decomposition determine the point of  $\mathcal{T}_{g,n}$  up to finitely many choices. Moreover, these lengths provide a local coordinate in a neighborhood of all points of  $\mathcal{T}_{g,n} \setminus X$  where  $X$  is a union of  $3g - 3 + n$  hypersurfaces. Furthermore, there exists a groupoid acting transitively on admissible double pants decompositions and generated by transformations exchanging only one curve of the decomposition. The local charts arising from different double pants decompositions compose an atlas covering the Teichmüller space. The gluings of the adjacent charts are coming from the elementary transformations of the decompositions, the gluing functions are algebraic. The same charts provide an atlas for a large part of the boundary strata in Deligne–Mumford compactification of the moduli space  $\mathcal{M}_{g,n}$ .

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### 0. Introduction

Consider a hyperbolic structure on a closed oriented surface  $S_{g,n}$ ,  $2g + n > 2$ , of genus  $g$  with  $n$  boundary components. In [5], Fricke and Klein proved that in case  $n = 0$  the Teichmüller space  $\mathcal{T} = \mathcal{T}_{g,0}$  for such a surface is homeomorphic to  $(6g - 6)$ -dimensional Euclidean space. Moreover, they specified a point of Teichmüller space by the lengths of closed geodesics contained in some (rather large) set.

After Fricke and Klein many authors investigated various sets of global parameters on the Teichmüller space. Fenchel and Nielsen [2] introduced “length-twists” coordinates which in case of closed surface consist of  $3g - 3$  lengths of mutually non-intersecting geodesics and  $3g - 3$  twist parameters along them. Natanzon [10] described a convenient set of parameters (including both lengths of geodesics and parameters of other nature), allowing to recover the Fuchsian group of the surface. A lot of efforts were spent on descriptions of purely length global parameters, especially, for the question of minimal possible number of geodesics whose lengths are sufficient to serve as a global coordinate on the Teichmüller space. First, it was shown that  $9g - 9$  length of geodesics may serve as global parameters in  $\mathcal{T}_{g,0}$  (see [15]). Later, Wolpert [16] used the construction of Fricke and Klein to show that  $6g - 6$  lengths are sufficient for a local coordinate in  $\mathcal{T}_{g,0}$  (but not for

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a global one). It was natural to expect that  $6g - 6$  lengths of geodesics can serve as a global coordinate on  $\mathcal{T}_{g,0}$ , however, Wolpert [17] showed that  $\mathcal{T}_{g,0}$  cannot be parametrized globally by lengths of  $6g - 6$  geodesics. Seppälä and Sorvali [13] presented a global parameterization of  $\mathcal{T}_{g,0}$  by  $6g - 4$  length functions (as a by-product they also gave an example of  $6g - 6$  length parameters defining the surface up to at most 4 possibilities). Finally, in [12] Schmutz obtained a global parameterization by  $6g - 5$  lengths of geodesics, which is due to [17] is minimal possible. Another example of such a minimal parameterization is given in [6] by Hamenstädt. In the case of surfaces with cusps or holes the situation is easier: the  $(6g - 6 + 2m + 3n)$ -dimensional Teichmüller space of surfaces with  $m$  cusps and  $n$  holes may be globally parametrized by  $(6g - 6 + 2m + 3n)$  length parameters (see [13,12] and [6]). Hamenstädt [7] also showed that such a parametrization may be extended to the Thurston boundary of  $\mathcal{T}$ .

In this paper, we consider the Teichmüller space  $\mathcal{T} = \mathcal{T}_{g,n}$  of marked hyperbolic structures on an oriented surface  $S = S_{g,n}$ ,  $2g + n > 2$  of genus  $g$  with  $n$  geodesic boundary components. The dimension of this space is  $6g - 6 + 3n$ , so we are interested in sets of  $6g - 6 + 3n$  curves on  $S$  whose lengths parametrize  $\mathcal{T}$ . We build a large family of the sets of  $6g - 6 + 3n$  curves such that the lengths of curves from each set determine a point of  $\mathcal{T}$  up to finitely many possibilities and provide a local coordinate in neighborhoods of most points of  $\mathcal{T}$ , the local charts of this type compose an atlas on  $\mathcal{T}$ , the transition functions between the charts are algebraic. Moreover, the same atlas works for regular points of the moduli space  $\mathcal{M} = \mathcal{T}/\text{Mod}$  (where Mod is a modular group) and covers also a large part of the Deligne–Mumford compactification of  $\mathcal{M}$ .

In more details, we build a large family of the sets of  $6g - 6 + 3n$  curves on  $S$  satisfying the following properties:

1. (Parameterizing property.) The lengths of the curves of each set determine a point of  $\mathcal{T}$  up to finitely many choices; they provide a local coordinate in the neighborhoods of almost all points of  $\mathcal{T}$ .
2. (Double pants decomposition property.) Each set compose an *admissible double pants decomposition* defined and studied recently in [1]; it consists of two pants decompositions (where a *pants decomposition* is a set of curves decomposing the surface into “pairs of pants”, i.e. into spheres with 3 holes). Each pants decomposition defines a handlebody with  $S$  as the boundary, so, two pants decompositions define a Heegaard splitting of some 3-manifold  $M^3$ . The *admissible* double pants decompositions are ones corresponding to Heegaard splittings of the 3-sphere (there exists also an equivalent combinatorial definition which is used throughout the proofs).
3. (Groupoid action.) There exists a groupoid acting on admissible double pants decompositions transitively and generated by simple transformations of two types (called “flips” and “handle-twists”), each of the generating transformations changes exactly one curve of a double pants decomposition. The length of the new curve is an algebraic function of the lengths of the initial curves.
4. (Atlas on  $\mathcal{T}$  with algebraic transition functions.) The charts arising from admissible double pants decompositions compose an atlas on  $\mathcal{T}$ ; the transition functions between the charts are algebraic.
5. (Extension to most strata of Deligne–Mumford compactification.) Let Mod be a modular group of  $S$  and let  $\mathcal{M} = \mathcal{T}/\text{Mod}$  be the corresponding moduli space. Each point of the Deligne–Mumford compactification  $\overline{\mathcal{M}}$  of  $\mathcal{M}$  is a boundary point for some chart coming from a double pants decomposition. Moreover, for most points of  $\overline{\mathcal{M}}$  (including almost all points of the strata of minimal codimension) there exists a chart coming from a double pants decomposition and covering a neighborhood of the point in the corresponding stratum as well as covering almost all point in the neighborhood of the point in  $\overline{\mathcal{M}}$ .

More precisely, let  $DP$  be an admissible double pants decomposition whose curves are closed geodesics in  $S$ . In principle, two pants decompositions contained in  $DP$  may have a common curve (called a *double curve*), we will be interested in double pants decompositions containing no double curves. Let  $l(DP)$  be the ordered set of lengths of curves composing  $DP$ . Then we prove the following:

**Theorem A.** (See Theorem 4.11 below.) *Let  $DP$  be an admissible double pants decomposition without double curves. Then  $DP$  together with the ordered set of lengths  $l(DP) = \{l(c_i) \mid c_i \in DP\}$  is a local coordinate in  $\mathcal{T} \setminus Z$  where  $Z$  is a union of finitely many codimension 1 subsurfaces in  $\mathcal{T}$  (each homeomorphic to a codimension 1 disk).*

Moreover, we show also

**Theorem B.** (See Theorem 5.1 below.) *Let  $DP$  be an admissible double pants decomposition containing no double curves. Then  $l(DP)$  determines a point of  $\mathcal{T}$  up to finitely many choices.*

Composing Theorems A and B with the fact (see [1]) that there exists a groupoid acting on admissible double pants decompositions transitively, we derive the following

**Theorem C.** (See Theorem 6.8 below.) (1) *The charts with coordinates  $l(DP)$ , where  $DP$  is an admissible double pants decomposition without double curves, provide an atlas on Teichmüller space  $\mathcal{T}$ .*

(2) The elementary transition functions of these charts are induced by elementary transformations of double pants decompositions, each elementary transition function change only one coordinate. This unique non-trivial transition function is algebraic.

(3) The compositions of elementary transition functions act transitively on the charts.

The entire construction looks similar to one in the theory of cluster algebras arising from triangulated surfaces [11,3,4]. In the latter, the hyperbolic metrics are encoded by suitably defined lengths of the arcs of triangulations (called lambda lengths), the set of lengths of arcs in one triangulation composing together Penner coordinates on the decorated Teihmüller space. The groupoid generated by flips of triangulations acts transitively on the triangulations, each of the flips changes exactly one of the arcs in the triangulation, so, changing exactly one component of the coordinates. In addition, in case of cluster algebras there are explicit formulas expressing changes in combinatorics of the triangulation (exchange matrices) and transition functions between the charts (exchange relations). For the case of pants decompositions explicit formulas are not known.

It is worth to mention that a single set of  $6g - 6 + 3n$  length (local) coordinates on  $\mathcal{T}$  may be easily obtained from Fenchel–Nielsen coordinates in the following way: for each curve  $c_i$  in a pants decomposition one takes the lengths of  $c_i$  and the length of  $c'_i$ , where  $c'$  is obtained from  $c_i$  by a flip. However, the coordinates obtained are not symmetric (have distinct geometrical origin), and the are not subject to transitive action of a groupoid.

The structure of double pants decomposition is convenient to work with Deligne–Mumford compactification of the moduli space. Let  $C$  be a set of mutually disjoint simple curves on  $S$ . Contracting the curves contained in  $C$  we obtain a point of the compactification, on the other hand, we stay in any chart arising from a double pants decomposition  $DP$  such that  $C \in DP$  (more precisely, the limit point belongs to the boundary of the chart), see [Theorem 7.1](#) and [Corollary 7.2](#).

Furthermore, contraction of the curves of  $C$  turns a conveniently chosen double pants decomposition  $DP$  into a double pants decomposition of the obtained surface with nodal singularities (provided that  $C \in DP$  and each curve of  $C$  is intersected by a unique other curve of  $DP$ ). There are some cases when such a convenient decomposition does not exist, however, for the most configuration of curves  $C$  we show that it does exist. In this case we say that the set  $C$  is *good* and the stratum  $S_C \in \overline{\mathcal{M}}$  is *good* (here  $S_C$  is the set of nodal surfaces obtained by shrinking all curves of  $C$ ,  $\mathcal{M}$  is the moduli space and  $\overline{\mathcal{M}}$  is its Deligne–Mumford compactification). In particular, all strata of minimal codimension (i.e. of codimension 2) are good strata. For a good set of curves  $C$  we define another length-type coordinates as  $\tilde{l}(DP, C) = \{l(c_i), \frac{1}{l(c_j)} \mid c_i \in C, c_j \in DP \setminus C\}$ . We show that the functions  $\tilde{l}(DP, C)$  produce *almost charts* covering the good strata of  $\overline{\mathcal{M}}$ , i.e. given a point  $\tau' \in S_C$  in a good stratum  $S_C$  there exists an admissible double pants decomposition  $DP$  and a neighborhood  $O(\tau') \subset \overline{\mathcal{M}}$  in a natural topology such that  $\tilde{l}(DP, C)$  produce a local coordinate in  $O(\tau') \cap S_C$  and give a local coordinate in some set  $O(\tau') \setminus Z \in \overline{\mathcal{M}}$ , where  $Z$  is a union of finitely many codimension 1 subsurfaces in  $\mathcal{M}$ . More precisely, we prove the following

**Theorem D.** (See [Theorem 7.13](#) below.) Let  $S$  be a nodal surface, let  $\mathcal{M}(S)$  be its moduli space and let  $\overline{\mathcal{M}}(S)$  be Deligne–Mumford compactification of  $\mathcal{M}$ . Let  $S_{\text{good}}^{\mathcal{M}} = S_{\text{good}} / \text{Mod}$  be the union of good strata in  $\mathcal{M}$ . Let  $O$  be a locus of orbifold points of  $\mathcal{M}$ , let  $\overline{O}$  be the closure of  $O$  in  $\overline{\mathcal{M}}$ . Then

- (1) the charts with coordinates  $\tilde{l}(DP, C)$  provide an atlas on  $\mathcal{M} \setminus O$  and on  $S_{\text{good}}^{\mathcal{M}} \setminus \overline{O}$ , (here  $C$  is a good set and  $DP$  is an admissible double pants decomposition without double curves);
- (2) each point  $\tau' \in S_{\text{good}}^{\mathcal{M}} \setminus \overline{O}$  is covered by some almost chart  $(O'(\tau'), \tilde{l}(DP, C))$ ;
- (3) the elementary transition functions of these charts (almost charts) change only one coordinate, this unique non-trivial transition function is algebraic;
- (4) the compositions of elementary transition functions act transitively on the union of charts and almost charts.

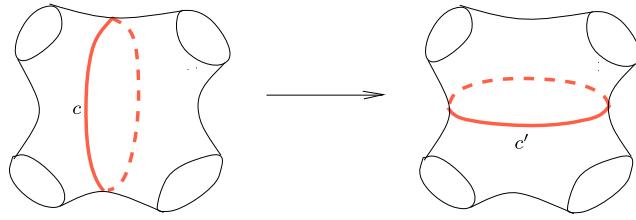
The paper is organized as follows. In Section 1, we recall from [1] the definition of double pants decompositions and their properties. In Sections 2 and 3, we discuss Fenchel–Nielsen coordinates on  $\mathcal{T}$ , and use them to prove some technical lemmas. In Section 4, we prove [Theorem A](#), i.e. we prove that double pants decompositions induce some local charts on  $\mathcal{T}$  (see [Theorem 4.11](#)). Section 5 is devoted to the proof of [Theorem B](#) (see [Theorem 5.1](#)). In Section 6, we collect the above mentioned local charts into an atlas on  $\mathcal{T}$ , this leads to [Theorem C](#) (see [Theorem 6.8](#)). Finally, in Section 7 we consider Deligne–Mumford compactification of the moduli space and prove [Theorem D](#) (see [Theorem 7.13](#)).

## 1. Preliminaries I: Double pants decompositions

In this section we recall from [1] the definition of double pants decompositions and their properties.

### 1.1. Pants decompositions

Let  $S = S_{g,n}$  be an oriented closed surface of genus  $g \geq 0$  with  $n$  boundary components. We assume  $2g + n > 2$ , which excludes spheres with less than 3 holes and the torus. The surface  $S$  is fixed throughout the paper.



**Fig. 1.** Flips of pants decomposition.

A curve  $c$  on  $S$  is an embedded closed non-contractible non-selfintersecting curve considered up to a homotopy of  $S$ .

Given a set of curves we always assume that there are no “unnecessary intersections”, so that if two curves of this set intersect each other in  $k$  points then there are no homotopy equivalent pair of curves intersecting in less than  $k$  points.

For a pair of curves  $c_1$  and  $c_2$  we denote by  $|c_1 \cap c_2|$  the number of (geometric) intersections of  $c_1$  with  $c_2$ .

**Definition 1.1** (*Pants decomposition*). A *pants decomposition* of  $S$  is a set of (non-oriented) mutually disjoint curves  $P = \{c_1, \dots, c_k\}$  decomposing  $S$  into pairs of pants (i.e. into spheres with 3 holes). In this paper, all boundary curves of  $S$  are considered as a part of each pants decomposition of  $S$ .

It is easy to see that any pants decomposition of  $S_{g,n}$  consists of  $3g - 3 + 2n$  (where  $3g - 3 + n$  curves decompose  $S$  and  $n$  curves are boundary curves). Note, that we do allow self-folded pants, two of whose boundary components are identified in  $S$ . A surface which consists of one self-folded pair of pants will be called *handle*.

A curve  $c \in P$ , is *regular* if  $c \notin \partial S$  and  $c$  is not a self-identified boundary curve of the self-folded pair of pants (i.e. if it is not lying inside a handle cut out by a curve  $c' \in P$ ).

**Definition 1.2** (*Flip*). Let  $P = \{c_1, \dots, c_{3g-3+2n}\}$  be a pants decomposition. Define a *flip of  $P$  in a regular curve  $c_i$*  as a replacing of  $c_i \subset P$  by any curve  $c'_i$  satisfying the following properties:

- $c'_i$  does not coincide with any of  $c_1, \dots, c_{3g-3+2n}$ ;
- $|c'_i \cap c_i| = 2$ ;
- $c'_i \cap c_j = \emptyset$  for all  $j \neq i$ .

See Fig. 1 for an example of a flip. Clearly, an inverse operation to a flip is also a flip (so that the set of flips compose a groupoid acting on pants decompositions).

**Definition 1.3** (*Standard decomposition*). A decomposition  $P$  of  $S_{g,n}$  is *standard* if  $P$  contains  $g$  curves  $c_1, \dots, c_g$  such that  $c_i$ ,  $i = 1, \dots, n$ , cuts out a handle.

## 1.2. Double pants decompositions

Let  $P = \{c_1, \dots, c_{3g-3+2n}\}$  be a pants decomposition. A *Lagrangian plane*  $\mathcal{L}(P) \subset H_1(S, \mathbb{Z})$  is a subspace spanned by the homology classes  $h(c_i)$ ,  $i = 1, \dots, 3g - 3 + 2n$  (here  $c_i$  is taken with any orientation).

Two Lagrangian planes  $\mathcal{L}(P_1)$  and  $\mathcal{L}(P_2)$  are in general position if  $\mathcal{L}_1 \cap \mathcal{L}_2 = 0$  and  $H_1(S, \mathbb{Z}) = \langle \mathcal{L}_1, \mathcal{L}_2 \rangle$  (where  $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$  denotes the sublattice of  $H_1(S, \mathbb{Z})$  spanned by  $\mathcal{L}_1$  and  $\mathcal{L}_2$ ).

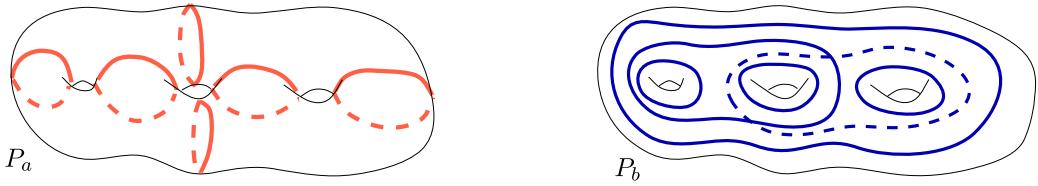
**Definition 1.4** (*Double pants decomposition*). A *double pants decomposition*  $DP = (P_a, P_b)$  is a pair of pants decompositions  $P_a$  and  $P_b$  of the same surface such that the Lagrangian planes  $\mathcal{L}_a = \mathcal{L}(P_a)$  and  $\mathcal{L}_b = \mathcal{L}(P_b)$  spanned by these pants decompositions are in general position.  $P_a$  and  $P_b$  are called *parts* of  $DP$ .

See Fig. 2 for an example of a double pants decomposition.

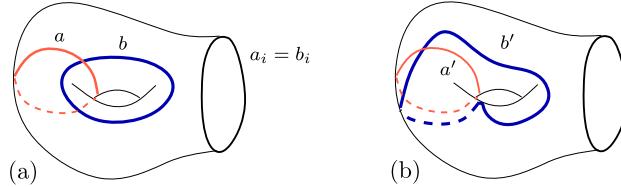
There are several natural transformations on the set of double pants decompositions:

- flips of  $P_a$ ;
- flips of  $P_b$ ;
- handle-twists (see Definition 1.5 below).

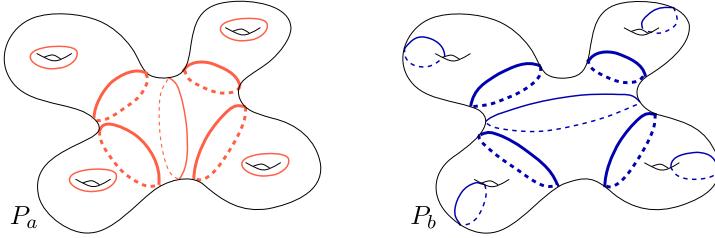
**Definition 1.5** (*Handle-twists*). Given a double pants decomposition  $DP = (P_a, P_b)$  we define an additional transformation which may be performed if both parts  $P_a$  and  $P_b$  contain the same curve  $a_i = b_i$  separating the same handle  $\mathfrak{h}$ , see Fig. 3(a). Let  $a \in \mathfrak{h}$  and  $b \in \mathfrak{h}$  be the only curves from  $P_a$  and  $P_b$  respectively. Then a *handle-twist*  $t_a(b)$  (respectively,  $t_b(a)$ ) is a Dehn twist along  $a$  (respectively,  $b$ ) in any of two directions (see Fig. 3(b)).



**Fig. 2.** A double pants decomposition ( $P_a, P_b$ ).



**Fig. 3.** Handle-twists: (a) Double self-folded pair of pants; (b) The same pair of pants after a handle-twist  $t_a(b)$ .



**Fig. 4.** A standard double pants decomposition ( $P_a, P_b$ ).

Notice that both flips and handle-twists are reversible transformations, so that flips and handle-twists generate a groupoid acting on the set of double pants decompositions.

**Definition 1.6** (*Double curve*). A curve  $c \in (P_a, P_b)$  is *double* if  $c \in (P_a \cap P_b)$  and  $c \notin \partial S$ .

**Definition 1.7** (*Standard decomposition*). A double pants decomposition  $(P_a, P_b)$  of  $S_{g,n}$  is *standard* if there exist  $g$  double curves  $c_1, \dots, c_g \in (P_a, P_b)$  such that  $c_i$  cuts out of  $S$  a handle  $\mathfrak{h}_i$ .

A standard double pants decomposition  $(P_a, P_b)$  is *strictly standard* if  $(P_a, P_b)$  contains  $2g - 3 + n$  double curves (i.e.  $c \in \{P_a \cup P_b\} \setminus \{P_a \cap P_b\}$  if and only if  $c$  is contained inside some handle).

See Fig. 4 for an example of a standard double pants decomposition (this decomposition may be turned into a strictly standard one in one flip).

**Definition 1.8** (*Admissible decomposition*). A double pants decomposition  $(P_a, P_b)$  is *admissible* if it is possible to transform  $(P_a, P_b)$  to a standard pants decomposition by a sequence of flips.

For example, the decomposition shown in Fig. 2 is admissible.

The following theorem is the main result of [1].

**Theorem 1.9.** (See [1].) A groupoid generated by flips and handle-twists acts transitively on admissible double pants decompositions of  $S = S_{g,n}$  (for any  $(g, n)$  such that  $2g + n > 2$ ).

**Remark 1.10** (*Admissible double pants decompositions and Heegaard splitting of  $\mathbb{S}^3$* ). A set of admissible double pants decompositions have an invariant topological description in terms of Heegaard splittings of 3-manifolds. For each pants decomposition  $P$  of  $S$  one may construct a handlebody  $S_+$  such that  $S$  is the boundary of  $S_+$  and all curves of  $P$  are contractible inside  $S_+$ . A union of two pants decompositions of the same surface define two different handlebodies bounded by  $S$ . Attaching this handlebodies along  $S$  one obtains a Heegaard splitting of some 3-manifold  $M^3(DP)$ . It is shown in [1] that a pants decomposition  $DP$  is admissible if and only if  $M^3(DP) = \mathbb{S}^3$ , where  $\mathbb{S}^3$  is a 3-sphere.

We will also use the following result proved in [1, Lemma 6.1].

**Proposition 1.11.** (See [1].) Let  $S = S_{g,n}$ ,  $2g + n > 2$ , and  $\text{Mod}(S)$  be its modular group. Let  $(P_a, P_b)$  be an admissible double pants decomposition without double curves. Then  $\gamma \in \text{Mod}(S)$  fixes  $(P_a, P_b)$  if and only if  $\gamma = \text{id}$ .

## 2. Preliminaries II: coordinates on Teichmüller space

Let  $S = S_{g,n}$  be a hyperbolic surface of genus  $g$  with  $n$  boundary components. Each boundary component is assumed to be a geodesic of finite length.

A Teichmüller space  $\mathcal{T} = \mathcal{T}_{g,n}$  is a parameter space of marked hyperbolic metrics on the surface  $S_{g,n}$ . For the marking on  $S$  we will usually use admissible double pants decompositions containing no double curves (this provides a correct marking since any elements  $\gamma \neq e$  of the modular group  $\text{Mod}(S_{g,n})$  acts non-trivially on the decomposition, see [1, Lemma 6.1]).

We will use Fenchel–Nielsen parameterization of the Teichmüller space. We shortly explain the parametrization below and refer to [14] for the details.

To build the parameterization one chooses a pants decomposition  $P$  of  $S$ . Each pair of pants is uniquely determined by the lengths of its boundary curves. To encode the concrete hyperbolic structure one need also to know how the adjacent pairs of pants a sewed together: one can choose an arbitrary way to attach them, and then rotate one piece along another by any real angle. More precisely, to determine the angle of the rotation one does the following:

- 1) for each pair of pants  $p^k \in P$  one chooses three disjoint segments  $s_{ij}^k$ ,  $i, j \in \{1, 2, 3\}$  orthogonal to the boundary components  $b_i^k$  and  $b_j^k$  of  $p^k$  (so that  $p^k$  is decomposed into two right-angled hexagons);
- 2) then one fixes some way to attach the adjacent pairs of pants  $p^k$  and  $p^{k'}$  so that the segments  $s_{ij}^k$  and  $s_{i'j'}^{k'}$  intersect the curve  $p^k \cap p^{k'}$  at the same points, this will produce some special gluing of pairs of pants, all other gluings (with other angles of rotation of  $p^k$  with respect to  $p^{k'}$ ) will be compared with this special gluing;
- 3) for arbitrary gluing the angles of rotation are compared with the chosen special gluing, when the angle is changed by  $2\pi$  one obtains the same hyperbolic structure on the surface, but the different point of the Teihmüller space.

So, the Fenchel–Nielsen coordinates on  $\mathcal{T}$  build from the pants decomposition  $P$  consist of  $3g - 3 + 2n$  length parameters  $l(c_i)$  (lengths of all the curves  $c_i \in P$  including the boundary curves of  $S$ ) and  $3g - 3 + n$  angle parameters  $\alpha(c_j)$  (angles along all non-boundary curves  $c_j \in P$ ,  $c_j \notin \partial S$ ). We denote

$$FN(P) = \{l(c_i), \alpha(c_j) \mid c_i \in P; c_j \in P, c_j \notin \partial S\}.$$

We will also assume that the Dehn twist along  $c_j$  changes  $\alpha(c_j)$  by  $2\pi$ .

The construction establishes the homeomorphism between  $\mathcal{T}$  and  $\mathbb{R}_{>0}^{3g-3+2n} \times \mathbb{R}^{3g-3+n}$  (where  $\mathbb{R}_{>0}$  stays for positive real numbers).

**Remark 2.1.** After the Teichmüller space  $\mathcal{T}$  is introduced using any given pants decomposition  $P_0$  (or even using a marking of other type), one can choose any pants decomposition  $P$  to introduce the coordinates  $FN(P)$  on the same space  $\mathcal{T}$ .

Our aim is to transform Fenchel–Nielsen coordinates to coordinates containing only length parameters.

**Definition 2.2** (*Locally parameterizing decomposition*). We say that a double pants decomposition  $DP$  is *locally parameterizing* at the point  $\tau \in \mathcal{T}$  if the functions  $l(DP) = \{l(c) \mid c \in DP\}$  provide a local homeomorphism from a neighborhood of  $\tau$  to a neighborhood of some point in  $\mathbb{R}^{6g-6+3n}$ . By a *chart*  $\mathfrak{C}(DP)$  we mean a pair  $(X, l(DP))$  where  $X$  is the set of points  $\tau \in \mathcal{T}$  such that  $DP$  is locally parameterizing at  $\tau$ .

Our first aim is to prove that admissible double pants decompositions are locally parameterizing. As an intermediate technical step in the proof we will use *mixed* coordinates, containing some angle-parameters (but less than Fenchel–Nielsen coordinates).

**Definition 2.3** (*Mixed coordinates*). Let  $DP = (P_a, P_b)$  be a double pants decomposition, possibly with some double curves. Let  $FN(P_b)$  be some Fenchel–Nielsen coordinates build from  $P_b$ . Denote by  $\text{mix}(DP, FN(P_b))$  the following set of functions:

$$\text{mix}(DP, FN(P_b)) = \{l(c), \alpha(c') \mid c \in DP, c' \in P_a \cap P_b\},$$

where  $\alpha(c')$  is the corresponding angle coordinate in  $FN(P_b)$ .

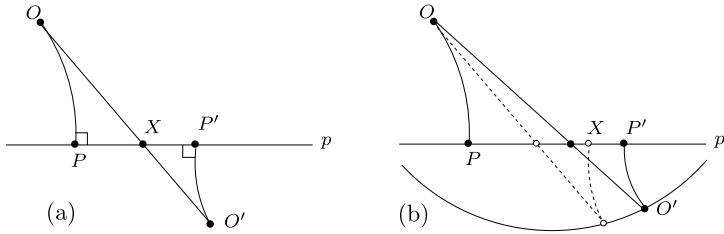


Fig. 5. To the proof of Lemma 3.2.

### 3. Some properties of length functions

In this section we prove several facts from hyperbolic geometry. In particular, Lemmas 3.4 and 3.6 will be crucial for the construction of locally parameterizing double pants decompositions. Lemmas 3.1–3.3 are preparatory. We will denote the hyperbolic plane by  $\mathbb{H}^2$ .

**Lemma 3.1.** Let  $S = S_{0,4}$ , let  $c, d \in S$  be two closed curves  $|d \cap c| = 2$ . Let  $P$  be a pants decomposition of  $S$ ,  $c \in P$ . Suppose that  $d' \in S$  is a curve obtained from  $c$  by a flip of  $P$ . Then  $d' = t_c^k(d)$  for some integer  $k$ , where  $t_c$  is a Dehn twist along  $c$ .

The lemma follows immediately from [1, Lemma 1.16].

**Lemma 3.2.** Let  $p \in \mathbb{H}^2$  be a line separating points  $O$  and  $O'$ . Given the distances from  $p$  to  $O$  and  $O'$ , the distance  $OO'$  is a monotonic function on the distance  $PP'$ , where  $P$  and  $P'$  are the orthogonal projections of points  $O$  and  $O'$  to  $p$ .

**Proof.** Suppose that the points  $P$  and  $O$  are fixed, and the point  $P'$  (together with  $O'$ ) glide away from  $P$ , see Fig. 5(b). Then the point  $X = OO' \cap p$  glide away from  $P$  which implies that the distance  $OX$  grows monotonically when  $PP'$  increases. By the similar reason  $O'X$  grows, and hence,  $OO'$  grows monotonically.  $\square$

**Lemma 3.3.** Let  $S = S_{0,3}$  be a three-holed sphere with a boundary  $\partial S = c_1 \cup c_2 \cup c_3$ , and let  $s_{ij}$  be a segment orthogonal to  $c_i$  and  $c_j$ , for  $i \neq j$ ,  $i, j \in \{1, 2, 3\}$ . Then the segments  $s_{12}, s_{13}, s_{2,3}$  decompose  $S$  into two congruent right-angled hexagons.

**Proof.** It is clear that the segments  $s_{ij}$  decompose  $S$  into two right-angled hexagons. Since a right-angled hexagon is determined (up to an isometry) by the lengths of three non-adjacent sides (the lengths of  $s_{12}, s_{13}, s_{2,3}$ ), the hexagons are congruent.  $\square$

If the curves  $a, b \in S$  are orthogonal to each other we will write " $a \perp b$ ".

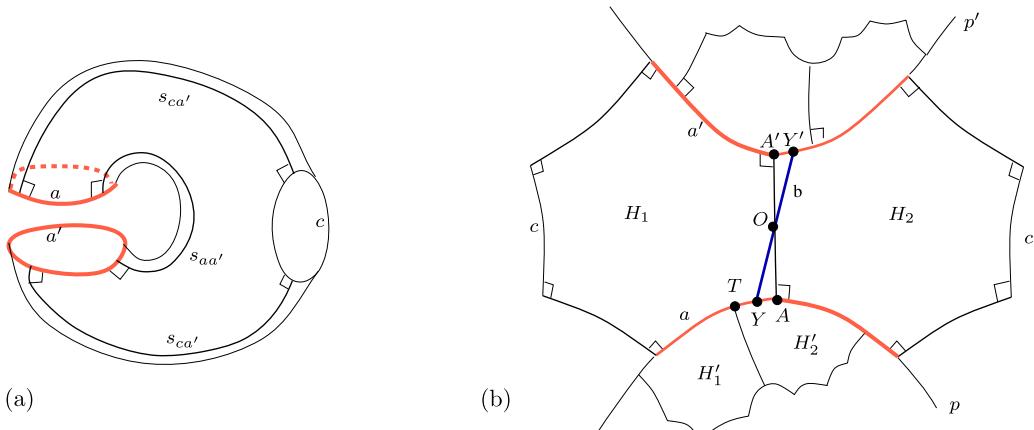
**Lemma 3.4.** Let  $S = S_{1,1}$  be a handle with a boundary curve  $c$ , let  $a, b \subset S$  be two curves  $|a \cap b| = 1$ . Then the set of functions  $\bar{x} = (l(a), l(b), l(c))$  is a local coordinate on  $\mathcal{T} \setminus X$  where  $X = \{\tau \in \mathcal{T} \mid a \perp b\}$ . Moreover,  $\bar{x}$  determines the point  $\tau \in \mathcal{T}$  up to at most two possibilities.

**Proof.** Shortly speaking, the coordinates  $\bar{x} = (l(a), l(b), l(c))$  will be produced from Fenchel–Nielsen coordinates. More precisely, we fix Fenchel–Nielsen coordinates  $FN(P) = (l(a), \alpha(a), l(c))$  arising from pants decomposition  $P = \{a, c\}$ . We fix some values of  $l(a)$  and  $l(c)$  and denote by  $\alpha_0$  the value of  $\alpha(a)$  at the point where  $l(a)$  and  $l(c)$  have the chosen values and  $a$  is orthogonal to  $b$ . We will show that  $l(b)$  is a monotonic function on the absolute value  $|\alpha(a) - \alpha_0|$ , which will imply all statements of the lemma. Below we explain this in more details.

First, we cut  $S$  along  $a$  and obtain a pair of pants  $S'$  with three boundary components  $c, a$  and  $a'$ . For each of the three pairs of boundary components of  $S'$  we draw a segment orthogonal to both of these two components. Denote these segments by  $s_{c,a}, s_{c,a'}, s_{a,a'}$ , see Fig. 6(a). The three segments decompose  $S'$  into two right-angled hexagons  $H_1$  and  $H_2$ . Similarly, together with the curve  $a$  the three segments decompose the initial handle  $S$  into two hexagons.

Consider the covering of  $S$  by hyperbolic plane. We are interested in the tiling of the plane by the images of  $H_1$  and  $H_2$ . Notice that the copies of  $H_1$  and  $H_2$  adjacent along the image of  $s_{a,a'}$  (or  $s_{c,a}$  or  $s_{c,a'}$ ) have this side in common, while the gluing along the images of  $a$  and  $a'$  depends on the angle parameter  $\alpha(a) \in FN(P)$ . More precisely, when  $\alpha(a) = \alpha_0$  the adjacent along  $a$  hexagons have a common side, otherwise the hexagons are shifted one along another as in Fig. 6(b). With growth of  $\alpha(a)$  the hexagons in one row glide monotonically along the hexagons of the other row. We denote by  $p$  and  $p'$  the lines separating the rows.

Now, consider the curve  $b \in S$ ,  $|b \cap a| = 1$ . First, suppose that  $b \perp a$ , i.e. the image  $\hat{b}$  of  $b$  in the hyperbolic plane coincide with the image  $AA'$  of  $s_{a,a'}$ . Now, we increase  $\alpha(a)$  and look at the image  $\hat{b} \in \mathbb{H}^2$  of  $b$ : since  $b$  is a closed geodesic on  $S$ ,



**Fig. 6.** Length coordinates on a handle.

$\hat{b}$  is a line forming the same angles with  $p$  and  $p'$ . This implies that  $\hat{b}$  passes through the midpoint  $O$  of  $AA'$ . Hence,  $AY = A'Y'$ , where  $Y = \hat{b} \cap p$  and  $Y' = \hat{b} \cap p'$ . Furthermore, the hexagon  $H'_2$  is shifted with respect to the hexagon  $H_2$  to the distance  $\rho = l(a) \frac{(\alpha(a) - \alpha_0)}{2\pi}$ . Denote by  $T$  the vertex of  $H'_1$  projecting to the same point of  $S$  as  $A'$  (as in Fig. 6(b)), then  $TY = AY = A'Y'$ . Hence,  $AY = 1/2\rho = l(a) \frac{(\alpha(a) - \alpha_0)}{4\pi}$ . The same formula holds for any positive value of  $(\alpha(a) - \alpha_0)$  as well as for any negative one (in the latter case the point  $Y \in l$  lies on the other side with respect to  $A$ ).

This implies that the distance  $YY' = l(b)$  grows monotonically with the growth of  $|\alpha(a) - \alpha_0|$ :

$$\cosh \frac{YY'}{2} = \cosh OY = \cosh OA \cosh AY = \cosh OA \cosh \left( l(a) \frac{\alpha(a) - \alpha_0}{4\pi} \right).$$

Hence,  $|\alpha(a) - \alpha_0|$  may be recovered from  $l(b)$ . So, given the lengths  $(l(a), l(b), l(c))$  one may find the Fenchel-Nielsen coordinates  $FN(P)$  up to two possibilities. In particular, in the neighborhood of a point  $\tau \in \mathcal{T}$  where  $a$  is not orthogonal to  $b$ , the sign of  $(\alpha(a) - \alpha_0)$  does not change, which implies that the functions  $(l(a), l(b), l(c))$  form a local coordinate in  $\mathcal{T} \setminus X$ ,  $X = \{\tau \in \mathcal{T} \mid a \perp b\}$ .  $\square$

**Remark 3.5.** Given Fenchel-Nielsen coordinates  $(l(a), \alpha(a), l(c))$  on the handle, for each pair of lengths  $l_0(a)$  and  $l_0(c)$  there exists a unique angle  $\alpha_0(a)$  such that  $a$  is orthogonal to  $b$ .

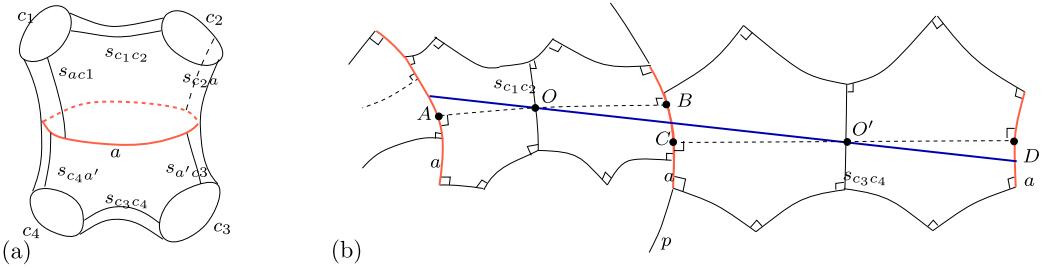
**Lemma 3.6.** Let  $S = S_{0,4}$  be a sphere with four holes, with boundary curves  $c_1, c_2, c_3, c_4$ . Let  $a \in S$  be a closed geodesic and let  $b \in S$  be a closed geodesic obtained from the curve  $a$  by a flip. Then

- (1) the angle formed by  $a$  and  $b$  is of the same size for both intersections of  $a$  and  $b$ ;
- (2) the set of functions  $\bar{x} = (l(a), l(b), l(c_1), l(c_2), l(c_3), l(c_4))$  is a local coordinate on  $\mathcal{T} \setminus X$  where  $X = \{\tau \in \mathcal{T} \mid a \perp b\}$ ;
- (3)  $\bar{x}$  determines the point  $\tau \in \mathcal{T}$  up to at most two possibilities.

**Proof.** The idea of the proof is the same as in the proof of Lemma 3.4: the coordinate  $\bar{x}$  is obtained from Fenchel-Nielsen coordinates  $FN(P)$  built from pants decomposition  $P = \{a, c_1, c_2, c_3, c_4\}$ . We show that given the values of  $(l(a), l(c_1), l(c_2), l(c_3), l(c_4))$  the length  $l(b)$  is a monotonic function on the absolute value  $|\alpha(a) - \alpha_0|$ , where  $\alpha_0$  is the value of  $\alpha(a) \in FN(P)$  at the point of  $\mathcal{T}$  such that  $a$  is orthogonal to  $b$  (and the values of  $(l(a), l(c_1), l(c_2), l(c_3), l(c_4))$  are the chosen ones). Hence,  $l(b)$  determines  $\alpha(a)$  up to 2 possibilities. Moreover, in the neighborhood of a point  $\tau \in \mathcal{T}$  where  $|\alpha(a) - \alpha_0| \neq 0$ , the sign of  $(\alpha(a) - \alpha_0)$  is determined uniquely by the sign at  $\tau$ .

In more details, the curve  $a$  decomposes  $S$  into two pairs of pants, and each pair of pants is decomposed into two right-angled hexagons (respectively, by the segments  $s_{ac_1}, s_{c_1c_2}, s_{c_2a}$  and  $s_{a'c_3}, s_{c_3c_4}, s_{c_4a'}$  orthogonal to a pair of boundary components), Fig. 7(a). The images of four right-angled hexagons tile the covering hyperbolic plane: two hexagons adjacent by the image of the side  $a$  are shifted by the distance  $\rho = l(a) \frac{\alpha(a) - \alpha_0}{2\pi}$  along the line containing the images of  $a$ , see Fig. 7(b).

Denote by  $O$  and  $O'$  the midpoints of images of  $s_{c_1c_2}$  and  $s_{c_3c_4}$ . Notice that the symmetry in the point  $O$  preserves the tiling of the hyperbolic plane by hexagons (compare with Lemma 3.3). The same holds for the symmetry in  $O'$ . Consider a line  $OO'$  and its intersection with the images of the curve  $a$ . It is easy to see that all angles made by  $OO'$  and images of  $a$  are equal. Furthermore,  $OO'$  intersects the images of  $s_{c_1c_2}$  and  $s_{c_3c_4}$  always in midpoints (to see that consider an image  $O''$  of  $O$  with respect to the symmetry in  $O'$ : it lies on  $OO'$  and in the midpoint of some image of  $s_{c_1c_2}$ , then consider the image of  $O'$  with respect to a symmetry in  $O''$  and so on). This implies that the line  $OO'$  is the union of images of some closed geodesic  $c \in S$ ,  $|c \cap a| = 2$ . Hence,  $c$  may be obtained from  $a$  by a flip. Notice that  $c$  intersects  $a$  in two points,



**Fig. 7.** Length coordinates on a four-holed sphere.

forming two angles of the same size. The length  $l(c) = 2 \cdot O O'$  increases as  $|\alpha(a) - \alpha_0|$  increases (the distances from the points  $O$  and  $O'$  to the line  $p$  remain constant, but one point glide along  $p$  with respect to the other, so that we may apply Lemma 3.2).

Increasing the angle  $\alpha(a)$ , we increase the shift between the adjacent hexagons. Increasing  $\alpha(a)$  by  $2\pi$  we obtain the initial tiling of the plane by hexagons, but the line  $O O'$  in the new picture is moved, so that it is an image of another closed curve  $c' \in S$  which may be obtained from  $a$  by a flip. Increasing (or decreasing)  $\alpha(a)$  by  $2\pi k$  we run through all curves on  $S$  which may be obtained by a flip from  $a$  (compare with Lemma 3.1). In particular, for some value of  $k$  we obtain the curve  $b$ . This implies statement (1). So, the length  $l(b)$  increases with growth of  $|\alpha(a) - \alpha_0|$ . Hence  $l(b)$  determines  $\alpha(a)$  up to two possibilities, which implies that the set of functions  $\bar{x}$  determines Fenchel–Nielsen coordinates  $FN(P)$  up to two possibilities. This proves statement (3). If  $b$  is not orthogonal to  $a$  at  $\tau \in \mathcal{T}$  then in the neighborhood of  $\tau$  the function  $l(b)$  (together with the chosen value of  $\alpha(a)$  at  $\tau$ ) determines completely the function  $\alpha(a)$ , which implies that  $\bar{x}$  is a set of local coordinates, and statement (2) is also proved.  $\square$

**Remark 3.7.** Given Fenchel–Nielsen coordinates on  $S_{0,4}$ , for each lengths  $l_0(a)$  together with fixed lengths of the boundary components of  $S_{0,4}$  there exists a unique angle  $\alpha_0(a)$  such that  $a$  is orthogonal to  $b$ .

#### 4. Locally parameterizing double pants decompositions

In this section we prove Theorem 4.11 which states that for an admissible double pants decomposition  $DP$  the functions  $I(DP)$  provide a local parameter in neighborhoods of almost all points  $\tau \in \mathcal{T}$ .

The proof of the theorem is inductive. In Section 4.1, we build some examples of locally parameterizing double pants decompositions. These examples called *special decompositions* will be the base of the induction. In Section 4.2, we show that any admissible double pants decomposition may be obtained from a special one by a sequence of flips. Finally, in Section 4.3 we show that flips preserve the parameterizing properties of double pants decompositions.

##### 4.1. Examples of locally parameterizing double pants decompositions

In this section we present an example of a locally parameterizing double pants decomposition for each surface  $S_{g,n}$ . This will provide a base for the inductive proof of Theorem 4.11. The construction is obtained as a modification of Fenchel–Nielsen coordinates.

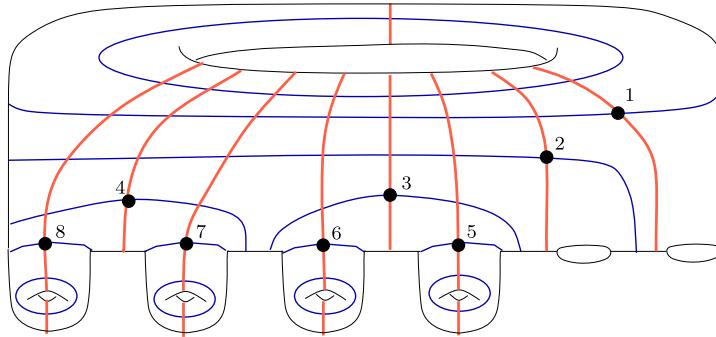
**Definition 4.1** (*Special decomposition, conjugate curves*). A double pants decomposition  $DP = (P_a, P_b)$  is *special* with the *standard part*  $P_b$  if the following holds:

- (1)  $DP$  contains no double curves;
- (2) the part  $P_b$  is standard;
- (3)  $DP$  may be obtained from a strictly standard double pants decomposition  $DP_0$  via a sequence of  $m = 3g - 3 + n$  flips  $f_1, \dots, f_m$  of the  $P_a$ -part.

For a special decomposition  $DP = (P_a, P_b)$  we will say that a curve  $a_i \in P_a$  is *conjugate* to a curve  $b_i \in P_b$  if either  $a_i$  is obtained by a flip  $f_i$  from  $b_i$  or  $a_i$  and  $b_i$  belong to the same handle in the standard decomposition  $P_b$ . In the former case  $(a_i, b_i)$  will called a *flip-conjugate pair*, in the latter case  $(a_i, b_i)$  will called a *handle-conjugate pair*.

See Fig. 8 for an example of a special decomposition. Notice, that any special double pants decomposition is admissible.

**Lemma 4.2.** For each standard pants decomposition  $P_b$  there exists a special double pants decomposition  $DP = (P_a, P_b)$ .



**Fig. 8.** Example of a special double pants decomposition. The black nodes show the intersections of the conjugate curves. The number near the nodes show the sequence of flips taking the strictly standard decomposition to the special one.

**Proof.** To build the required decomposition we consider a strictly standard double pants decomposition  $DP' = (P'_a, P_b)$  containing  $P_b$  and apply a flip of the  $P_a$ -part to each of the double curves.  $\square$

**Notation 4.3.** Let  $DP = (P_a, P_b)$  be a special double pants decomposition. Denote by  $Z(DP) \in \mathcal{T}$  the locus of points where  $a_i$  is orthogonal to  $b_i$  for at least one pair of conjugate curves  $(a_i, b_i) \in DP$ .

**Remark 4.4.** Let  $(a_i, b_i)$  be a pair of conjugate curves in a special double pants decomposition. Remarks 3.5 and 3.7 imply that the locus of points where  $a_i$  is orthogonal to  $b_i$  is homeomorphic to a hyperplane in  $\mathcal{T} = \mathbb{R}_{>0}^{3g-3+2n} \times \mathbb{R}^{3g-3+n}$  (here Remarks 3.5 and 3.7 work for cases of handle-conjugate and flip-conjugate pairs respectively). Therefore, the set  $Z(DP) \in \mathcal{T}$  is homeomorphic to a union of  $3g - 3 + n$  hyperplanes in  $\mathcal{T} = \mathbb{R}_{>0}^{3g-3+2n} \times \mathbb{R}^{3g-3+n}$ .

**Lemma 4.5.** Let  $DP = (P_a, P_b)$  be a special double pants decomposition. Then

- (1)  $l(DP)$  is a local coordinate in  $\mathcal{T} \setminus Z(DP)$ ;
- (2)  $l(DP)$  determine the point in  $\mathcal{T}$  up to at most  $2^{3g-3+n}$  choices.

**Proof.** Suppose that  $P_b$  is a standard part of  $DP$ . Choose Fenchel–Nielsen coordinates  $FN(P_b)$  based on the pants decomposition  $P_b$ . It is a global coordinate on  $\mathcal{T}$ . We will substitute angle coordinates of  $FN(P_b)$  by length coordinates one by one.

Let  $f_1, \dots, f_m$  be the sequence of flips described in the Definition 4.2, let  $b_1, \dots, b_m$  be the curves of  $P_b$  such that  $f_i$  is a flip applied to  $b_i$ . Let  $DP_i = f_i \circ \dots \circ f_1(DP_0)$ , where  $DP_0$  is the corresponding strictly standard double pants decomposition. Applying Lemma 3.4 sufficiently to all handle-conjugate pairs of curves  $a_i, b_i \in DP$  we see that  $\text{mix}(DP_0, FN(P_b))$  is a local coordinate away from  $Z(DP_0)$  and defines the coordinate  $FN(P_b)$  up to  $2^g$  choices. Then, applying Lemma 3.6 to each pair of flip-conjugate curves successively (more precisely, to the subsurface  $S_{0,4}$  obtained by a union of two pairs of pants adjacent to  $b_i$  in  $P_a$ -part of  $DP_i$ ), we see that  $\text{mix}(DP_i, FN(P_b))$  is a local coordinate away from  $Z(DP_i)$  and defines  $\text{mix}(DP_{i-1}, FN(P_b))$  up to 2 choices. This implies the lemma.  $\square$

#### 4.2. Induction step: Reduction to flips

**Lemma 4.6.** Let  $DP$  be an admissible double pants decomposition. Then there exists a sequence of flips  $f_1, \dots, f_k$  such that  $DP_0 = f_k \circ \dots \circ f_1(DP)$  is a strictly standard double pants decomposition.

**Proof.** Since  $DP$  is an admissible decomposition, there exists a sequence of flips taking  $DP$  to a standard double pants decomposition. It is known that flips act transitively on pants decompositions of  $S_{0,k}$  (see [8]), which implies that any strictly standard double pants decomposition may be transformed to a strictly standard ones by flips.  $\square$

**Lemma 4.7.** Let  $DP$  be a double pants decomposition containing no double curves. Suppose that  $DP' = f_k \circ \dots \circ f_1(DP)$ , where  $f_i, i = 1, \dots, k$ , is a flip. If  $DP'$  contains no double curves then there exists a sequence of flips  $g_1, \dots, g_r$  such that  $DP' = g_r \circ \dots \circ g_1(DP)$  and no of the decompositions  $g_i \circ \dots \circ g_1(DP)$ ,  $i = 1, \dots, r$  contains double curves.

**Proof.** Denote  $DP = (P_a, P_b)$  and  $DP' = (P'_a, P'_b)$ . We will use the fact that flips of the  $P_a$ -part commute with flips of the  $P_b$ -part.

Let  $C = \{c \mid c \in DP_i = f_i \circ \dots \circ f_1(DP), 0 \leq i \leq k\}$  be a set of all curves appearing during the transformation from  $DP$  to  $DP' = f_k \circ \dots \circ f_1(DP)$ .

First, for each of the curves  $a_i \in P_a$  we apply a flip  $g_i$  so that  $g_i(a_i) \notin C$ : this is possible, since  $C$  is a finite set, while a set of flips for a given curve  $a_i$  in a given pants decomposition is either infinite or empty (in the later case,  $a_i$  lies in a handle bounded by some other curve  $a_j$ , so we can first destroy the handle applying a flip to  $a_j$ , and then apply a flip to  $a_i$ ). Denote by  $P''_a$  the obtained  $P_a$ -part of the decomposition.

Second, we transform  $P_b$  to  $P'_b$  by the same sequence of flips as in  $f_1, \dots, f_k$ .

Third, there exists a sequence  $f'_1, \dots, f'_l$  of flips taking  $P''_a$  to  $P'_a$ . Denote  $\xi = f'_l \circ \dots \circ f'_1$ . Denote  $C' = \{c \mid c \in DP'' = f'_l \circ \dots \circ f'_1(DP), 0 \leq i \leq l\}$ . For each of the curves  $b_i \in P'_b$  we apply a flip  $g'_i$  so that  $g'_i(b_i) \notin C'$ .

Next, we transform  $P_b$  to  $P'_b$  by the same sequence of flips as in  $f_1, \dots, f_k$ .

Finally, we apply the inverse sequence  $\xi^{-1}$  to take the  $P_b$ -part back to the state  $P'_b$ .

Clearly, we cannot obtain double curves at any stage of the transformation, so the lemma is proved.  $\square$

**Lemma 4.6** together with **Lemma 4.7** imply the following lemma.

**Lemma 4.8.** *Let  $DP$  be an admissible double pants decomposition without double curves. Then there exists a special double pants decomposition  $DP'$  and a sequence of flips  $f_1, \dots, f_k$  such that  $DP_0 = f_k \circ \dots \circ f_1(DP)$  and no of the decompositions  $f_i \circ \dots \circ f_1(DP)$ ,  $i = 1, \dots, k$ , contains double curves.*

#### 4.3. Induction step: Flips

In this section we show that flips take locally parameterizing double pants decompositions to locally parameterizing ones.

In the next lemma we show this property for almost all flips.

**Lemma 4.9.** *Let  $DP$  be a parameterizing double pants decomposition at  $\tau \in \mathcal{T}$ . Let  $f'$  and  $f''$  be two different flips of the same curve  $c \in DP$ , such that neither  $DP' = f'(DP)$  nor  $DP'' = f''(DP)$  contain double curves. If  $DP'$  is not parameterizing at  $\tau \in \mathcal{T}$  then  $DP''$  is parameterizing at  $\tau$ .*

**Proof.** Let  $DP = (P_a, P_b)$ ,  $c \in P_a$ . Let  $DP' = (P'_a, P_b)$ ,  $DP'' = (P''_a, P_b)$ . Denote by  $c'$  and  $c''$  the curves of  $P'_a$  and  $P''_a$  obtained from  $c$  by flips  $f'$  and  $f''$  respectively. In addition, denote by  $S_*$  a subsurface of  $S$  composed of two pairs of pants in  $P_a$  adjacent to the curve  $c$ .

Suppose that  $DP'$  is not a parameterizing double pants decomposition at  $\tau \in \mathcal{T}$ . By definition, this means that there exists a non-trivial deformation  $\xi(\tau)$  of the hyperbolic structure, where  $\xi$  preserves all lengths of curves contained in  $(P'_a, P_b)$ . This deformation may be described as a set of simultaneous small twists along the curves of  $P'_a$  (the rates of the twists need not coincide or to be constant).

Suppose that  $\xi$  contains no twist along  $c'$  (i.e. the twist along this curve is trivial, zero). Then the subsurface  $S_*$  is not changed, and the length of the curve  $c$  is preserved by  $\xi$ . Hence,  $\xi$  preserves the lengths of all curves in  $(P_a, P_b) = DP$ . By assumption, these lengths provide a local coordinate at  $\tau$ , so the deformation  $\xi$  is trivial (does not change the point of Teichmüller space). The contradiction shows that  $\xi$  contains a non-trivial twist along  $c'$ .

On the other hand, consider another deformation  $\eta$  of the initial hyperbolic structure  $\tau \in \mathcal{T}$ , where  $\eta$  preserves all lengths of curves from  $(P_a, P_b)$  except the length of  $c$ . A locus of points of  $\mathcal{T}$  obtained by  $\eta$  from  $\tau$  is a 1-dimensional curve in a neighborhood of  $\tau$ . This implies that  $\eta = \xi$ .

Suppose now that  $DP''$  also is not parameterizing at  $\tau$ . Similarly to the case of  $DP'$ , this implies that there exists a deformation  $\psi$  preserving all lengths of curves from  $P''_a$  and containing a non-trivial twist along the curve  $c'' \in P_a$ . Similarly to  $\xi$ , the deformation  $\psi$  should coincide with  $\eta$ , so,  $\xi = \psi$ . However, these two transformations do not coincide in the subsurface  $S_*$ : one twists along  $c'$ , another along  $c'' \neq c'$ . The contradiction shows that the double pants decomposition  $DP''$  is parameterizing at  $\tau$ .  $\square$

**Lemma 4.10.** *Let  $DP$  be a locally parameterizing double pants decomposition at  $\tau \in \mathcal{T}$ . Let  $f_0$  be a flip of  $DP$  such that the double pants decomposition  $DP^{(0)} = f_0(DP)$  contains no double curves. Then  $DP^{(0)}$  is a parameterizing double pants decomposition at  $\tau \in \mathcal{T}$ .*

**Proof.** Let  $DP = (P_a, P_b)$  be a parameterizing double pants decomposition at  $\tau \in \mathcal{T}$ . Let  $c \in DP$  be a curve flipped by  $f_0$ . Without loss of generality we may assume that  $c \in P_a$ . Denote  $m = 3g - 3 + n$ .

Consider an  $m$ -dimensional surface  $C_a$  through  $\tau \in \mathcal{T}$  such that the lengths of all curves contained in  $P_a \setminus P_b$  are constant in  $C_a$ . Let  $C_b$  be a similar surface for  $P_b$ . Denote by  $\Pi_a$  and  $\Pi_b$  the tangent planes to  $C_a$  and  $C_b$  in  $\tau$ . Let  $C_{\partial S}$  be an  $n$ -dimensional surface through  $\tau$  such that all curves contained in  $\partial S$  have constant lengths in  $C_{\partial S}$ , let  $\Pi_{\partial S}$  be the corresponding tangent plane. Since  $DP = (P_a, P_b)$  is parameterizing at  $\tau$ , the planes  $\Pi_a$ ,  $\Pi_b$  and  $\Pi_{\partial S}$  intersect each other in  $\tau$  only (and span the whole tangent space at  $\tau$ ).

Let  $\psi_i$ ,  $i = 1, \dots, m$ , be the curves in  $\mathcal{T}$  on which all lengths of curves of  $DP$  are preserved except for the length of one curve  $b_i \in P_b \setminus P_a$ . Let  $\bar{b}_1, \dots, \bar{b}_m$  be the tangent vectors to  $\psi_1, \dots, \psi_m$  at  $\tau$ . Clearly, the plane  $\Pi_a$  is spanned by the vectors  $\bar{b}_1, \dots, \bar{b}_m$ .

Now, consider a series of flips  $f_i$  of the curve  $c \in P_a$  (including the flip  $f_0$  described in the lemma): we will assume that the flip  $f_i$  takes  $c$  to the curves  $c_i$  of the same homology class; moreover, we assume that  $c_{i+1}$  may be obtained from  $c_i$  by a Dehn twist along  $c$ . For each of the flips  $f_i$  we denote  $P_a^{(i)} = f_i(P_a)$ . Denote by  $\Pi_a^{(i)}$  the tangent planes at  $\tau$  to the surfaces of the constant lengths of curves from  $P_a^{(i)} \setminus P_b$ .

If the double pants decomposition  $DP^{(0)} = (P_a^{(0)}, P_b)$  is parameterizing at  $\tau$ , then there is nothing to prove. So, suppose that  $DP^{(0)}$  is not parameterizing at  $\tau$ . By Lemma 4.9, this implies that all other double pants decompositions  $DP^{(i)} = (P_a^{(i)}, P_b)$  are parameterizing at  $\tau$  (with possible exclusion of at most one decomposition  $DP^{(j)}$ : at most one of these decompositions may contain a double curve  $c_i$ ). Reasoning as above with  $\Pi_a$ , we show that the plane  $\Pi_a^{(i)}$  is spanned by  $b_1, \dots, b_m$ . This implies that for  $i \notin \{0, j\}$  all planes  $\Pi_a^{(i)}$  coincide with  $\Pi_a$ .

Now, our aim is to show that  $\Pi_a^{(0)} = \Pi_a$ . Let  $t_c$  be a Dehn twist along  $c$ . The twist  $t_c$  takes  $c_i$  to  $c_{i+1}$ . On the other hand,  $t_c$  acts on  $\mathcal{T}$  and takes  $\Pi_a^i$  to  $\Pi_a^{i+1}$ . Since  $\Pi_a^i = \Pi_a$  for  $i \notin \{0, j\}$ ,  $t_c$  preserves  $\Pi_a$ . Hence,  $\Pi_a^i = \Pi_a$  for all  $i \in \mathbb{Z}$ .

Since  $\Pi_a^{(0)} = \Pi_a$ , the planes  $\Pi_a^{(0)}$ ,  $\Pi_b$  and  $\Pi_{\partial S}$  span the tangent space at  $\tau$ , which implies that  $DP^{(0)} = (P_a^{(0)}, P_b)$  is a parameterizing double pants decomposition at  $\tau$ .  $\square$

**Theorem 4.11.** *Let  $DP$  be an admissible double pants decomposition without double curves. Then  $DP$  together with the ordered set of lengths  $l(DP) = \{l(c_i) \mid c_i \in DP\}$  is a local coordinate in  $\mathcal{T} \setminus Z(DP')$  for some special double pants decomposition  $DP'$ .*

**Proof.** By Lemma 4.8 there exists a special double pants decomposition  $DP' = (P'_a, P'_b)$ , and a sequence  $\psi$  of flips taking  $DP$  to  $DP'$  and producing no double curves on its way. By Lemma 4.5 the lengths  $l(DP')$  form a local coordinates in  $\mathcal{T} \setminus Z(DP')$ . By Lemma 4.10 each of the flips in the sequence  $\psi$  preserve the parameterizing property of double pants decomposition (i.e. the obtained decomposition provides a local parameter in  $\mathcal{T} \setminus Z(DP')$ ). Hence,  $DP$  is parameterizing in  $\mathcal{T} \setminus Z(DP')$ .  $\square$

## 5. Finite number of choices

In Section 4, we proved that the set of functions  $l(DP)$  is a local parameter in almost all points of  $\mathcal{T}$ . In this section, we prove that the functions  $l(DP)$  determine the point of  $\mathcal{T}$  up to finitely many choices.

Consider an universal covering  $\pi$  of  $S$  by a hyperbolic plane  $\mathbb{H}^2$ , so that  $S = \mathbb{H}^2/G$  where  $G \in SL_2(\mathbb{R})$  is some finitely generated discrete group. Let  $M_1, \dots, M_s$  be a finite set of matrices generating  $G$ . Let  $r_1(M_1, \dots, M_s) = \dots = r_n(M_1, \dots, M_s) = E$  be the defining relations, where  $r_i$  is a word in the alphabet  $A = \{M_1, M_1^{-1}, \dots, M_s, M_s^{-1}\}$ .

For each closed geodesic  $c \subset S$  each connected component of the preimage  $\pi^{-1}c$  is a line (denote it by  $L_i(c)$ , where integer index stays to emphasize that there are countably many of these preimages). The group  $G$  contains a hyperbolic transformation  $\gamma(c)$  shifting  $\mathbb{H}^2$  along  $L_i(c)$  for the distance equal to  $l(c)$ . So, we have

$$\text{tr}(\gamma(c)) = 2 \cosh(l(c)/2). \quad (1)$$

Notice that  $\gamma(c) = w(M_1, \dots, M_s)$  for some word  $w$  in the same alphabet  $A$ . So, Eq. (1) may be considered as a finite set of polynomials in matrix elements of  $M_1, \dots, M_s$  with coefficients

$$\hat{l}(c) = 2 \cosh(l(c)/2).$$

**Theorem 5.1.** *Let  $DP$  be an admissible double pants decomposition containing no double curves. Then  $l(DP)$  determines a point of  $\mathcal{T}$  up to finitely many choices.*

**Proof.** For each of the curves  $c_i \in DP$  we consider one of its preimages on  $\mathbb{H}^2$  together with the hyperbolic transformation  $\gamma(c_i)$ . Taking in account Eq. (1), we obtain a system of polynomial equations in elements of  $M_i$ : the system consists of the equations arising from the following three sources:

- (1)  $M_i \in SL_2(\mathbb{R})$ ;
- (2)  $r_j(M_1, \dots, M_s) = E$ , where  $r_k$  is one of the defining relations;
- (3)  $\text{tr}(\gamma(c)) = \hat{l}(c)$ .

The matrix equations of the second type are considered as four scalar equations in matrix elements. Notice, that the equations of all three types are polynomial (here we use the fact that  $M_i \in SL_2(\mathbb{R})$ , and hence, all elements of  $M_i^{-1}$  are also elements of  $M_i$ ). So, the three types compose a system of finitely many polynomial equations in matrix elements of  $M_i$  with integer coefficients and constant terms in  $\mathbb{Z} \cup \{\hat{l}(c_1), \dots, \hat{l}(c_m)\}$ . Suppose in addition that the values of  $(\hat{l}(c_1), \dots, \hat{l}(c_m))$  correspond to at least one hyperbolic structure  $\tau \in \mathcal{T}$  on  $S$ . Then the system of equations is solvable. On the other hand, Theorem 4.11 implies that the system is non-generate. Thus, there are finitely many solutions of this system.

In other words, for each set of values  $l(DP)$  we can write a unique set of values  $\hat{l}(DP) = \{\hat{l}(c_1), \dots, \hat{l}(c_m)\}$ ; for this set  $\hat{l}(DP)$  there are finitely many possible values of matrix elements of  $M_i$ . So, for each value of  $l(DP)$  there are finitely many distinct points in  $\mathcal{T}$ .  $\square$

**Corollary 5.2.** Let  $DP$  be an admissible double pants decomposition of  $S$  without double curves. Let  $c \in S$  be a closed curve. Then  $\hat{l}(c)$  is an algebraic function of  $\hat{l}(DP)$ .

**Proof.** By [Theorem 5.1](#) the value of  $l(DP)$  determines the point of  $\mathcal{T}$  up to finitely many choices. Each of these choices correspond to a unique (modulo conjugation) discrete subgroup  $G \in SL_2(\mathbb{R})$  acting on  $\mathbb{H}^2$ . Consider the group  $G$  for one of these possibilities.

Following the proof of [Theorem 5.1](#) consider a preimage  $L(c)$  of  $c$  in  $\mathbb{H}^2$  and a hyperbolic transformation  $\gamma(c)$  which shifts along  $L(c)$  by the distance  $l(c)$ . Then  $\gamma(c) = w(M_1, \dots, M_s)$  where  $w$  is a word in the alphabet  $\{M_i, M_i^{-1} \mid i = 1, \dots, s\}$ . So,  $\hat{l}(c) = \text{tr } \gamma(c)$  is a polynomial in the matrix elements of  $M_1, \dots, M_s$ . Since the elements of matrices  $M_i$  are the solution of a system of polynomial equations, these elements are algebraic functions of  $\hat{l}(DP)$ . This implies, that  $\hat{l}(c)$  is an algebraic function of  $\hat{l}(DP)$  either.  $\square$

## 6. An atlas on the Teichmüller space

In [Section 4.3](#), we proved that for each admissible double pants decomposition  $DP$  the function  $l(DP)$  provides a local coordinate in neighborhoods of almost all points in  $\mathcal{T}$  (more precisely, away from a set of measure 0 formed by a finite union of hypersurfaces). In this section, we show that the coordinate charts with coordinates  $l(DP)$  compose an atlas on  $\mathcal{T}$ . Moreover, the transition functions between the adjacent chart change exactly one coordinate (and correspond to flips and handle twists of double pants decompositions).

**Lemma 6.1.** Let  $S$  be a surface with a fixed hyperbolic structure. Let  $DP = (P_a, P_b)$  be a special double pants decomposition with a standard part  $P_b$ . Let  $a_i, b_i \in DP$  be a pair of conjugate curves in  $DP$ . Let  $b_j \in P_b$  be a curve such that  $b_j \cap a_i \neq \emptyset$  and let  $t_{b_j}$  be a Dehn twist along  $b_j$ . If  $a_i$  is orthogonal to  $b_i$  then  $t_{b_j}^k(a_i)$  is not orthogonal to  $b_i$  for all  $k \in \mathbb{Z} \setminus 0$ .

**Proof.** First, notice that if  $i = j$  than there is nothing to prove (the statement follows than from [Lemma 3.4](#) in case of handle-conjugate curves and from [Lemma 3.6](#) in case of flip-conjugate curves). From now on we assume  $i \neq j$ .

Notice that by construction of special decompositions, the condition  $b_j \cap a_i \neq \emptyset$  implies that  $(a_i, b_i)$  cannot be a pair of handle-conjugate curves. So,  $(a_i, b_i)$  is a pair of flip-conjugate curves, and the curve  $b_i$  is homologically trivial. Suppose  $b_i$  is orthogonal to  $a_i$  as well as to  $t_{b_j}^k(a_i)$ , where  $k \neq 0$ . Since  $b_i$  is homologically trivial,  $b_i$  cuts  $S$  into two connected components  $S_1$  and  $S_2$ . Let  $S_1$  be the component containing the curve  $b_j$ . Denote  $s = a_i \cap S_2$  and  $s' = t_{b_j}^k(a_i) \cap S_2$ . In view of [Lemma 3.6](#) all ends of  $s$  and  $s'$  are orthogonal to  $b_i$ .

Since  $b_j \in S_1$ , and  $b_j \cap b_i = \emptyset$ , the topology of the decomposition of  $S_2$  is not changed by  $t_{b_j}$  (however, geometrically  $s \neq s'$ ). This implies that there exists an isotopy  $\gamma_x$  of  $s$  to  $s'$  (where  $x \in [0, 1]$ ,  $\gamma_0 = s$ ,  $\gamma_1 = s'$ ) such that the ends of the segment  $\gamma_x(s)$  belong to  $b_i$ . Notice that  $s$  cannot intersect  $s'$ , otherwise the segments  $s, s'$  and a part of  $b_j$  bound a hyperbolic triangle with two right angles  $b_js$  and  $b_js'$ , which is impossible. On the other hand, if  $s \cap s' = \emptyset$  then two parts of  $b_j$ ,  $s$  and  $s'$  bound a hyperbolic quadrilateral with four right angles, which is also impossible. The contradiction shows the lemma.  $\square$

**Lemma 6.2.** For each point  $\tau \in \mathcal{T}$  there exists a double pants decomposition  $DP_\tau$  such that  $l(DP_\tau)$  is a local coordinate in a neighborhood of  $\tau$ .

**Proof.** Consider an arbitrary special double pants decomposition  $DP = (P_a, P_b)$  with a standard part  $P_b$ . By [Theorem 4.11](#),  $l(DP)$  is a local coordinate in  $\mathcal{T} \setminus Z(DP)$ . So, if  $\tau \notin Z(DP)$  then there is nothing to prove. Suppose that  $\tau \in Z(DP)$ , i.e. there exists an orthogonal conjugate pair of curves  $a_i, b_i \in DP$ , (a pair of conjugate curves such that  $a_i$  is orthogonal to  $b_i$  in  $\tau$ ). We will apply to  $DP$  a twist  $t_{b_j}$  in some of the curves  $b_j \in DP$  in order to reduce the number of orthogonal conjugate pairs.

To see that it is always possible, suppose that  $a_i, b_i \in DP$  is an orthogonal conjugate pair. In this case there exists an integer  $k$  such that the special decomposition  $t_{b_i}^k(DP)$  contains less orthogonal conjugate pairs than  $DP$  has (the pair  $a_i, b_i$  of this twisted decomposition is not orthogonal for each  $k \neq 0$ , [Lemma 6.1](#) implies that for all but finitely many values of  $k$  the  $k$ -th degree of the twist will not produce new orthogonalities for other conjugate pairs).  $\square$

[Lemma 6.2](#) shows that the charts with coordinates  $l(DP)$  cover the space  $\mathcal{T}$ . Now, we consider the transition functions between the charts. In view of [Theorem 1.9](#), it is natural to choose these transition functions as ones induced by flips and handle-twists of admissible double pants decompositions.

The case of flip is considered in [Lemma 4.10](#): it is shown that as long as a flip  $f$  produces no double curves,  $f$  preserves the locus of points where the set of functions  $l(DP)$  is a local coordinate. We have also shown in [Lemma 4.8](#) that if  $DP$  and  $DP'$  are two double pants decompositions containing no double curves and  $DP$  can be turned into  $DP'$  by a sequence of flips, than one can choose this sequence of flips so that no double curves are produced on the way.

It is impossible to treat handle-twists directly in the same way: by definition no handle-twist can be applied to a double pants decomposition containing no double curves. To overcome this obstacle, we introduce the notion of a *quasi-handle-twist*.

**Definition 6.3** (*Quasi-handle-twist*). Let  $DP$  be a double pants decomposition without double curves. Let  $c \in DP$  be a curve such that there exists a flip  $f(c)$  producing a handle  $\mathfrak{h}$  in the decomposition  $f(DP)$  (so that  $f(c)$  is a double curve which cuts out the handle). Let  $a \in DP \cap f(DP)$  be a curve contained in the handle  $\mathfrak{h}$ . By a *quasi-handle-twist*  $t_a$  of  $DP$  we mean a Dehn twist along  $a$ .

**Remark 6.4.** The quasi-handle-twist  $t_a$  may be written as  $t_a = f^{-1} \circ \hat{t}_a \circ f$ , where  $f$  is a flip as in Definition 6.3 and  $\hat{t}_a$  is a handle twist in the handle  $\mathfrak{h}$ .

**Remark 6.5.** Since  $t_a$  is a Dehn twist,  $t_a$  acts on the Teichmüller space  $\mathcal{T}$ . We denote by  $t_a(\tau)$  the point of  $\mathcal{T}$  obtained from  $\tau$  by the Dehn twist  $t_a$ .

Now, we will prove the counterparts to the Lemmas 4.10 and 4.8 for the case of quasi-handle-twists.

The next lemma follows immediately from Definition 6.3 and Remarks 6.4 and 6.5.

**Lemma 6.6.** Let  $DP$  be an admissible double pants decomposition without double curves. Let  $\tau \in \mathcal{T}$  be a point such that  $l(DP)$  is a local coordinate in  $\tau$ . Let  $t$  be a quasi-handle-twist along the curve  $c \in DP$ . Then  $l(t(DP))$  is a local coordinate in  $\tau' = t(\tau)$ .

**Lemma 6.7.** Let  $DP$  and  $DP'$  be two admissible double pants decompositions containing no double curves. Then there exists a sequence of flips and quasi-handle-twists which takes  $DP$  to  $DP'$  and produces no double curves on its way.

**Proof.** By Theorem 1.9 there exists a sequence  $\psi$  of flips and handle-twists taking  $DP$  to  $DP'$ . In view of Lemma 4.7, each subsequence containing no handle-twist may be realized without producing double curves. It is sufficient to prove the lemma for the case when  $\psi$  contains one handle-twist only (and then apply inductional reasoning). Suppose that this unique handle-twist  $\hat{t}_c$  is a twist in a curve  $c \in DP^{st}$  where  $DP^{st}$  is a standard double pants decomposition flip-equivalent to  $DP$  (it is shown in [1, Lemma 4.1] that handle-twists in standard decompositions are sufficient for obtaining the transitivity theorem).

Let  $DP^{st} = (P_a^{st}, P_b^{st})$  be the two parts, suppose that  $c \in P_a^{st}$ . Let  $DP^{sp} = (P_a^{sp}, P_b^{sp})$  be a special decomposition with the standard part  $P_b^{sp} = P_b^{st}$ . By Lemma 4.7, there exists a sequence of flips taking  $DP$  to  $DP^{sp}$  without producing double curves. Then we apply a quasi-handle-twist  $t_c$  in  $c$ , so that we obtain another special decomposition  $DP_*^{sp}$ . In view of Remark 6.4,  $DP_*^{sp}$  is flip-equivalent to  $DP'$ . The sequence of flips taking  $DP_*^{sp}$  to  $DP'$  without producing double curves does exist in view of Lemma 4.7.  $\square$

Summarizing results of Lemmas 6.2, 6.7 and Corollary 5.2 we obtain the following theorem.

**Theorem 6.8.** (1) The charts  $\mathbb{C}(DP)$  with coordinates  $l(DP)$ , where  $DP$  is an admissible double pants decomposition without double curves, provide an atlas on Teichmüller space  $\mathcal{T}$ .

(2) The elementary transition functions of these charts are induced by flips and quasi-handle-twists of double pants decompositions, each elementary transition function changes only one coordinate. This unique non-trivial transition function is algebraic.

(3) The compositions of elementary transition functions act transitively on the charts.

## 7. Deligne–Mumford compactification of moduli space

In Section 6, we showed that the Teichmüller space is covered by coordinate charts arising from admissible double pants decompositions. Since local coordinates on Teichmüller space are also local coordinates on the moduli space, the charts with coordinate  $l(DP)$  also compose an atlas on the moduli space. In this section, we show that this atlas works also for most strata in Deligne–Mumford compactification of the moduli space.

Consider some Fenchel–Nielsen coordinates  $FN(P)$  on the Teichmüller space

$$\mathcal{T} = \{l(c_i) > 0, \alpha(c_j) \in \mathbb{R} \mid c_i, c_j \in P, c_j \notin \partial S\}.$$

Given a pants decomposition  $P$  denote

$$\mathcal{T}_P = \{l(c_i) \geq 0, \alpha(c_j) \in \mathbb{R} \mid c_i, c_j \in P, c_j \notin \partial S\}.$$

The augmented Teichmüller space  $\overline{\mathcal{T}}$  is the following closure of  $\mathcal{T}$ :

$$\bar{\mathcal{T}} = \bigcup_P \mathcal{T}_P$$

where the union is taken by all pants decompositions of the surface. The points of  $\bar{\mathcal{T}} \setminus \mathcal{T}$  correspond to *nodal surfaces*, i.e. to the surfaces with *nodal singularities*: a nodal singularity arises when a non-trivial closed curve  $c$  in  $S$  is degenerated to a point (i.e.  $l(c) \rightarrow 0$ ). A nodal surface is not a surface: a neighborhood of a nodal point is not homeomorphic to a disk. We denote by  $N$  the set of all nodal points on the nodal surface. It is known that  $\bar{\mathcal{T}}/\text{Mod} = \bar{\mathcal{M}}$ , where  $\text{Mod}$  is the modular group and  $\bar{\mathcal{M}}$  is the Deligne–Mumford compactification of the modular space  $\mathcal{M} = \mathcal{T}/\text{Mod}$ .

The space  $\bar{\mathcal{T}}$  inherits topology from  $\bigcup_P T_p = \bigcup_{p \geq 0} (\mathbb{R}^{3g-3+2n} \times \mathbb{R}^{3g-3+n})$ .

Given an admissible double pants decomposition  $DP$  without double curves, we say that the *boundary of the chart*  $\mathcal{C}(DP)$  is the locus of points  $\tau' \in \bar{\mathcal{T}}$  where  $l(c) = 0$  for at least one  $c \in DP$ .

**Theorem 7.1.** *For each point  $\tau' \in \bar{\mathcal{T}}$  there exists an admissible double pants decomposition  $DP$  containing no double curves and such that  $\tau'$  belongs to the boundary of the chart  $\mathcal{C}(DP)$  with coordinates  $l(DP)$ .*

**Proof.** If  $\tau' \in \mathcal{T}$ , then there is nothing to prove in view of [Theorem 6.8](#). Suppose that  $\tau' \in (\bar{\mathcal{T}} \setminus \mathcal{T})$ . Then there exists a set of mutually disjoint curves  $C$  on  $S$  such that the surface  $S'$  corresponding to  $\tau'$  is obtained by contracting all curves  $c_i \in C$ . It is sufficient to show that there exists an admissible double pants decomposition  $DP$  containing no double curves and such that  $C \in DP$ .

Consider any pants decomposition  $P_a$  containing the set  $C$ . We will build the required decomposition  $DP = (P_a, P_b)$  in the following four steps: first, we transform  $P_a$  by a sequence of flips to a standard decomposition  $P'_a$ ; second, we build a standard double pants decomposition  $(P'_a, P'_b)$ ; next, we transform  $P'_a$  back to  $P_a$  by flips; finally, we apply (if necessary) several flips to  $P'_b$  to avoid double curves.  $\square$

Factorizing by the modular group  $\text{Mod}$  we obtain the charts on the Deligne–Mumford compactification of the modular space (with the natural notion of the *boundary of the chart* on  $\bar{\mathcal{M}}$  defined as the boundary of the same chart on  $\bar{\mathcal{T}}$  factorized by  $\text{Mod}$ ). Applying the same reasoning as in [Theorem 7.1](#) we obtain the following corollary.

**Corollary 7.2.** *For each point  $\tau' \in \bar{\mathcal{M}}$  there exists an admissible double pants decomposition  $DP$  containing no double curves and such that  $\tau'$  belongs to the boundary of the chart  $\mathcal{C}(DP)$  with coordinates  $l(DP)$ .*

**Remark 7.3.** For many of the points  $\tau' \in \partial \bar{\mathcal{T}}$  the coordinates  $l(DP)$  provide also a chart in a neighborhood  $O'(\tau') = O(\tau') \cap \partial \bar{\mathcal{T}}$  where  $O(\tau')$  is some neighborhood of  $\tau'$  in  $\bar{\mathcal{T}}$ . It would be natural to try to cover  $\partial \bar{\mathcal{T}}$  (respectively the whole boundary of  $\bar{\mathcal{M}}$ ) by these charts. However, in general it turns to be impossible (see [Remark 7.14](#)).

Below, we define a large subset of “good” points in the boundary and show that all points of this subset are covered by the charts  $\mathcal{C}(DP)$ .

The boundary  $\partial \bar{\mathcal{T}}$  is stratified: given a set  $C$  of mutually non-intersecting curves in  $S$ , a stratum  $\mathcal{S}_C$  is a locus  $\{l(c_i) = 0 \mid c_i \in C\}$ . All nodal surfaces (of genus  $g$  with  $n$  boundary parts) with  $k$  nodal singularities compose a union  $\mathcal{S}_{2k}$  of codimension  $2k$  strata.

By a pants decomposition of a surface  $S$  with punctures we mean a decomposition into generalized pairs of pants (where a generalized pair of pants is either a sphere with three holes, or a sphere with two holes and a puncture, or a sphere with a hole and two punctures, or a sphere with three punctures).

By a *pants decomposition* (respectively, *double pants decomposition*) of a nodal surface  $S$  we mean a set of curves  $P$  composing a pants decomposition (respectively double pants decomposition) in all components of  $S$  (connected components of  $S \setminus N$ ). The nodal points are not considered as curves of the pants decomposition.

A (double) pants decomposition of  $S$  is *standard* if the decompositions of all components are standard. Similarly, a double pants decomposition is *special* if decompositions of all components are special.

Let  $DP = (P_a, P_b)$  be a double pants decomposition of  $S$  containing no double curves. Let  $c \in DP$ . Denote by  $S'$  the nodal surface obtained from  $S$  by collapsing  $c$  to a nodal singularity. Consider a set  $DP'$  of curves on  $S'$  obtained as a union of images of curves of  $DP$  which do not intersect  $c$ . Notice that  $DP'$  is not necessarily a double pants decomposition of  $S'$ ; however, if  $c \in P_b$  intersects only one other curve of  $DP$  then  $DP'$  is.

**Lemma 7.4 (Collar lemma).** (See [9].) *Let  $c \in S$  be a simple closed geodesic on hyperbolic surface  $S$  of lengths  $l = l(c)$ . Define  $w$  by the relation*

$$\sinh l \sinh w = 1.$$

*Then  $S$  contains a collar  $\text{Col}(c)$  of width  $w$  defined by  $\text{Col}(c) = \{x \in S \mid \rho_S(x, c) < w/2\}$ , where  $\rho_S(A, B)$  is the distance in  $S$  from the set  $A$  to the set  $B$ .*

It follows immediately from the Collar lemma that if  $a, b \in S$  are closed geodesics  $b \cap a \neq \emptyset$  then contracting  $a$  so that  $l(a) \rightarrow 0$  implies  $l(b) \rightarrow \infty$ .

The Collar lemma implies that the local coordinates  $l(DP)$  degenerate while the curves  $c_i$  are collapsing: if  $a_i \cap c_i \neq \emptyset$  then  $l(a_i) \rightarrow \infty$  while  $c_i \rightarrow 0$  (the curve  $a_i$  intersecting  $c_i$  do exists since  $c_i \notin P_a$  and  $P_a$  is a maximal set of disjoint curves in  $S$ ). For the case  $C \subset DP$  we define the new set of functions  $\tilde{l}(DP, C)$  as follows:

$$\tilde{l}(DP, C) = \left\{ l(c_i), \frac{1}{l(c_j)} \mid c_i \in C, c_j \in DP \setminus C \right\}.$$

Clearly,  $\tilde{l}(DP, C)$  is a local coordinate in all points of  $T$ , where  $l(DP)$  is a local coordinate. Moreover, this set of functions remains correctly defined while the curves of the set  $C$  are collapsed.

**Definition 7.5 (Inversion).** An *inversion* of a  $k$ -th function of  $\tilde{l}(DP, C)$  is an exchange of  $l(c_k)$  or  $\frac{1}{l(c_k)}$  (where  $c_k \in DP$ ) by  $\frac{1}{l(c_k)}$  or  $l(c_k)$  respectively.

It is clear the transformation from a set of functions  $\tilde{l}(DP, C)$  to any other set of functions  $\tilde{l}(DP', C')$  may be obtained as a composition of inversions and transformations induced by flips and quasi-handle-twists of double pants decompositions (here  $DP$  and  $DP'$  are admissible double pants decompositions containing no double curves,  $C$  and  $C'$  are sets of disjoint curves).

**Definition 7.6 (Strong and weak curves).** Let  $C = \{c_1, \dots, c_k\}$  be a set of mutually disjoint curves on  $S$ . Each curve  $c \in C$  appears two times in the boundary of  $S \setminus C$ . We say that  $c$  is a *strong curve* of  $C$  if two copies of  $c$  appear in two different connected components of  $S \setminus C$ . Otherwise, we say that  $c$  is *weak*.

We denote by  $C_{\text{strong}} \subset C$  the subset of all strong curves.

We denote by  $S^1, \dots, S^l$  the connected components of  $S \setminus C$ . By  $\hat{S}^i$  we denote the connected component of  $S \setminus C_{\text{strong}}$  corresponding to the component  $S^i$  of  $S \setminus C$  ( $\hat{S}^i$  is obtained from  $S^i$  by gluing along the pairs of boundary components arising from the weak curves).

**Definition 7.7 (Good set of curves).** We say that a set  $C = \{c_1, \dots, c_k\}$  of mutually disjoint curves on  $S$  is *good* if each connected component  $\hat{S}^i$  of  $S \setminus C_{\text{strong}}$  is either a surface of positive genus or has at least two boundary components contained in  $\partial S$ .

Let  $\mathcal{S}_{\text{good}}$  be a union of all strata  $\mathcal{S}_C$  where  $C$  is a good set.

**Remark 7.8.** It is easy to see that  $\mathcal{S}_2 \subset \mathcal{S}_{\text{good}}$ , where  $\mathcal{S}_2$  is a union of all codimension 2 strata.

**Lemma 7.9.** Let  $C = \{c_1, \dots, c_k\}$  be a good set of curves. Then there exists a special double pants decomposition  $DP$  of  $S$  such that  $C \subset DP$  and each curve  $c_i \in C$  is intersected by a unique curve of  $DP \setminus C$ . Moreover, collapsing any curve  $c_i \in C$  to a nodal singularity leads to a special double pants decomposition of the obtained nodal surface.

**Proof.** We build a special double pants decomposition  $DP = (P_a, P_b)$  with a standard part  $P_b$  such that  $P_b$  contains all strong curves of  $C$  and  $P_a$  contains all weak curves of  $C$ . We construct the decomposition  $DP$  separately for each connected component  $\hat{S}^i$  of  $S \setminus C_{\text{strong}}$ .

If  $\hat{S}^i$  is a sphere with holes, then we build the decomposition  $DP$  as shown in Fig. 9: since  $C$  is a good set of curves, at least two boundary components of  $\hat{S}^i$  do not belong to  $C$  (the two bottom boundary components in the figure). In Fig. 9(a) we show the part  $P_a$  of  $DP$ , in Fig. 9(b) we show the whole decomposition  $DP = (P_a, P_b)$ , notice that each curve of  $C$  is intersected by a unique curve of  $P_b$ .

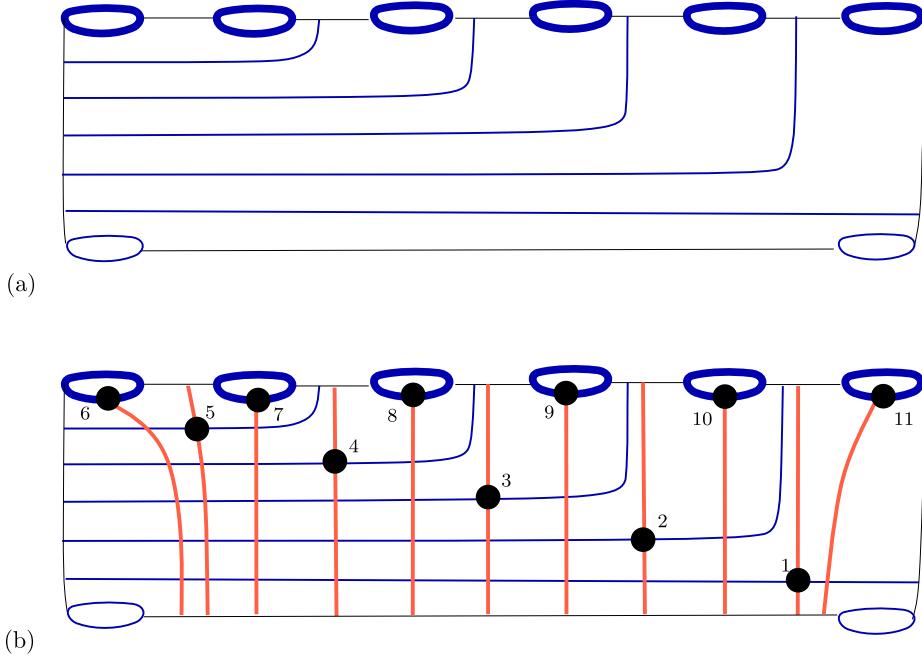
Now, suppose that  $S^i$  contains at least one handle.

First, we build a standard pants decomposition  $P$  containing the set  $C$ . To do this for the component  $\hat{S}^i$ , we build a standard decomposition with a linear structure as in Fig. 10(a): first come all handles than come all holes. Moreover, for each strong curve  $c_j \in \hat{S}^i$  the curve  $c_j$  is contained inside one of the handles (more precisely, first we build the curves  $\tilde{c}_j \in S^i$  which together with both copies of  $c_j$  bounds a pair of pants in  $S^i$ , then in  $\hat{S}^i$  the curve  $\tilde{c}_j$  cuts out a handle  $h_j$  containing  $c_j$ ).

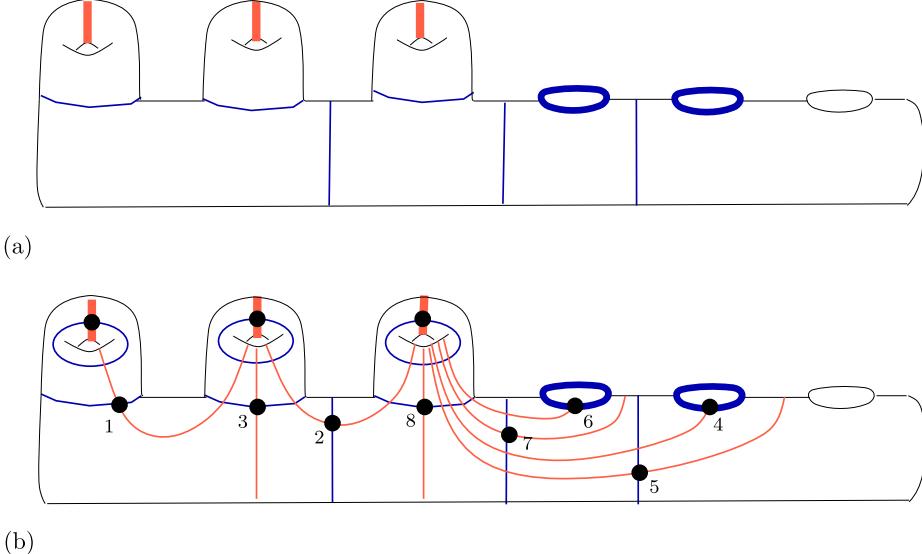
Next, we build the standard part  $P_b$  of the special decomposition  $DP = (P_a, P_b)$ : we take the standard decomposition  $P$  and for each handle  $h_j$  of  $P$  we substitute the curve  $c_j \in P \cap C$  by any other curve  $c'_j \in h_j$  such that  $|c_j \cap c'_j| = 1$  (in the handles containing no curves of  $C$  we do nothing).

Now, we build the part  $P_a$  of the special decomposition  $DP = (P_a, P_b)$ . We build the restriction of  $P_a$  to  $\hat{S}^i$  as it is shown in Fig. 10(b): namely, each of the weak curves  $c_j \in C$  is intersected only by a unique curve of  $P_a$  lying in the same handle as  $c_j$ ; each of the strong curves is intersected only by a curve passing through the handle  $h_0$ .

The obtained decomposition  $DP$  is special: it may be transformed to a standard decomposition by a sequence of flips as shown in Figs. 9 and 10 (we show the order of flips by numbering the intersection points of conjugate curves). It is easy



**Fig. 9.** Special double pants decomposition containing  $C$ : case  $\hat{S}^i = S_{0,r}$ . The curves of  $C$  are bold, each intersects a unique other curve of  $DP$ . The figure shows only the front part of the surface, the decomposition of the back part is the same. The black nodes show the intersections of the conjugate curves.

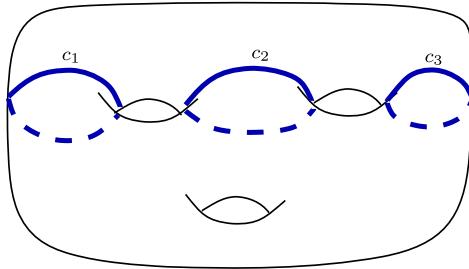


**Fig. 10.** Special double pants decomposition containing  $C$  (the curves of  $C$  are bold, each intersects a unique other curve of  $DP$ ). The figure shows only the front part of the surface, the decomposition of the back part is the same. The black nodes show the intersections of the conjugate curves.

to see that collapsing any curve  $c_i \in C$  to a point we get a special double pants decomposition  $DP'$  of the obtained nodal surface: the sequence of flips taking  $DP'$  to a standard decomposition almost coincide with the corresponding sequence for  $DP$  (the only difference is that in case of strong curve  $c_i$  one needs to omit the flip in the curve conjugated to  $c_i$ ).  $\square$

Let  $DP$  be a special double pants decomposition of  $S$ , let  $C \in DP$  be a good set of curves. Denote by  $\bar{Z}(DP, C)$  the locus in  $\bar{\mathcal{T}}$  where at least one of the conjugate pairs of  $DP \setminus C$  is an orthogonal pair.

**Remark 7.10.** Let  $(a_i, b_i)$  be a conjugate pair of  $DP$  and let  $b_i \in C$ . It is easy to see that while  $b_i$  is collapsed, the angle formed up by  $a_i$  and  $b_i$  tends to the right angle (if lengths of other curves of  $P_b$  remain fixed). This implies that  $\mathcal{S}_C$  belongs



**Fig. 11.** Example of the stratum not covered by the atlas:  $S = S_{3,0}$ ,  $C = \{c_1, c_2, c_3\}$ .

to the closure of  $Z(DP)$  in  $\bar{\mathcal{T}}$ . Therefore, we cannot hope that the set of functions  $\tilde{l}(DP)$  will provide a local coordinate in the whole neighborhood of a given point  $\tau' \in \mathcal{S}_C$ .

Instead, we will show that for any point  $\tau' \in \mathcal{S}_C$  there exists a suitable special double pants decomposition  $DP$  such that  $\tilde{l}(DP)$  is a local coordinate in the neighborhood of  $\tau'$  in  $\mathcal{S}_C$  as well as a local coordinate in almost all points of the neighborhood of  $\tau'$  in  $\bar{\mathcal{T}}$  (more precisely,  $\tilde{l}(DP)$  is a local coordinate in  $O(\tau') \setminus Z(DP)$  where  $O(\tau')$  is a neighborhood of  $\tau'$  in  $\bar{\mathcal{T}}$ ).

This motivates the following definition:

**Definition 7.11** (*Almost chart*). Let  $C$  be a good set of curves, let  $\mathcal{S}_C \subset \bar{\mathcal{T}}$  be the corresponding stratum and let  $\tau' \in \mathcal{S}_C$  be a point. An *almost chart* centered at  $\tau'$  is a pair  $(O(\tau'), f)$  where  $O(\tau') \subset \bar{\mathcal{T}}$  is a neighborhood of  $\tau'$  and  $f = (f_1, \dots, f_k)$  is a set of  $k$  functions,  $k = \dim \mathcal{T} = 6g - 6 + 3n$  satisfying the following conditions:

- 1) the functions  $f$  are defined and continuous in  $O(\tau')$ ;
- 2)  $f$  is a local coordinate in a neighborhood  $O'(\tau') = O(\tau') \cap \mathcal{S}_C$ ;
- 3) there exists a finite set  $X$  of codimension 1 surfaces in  $\bar{\mathcal{T}}$  such that  $f$  is a local coordinate in a neighborhood of each point  $\tau \in O(\tau') \cap (\bar{\mathcal{T}} \setminus X)$ .

**Lemma 7.12.** Let  $S$  be a marked hyperbolic surface considered as a point of  $\mathcal{T} = \mathcal{T}(S)$ . Let  $S_{good} \subset \bar{\mathcal{T}}$  be a union of the good strata. Let  $S'$  be a nodal surface with nodal singularities, such that the marked hyperbolic structure  $\tau'$  of  $S'$  belongs to  $S_{good}$ .

Then there exists an admissible double pants decomposition  $DP$  of  $S$  which degenerates to an admissible double pants decomposition  $DP'$  of  $S'$  such that  $\tilde{l}(DP', C)$  provides an almost chart centered in  $\tau'$ .

**Proof.** Since  $S'$  belongs to  $S_{good}$ , the nodal surface  $S'$  is obtained from  $S$  by collapsing the curves contained in some good set  $C$ .

By Lemma 7.9 there exists a special double pants decomposition  $DP = (P_a, P_b)$  with standard part  $P_b$ , such that

- (1)  $c_i \in P_b$  for all strong curves  $c_i$  of  $C$ ;
- (2)  $c_i \in P_a$  for all weak curves  $c_i$  of  $C$ ;
- (3) for each  $c_i \in C$  the decomposition  $(P_a, P_b)$  contains a unique curve  $d_i$  intersecting  $c_i$ .

By Lemma 7.9 by collapsing a curve  $c_i \in C$  one obtains a special double pants decomposition of the obtained nodal surface, and, after collapsing all curves  $c_i \in C$ , we obtain a special double pants decomposition  $DP'$  of  $S'$ . Clearly, the set of functions  $\tilde{l}(DP, C)$  is defined and continuous in a neighborhood  $O'$  of  $\tau'$ .

Using Lemma 6.1 (as in the proof of Lemma 6.2) we may apply to  $DP$  several twists (along the curves of  $P_b$ ) so that the resulting special decomposition  $DP_* = t_{c_m}^{k_m} \circ \dots \circ t_{c_1}^{k_1}(DP)$  satisfies  $\tau' \notin \bar{Z}(DP'_*)$ .

Suppose that some of the twists  $t_{c_j}$  changes a curve  $c \in C$ . Then  $c \in P_a$ , so  $c$  is a weak curve of  $C$ . The curve  $c_j$  then is the curve conjugated to  $c$  in  $DP$ . Clearly, we may substitute a degree of the twist  $t_{c_j}$  by a degree of the twist  $t_c$  so that in the resulting double pants decomposition the images of curves  $c$  and  $c_j$  are not orthogonal to each other. So, after several substitutions we transform  $DP_*$  to a special decomposition  $DP_{**}$  such that  $\tau' \notin \bar{Z}(DP_{**})$  and  $C \in DP_{**}$ . This implies the conditions 2) and 3) of Definition 7.11. The condition 1) of the same definition holds for  $\tilde{l}(DP_{**}, C)$  in some neighborhood  $O'(\tau') \subset \bar{\mathcal{T}}$  trivially. Hence, the pair  $(O'(\tau'), \tilde{l}(DP_{**}, C))$  provides an almost chart centered at  $\tau'$ .  $\square$

Now, consider the moduli space  $\mathcal{M} = \mathcal{T}/\text{Mod}$ . A local chart in a neighborhood of  $\tau \in \mathcal{T}$  projects to a local chart in a neighborhood of  $\pi(\tau) \in \mathcal{M}$  (where  $\pi$  is a factorization by  $\text{Mod}$ ) unless  $\tau$  is a hyperbolic structure with non-trivial automorphism group, or, equivalently, unless  $\pi(\tau)$  is an orbifold point of  $\mathcal{M}$ . Composing this with Lemma 7.12 and Theorem 6.8 we obtain the following theorem:

**Theorem 7.13.** Let  $S$  be a nodal surface, let  $\mathcal{M}(S)$  be its moduli space and let  $\overline{\mathcal{M}}(S)$  be the Deligne–Mumford compactification of  $\mathcal{M}$ . Let  $\mathcal{S}_{\text{good}}^{\mathcal{M}} = \mathcal{S}_{\text{good}} / \text{Mod}$  be the union of good strata in  $\mathcal{M}$ . Let  $O$  be a locus of orbifold points of  $\mathcal{M}$ , let  $\overline{O}$  be the closure of  $O$  in  $\overline{\mathcal{M}}$ . Then

- (1) the charts with coordinates  $\tilde{l}(DP, C)$  provide an atlas on  $\mathcal{M} \setminus O$  and on  $\mathcal{S}_{\text{good}}^{\mathcal{M}} \setminus \overline{O}$ , (here  $C$  is a good set and  $DP$  is an admissible double pants decomposition without double curves);
- (2) each point  $\tau' \in \mathcal{S}_{\text{good}}^{\mathcal{M}} \setminus \overline{O}$  is covered by some almost chart  $(O'(\tau'), \tilde{l}(DP, C))$ ;
- (3) the elementary transition functions of these charts (almost charts) are inversions and transformations induced by flips and quasi-handle-twists of double pants decompositions; each elementary transition function change only one coordinate; this unique non-trivial transition function is algebraic;
- (4) the compositions of elementary transition functions act transitively on the union of charts and almost charts.

**Remark 7.14.** We do not claim that Definition 7.7 of the good strata exhaust all the points of  $\partial\overline{\mathcal{T}}$  (respectively  $\partial\overline{\mathcal{M}}$ ) covered by the almost charts of our atlas. However, some restrictions for the “good” points covered by the atlas are indispensable. For example, if  $S = S_{3,0}$  and  $C$  is a set of three curves cutting a pair of pants out of  $C$  (see Fig. 11) then it is possible to prove that in each admissible double pants decomposition  $DP$  such that  $C \in DP$  the set of curves  $\{c_i \in DP \setminus C, c_i \cap C \neq \emptyset\}$  contains more than three curves. Hence, after retracting the curves of  $C$ , any decomposition  $DP$  contains less curves (of finite non-zero length) than required. This implies that the points  $\tau' \in \mathcal{S}_C$  cannot be covered by any chart of our atlas.

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