

Computing α -Invariants of Singular del Pezzo Surfaces

Ivan Cheltsov · Dimitra Kosta

Received: 7 March 2012 / Published online: 11 October 2012
© Mathematica Josephina, Inc. 2012

Abstract We prove a new local inequality for divisors on surfaces and utilize it to compute α -invariants of singular del Pezzo surfaces, which implies that del Pezzo surfaces of degree one whose singular points are of type \mathbb{A}_1 , \mathbb{A}_2 , \mathbb{A}_3 , \mathbb{A}_4 , \mathbb{A}_5 , or \mathbb{A}_6 are Kähler-Einstein.

Keywords Fano · del Pezzo · Kähler-Einstein · α -invariant · Tian · du Val

Mathematics Subject Classification 14J45 · 14J26 · 32Q20

We assume that all varieties are projective, normal, and defined over \mathbb{C} .

1 Introduction

Let X be a Fano variety with at most quotient singularities (a Fano orbifold).

Theorem 1.1 [37] *If $\dim(X) = 2$ and X is smooth, then*

the surface X is Kähler-Einstein \iff the group $\text{Aut}(X)$ is reductive.

Communicated by Gang Tian.

This paper was completed under financial support provided by IKY (Greek State Scholarship Foundation).

I. Cheltsov (✉)
School of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, UK
e-mail: I.Cheltsov@ed.ac.uk

D. Kosta
e-mail: dimitrakosta@hotmail.com

An important role in the proof of Theorem 1.1 is played by several holomorphic invariants, which are now known as α -invariants. Let us describe their algebraic counterparts.

Let D be an effective \mathbb{Q} -divisor on the variety X . Then the number

$$c(X, D) = \sup\{\epsilon \in \mathbb{Q} \mid \text{the log pair } (X, \epsilon D) \text{ is log canonical}\} \in \mathbb{Q} \cup \{+\infty\}$$

is called the log canonical threshold of the divisor D (see [21, Definition 8.1]). Put

$$\text{lct}_n(X) = \inf\left\{c\left(X, \frac{1}{n}B\right) \mid B \text{ is a divisor in } |-nK_X|\right\}$$

for every $n \in \mathbb{N}$. For small n , the number $\text{lct}_n(X)$ is usually not very hard to compute.

Example 1.2 [28] If X is a smooth surface in \mathbb{P}^3 of degree 3, then

$$\text{lct}_1(X) = \begin{cases} 2/3 & \text{if } X \text{ has an Eckardt point,} \\ 3/4 & \text{if } X \text{ has no Eckardt points.} \end{cases}$$

The number $\text{lct}_n(X)$ is denoted by $\alpha_n(X)$ in [38].

Remark 1.3 It follows from [27, Lemma 4.8] that the set

$$\left\{c\left(X, \frac{1}{n}B\right) \mid B \text{ is a divisor in } |-nK_X|\right\}$$

is finite (cf. [23]). Thus, there exists a divisor $B \in |-nK_X|$ such that $\text{lct}_n(X) = c(X, B/n) \in \mathbb{Q}$.

If the variety X is smooth, then it is proved by Demailly (see [6, Theorem A.3]) that

$$\inf\{\text{lct}_n(X) \mid n \in \mathbb{N}\} = \alpha(X),$$

where $\alpha(X)$ is the α -invariant introduced by Tian in [36]. Put $\text{lct}(X) = \inf\{\text{lct}_n(X) \mid n \in \mathbb{N}\}$.

Conjecture 1.4 [38, Question 1] *There is an $n \in \mathbb{N}$ such that $\text{lct}(X) = \text{lct}_n(X)$.*

The proof of Theorem 1.1 uses (at least implicitly) the following result.

Theorem 1.5 [10, 36] *The Fano orbifold X is Kähler–Einstein if*

$$\text{lct}(X) > \frac{\dim(X)}{\dim(X) + 1}.$$

Note that there are many well-known obstructions to the existence of Kähler–Einstein metrics on smooth Fano manifolds and Fano orbifolds (see [14, 15, 25, 34]).

Example 1.6 If $X \cong \mathbb{P}(1, 2, 3)$, then X is not Kähler–Einstein (see [15, 34]).

Let us describe one more α -invariant that took its origin in [37].

Let \mathcal{M} be a linear system on the variety X . Then the number

$$c(X, \mathcal{M}) = \sup\{\epsilon \in \mathbb{Q} \mid \text{the log pair } (X, \epsilon\mathcal{M}) \text{ is log canonical}\} \in \mathbb{Q} \cup \{+\infty\}.$$

is called the log canonical threshold of the linear system \mathcal{M} (cf. [21, Theorem 4.8]).

Put

$$\text{lct}_{n,2}(X) = \inf\left\{c\left(X, \frac{1}{n}\mathcal{B}\right) \mid \mathcal{B} \text{ is a pencil in } |-nK_X|\right\}$$

for every $n \in \mathbb{N}$. The number $\text{lct}_{n,2}(X)$ is denoted by $\alpha_{n,2}(X)$ in [8] and [41]. Note that

$$\text{lct}(X) = \inf\{\text{lct}_{n,2}(X) \mid n \in \mathbb{N}\}, \tag{1}$$

and it follows from [21, Theorem 4.8] that $\text{lct}_n(X) \leq \text{lct}_{n,2}(X)$ for every $n \in \mathbb{N}$.

Remark 1.7 It follows from [27, Lemma 4.8] and [21, Theorem 4.8] that the set

$$\left\{c\left(X, \frac{1}{n}\mathcal{B}\right) \mid \mathcal{B} \text{ is a pencil in } |-nK_X|\right\}$$

is finite. Thus, there is a pencil \mathcal{B} in $|-nK_X|$ such that the equality $\text{lct}_{n,2}(X) = c(X, \mathcal{B}/n)$. Then

$$\text{lct}_{n,2}(X) > \text{lct}(X)$$

if there exist at most finitely many effective \mathbb{Q} -divisors D_1, D_2, \dots, D_r on the variety X such that

$$c(X, D_1) = c(X, D_2) = \dots = c(X, D_r) = \text{lct}(X)$$

and $D_1 \sim_{\mathbb{Q}} D_2 \sim_{\mathbb{Q}} \dots \sim_{\mathbb{Q}} D_r \sim_{\mathbb{Q}} -K_X$.

The importance of the number $\text{lct}_{n,2}(X)$ is due to the following conjecture.

Conjecture 1.8 (cf. [8, Theorem 2], [41, Theorem 1]) *Suppose that*

$$\text{lct}_{n,2}(X) > \frac{\dim(X)}{\dim(X) + 1}$$

for every $n \in \mathbb{N}$. Then X is Kähler–Einstein.

Note that Conjecture 1.8 is not much stronger than Theorem 1.5 by (1).

Example 1.9 Suppose that X is a smooth hypersurface in \mathbb{P}^m of degree $m \geq 3$. Then

$$\text{lct}_n(X) \geq 1 - \frac{1}{m} = \frac{\dim(X)}{\dim(X) + 1}$$

for every $n \in \mathbb{N}$ by [2]. The equality $\text{lct}_n(X) = 1 - 1/m$ holds \iff the hypersurface X contains a cone of dimension $m - 2$ (see [2, Theorem 1.3], [2, Theorem 4.1], [13, Theorem 0.2]). Then

$$\text{lct}_{n,2}(X) > \frac{\dim(X)}{\dim(X) + 1}$$

by Remark 1.7, [2, Remark 1.6], [2, Theorem 4.1], [2, Theorem 5.2], and [13, Theorem 0.2], because X contains at most finitely many cones by [9, Theorem 4.2]. If X is general, then

$$1 = \text{lct}_1(X) \geq \text{lct}(X) \geq \begin{cases} 3/4 & \text{if } m = 3, \\ 7/9 & \text{if } m = 4, \\ 5/6 & \text{if } m = 5, \\ 1 & \text{if } m \geq 5, \end{cases}$$

by [3, 5, 33]. Thus, if X is general, then it is Kähler–Einstein by Theorem 1.5.

The assertion of Conjecture 1.8 follows from [8, Theorem 2] and [41, Theorem 1] under an additional assumption that the Kähler–Ricci flow on X is tamed (see [8] and [41]).

Theorem 1.10 [8, 41] *If $\dim(X) = 2$, then the Kähler–Ricci flow on X is tamed.*

Corollary 1.11 *Suppose that $\dim(X) = 2$ and*

$$\text{lct}_{n,2}(X) > \frac{2}{3}$$

for every $n \in \mathbb{N}$. Then X is Kähler–Einstein.

Two-dimensional Fano orbifolds are called del Pezzo surfaces.

Remark 1.12 Del Pezzo surfaces with quotient singularities are not classified (cf. [20]). But

- del Pezzo surfaces with canonical singularities are classified (see [18]),
- del Pezzo surfaces with 2-Gorenstein quotient singularities are classified (see [1]),
- smoothable del Pezzo surfaces with quotient singularities of Picard rank 1 are classified (see [17]).

Del Pezzo surfaces with canonical singularities form a very natural class of del Pezzo surfaces.

Problem 1.13 Describe all Kähler–Einstein del Pezzo surface with canonical singularities.

Recall that if X is a del Pezzo surface with canonical singularities, then

- either the inequality $K_X^2 \geq 5$ holds,
- or one of the following possible cases occurs:
 - the equality $K_X^2 = 1$ holds and X is a sextic surface in $\mathbb{P}(1, 1, 2, 3)$,
 - the equality $K_X^2 = 2$ holds and X is a quartic surface in $\mathbb{P}(1, 1, 1, 2)$,
 - the equality $K_X^2 = 3$ holds and X is a cubic surface in \mathbb{P}^3 ,
 - the equality $K_X^2 = 4$ holds and X is a complete intersection in \mathbb{P}^4 of two quadrics.

Let us consider few examples to illustrate the expected answer to Problem 1.13.

Example 1.14 Suppose that X is a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ such that its singular locus consists of singular points of type \mathbb{A}_1 or \mathbb{A}_2 . Arguing as in the proof of [3, Lemma 4.1], we see that

$$\text{lct}_{n,2}(X) > \frac{2}{3}$$

for every $n \in \mathbb{N}$. Thus, the surface X is Kähler–Einstein by Corollary 1.11.

Example 1.15 Suppose that X is a quartic surface in $\mathbb{P}(1, 1, 1, 2)$ such that its singular locus consists of singular points of type \mathbb{A}_1 or \mathbb{A}_2 . Then X is Kähler–Einstein by [16, Theorem 2].

Example 1.16 Suppose that X is a cubic surface in \mathbb{P}^3 that is not a cone. Then

- if X is smooth, then X is Kähler–Einstein by Theorem 1.1,
- if $\text{Sing}(X)$ consists of one point of type \mathbb{A}_1 , then it follows from [35, Theorem 5.1] that

$$\text{lct}_{n,2}(X) > \frac{2}{3} = \text{lct}_1(X) = \text{lct}(X)$$

for every $n \in \mathbb{N}$, which implies that X is Kähler–Einstein by Corollary 1.11,

- if the cubic surface X has a singular point that is not a singular point of type \mathbb{A}_1 or \mathbb{A}_2 , then the surface X is not Kähler–Einstein by [11, Proposition 4.2].

Example 1.17 Suppose that X is a complete intersection in \mathbb{P}^4 of two quadrics. Then

- if X is smooth, then X is Kähler–Einstein by Theorem 1.1,
- if X is Kähler–Einstein, then X has at most singular points of type \mathbb{A}_1 (see [19]),
- it follows from [24] or [16, Theorem 44] that X is Kähler–Einstein if it is given by

$$\sum_{i=0}^4 x_i^2 = \sum_{i=0}^4 \lambda_i x_i^2 = 0 \subseteq \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x_0, \dots, x_4]),$$

and X has at most singular points of type \mathbb{A}_1 , where $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4) \in \mathbb{P}^4$.

Keeping in mind Examples 1.14, 1.15, 1.16, 1.17, [4, Example 1.12], and [26, Table 1], it is very natural to expect that the following answer to Problem 1.13 is true (cf. Example 1.6).

Conjecture 1.18 *If the orbifold X is a del Pezzo surface with at most canonical singularities, then the surface X is Kähler–Einstein \iff it satisfies one of the following conditions:*

- $K_X^2 = 1$ and $\text{Sing}(X)$ consists of points of type $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4, \mathbb{A}_5, \mathbb{A}_6, \mathbb{A}_7$, or \mathbb{D}_4 ,
- $K_X^2 = 2$ and $\text{Sing}(X)$ consists of points of type $\mathbb{A}_1, \mathbb{A}_2$, or \mathbb{A}_3 ,
- $K_X^2 = 3$ and $\text{Sing}(X)$ consists of points of type \mathbb{A}_1 , or \mathbb{A}_2 ,
- $K_X^2 = 4$ and $\text{Sing}(X)$ consists of points of type \mathbb{A}_1 ,
- the surface X is smooth and $6 \geq K_X^2 \geq 5$,
- either $X \cong \mathbb{P}^2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

In this paper, we prove the following result.

Theorem 1.19 *Suppose that X is a sextic surface in $\mathbb{P}(1, 1, 2, 3)$. Then*

$$\text{lct}_{n,2}(X) > \frac{2}{3}$$

for every $n \in \mathbb{N}$ if $\text{Sing}(X)$ consists of points of type $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4, \mathbb{A}_5$ or \mathbb{A}_6 .

Corollary 1.20 *Suppose that X is a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ such that its singular locus consists of singular points of type $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4, \mathbb{A}_5$, or \mathbb{A}_6 . Then X is Kähler–Einstein.*

It should be pointed out that Corollary 1.20 and Examples 1.14, 1.15, 1.16, 1.17 illustrate a general philosophy that the existence of Kähler–Einstein metrics on Fano orbifolds is related to an algebro-geometric notion of stability (see [11, 12, 39, Theorem 4.1]).

Remark 1.21 If X is a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ with canonical singularities, then either

$\text{Sing}(X)$

$$\in \left\{ \begin{array}{l} \mathbb{E}_8, \mathbb{E}_7, \mathbb{E}_7 + \mathbb{A}_1, \mathbb{E}_6, \mathbb{E}_6 + \mathbb{A}_2, \mathbb{E}_6 + \mathbb{A}_1, \mathbb{D}_8, \mathbb{D}_7, \mathbb{D}_6, \mathbb{D}_6 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{D}_6 + \mathbb{A}_1, \\ \mathbb{D}_5, \mathbb{D}_5 + \mathbb{A}_3, \mathbb{D}_5 + \mathbb{A}_2, \mathbb{D}_5 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{D}_5 + \mathbb{A}_1, \mathbb{D}_4, \mathbb{D}_4 + \mathbb{D}_4, \mathbb{D}_4 + \mathbb{A}_3, \mathbb{D}_4 + \mathbb{A}_2, \\ \mathbb{D}_4 + \mathbb{A}_1 + \mathbb{A}_1 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{D}_4 + \mathbb{A}_1 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{D}_4 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{D}_4 + \mathbb{A}_1, \mathbb{A}_8, \\ \mathbb{A}_7, \mathbb{A}_7 + \mathbb{A}_1, \mathbb{A}_6, \mathbb{A}_6 + \mathbb{A}_1, \mathbb{A}_5, \mathbb{A}_5 + \mathbb{A}_1, \mathbb{A}_5 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{A}_5 + \mathbb{A}_2, \mathbb{A}_5 + \mathbb{A}_2 + \mathbb{A}_1, \\ \mathbb{A}_4, \mathbb{A}_4 + \mathbb{A}_4, \mathbb{A}_4 + \mathbb{A}_3, \mathbb{A}_4 + \mathbb{A}_2 + \mathbb{A}_1, \mathbb{A}_4 + \mathbb{A}_2, \mathbb{A}_4 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{A}_4 + \mathbb{A}_1, \\ \mathbb{A}_3, \mathbb{A}_3 + \mathbb{A}_3, \mathbb{A}_3 + \mathbb{A}_3 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_2, \mathbb{A}_3 + \mathbb{A}_2 + \mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_2 + \mathbb{A}_1 + \mathbb{A}_1, \\ \mathbb{A}_3 + \mathbb{A}_1 + \mathbb{A}_1 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_1 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_1 \end{array} \right\}$$

or $\text{Sing}(X)$ consists only of points of type \mathbb{A}_1 and \mathbb{A}_2 (see [40]).

What is known about α -invariants of del Pezzo surfaces with canonical singularities?

Theorem 1.22 [3] *If X is a smooth del Pezzo surface, then $\text{lct}(X) = \text{lct}_1(X)$.*

Theorem 1.23 [3, 31] *If X is a del Pezzo surface with canonical singularities, then*

$$\text{lct}(X) = \text{lct}_1(X)$$

in the case when $K_X^2 \geq 3$.

Theorem 1.24 [31] *If X is a quartic surface in $\mathbb{P}(1, 1, 1, 2)$ with canonical singularities, then*

$$\text{lct}(X) = \begin{cases} \text{lct}_2(X) = 1/3 & \text{if } X \text{ has a singular point of type } \mathbb{A}_7, \\ \text{lct}_2(X) = 2/5 & \text{if } X \text{ has a singular point of type } \mathbb{A}_6, \\ \text{lct}_1(X) & \text{in the remaining cases.} \end{cases}$$

In this paper, we prove the following result (cf. Example 1.14).

Theorem 1.25 *Suppose that X is a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ with canonical singularities, let $\omega: X \rightarrow \mathbb{P}(1, 1, 2)$ be a natural double cover, and let R be its branch curve in $\mathbb{P}(1, 1, 2)$. Then*

$$\text{lct}(X) = \begin{cases} \text{lct}_2(X) = 1/3 & \text{if } \text{Sing}(X) \text{ consists of a point of type } \mathbb{D}_8, \\ \text{lct}_2(X) = 2/5 & \text{if } \text{Sing}(X) \text{ consists of a point of type } \mathbb{D}_7, \\ \text{lct}_3(X) = 1/2 & \text{if } \text{Sing}(X) \text{ consists of a point of type } \mathbb{A}_8, \\ \text{lct}_2(X) = 1/2 & \text{if } \text{Sing}(X) \text{ consists of a point of type } \mathbb{A}_7 \text{ and a point of type } \mathbb{A}_1, \\ \text{lct}_2(X) = 1/2 & \text{if } \text{Sing}(X) \text{ consists of a point of type } \mathbb{A}_7 \text{ and } R \text{ is reducible,} \\ \text{lct}_3(X) = 3/5 & \text{if } X \text{ has a singular point of type } \mathbb{A}_7 \text{ and } R \text{ is irreducible,} \\ \text{lct}_2(X) = 2/3 & \text{if } X \text{ has a singular point of type } \mathbb{A}_6, \\ \text{lct}_2(X) = 2/3 & \text{if } X \text{ has a singular point of type } \mathbb{A}_5, \\ \text{lct}_2(X) = \min(\text{lct}_1(X), 4/5) & \text{if } X \text{ has a singular point of type } \mathbb{A}_4, \\ \text{lct}_1(X) & \text{in the remaining cases.} \end{cases}$$

It should be pointed out that if X is a del Pezzo surface with at most canonical singularities, then all possible values of the number $\text{lct}_1(X)$ are computed in [28–30].

Example 1.26 *If X is a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ with canonical singularities, then*

- $\text{lct}_1(X) = 1/6 \iff$ the surface X has a singular point of type \mathbb{E}_8 ,
- $\text{lct}_1(X) = 1/4 \iff$ the surface X has a singular point of type \mathbb{E}_7 ,
- $\text{lct}_1(X) = 1/3 \iff$ the surface X has a singular point of type \mathbb{E}_6 ,
- $\text{lct}_1(X) = 1/2 \iff$ the surface X has a singular point of type $\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{D}_7$ or \mathbb{D}_8 ,

- $\text{lct}_1(X) = 2/3 \iff$ the following two conditions are satisfied:
 - the surface X has no singular points of type $\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{D}_7, \mathbb{D}_8, \mathbb{E}_6, \mathbb{E}_7,$ or $\mathbb{E}_8,$
 - there is a curve in $|-K_X|$ that has a cusp at a point in $\text{Sing}(X)$ of type $\mathbb{A}_2,$
- $\text{lct}_1(X) = 3/4 \iff$ the following three conditions are satisfied:
 - the surface X has no singular points of type $\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{D}_7, \mathbb{D}_8, \mathbb{E}_6, \mathbb{E}_7,$ or $\mathbb{E}_8,$
 - there is no curve in $|-K_X|$ that has a cusp at a point in $\text{Sing}(X)$ of type $\mathbb{A}_2,$
 - there is a curve in $|-K_X|$ that has a cusp at a point in $\text{Sing}(X)$ of type $\mathbb{A}_1,$
- $\text{lct}_1(X) = 5/6 \iff$ the following three conditions are satisfied:
 - the surface X has no singular points of type $\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{D}_7, \mathbb{D}_8, \mathbb{E}_6, \mathbb{E}_7,$ or $\mathbb{E}_8,$
 - there is no curve in $|-K_X|$ that have a cusp at a point in $\text{Sing}(X),$
 - there is a curve in $|-K_X|$ that has a cusp,
- $\text{lct}_1(X) = 1 \iff$ there are no cuspidal curves in $|-K_X|.$

A crucial role in the proofs of both Theorems 1.25 and 1.19 is played by a new local inequality that we discovered. This inequality is a technical tool, but let us describe it now.

Let S be a surface, let D be an arbitrary effective \mathbb{Q} -divisor on the surface $S,$ let O be a smooth point of the surface $S,$ let Δ_1 and Δ_2 be reduced irreducible curves on S such that

$$\Delta_1 \not\subseteq \text{Supp}(D) \not\supseteq \Delta_2,$$

and the divisor $\Delta_1 + \Delta_2$ has a simple normal crossing singularity at the smooth point $O \in \Delta_1 \cap \Delta_2,$ let a_1 and a_2 be some non-negative rational numbers. Suppose that the log pair

$$(S, D + a_1\Delta_1 + a_2\Delta_2)$$

is not Kawamata log terminal at $O,$ but $(S, D + a_1\Delta_1 + a_2\Delta_2)$ is Kawamata log terminal in a punctured neighborhood of the point $O.$

Theorem 1.27 *Let $A, B, M, N, \alpha, \beta$ be non-negative rational numbers. Then*

$$\text{mult}_O(D \cdot \Delta_1) \geq M + Aa_1 - a_2 \quad \text{or} \quad \text{mult}_O(D \cdot \Delta_2) \geq N + Ba_2 - a_1$$

in the case when the following conditions are satisfied:

- *the inequality $\alpha a_1 + \beta a_2 \leq 1$ holds,*
- *the inequalities $A(B - 1) \geq 1 \geq \max(M, N)$ hold,*
- *the inequalities $\alpha(A + M - 1) \geq A^2(B + N - 1)\beta$ and $\alpha(1 - M) + A\beta \geq A$ hold,*
- *either the inequality $2M + AN \leq 2$ holds or*

$$\alpha(B + 1 - MB - N) + \beta(A + 1 - AN - M) \geq AB - 1.$$

Corollary 1.28 *Suppose that*

$$\frac{2m - 2}{m + 1}a_1 + \frac{2}{m + 1}a_2 \leq 1$$

for some integer m such that $m \geq 3$. Then

$$\text{mult}_O(D \cdot \Delta_1) \geq 2a_1 - a_2 \quad \text{or} \quad \text{mult}_O(D \cdot \Delta_2) \geq \frac{m}{m-1}a_2 - a_1.$$

Proof To prove the required assertion, let us put

$$A = 2, \quad B = \frac{m}{m-1}, \quad M = 0, \quad N = 0, \quad \alpha = \frac{2m-2}{m+1}, \quad \beta = \frac{2}{m+1}a_2,$$

and let us check that all hypotheses of Theorem 1.27 are satisfied.

We have $\alpha a_1 + \beta a_2 \leq 1$ by assumption. We have

$$A(B-1) = \frac{2}{m-1} \geq 1 \geq 0 = \max(M, N),$$

since $m \geq 3$. We have

$$\alpha(A+M-1) = \frac{2m-2}{m+1} \geq \frac{8}{m^2-1} = A^2(B+N-1)\beta,$$

since $m \geq 3$. We have $\alpha(1-M) + A\beta = 2 \geq 2 = A$ and $2M + AN = 0 \leq 2$.

Thus, we see that all hypotheses of Theorem 1.27 are satisfied. Then

$$\begin{aligned} \text{mult}_O(D \cdot \Delta_1) &\geq M + Aa_1 - a_2 = 2a_1 - a_2 \quad \text{or} \\ \text{mult}_O(D \cdot \Delta_2) &\geq N + Ba_2 - a_1 = \frac{m}{m-1}a_2 - a_1 \end{aligned}$$

by Theorem 1.27. □

For the convenience of the reader, we organize the paper in the following way:

- in Sect. 2, we collect auxiliary results,
- in Sect. 3, we prove Theorem 1.27,
- in Sect. 4, we prove Theorem 4.1,
- in Sect. 5, we prove Theorem 5.1,
- in Sect. 6, we prove Theorem 6.1.

By Remark 1.21, both Theorems 1.19 and 1.25 follow from Theorems 4.1, 5.1, and 6.1.

2 Preliminaries

Let S be a surface with canonical singularities, and let D be an effective \mathbb{Q} -divisor on S . Put

$$D = \sum_{i=1}^r a_i D_i,$$

where D_i is an irreducible curve, and $a_i \in \mathbb{Q}_{>0}$. We assume that $D_i \neq D_j \iff i \neq j$.

Suppose that (S, D) is log canonical, but (S, D) is not Kawamata log terminal.

Remark 2.1 Let \bar{D} be an effective \mathbb{Q} -divisor on the surface S such that

$$\bar{D} = \sum_{i=1}^r \bar{a}_i D_i \sim_{\mathbb{Q}} D,$$

and the log pair (S, \bar{D}) is log canonical, where \bar{a}_i is a non-negative rational number. Put

$$\alpha = \min \left\{ \frac{a_i}{\bar{a}_i} \mid \bar{a}_i \neq 0 \right\},$$

where α is well defined and $\alpha \leq 1$. Then $\alpha = 1 \iff D = \bar{D}$. Suppose that $D \neq \bar{D}$. Put

$$D' = \sum_{i=1}^r \frac{a_i - \alpha \bar{a}_i}{1 - \alpha} D_i,$$

and choose $k \in \{1, \dots, r\}$ such that $\alpha = a_k / \bar{a}_k$. Then $D_k \not\subset \text{Supp}(D')$ and $D' \sim_{\mathbb{Q}} \bar{D} \sim_{\mathbb{Q}} D$, but the log pair (S, D') is not Kawamata log terminal.

Let $\text{LCS}(S, D)$ be the locus of log canonical singularities of the log pair (S, D) (see [6]).

Theorem 2.2 [22, Theorem 17.4] *If $-(K_S + D)$ is nef and big, then $\text{LCS}(S, D)$ is connected.*

Take a point $P \in \text{LCS}(S, D)$. Suppose that $\text{LCS}(S, D)$ contains no curves that pass through P .

Lemma 2.3 *Suppose that $P \notin \text{Sing}(S)$ and $P \notin \text{Sing}(D_1)$. Then*

$$D_1 \cdot \left(\sum_{i=2}^r a_i D_i \right) \geq \sum_{i=2}^r a_i \text{mult}_P(D_1 \cdot D_i) > 1.$$

Proof The log pair $(S, D_1 + \sum_{i=2}^r a_i D_i)$ is not log canonical at P , since $a_1 < 1$. Then

$$D_1 \cdot \sum_{i=2}^r a_i D_i \geq \sum_{i=2}^r a_i \text{mult}_P(D_1 \cdot D_i) \geq \text{mult}_P \left(\sum_{i=2}^r a_i D_i \mid_{D_1} \right) > 1$$

by [22, Theorem 17.6]. □

Let $\pi : \bar{S} \rightarrow S$ be a birational morphism, and let \bar{D} be a proper transform of D via π . Then

$$K_{\bar{S}} + \bar{D} + \sum_{i=1}^s e_i E_i \sim_{\mathbb{Q}} \pi^*(K_S + D),$$

where E_i is an irreducible π -exceptional curve, and $e_i \in \mathbb{Q}$. We assume that $E_i = E_j \iff i = j$.

Suppose, in addition, that the birational morphism π induces an isomorphism

$$\bar{S} \setminus \left(\bigcup_{i=1}^s E_i \right) \cong S \setminus P.$$

Remark 2.4 The log pair $(\bar{S}, \bar{D} + \sum_{i=1}^s e_i E_i)$ is not Kawamata log terminal at a point in $\bigcup_{i=1}^s E_i$.

Suppose that S is singular at P , and either P is a singular point of type \mathbb{D}_n for some $n \in \mathbb{N}_{\geq 4}$, or the point P is a singular point of type \mathbb{E}_m for some $m \in \{6, 7, 8\}$.

Lemma 2.5 *Suppose that $E_1^2 = E_2^2 = \dots = E_s^2 = -2$. Then $e_1 = 1$ if*

$$E_1 \cdot \left(\sum_{i=2}^s E_i \right) = 3.$$

Proof This follows from [32, Proposition 2.9], because $(S \ni P)$ is a weakly exceptional singularity (see [32, Example 4.7], [7, Example 3.4], [7, Theorem 3.15]). \square

Lemma 2.6 *Suppose that S is a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ that has canonical singularities, and suppose that $D \sim_{\mathbb{Q}} -K_X$. Let μ be a positive rational number such that either*

$$\mu < \text{lct}_1(S)$$

or $\mu = 2/3$ and D is not a curve in $|-K_X|$ with a cusp at a point in $\text{Sing}(S)$ of type \mathbb{A}_2 . Then

$$\text{LCS}(S, \mu D) \subseteq \text{Sing}(S),$$

the locus $\text{LCS}(S, \mu D)$ contains no points of type \mathbb{A}_1 or \mathbb{A}_2 , and $|\text{LCS}(S, \mu D)| \leq 1$.

Proof This follows from Theorem 2.2 and the proof of [3, Lemma 4.1]. \square

Most of the described results are valid in much more general settings (cf. [22] and [21]).

3 Local Inequality

The purpose of this section is to prove Theorem 1.27.

Let S be a surface, let D be an arbitrary effective \mathbb{Q} -divisor on the surface S , let O be a smooth point of the surface S , let Δ_1 and Δ_2 be reduced irreducible curves on S such that

$$\Delta_1 \not\subseteq \text{Supp}(D) \not\supseteq \Delta_2,$$

and the divisor $\Delta_1 + \Delta_2$ has a simple normal crossing singularity at the smooth point $O \in \Delta_1 \cap \Delta_2$, let a_1 and a_2 be some non-negative rational numbers. Suppose that the log pair

$$(S, D + a_1\Delta_1 + a_2\Delta_2)$$

is not Kawamata log terminal at O , but $(S, D + a_1\Delta_1 + a_2\Delta_2)$ is Kawamata log terminal in a punctured neighborhood of the point O . In particular, we must have $a_1 < 1$ and $a_2 < 1$.

Let $A, B, M, N, \alpha, \beta$ be non-negative rational numbers such that

- the inequality $\alpha a_1 + \beta a_2 \leq 1$ holds,
- the inequalities $A(B - 1) \geq 1 \geq \max(M, N)$ hold,
- the inequalities $\alpha(A + M - 1) \geq A^2(B + N - 1)\beta$ and $\alpha(1 - M) + A\beta \geq A$ holds,
- either the inequality $2M + AN \leq 2$ holds or

$$\alpha(B + 1 - MB - N) + \beta(A + 1 - AN - M) \geq AB - 1.$$

Lemma 3.1 *The inequalities $A + M \geq 1$ and $B > 1$ holds. The inequality*

$$\alpha(B + 1 - MB - N) + \beta(A + 1 - AN - M) \geq AB - 1$$

holds. The inequality $\beta(1 - N) + B\alpha \geq B$ holds. The inequalities

$$\frac{\alpha(2 - M)}{A + 1} + \frac{\beta(2 - N)}{B + 1} \geq 1$$

and $\alpha(2 - M)B + \beta(1 - N)(A + 1) \geq B(A + 1)$ hold.

Proof The inequality $B > 1$ follows from the inequality $A(B - 1) \geq 1$. Then

$$\frac{\alpha}{A + 1} + \frac{\beta}{B + 1} \geq \frac{\alpha}{A + 1} + \frac{\beta}{2B} \geq \frac{1}{2}$$

because $2B \geq B + 1$. Similarly, we see that $A + M \geq 1$, because

$$\frac{\alpha(A + M - 1)}{A^2(B + N - 1)} \geq \beta \geq 0$$

and $B + N - 1 \geq 0$. The inequality $\beta(1 - N) + B\alpha \geq B$ follows from the inequalities

$$\alpha + \frac{\beta(1 - N)}{B} \geq \frac{2 - M}{A + 1}\alpha + \frac{\beta(1 - N)}{B} \geq 1,$$

because $A + 1 \geq 2 - M$.

Let us show that the inequality

$$\alpha(2 - M)B + \beta(1 - N)(A + 1) \geq B(A + 1)$$

holds. Let L_1 be the line in \mathbb{R}^2 given by the equation

$$x(2 - M)B + y(1 - N)(A + 1) - B(A + 1) = 0$$

and let L_2 be the line that is given by the equation

$$x(1 - M) + Ay - A = 0,$$

where (x, y) are coordinates on \mathbb{R}^2 . Then L_1 intersects the line $y = 0$ at the point

$$\left(\frac{A + 1}{2 - M}, 0 \right)$$

and L_2 intersects the line $y = 0$ at the point $(A/(1 - M), 0)$. But

$$\frac{A + 1}{2 - M} < \frac{A}{1 - M},$$

which implies that $\alpha(2 - M)B + \beta(1 - N)(A + 1) \geq B(A + 1)$ if

$$A^2\beta_0(B + N - 1) \geq \alpha_0(A + M - 1),$$

where (α_0, β_0) is the intersection point of the lines L_1 and L_2 . But

$$(\alpha_0, \beta_0) = \left(\frac{A(A + 1)(B + N - 1)}{\Delta}, \frac{B(A - 1 + M)}{\Delta} \right),$$

where $\Delta = 2AB - ABM - A + AM - 1 + M + NA - NAM + N - NM$. But

$$A^2(B(A - 1 + M))(B + N - 1) \geq (A(A + 1)(B + N - 1))(A + M - 1),$$

because $A(B - 1) \geq 1$, which implies that $A^2\beta_0(B + N - 1) \geq \alpha_0(A + M - 1)$.

Finally, let us show that the inequality

$$\alpha(B + 1 - MB - N) + \beta(A + 1 - AN - M) \geq AB - 1$$

holds. Let L'_1 be the line in \mathbb{R}^2 given by the equation

$$x(B + 1 - MB - N) + y\beta(A + 1 - AN - M) - AB + 1 = 0,$$

where (x, y) are coordinates on \mathbb{R}^2 . Then L'_1 intersects the line $y = 0$ at the point

$$\left(\frac{AB - 1}{B + 1 - MB - N}, 0 \right)$$

and L_2 intersects the line $y = 0$ at the point $(A/(1 - M), 0)$. But

$$\frac{AB - 1}{B + 1 - MB - N} < \frac{A}{1 - M},$$

which implies that $\alpha(B + 1 - MB - N) + \beta(A + 1 - AN - M) \geq AB - 1$ if

$$A^2\beta_1(B + N - 1) \geq \alpha_1(A + M - 1),$$

where (α_1, β_1) is the intersection point of the lines L'_1 and L_2 . Note that

$$(\alpha_1, \beta_1) = \left(\frac{A(AB - A - 2 + NA + M)}{\Delta'}, \frac{A + 1 - NA - M}{\Delta'} \right),$$

where $\Delta' = AB - 1 - ABM + AM + 2M - NAM - M^2$.

To complete the proof, it is enough to show that the inequality

$$A^2(A + 1 - NA - M)(B + N - 1) \geq (A(AB - A - 2 + NA + M))(A + M - 1)$$

holds. This inequality is equivalent to the inequality

$$(2 - M)(A + M - 1) \geq A(AN + 2M - 2)(B + N - 1),$$

which is true, because $M \leq 1$ and $AN + 2M - 2 \leq 0$. □

Let us prove Theorem 1.27 by reductio ad absurdum. Suppose that the inequalities

$$\text{mult}_O(D \cdot \Delta_1) < M + Aa_1 - a_2 \quad \text{and} \quad \text{mult}_O(D \cdot \Delta_2) < N + Ba_2 - a_1$$

hold. Let us show that this assumption leads to a contradiction.

Lemma 3.2 *The inequalities $a_1 > (1 - M)/A$ and $a_2 > (1 - N)/B$ hold.*

Proof It follows from Lemma 2.3 that

$$M + Aa_1 - a_2 > \text{mult}_O(D \cdot \Delta_1) > 1 - a_2,$$

which implies that $a_1 > (1 - M)/A$. Similarly, we see that $a_2 > (1 - N)/B$. □

Put $m_0 = \text{mult}_O(D)$. Then m_0 is a positive rational number.

Remark 3.3 The inequalities $m_0 < M + Aa_1 - a_2$ and $m_0 < N + Ba_2 - a_1$ hold.

Lemma 3.4 *The inequality $m_0 + a_1 + a_2 < 2$ holds.*

Proof We know that $m_0 + a_1 + a_2 < M + (A + 1)a_1$ and $m_0 + a_1 + a_2 < N + (B + 1)a_2$. Then

$$\begin{aligned} (m_0 + a_1 + a_2) \left(\frac{\alpha}{A + 1} + \frac{\beta}{B + 1} \right) &< \alpha a_1 + \beta a_2 + \frac{\alpha M}{A + 1} + \frac{\beta N}{B + 1} \\ &\leq 1 + \frac{\alpha M}{A + 1} + \frac{\beta N}{B + 1}, \end{aligned}$$

which implies that $m_0 + a_1 + a_2 < 2$ by Lemma 3.1. □

Let $\pi_1: S_1 \rightarrow S$ be the blow-up of the point O , and let F_1 be the π_1 -exceptional curve. Then

$$\begin{aligned} K_{S_1} + D^1 + a_1 \Delta_1^1 + a_2 \Delta_2^1 + (m_0 + a_1 + a_2 - 1)F_1 \\ \sim_{\mathbb{Q}} \pi_1^*(K_S + D + a_1 \Delta_1 + a_2 \Delta_2), \end{aligned}$$

where $D^1, \Delta_1^1, \Delta_2^1$ are proper transforms of the divisors D, Δ_1, Δ_2 via π_1 , respectively. Then

$$(S_1, D^1 + a_1\Delta_1^1 + a_2\Delta_2^1 + (m_0 + a_1 + a_2 - 1)F_1)$$

is not Kawamata log terminal at some point $O_1 \in F_1$ (see Remark 2.4), where $m_0 + a_1 + a_2 \geq 1$.

Lemma 3.5 *Either $O_1 = F_1 \cap \Delta_1^1$ or $O_1 = F_1 \cap \Delta_2^1$.*

Proof Suppose that $O_1 \notin \Delta_1^1 \cup \Delta_2^1$. Then $m_0 = D^1 \cdot F_1 > 1$ by Lemma 2.3. But

$$m_0 \left(\frac{\beta + B\alpha}{AB - 1} + \frac{\alpha + A\beta}{AB - 1} \right) < (M + Aa_1 - a_2) \frac{\beta + B\alpha}{AB - 1} + (N + Ba_2 - a_1) \frac{\alpha + A\beta}{AB - 1},$$

because $m_0 < M + Aa_1 - a_2$ and $m_0 < N + Ba_2 - a_1$. On the other hand, we have

$$\begin{aligned} & (M + Aa_1 - a_2) \frac{\beta + B\alpha}{AB - 1} + (N + Ba_2 - a_1) \frac{\alpha + A\beta}{AB - 1} \\ & \leq 1 + \frac{M\beta + MB\alpha + N\alpha + AN\beta}{AB - 1}, \end{aligned}$$

because $\alpha a_1 + \beta a_2 \leq 1$ and $AB - 1 > 0$. But we already proved that $m_0 > 1$. Thus, we see that

$$\beta + B\alpha + \alpha + A\beta < AB - 1 + M\beta + MB\alpha + N\alpha + AN\beta,$$

which is impossible by Lemma 3.1. □

Lemma 3.6 *The inequality $O_1 \neq F_1 \cap \Delta_1^1$ holds.*

Proof Suppose that $O_1 = F_1 \cap \Delta_1^1$. It follows from Lemma 2.3 that

$$M + Aa_1 - a_2 - m_0 > \text{mult}_{O_1}(D^1 \cdot \Delta_1^1) > 1 - (m_0 + a_1 + a_2 - 1),$$

which implies that $a_1 > (2 - M)/(A + 1)$. Then

$$\frac{(2 - M)\alpha}{A + 1} + \frac{\beta(1 - N)}{B} < \alpha a_1 + \beta a_2 \leq 1,$$

because $a_2 > (1 - N)/B$ by Lemma 3.2. Thus, we see that

$$\frac{(2 - M)\alpha}{A + 1} + \frac{\beta(1 - N)}{B} < 1,$$

which is impossible by Lemma 3.1. □

Therefore, we see that $O_1 = F_1 \cap \Delta_2^1$. Then the log pair

$$(S_1, D^1 + a_1\Delta_1^1 + a_2\Delta_2^1 + (m_0 + a_1 + a_2 - 1)F_1)$$

is not Kawamata log terminal at the point O_1 . We know that $1 > m_0 + a_1 + a_2 - 1 \geq 0$.

We have a blow-up $\pi_1 : S_1 \rightarrow S$. For any $n \in \mathbb{N}$, consider a sequence of blow-ups

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S$$

such that $\pi_{i+1} : S_{i+1} \rightarrow S_i$ is a blow-up of the point $F_i \cap \Delta_2^i$ for every $i \in \{1, \dots, n - 1\}$, where

- we denote by F_i the exceptional curve of the morphism π_i ,
- we denote by Δ_2^i the proper transform of the curve Δ_2 on the surface S_i .

For every $k \in \{1, \dots, n\}$ and for every $i \in \{1, \dots, k\}$, let D^k , Δ_1^k , and F_i^k be the proper transforms on the surface S_k of the divisors D , Δ_1 and F_i , respectively. Then

$$K_{S_n} + D^n + a_1 \Delta_1^n + a_2 \Delta_2^n + \sum_{i=1}^n \left(a_1 + ia_2 - i + \sum_{j=0}^{i-1} m_j \right) F_i^n \sim_{\mathbb{Q}} \pi^*(K_S + D + a_1 \Delta_1 + a_2 \Delta_2),$$

where $\pi = \pi_n \circ \cdots \circ \pi_2 \circ \pi_1$ and $m_i = \text{mult}_{O_i}(D^i)$ for every $i \in \{1, \dots, n\}$. Then the log pair

$$\left(S_n, D^n + a_1 \Delta_1^n + a_2 \Delta_2^n + \sum_{i=1}^n \left(a_1 + ia_2 - i + \sum_{j=0}^{i-1} m_j \right) F_i^n \right) \tag{2}$$

is not Kawamata log terminal at some point of the set $F_1^n \cup F_2^n \cup \cdots \cup F_n^n$ (see Remark 2.4).

Put $O_k = F_k \cap \Delta_2^k$ for every $k \in \{1, \dots, n\}$.

Lemma 3.7 *For every $i \in \{1, \dots, n\}$, we have*

$$1 > a_1 + ia_2 - i + \sum_{j=0}^{i-1} m_j \geq 0,$$

and (2) is Kawamata log terminal at every point of the set $(F_1^n \cup F_2^n \cup \cdots \cup F_n^n) \setminus O_n$.

Since $\text{mult}_O(D \cdot \Delta_2) < N + Ba_2 - a_1$ by assumption, it follows from Lemma 3.7 that

$$N + Ba_2 - a_1 > \text{mult}_O(D \cdot \Delta_2) \geq \sum_{i=0}^{n-1} m_i \geq (n - 1)(1 - a_2) - a_1,$$

which implies that $n \leq (N + Ba_2)/(1 - a_2)$. On the other hand, the assertion of Lemma 3.7 holds for arbitrary $n \in \mathbb{N}$. So, taking any $n > (N + Ba_2)/(1 - a_2)$, we obtain a contradiction.

We see that to prove Theorem 1.27, it is enough to prove Lemma 3.7.

Let us prove Lemma 3.7 by induction on $n \in \mathbb{N}$. The case $n = 1$ is already done. We may assume that $n \geq 2$. For every $k \in \{1, \dots, n - 1\}$, we may assume that

$$1 > a_1 + ka_2 - k + \sum_{j=0}^{k-1} m_j \geq 0,$$

the singularities of the log pair

$$\left(S_k, D^k + a_1 \Delta_1^k + a_2 \Delta_2^k + \sum_{i=1}^k \left(a_1 + ka_2 - k + \sum_{j=0}^{i-1} m_j \right) F_i^k \right)$$

are Kawamata log terminal along $(F_1^k \cup F_2^k \cup \dots \cup F_k^k) \setminus O_k$ and not Kawamata log terminal at O_k .

Lemma 3.8 *The inequality $a_2 > (n - N)/(B + n - 1)$ holds.*

Proof The singularities of the log pair

$$\left(S_{n-1}, D^{n-1} + a_2 \Delta_2^{n-1} + \left(a_1 + (n - 1)a_2 - (n - 1) + \sum_{j=0}^{n-2} m_j \right) F_{n-1}^{n-1} \right)$$

are not Kawamata log terminal at the point O_{n-1} . Then it follows from Lemma 2.3 that

$$\begin{aligned} N + Ba_2 - a_1 - \sum_{j=0}^{n-2} m_j &> \text{mult}_{O_{n-1}}(D^{n-1} \cdot \Delta_2^{n-1}) \\ &> 1 - \left(a_1 + (n - 1)a_2 - (n - 1) + \sum_{j=0}^{n-2} m_j \right), \end{aligned}$$

which implies that $a_2 > (n - N)/(B + n - 1)$. □

Lemma 3.9 *The inequalities $1 > a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \geq 0$ hold.*

Proof The inequality $a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \geq 0$ follows from the fact that the log pair

$$\left(S_{n-1}, D^{n-1} + a_2 \Delta_2^{n-1} + \left(a_1 + (n - 1)a_2 - (n - 1) + \sum_{j=0}^{n-2} m_j \right) F_{n-1}^{n-1} \right)$$

is not Kawamata log terminal at the point O_{n-1} .

Suppose that $a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \geq 1$. Let us derive a contradiction.

It follows from Remark 3.3 that $m_0 + a_2 \leq M + Aa_1$. Then

$$a_1 + nM + nAa_1 - n \geq a_1 + na_2 - n + nm_0 \geq a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \geq 1,$$

which implies that $a_1 \geq (n + 1 - Mn)/(nA + 1)$. But $a_2 > (n - N)/(B + n - 1)$ by Lemma 3.8. Then

$$\begin{aligned} & \left(\frac{\alpha(1 - M)}{A} + \beta \right) + \alpha \frac{A - 1 + M}{A(An + 1)} + \beta \frac{1 - B - N}{B + n - 1} \\ &= \alpha \frac{n + 1 - Mn}{nA + 1} + \beta \frac{n - N}{B + n - 1} < \alpha a_1 + \beta a_2 \leq 1, \end{aligned}$$

where $\alpha(1 - M)/A + \beta \geq 1$ by assumption. Therefore, we see that

$$\alpha \frac{A + M - 1}{A(An + 1)} < \beta \frac{B + N - 1}{B + n - 1},$$

where $n \geq 2$. But $A + M > 1$ and $B + N > 1$ by Lemma 3.2, since $a_1 < 1$ and $a_2 < 1$. Then

$$\frac{A(An + 1)}{\alpha(A + M - 1)} > \frac{B + n - 1}{\beta(B + N - 1)},$$

but $A^2(B + N - 1)\beta \leq \alpha(A + M - 1)$ by assumption. Then

$$\begin{aligned} \frac{A}{\alpha(A + M - 1)} - \frac{B - 1}{\beta(B + N - 1)} &\geq \left(\frac{A^2}{\alpha(A + M - 1)} - \frac{1}{\beta(B + N - 1)} \right) n \\ &+ \frac{A}{\alpha(A + M - 1)} - \frac{B - 1}{\beta(B + N - 1)} > 0, \end{aligned}$$

which implies that $\beta A(B + N - 1) > \alpha(B - 1)(A + M - 1)$. Then

$$\frac{\alpha(A + M - 1)}{A} \geq \beta A(B + N - 1) > \alpha(B - 1)(A + M - 1),$$

because $A^2(B + N - 1)\beta \leq \alpha(A + M - 1)$ by assumption. Then we have $\alpha \neq 0$ and $A(B - 1) < 1$, which is impossible, because $A(B - 1) \geq 1$ by assumption. \square

Lemma 3.10 *The log pair (2) is Kawamata log terminal at every point of the set*

$$F_n \setminus ((F_n \cap F_{n-1}^n) \cup (F_n \cap \Delta_2^n)).$$

Proof Suppose that there is a point $Q \in F_n$ such that

$$F_n \cap F_{n-1}^n \neq Q \neq F_n \cap \Delta_2^n,$$

but (2) is not Kawamata log terminal at the point Q . Then the log pair

$$\left(S_n, D^n + \left(a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \right) F_n \right)$$

is not Kawamata log terminal at the point Q as well. Then

$$m_0 \geq m_{n-1} = D^n \cdot F_n > 1$$

by Lemma 2.3, because $a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j < 1$ by Lemma 3.9. Then

$$m_0 \left(\frac{\beta + B\alpha}{AB - 1} + \frac{\alpha + A\beta}{AB - 1} \right) < (M + Aa_1 - a_2) \frac{\beta + B\alpha}{AB - 1} + (N + Ba_2 - a_1) \frac{\alpha + A\beta}{AB - 1},$$

because $m_0 < M + Aa_1 - a_2$ and $m_0 < N + Ba_2 - a_1$ by Remark 3.3. We have

$$\begin{aligned} & (M + Aa_1 - a_2) \frac{\beta + B\alpha}{AB - 1} + (N + Ba_2 - a_1) \frac{\alpha + A\beta}{AB - 1} \\ & \leq 1 + \frac{M\beta + MB\alpha + N\alpha + AN\beta}{AB - 1}, \end{aligned}$$

because $\alpha a_1 + \beta a_2 \leq 1$ and $AB - 1 > 0$. But $m_0 > 1$. Thus, we see that

$$\beta + B\alpha + \alpha + A\beta < AB - 1 + M\beta + MB\alpha + N\alpha + AN\beta,$$

which contradicts our initial assumptions. □

Lemma 3.11 *The log pair (2) is Kawamata log terminal at the point $F_n \cap F_{n-1}^n$.*

Proof Suppose that (2) is not Kawamata log terminal at $F_n \cap F_{n-1}^n$. Then the log pair

$$\begin{aligned} & \left(S_n, D^n + \left(a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j \right) F_{n-1}^n \right) \\ & + \left(a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \right) F_n \end{aligned}$$

is not Kawamata log terminal at the point $F_n \cap F_{n-1}^n$ as well. Then

$$m_{n-2} - m_{n-1} = D^n \cdot F_{n-2} > 1 - \left(a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \right)$$

by Lemma 2.3, because $a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j < 1$. Note that

$$M + Aa_1 - a_2 - m_0 > \text{mult}_O(D \cdot \Delta_1) - m_0 \geq \text{mult}_O(D) \text{mult}_O(\Delta_1) - m_0 = 0,$$

which implies that $m_0 + a_2 < Aa_1 + M$. Then

$$nM + nAa_1 - na_2 > nm_0 \geq m_{n-2} - m_{n-1} + \sum_{j=0}^{n-1} m_j > n + 1 - a_1 - na_2,$$

which gives $a_1 > (n + 1 - nM)/(An + 1)$.

Now arguing as in the proof of Lemma 3.9, we obtain a contradiction. □

The assertion of Lemma 3.7 is proved. The assertion of Theorem 1.27 is proved.

4 One Cyclic Singular Point

Let X be a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ with canonical singularities such that $|\text{Sing}(X)| = 1$, let $\omega: X \rightarrow \mathbb{P}(1, 1, 2)$ be the natural double cover, let R be its ramification curve in $\mathbb{P}(1, 1, 2)$, and suppose that $\text{Sing}(X)$ consists of one singular point of type \mathbb{A}_m , where $m \in \{1, \dots, 8\}$.

Theorem 4.1 *The following equality holds:*

$$\text{lct}(X) = \begin{cases} \text{lct}_3(X) = 1/2 & \text{if } m = 8, \\ \text{lct}_2(X) = 1/2 & \text{if } m = 7 \text{ and } R \text{ is reducible,} \\ \text{lct}_3(X) = 3/5 & \text{if } m = 7 \text{ and } R \text{ is irreducible,} \\ \text{lct}_2(X) = 2/3 & \text{if } m = 6, \\ \text{lct}_2(X) = 2/3 & \text{if } m = 5, \\ \text{lct}_2(X) = 4/5 & \text{if } m = 4, \\ \text{lct}_1(X) & \text{in the remaining cases,} \end{cases}$$

and if $\text{lct}(X) = 2/3$, then there is a unique effective \mathbb{Q} -divisor D on X such that $D \sim_{\mathbb{Q}} -K_X$ and

$$c(X, D) = \text{lct}(X) = \frac{2}{3}.$$

By Theorem 1.5, Corollary 1.11, and Remark 1.7, we obtain the following two corollaries.

Corollary 4.2 *If $m \leq 6$, then $\text{lct}_{n,2}(X) > 2/3$ for every $n \in \mathbb{N}$.*

Corollary 4.3 *If $m \leq 6$, then X is Kähler–Einstein.*

In the rest of this section, we will prove Theorem 4.1.

Let D be an arbitrary effective \mathbb{Q} -divisor on the surface X such that

$$D \sim_{\mathbb{Q}} -K_X,$$

and put $\mu = c(X, D)$. To prove Theorem 4.1, it is enough to show that

$$\mu \geq \begin{cases} \text{lct}_3(X) = 1/2 & \text{if } m = 8, \\ \text{lct}_2(X) = 1/2 & \text{if } m = 7 \text{ and } R \text{ is reducible,} \\ \text{lct}_3(X) = 3/5 & \text{if } m = 7 \text{ and } R \text{ is irreducible,} \\ \text{lct}_2(X) = 2/3 & \text{if } m = 6, \\ \text{lct}_2(X) = 2/3 & \text{if } m = 5, \\ \text{lct}_2(X) = 4/5 & \text{if } m = 4, \\ \text{lct}_1(X) & \text{in the remaining cases,} \end{cases}$$

and if $\mu = \text{lct}(X) = 2/3$, then D is uniquely defined. Note that $\text{lct}_1(X) \geq 5/6$ if $m \geq 3$ (see [30]).

Let us prove Theorem 4.1. By Lemma 2.6, we may assume that $m \geq 3$ and $\mu < \text{lct}_1(X)$. Then

$$\text{LCS}(X, \mu D) = \text{Sing}(X)$$

by Lemma 2.6. Put $P = \text{Sing}(X)$.

Let $\pi: \bar{X} \rightarrow X$ be a minimal resolution, let E_1, E_2, \dots, E_m be π -exceptional curves such that

$$E_i \cdot E_j \neq 0 \iff |i - j| \leq 1,$$

let C be the curve in $|-K_X|$ such that $P \in C$, and let \bar{C} be its proper transform on \bar{X} . Then

$$\bar{C} \sim_{\mathbb{Q}} \pi^*(C) - \sum_{i=1}^m E_i,$$

and the curve C is irreducible. We may assume that $D \neq C$, because $\mu \geq \text{lct}_1(X)$ if $D = C$.

By Remark 2.1, we may assume that $C \not\subset \text{Supp}(D)$.

Let \bar{D} be the proper transform of the divisor D on the surface \bar{X} . Then

$$\bar{D} \sim_{\mathbb{Q}} \pi^*(D) - \sum_{i=1}^m a_i E_i,$$

where a_i is a non-negative rational number. Then the log pair

$$\left(\bar{X}, \mu \bar{D} + \sum_{i=1}^m \mu a_i E_i \right) \tag{3}$$

is not Kawamata log terminal (by Remark 2.4). On the other hand, we have

$$\begin{aligned} \bar{D} \cdot E_1 &= 2a_1 - a_2, & \bar{D} \cdot E_2 &= 2a_2 - a_1 - a_3, & \dots, \\ \bar{D} \cdot E_{m-1} &= 2a_{m-1} - a_{m-2} - a_m, & \bar{D} \cdot E_m &= 2a_m - a_{m-1}, \end{aligned}$$

where all intersections $\bar{D} \cdot E_1, \bar{D} \cdot E_2, \dots, \bar{D} \cdot E_m$ are non-negative. Moreover, we have

$$\bar{D} \cdot \bar{C} = 1 - a_1 - a_m,$$

where the intersection $\bar{D} \cdot \bar{C}$ is non-negative, since $C \not\subset \text{Supp}(D)$ by assumption. Hence, we have

$$\left\{ \begin{array}{l} a_1 \geq \frac{a_2}{2}, \\ a_2 \geq \frac{a_1 + a_3}{2}, \\ a_3 \geq \frac{a_2 + a_4}{2}, \\ \vdots \\ a_{m-1} \geq \frac{a_{m-2} + a_m}{2}, \\ a_m \geq \frac{a_{m-1}}{2}, \\ 1 \geq a_1 + a_m. \end{array} \right. \tag{4}$$

It should be pointed out that at least one inequality in (4) must be strict, since $\bar{D} \cdot E_i > 0$ for at least one $i \in \{1, \dots, m\}$, because $P \in \text{Supp}(D)$. Then $a_i > 0$ for some $i \in \{1, \dots, m\}$.

Note that $a_1 \geq a_2/2$ by (4). Similarly, it follows from (4) that

$$a_2 \geq \frac{a_1 + a_3}{2} \geq \frac{a_1}{4} + \frac{a_3}{4},$$

which implies that $a_2 \geq 2a_3/3$. Arguing in the same way, we see that

$$a_k \geq \frac{k}{k+1} a_{k+1}$$

for every $k \in \{1, \dots, m-1\}$ (use (4) and induction on k). Using symmetry, we see that

$$a_{k+1} \geq \frac{m-k}{m-k+1} a_k$$

for every $k \in \{1, \dots, m-1\}$. In particular, the inequality $a_k > 0$ holds for every $k \in \{1, \dots, m\}$, since we already know that $a_i > 0$ for some $i \in \{1, \dots, m\}$.

Lemma 4.4 *Suppose that $\mu a_i < 1$ for every $i \in \{1, \dots, m\}$. Then*

- *there exists a point*

$$Q \in \{E_1 \cap E_2, E_2 \cap E_3, \dots, E_{m-1} \cap E_m\}$$

such that the log pair (3) is not Kawamata log terminal at Q ,

- *the log pair (3) is Kawamata log terminal outside of the point Q ,*
- *if $\mu < (m + 1)/(2m - 2)$, then $Q \neq E_1 \cap E_2$ and $Q \neq E_{m-1} \cap E_m$.*

Proof It follows from Remark 2.4 and Theorem 2.2 that there is a point $Q \in \bigcup_{i=1}^m E_i$ such that the log pair (3) is not Kawamata log terminal at Q and is Kawamata log terminal elsewhere.

Suppose that $Q \in E_1$ and $Q \notin E_2$. Then

$$2a_1 - a_2 = \bar{D} \cdot E_i > 1$$

by Lemma 2.3. Taking (4) into account, we get

$$\left\{ \begin{array}{l} a_1 > \frac{1}{2} + \frac{a_2}{2}, \\ a_2 \geq \frac{a_1 + a_3}{2}, \\ a_3 \geq \frac{a_2 + a_4}{2}, \\ \vdots \\ a_{m-1} \geq \frac{a_{m-2} + a_m}{2}, \\ a_m \geq \frac{a_{m-1}}{2}, \end{array} \right.$$

and adding all these inequalities together we get

$$\sum_{i=1}^m a_i > \frac{1}{2} + \frac{a_1}{2} + \sum_{i=2}^{m-1} a_i + \frac{a_m}{2},$$

which implies that $a_1 + a_m > 1$. However, the latter is impossible, since $a_1 + a_m \leq 1$ by (4).

We see that if $Q \in E_1$, then $Q = E_1 \cap E_2$. Similarly, we see that $Q = E_{m-1} \cap E_m$ if $Q \in E_m$.

Suppose that $Q \in E_i$ and $Q \notin E_j$ for every $j \neq i$. Then $i \neq 1$ and $i \neq m$. We have

$$2a_i - a_{i-1} - a_{i+1} = \bar{D} \cdot E_i > 1$$

by Lemma 2.3. Taking (4) into account, we get

$$\left\{ \begin{array}{l} a_1 > \frac{a_2}{2}, \\ a_2 \geq \frac{a_1 + a_3}{2}, \\ a_3 \geq \frac{a_2 + a_4}{2}, \\ \vdots \\ a_i \geq \frac{1}{2} + \frac{a_{i-1} + a_{i+1}}{2}, \\ \vdots \\ a_{m-1} \geq \frac{a_{m-2} + a_m}{2}, \\ a_m \geq \frac{a_{m-1}}{2}, \end{array} \right.$$

and adding all these inequalities together we get

$$\sum_{i=1}^m a_i > \frac{1}{2} + \frac{a_1}{2} + \sum_{i=2}^{m-1} a_i + \frac{a_m}{2},$$

which implies that $a_1 + a_m > 1$. However, the latter is impossible, since $a_1 + a_m \leq 1$ by (4).

Thus, we see that there is $k \in \{1, \dots, m - 1\}$ such that $Q = E_k \cap E_{k+1}$.

Suppose that $\mu < (m + 1)/(2m - 2)$. Let us show that $k \neq 1$ and $k \neq m - 1$.

Due to symmetry, it is enough to show that $k \neq 1$. Recall that $m \geq 3$.

Suppose that $k = 1$. Then $Q = E_1 \cap E_2$. Take $\bar{\mu} \in \mathbb{Q}$ such that $(m + 1)/(2m - 2) > \bar{\mu} > \mu$ and

$$(\bar{X}, \mu \bar{D} + \bar{\mu} a_1 E_1 + \bar{\mu} a_2 E_2)$$

is not Kawamata log terminal at Q and is Kawamata log terminal outside of the point Q . Then

$$\frac{2m - 2}{m + 1} \bar{\mu} a_1 + \frac{2}{m + 1} \bar{\mu} a_2 < a_1 + \frac{1}{m - 1} a_2 \leq 1,$$

by (4), since $a_1 \neq 0$ and $a_2 \neq 0$. On the other hand, we have

$$\text{mult}_Q(\mu \bar{D} \cdot E_1) \leq \mu \bar{D} \cdot E_1 = \mu(2a_1 - a_2) < \bar{\mu}(2a_1 - a_2),$$

since $\mu < \bar{\mu}$. Therefore, it follows from Corollary 1.28 that

$$\mu(2a_2 - a_1 - a_3) = \mu \bar{D} \cdot E_2 \geq \text{mult}_Q(\mu \bar{D} \cdot E_2) \geq \frac{m}{m - 1} \bar{\mu} a_2 - \bar{\mu} a_1,$$

which implies that $a_2(m - 2) > a_3(m - 1)$, since $\mu < \bar{\mu}$. But we proved earlier that

$$a_3 \geq \frac{m - 2}{m - 1} a_2,$$

which is impossible, since $a_2(m - 2) > a_3(m - 1)$. Thus, we see that $k \neq 1$. □

If $m = 3$, then it follows from (4) that $a_1 \leq 3/4, a_2 \leq 1, a_3 \leq 3/4$.

Corollary 4.5 *If $m = 3$, then $\mu \geq \text{lct}_1(X) \geq 5/6$.*

Lemma 4.6 *Suppose that $m = 4$. Then $\mu \geq \text{lct}_2(X) = 4/5$.*

Proof There is a unique smooth irreducible curve $\bar{Z} \subset \bar{X}$ such that

$$\bar{Z} \sim \pi^*(-2K_X) - E_1 - 2E_2 - 2E_3 - E_4$$

and $E_2 \cap E_3 \in \bar{Z}$ (cf. the proof of Lemma 6.8). Put $Z = \pi(\bar{Z})$. Then

$$\text{lct}_2(X) \leq c\left(X, \frac{1}{2}Z\right) = \frac{4}{5}.$$

To complete the proof, it is enough to show that $\mu \geq 4/5$. Suppose that $\mu < 4/5$.

By Remark 2.1, we may assume that $Z \not\subset \text{Supp}(D)$, because Z is irreducible.

It follows from (4) that $a_1 \leq 4/5, a_2 \leq 6/5, a_3 \leq 6/5, a_4 \leq 4/5$.

Put $Q = E_2 \cap E_3$. Then it follows from Lemma 4.4 that (3) is not Kawamata log terminal at the point Q and is Kawamata log terminal outside of the point Q . Then

$$2a_2 - \frac{1}{2}a_2 - a_3 \geq 2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 \geq \text{mult}_Q(\bar{D} \cdot E_2) > \frac{5}{4} - a_3,$$

by Lemma 2.3. Similarly, we see that

$$2a_3 - a_2 - a_4 = \bar{D} \cdot E_3 \geq \text{mult}_Q(\bar{D} \cdot E_3) > \frac{5}{4} - a_2,$$

which implies that $a_2 > 5/6$ and $a_3 > 5/6$.

Let $\xi: \tilde{X} \rightarrow \bar{X}$ be a blow-up of the point Q , let E be the exceptional curve of the blow-up ξ , and let \tilde{D} be the proper transform of the divisor \bar{D} on the surface \tilde{X} . Put $\delta = \text{mult}_Q(\bar{D})$.

Let $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ be the proper transforms on \tilde{X} of E_1, E_2, E_3, E_4 , respectively. Then

$$(\tilde{X}, \mu\tilde{D} + \mu a_2 \tilde{E}_2 + \mu a_3 \tilde{E}_3 + (\mu a_2 + \mu a_3 + \mu\delta - 1)E) \tag{5}$$

is not Kawamata log terminal at some point $O \in E$.

Let \tilde{Z} be the proper transform on \tilde{X} of the curve \bar{Z} . Then

$$0 \leq \tilde{Z} \cdot \tilde{D} = 2 - a_2 - a_3 - \text{mult}_Q(\bar{D}) = 2 - a_2 - a_3 - \delta,$$

which implies that $\delta + a_2 + a_3 \leq 2$. We have $\mu a_2 + \mu a_3 + \mu \delta - 1 \leq 2\mu - 1 \leq 3/5$, which implies that (5) is Kawamata log terminal outside of the point O by Theorem 2.2. We have

$$\begin{cases} 2a_3 - a_2 - a_4 - \delta = \tilde{E}_3 \cdot \tilde{D} \geq 0, \\ 2a_2 - a_1 - a_3 - \delta = \tilde{E}_2 \cdot \tilde{D} \geq 0, \end{cases}$$

which implies that $\delta \leq 1$. If $O \notin \tilde{E}_2 \cup \tilde{E}_3$, then

$$1 \geq \delta = \tilde{D} \cdot E \geq \text{mult}_O(\tilde{D} \cdot E) > \frac{5}{4}$$

by Lemma 2.3. Thus, we see that either $O = \tilde{E}_2 \cap E$ or $O = \tilde{E}_3 \cap E$.

Without loss of generality, we may assume that $O = \tilde{E}_2 \cap E$. By Lemma 2.3, one has

$$\begin{aligned} \frac{5}{4} - a_2 &> \frac{7}{6} - a_2 = 2 - \frac{5}{6} - a_2 > 2 - a_2 - a_3 \geq \delta = \tilde{D} \cdot E \geq \text{mult}_O(\tilde{D} \cdot E) \\ &> \frac{5}{4} - a_2, \end{aligned}$$

since $\delta + a_2 + a_3 \leq 2$ and $a_3 > 5/6$. The obtained contradiction concludes the proof. \square

Let τ be a biregular involution of the surface \bar{X} that is induced by the double cover ω .

Lemma 4.7 *Suppose that $m = 5$. Then there exists a unique curve $Z \in |-2K_X|$ such that*

$$c\left(X, \frac{1}{2}Z\right) = \text{lct}_2(X) = \frac{2}{3},$$

and either $D = Z/2$ or $\mu > 2/3$.

Proof Let $\alpha: \bar{X} \rightarrow \check{X}$ be a contraction of the curves \bar{C}, E_5, E_4, E_3 . Then

$$\alpha(E_1) \cdot \alpha(E_1) = \alpha(E_2) \cdot \alpha(E_2) = -1,$$

and \check{X} is a smooth del Pezzo surface such that $K_{\check{X}}^2 = 5$, which implies that there is a smooth irreducible rational curve \check{L}_2 on the surface \check{X} such that $\check{L}_2 \cdot \alpha(E_2) = 1$ and $\check{L}_2 \cdot \check{L}_2 = -1$.

Let \bar{L}_2 be the proper transform of the curve \check{L}_2 on the surface \bar{X} . Then $\bar{L}_2 \cdot \bar{L}_2 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_2 = E_2 \cdot \bar{L}_2 = 1,$$

which implies that $E_1 \cdot \bar{L}_2 = E_3 \cdot \bar{L}_2 = E_4 \cdot \bar{L}_2 = E_5 \cdot \bar{L}_2 = \bar{C} \cdot \bar{L}_2 = 0$.

Let $\beta: \bar{X} \rightarrow \check{X}$ be a contraction of the curves $\bar{L}_2, \bar{C}, E_5, E_4$. Then

$$\beta(E_2) \cdot \beta(E_2) = \beta(E_3) \cdot \beta(E_3) = -1,$$

and \check{X} is a smooth del Pezzo surface such that $K_{\check{X}}^2 = 5$, which implies that there is an irreducible smooth curve $\check{L}_3 \subset \check{X}$ such that $\check{L}_3 \cdot \beta(E_3) = 1$ and $\check{L}_3 \cdot \check{L}_3 = -1$ (cf. the proof of Lemma 6.7).

Let \bar{L}_3 be the proper transform of the curve \check{L}_3 on the surface \bar{X} . Then $\bar{L}_3 \cdot \bar{L}_3 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_3 = E_3 \cdot \bar{L}_3 = 1,$$

which implies that $E_1 \cdot \bar{L}_3 = E_2 \cdot \bar{L}_3 = E_4 \cdot \bar{L}_3 = E_5 \cdot \bar{L}_3 = \bar{C} \cdot \bar{L}_3 = 0$.

If $\tau(\bar{L}_3) = \bar{L}_3$, then $2\pi(\bar{L}_3) \sim -2K_X$, but $\pi(\bar{L}_3)$ is not a Cartier divisor.

Put $Z = \pi(\bar{L}_3 + \tau(\bar{L}_3))$. Then $Z \sim -2K_X$ and $c(X, Z) = 1/3$. We see that $\text{lct}_2(X) \leq 2/3$.

Suppose that $D \neq Z/2$. To complete the proof, it is enough to show that $\mu > 2/3$.

Suppose that $\mu \leq 2/3$. Let us derive a contradiction. It follows from (4) that

$$a_1 \leq \frac{5}{6}, \quad a_2 \leq \frac{4}{3}, \quad a_3 \leq \frac{3}{2}, \quad a_4 \leq \frac{4}{3}, \quad a_5 \leq \frac{5}{6}.$$

By Remark 2.1, without loss of generality we may assume that $\pi(\bar{L}_3) \not\subset \text{Supp}(D)$. Then

$$1 - a_3 = \bar{L}_3 \cdot \bar{D} \geq 0,$$

which implies that $a_3 \leq 1$.

Put $Q = E_2 \cap E_3$. By Lemma 4.4, we may assume that (3) is not Kawamata log terminal at the point Q and is Kawamata log terminal outside of the point Q . Then

$$2a_3 - a_2 - a_4 = \bar{D} \cdot E_3 \geq \text{mult}_Q(\bar{D} \cdot E_3) \geq \frac{1}{\mu} - a_2 > \frac{3}{2} - a_2$$

by Lemma 2.3, which implies that $a_3 > 9/8$ by (4). But $a_3 \leq 1$. □

Lemma 4.8 *Suppose that $m = 6$. Then there exists a unique curve $Z \in |-2K_X|$ such that*

$$c\left(X, \frac{1}{2}Z\right) = \text{lct}_2(X) = \frac{2}{3}$$

and either $D = Z/2$ or $\mu > 2/3$.

Proof Let $\alpha: \bar{X} \rightarrow \check{X}$ be a contraction of the curves \bar{C}, E_6, E_5, E_4 , and E_3 . Then

$$\alpha(E_1) \cdot \alpha(E_1) = \alpha(E_2) \cdot \alpha(E_2) = -1,$$

and \check{X} is a smooth del Pezzo surface such that $K_{\check{X}}^2 = 6$, which implies that there is a smooth irreducible rational curve \check{L}_2 on the surface \check{X} such that $\check{L}_2 \cdot \alpha(E_2) = 1$ and $\check{L}_2 \cdot \check{L}_2 = -1$.

Let \bar{L}_2 be the proper transform of the curve \check{L}_2 on the surface \bar{X} . Then $\bar{L}_2 \cdot \bar{L}_2 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_2 = E_2 \cdot \bar{L}_2 = 1,$$

which implies that $E_1 \cdot \bar{L}_2 = E_3 \cdot \bar{L}_2 = E_4 \cdot \bar{L}_2 = E_5 \cdot \bar{L}_2 = E_6 \cdot \bar{L}_2 = \bar{C} \cdot \bar{L}_2 = 0$.

Let $\beta: \bar{X} \rightarrow \check{X}$ be a contraction of the curves $\bar{L}_2, \bar{C}, E_6, E_5,$ and E_4 . Then

$$\beta(E_2) \cdot \beta(E_2) = \beta(E_3) \cdot \beta(E_3) = -1,$$

and \check{X} is a smooth del Pezzo surface such that $K_{\check{X}}^2 = 6$, which implies that there are irreducible smooth rational curves \check{L}_3 and \check{L}'_2 on the surface \check{X} such that

$$\check{L}_3 \cdot \beta(E_3) = \check{L}'_2 \cdot \beta(E_2) = 1$$

and $\check{L}_3 \cdot \check{L}_3 = \check{L}'_2 \cdot \check{L}'_2 = -1$. Let \bar{L}_3 and \bar{L}'_2 be the proper transforms of the curves \check{L}_3 and \check{L}'_2 on the surface \bar{X} , respectively. Then $\bar{L}_3 \cdot \bar{L}_3 = \bar{L}'_2 \cdot \bar{L}'_2 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_3 = -K_{\bar{X}} \cdot \bar{L}'_2 = E_3 \cdot \bar{L}_3 = E_2 \cdot \bar{L}'_2 = 1,$$

which implies that $\bar{C} \cdot \bar{L}_3 = \bar{C} \cdot \bar{L}'_2 = 0$, and $E_i \cdot \bar{L}_3 = E_j \cdot \bar{L}'_2 = 0$ for every $i \neq 3$ and $j \neq 2$,

Put $\bar{L}_4 = \tau(\bar{L}_3), \bar{L}_5 = \tau(\bar{L}_2), \bar{L}'_5 = \tau(\bar{L}'_2)$. Then $\bar{C} \cdot \bar{L}_4 = \bar{C} \cdot \bar{L}_5 = \bar{C} \cdot \bar{L}'_5 = 0$ and

$$-K_{\bar{X}} \cdot \bar{L}_4 = -K_{\bar{X}} \cdot \bar{L}_5 = -K_{\bar{X}} \cdot \bar{L}'_5 = E_4 \cdot \bar{L}_4 = E_5 \cdot \bar{L}_5 = E_5 \cdot \bar{L}'_5 = 1,$$

which implies that $E_i \cdot \bar{L}_5 = E_i \cdot \bar{L}'_5 = E_j \cdot \bar{L}_4 = 0$ for every $i \neq 5$ and $j \neq 4$.

Put $L_3 = \pi(\bar{L}_3), L_4 = \pi(\bar{L}_4), L_2 = \pi(\bar{L}_2), L'_2 = \pi(\bar{L}'_2), L_5 = \pi(\bar{L}_5), L'_5 = \pi(\bar{L}'_5)$. Then

$$L_3 + L_4 \sim L_2 + L_5 \sim L'_2 + L'_5 \sim -2K_X,$$

and $c(X, L_3 + L_4) = 1/3$, which implies that $\text{lct}_2(X) \leq 2/3$.

Note that $c(X, L_2 + L_5) = c(X, L'_2 + L'_5) = 1/2$.

Suppose that $D \neq (L_3 + L_4)/2$. To complete the proof, it is enough to show that $\mu > 2/3$.

Suppose that $\mu \leq 2/3$. Let us derive a contradiction.

It follows from (4) that $a_1 \leq 6/7, a_2 \leq 10/7, a_3 \leq 12/7, a_4 \leq 12/7, a_5 \leq 10/7, a_6 \leq 6/7$.

By Remark 2.1, without loss of generality we may assume that $\bar{L}_4 \not\subset \text{Supp}(D)$. Then

$$1 - a_4 = \bar{L}_3 \cdot \bar{D} \geq 0,$$

which gives us $a_4 \leq 1$. Similarly, we may assume that either $\bar{L}_2 \not\subset \text{Supp}(D)$ or $\bar{L}_5 \not\subset \text{Supp}(D)$, which implies that either $a_2 \leq 1$ or $a_5 \leq 1$, respectively.

Let us show that $L_2 + L'_2 + L_3 \sim -3K_X$. We can easily see that

$$\begin{aligned} \bar{L}_2 &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{5}{7}E_1 - \frac{10}{7}E_2 - \frac{8}{7}E_3 - \frac{6}{7}E_4 - \frac{4}{7}E_5 - \frac{2}{7}E_6, \\ \bar{L}'_2 &\sim_{\mathbb{Q}} \pi^*(L'_2) - \frac{5}{7}E_1 - \frac{10}{7}E_2 - \frac{8}{7}E_3 - \frac{6}{7}E_4 - \frac{4}{7}E_5 - \frac{2}{7}E_6, \\ \bar{L}_3 &\sim_{\mathbb{Q}} \pi^*(L_3) - \frac{4}{7}E_1 - \frac{8}{7}E_2 - \frac{12}{7}E_3 - \frac{9}{7}E_4 - \frac{6}{7}E_5 - \frac{3}{7}E_6, \end{aligned}$$

which implies that $L_2 + L'_2 + L_3 \sim_{\mathbb{Q}} -3K_X$, since $\text{Pic}(X) \cong \mathbb{Z}^3$ and

$$\begin{aligned} L_2 \cdot L_2 &= \frac{3}{7}, & L'_2 \cdot L'_2 &= \frac{3}{7}, & L_3 \cdot L_3 &= \frac{5}{7}, \\ L'_2 \cdot L_3 &= \frac{8}{7}, & L_2 \cdot L_3 &= \frac{8}{7}, & L_2 \cdot L'_2 &= \frac{10}{7}, \end{aligned}$$

but $L_2 + L'_2 + L_3$ is a Cartier divisor, which implies that $L_2 + L'_2 + L_3 \sim -3K_X$.

Since $c(X, L_2 + L'_2 + L_3) = 1/4$, we may assume that $\text{Supp}(D)$ does not contain at least one curve among L_2, L'_2 , and L_3 by Remark 2.1, which implies that either $a_2 \leq 1$ or $a_3 \leq 1$.

It follows from (4) and $a_4 \leq 2$ that $\mu a_i < 1$ for every i . By Lemma 4.4, there exists a point

$$Q \in \{E_2 \cap E_3, E_3 \cap E_4, E_4 \cap E_5\},$$

such that (3) is not Kawamata log terminal at the point $Q \in \bar{X}$, but it is Kawamata log terminal elsewhere. Take $k \in \{2, 3, 4\}$ such that $Q = E_k \cap E_{k+1}$. It follows from Lemma 2.3 that

$$\begin{cases} 2a_k - a_{k-1} - a_{k+1} = \bar{D} \cdot E_k \geq \text{mult}_Q(\bar{D} \cdot E_k) > \frac{1}{\mu} - a_{k+1} > \frac{3}{2} - a_{k+1}, \\ 2a_{k+1} - a_k - a_{k+2} = \bar{D} \cdot E_{k+1} \geq \text{mult}_Q(\bar{D} \cdot E_{k+1}) > \frac{1}{\mu} - a_k \geq \frac{3}{2} - a_k, \end{cases}$$

which is impossible by (4), since $a_4 \leq 1$, and either $a_2 \leq 1$ or $a_3 \leq 1$. □

Lemma 4.9 *Suppose that $m = 7$. Then the following conditions are equivalent:*

- the curve R is irreducible,
- the surface \bar{X} contains an irreducible curve \bar{L}_4 such that $\bar{L}_4 \cdot \bar{L}_4 = -1$ and $\bar{L}_4 \cdot E_4 = 1$.
- the surface \bar{X} contains an irreducible curve \bar{L}_4 such that $\bar{L}_4 \cdot \bar{L}_4 = -1$, $\bar{L}_4 \cdot E_4 = 1$ and

$$\omega \circ \pi(\bar{L}_4) \subset \text{Supp}(R).$$

curve Z . Then

$$\text{mult}_{\omega'(P')}(L \cdot Z) = 3,$$

because X' has a singularity of type \mathbb{A}_5 at the point P' . Then \bar{X} contains a curve \bar{L}_4 such that

$$\omega' \circ \pi' \circ \eta(\bar{L}_4) = L,$$

and \bar{L}_4 is irreducible. Then $\bar{L}_4 \cdot \bar{L}_4 = -1$ and $\bar{L}_4 \cdot E_4 = 1$. □

The proof of Lemma 4.9 can be simplified using the results obtained in [31, Sect. 2].

Lemma 4.10 *Suppose that $m = 7$ and R is irreducible. Then $\mu \geq \text{lct}_3(X) = 3/5$.*

Proof Arguing as in the proofs of Lemmas 4.7 and 4.8, we see that there is an irreducible smooth rational curve \bar{L}_2 on the surface \bar{X} such that $\bar{L}_2 \cdot \bar{L}_2 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_2 = E_2 \cdot \bar{L}_2 = 1,$$

which implies that $E_1 \cdot \bar{L}_2 = E_3 \cdot \bar{L}_2 = E_4 \cdot \bar{L}_2 = E_5 \cdot \bar{L}_2 = E_6 \cdot \bar{L}_2 = E_7 \cdot \bar{L}_2 = \bar{C} \cdot \bar{L}_2 = 0$.

Put $\bar{L}_5 = \tau(\bar{L}_2)$. Then $\bar{L}_5 \cdot \bar{L}_5 = -1$ and $-K_{\bar{X}} \cdot \bar{L}_5 = E_5 \cdot \bar{L}_5 = 1$, which implies that

$$E_1 \cdot \bar{L}_5 = E_2 \cdot \bar{L}_5 = E_3 \cdot \bar{L}_5 = E_4 \cdot \bar{L}_5 = E_6 \cdot \bar{L}_5 = E_7 \cdot \bar{L}_5 = \bar{C} \cdot \bar{L}_5 = 0.$$

Since the branch curve R is reducible by Lemma 4.9, one can show that there exists an irreducible smooth rational curve \bar{L}_3 on the surface \bar{X} such that $\bar{L}_3 \cdot \bar{L}_3 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_3 = E_3 \cdot \bar{L}_3 = 1,$$

which implies that $E_1 \cdot \bar{L}_3 = E_2 \cdot \bar{L}_3 = E_4 \cdot \bar{L}_3 = E_5 \cdot \bar{L}_3 = E_6 \cdot \bar{L}_3 = E_7 \cdot \bar{L}_3 = \bar{C} \cdot \bar{L}_3 = 0$.

Put $\bar{L}_6 = \tau(\bar{L}_2)$, $\bar{L}_5 = \tau(\bar{L}_3)$, $L_2 = \pi(\bar{L}_2)$, $L_3 = \pi(\bar{L}_4)$, $L_5 = \pi(\bar{L}_5)$ and $L_6 = \pi(\bar{L}_6)$. Then

$$\begin{aligned} \bar{L}_2 &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{3}{4}E_1 - \frac{3}{2}E_2 - \frac{5}{4}E_3 - E_4 - \frac{3}{4}E_5 - \frac{1}{2}E_6 - \frac{1}{4}E_7, \\ \bar{L}_3 &\sim_{\mathbb{Q}} \pi^*(L_3) - \frac{5}{8}E_1 - \frac{5}{4}E_2 - \frac{15}{8}E_3 - \frac{3}{2}E_4 - \frac{9}{8}E_5 - \frac{3}{4}E_6 - \frac{3}{8}E_7, \\ \bar{L}_5 &\sim_{\mathbb{Q}} \pi^*(L_5) - \frac{3}{8}E_1 - \frac{3}{4}E_2 - \frac{9}{8}E_3 - \frac{3}{2}E_4 - \frac{15}{8}E_5 - \frac{5}{4}E_6 - \frac{5}{8}E_7, \\ \bar{L}_6 &\sim_{\mathbb{Q}} \pi^*(L_6) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - E_4 - \frac{5}{4}E_5 - \frac{3}{2}E_6 - \frac{3}{4}E_7, \end{aligned}$$

which implies that $L_2 + 2L_3 \sim -3K_X$. Indeed, we have $L_2 + 2L_3 \sim_{\mathbb{Q}} -3K_X$, since

$$L_2 \cdot L_2 = \frac{1}{2}, \quad L_3 \cdot L_3 = \frac{7}{8}, \quad L_2 \cdot L_3 = \frac{5}{4},$$

and $\text{Pic}(X) \cong \mathbb{Z}^3$. But $L_2 + 2L_3$ is a Cartier divisor, which implies that $L_2 + 2L_3 \sim -3K_X$.

We have $c(X, L_2 + 2L_3) = 3/15$ and $L_2 + 2L_3 \sim -3K_X$, which implies that $\text{lct}_3(X) \leq 3/5$.

To complete the proof, it is enough to show that $\mu \geq 3/5$.

Suppose that $\mu < 3/5$. Let us derive a contradiction.

By Remark 2.1, we may assume that the support of the divisor \bar{D} does not contain at least one component of every curve $\bar{L}_2 + \bar{L}_6, \bar{L}_2 + 2\bar{L}_3, \bar{L}_3 + \bar{L}_5$. But

$$\bar{D} \cdot \bar{L}_i = 1 - a_i,$$

which implies that $a_i \leq 1$ if $\bar{L}_i \not\subset \text{Supp}(\bar{D})$. Therefore, either $a_3 \leq 1$ or $a_2 \leq 1$ and $a_5 \leq 1$.

If $a_3 \leq 1$, then it follows from (4) that

$$a_1 \leq \frac{7}{8}, \quad a_2 \leq \frac{6}{5}, \quad a_3 \leq 1, \quad a_4 \leq \frac{4}{3}, \quad a_5 \leq \frac{5}{3}, \quad a_6 \leq \frac{3}{2}, \quad a_7 \leq \frac{7}{8}.$$

If $a_2 \leq 1$ and $a_5 \leq 1$, then it follows from (4) that

$$a_1 \leq \frac{7}{8}, \quad a_2 \leq 1, \quad a_3 \leq \frac{3}{2}, \quad a_4 \leq \frac{4}{3}, \quad a_5 \leq 1, \quad a_6 \leq \frac{6}{5}, \quad a_7 \leq \frac{7}{8}.$$

By Lemma 4.4, there exists $k \in \{2, 3, 4, 5\}$ such that (3) is not Kawamata log terminal at the point $E_k \cap E_{k+1}$ and is Kawamata log terminal outside of $E_k \cap E_{k+1}$.

Put $Q = E_k \cap E_{k+1}$. Then it follows from Lemma 2.3 that

$$\begin{cases} 2a_k - a_{k-1} - a_{k+1} = \bar{D} \cdot E_k \geq \text{mult}_Q(\bar{D} \cdot E_k) > \frac{1}{\mu} - a_{k+1} > \frac{5}{3} - a_{k+1}, \\ 2a_{k+1} - a_k - a_{k+2} = \bar{D} \cdot E_{k+1} \geq \text{mult}_Q(\bar{D} \cdot E_{k+1}) > \frac{1}{\mu} - a_k \geq \frac{5}{3} - a_k, \end{cases}$$

which is impossible by (4), since we assume that either $a_3 \leq 1$ or $a_2 \leq 1$ and $a_5 \leq 1$. □

Lemma 4.11 *Suppose that $m = 7$ and R is reducible. Then $\mu \geq \text{lct}_2(X) = 1/2$.*

Proof By Lemma 4.9, the surface X contains an irreducible curve \bar{L}_4 such that

$$\omega \circ \pi(\bar{L}_4) \subset \text{Supp}(R)$$

and $-\bar{L}_4 \cdot \bar{L}_4 = \bar{L}_4 \cdot E_4 = 1$. Then $-K_{\bar{X}} \cdot \bar{L}_4 = 1$, which implies that

$$E_1 \cdot \bar{L}_4 = E_2 \cdot \bar{L}_4 = E_3 \cdot \bar{L}_4 = E_5 \cdot \bar{L}_4 = E_6 \cdot \bar{L}_4 = E_7 \cdot \bar{L}_4 = \bar{C} \cdot \bar{L}_4 = 0.$$

Put $L_4 = \pi(\bar{L}_4)$. Then $2L_4 \sim -2K_X$ and

$$\bar{L}_4 \sim_{\mathbb{Q}} \pi^*(L_4) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - 2E_4 - \frac{3}{2}E_5 - E_6 - \frac{1}{2}E_7,$$

which implies that $\text{lct}_2(X) \leq c(X, L_4) = 1/2$.

To complete the proof, it is enough to show that $\mu \geq 1/2$.

Suppose that $\mu < 1/2$. Let us derive a contradiction.

By Remark 2.1, we may assume that $L_4 \not\subset \text{Supp}(D)$. Then

$$0 \leq \bar{L}_4 \cdot \bar{D} = 1 - a_4,$$

which implies that $a_4 \leq 1$. Thus, it follows from (4) that

$$a_1 \leq \frac{7}{8}, \quad a_2 \leq \frac{3}{2}, \quad a_3 \leq \frac{5}{4}, \quad a_4 \leq 1, \quad a_5 \leq \frac{5}{4}, \quad a_6 \leq \frac{3}{2}, \quad a_7 \leq \frac{7}{8}.$$

It follows from Lemma 4.4 that there exists a point

$$Q \in \{E_2 \cap E_3, E_3 \cap E_4, E_4 \cap E_5, E_5 \cap E_6\}$$

such that $\text{LCS}(\bar{X}, \mu\bar{D} + \sum_{i=1}^7 \mu a_i E_i) = Q$.

Without loss of generality, we may assume that either $Q = E_2 \cap E_3$ or $Q = E_3 \cap E_4$.

If $Q = E_3 \cap E_4$, then it follows from Lemma 2.3 that

$$2a_4 - a_3 - a_5 = \bar{D} \cdot E_4 \geq \text{mult}_Q(\bar{D} \cdot E_4) > \frac{1}{\mu} - a_3 > 2 - a_3,$$

which together with (4) imply that $a_4 > 1$, which is a contradiction.

If $Q = E_2 \cap E_3$, then it follows from Lemma 2.3 that

$$2a_3 - a_2 - a_4 = \bar{D} \cdot E_3 \geq \text{mult}_Q(\bar{D} \cdot E_3) > \frac{1}{\mu} - a_2 > 2 - a_2,$$

which together with (4) immediately leads to a contradiction. □

Lemma 4.12 *Suppose that $m = 8$. Then $\mu \geq \text{lct}_3(X) = 1/2$.*

Proof Arguing as in the proofs of Lemmas 4.7 and 4.8, we see that there is an irreducible smooth rational curve \bar{L}_3 on the surface \bar{X} such that $\bar{L}_3 \cdot \bar{L}_3 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_3 = E_3 \cdot \bar{L}_3 = 1,$$

which implies that $E_1 \cdot \bar{L}_3 = E_2 \cdot \bar{L}_3 = E_4 \cdot \bar{L}_3 = E_5 \cdot \bar{L}_3 = E_6 \cdot \bar{L}_3 = E_7 \cdot \bar{L}_3 = \bar{C} \cdot \bar{L}_3 = 0$.

Put $\bar{L}_6 = \tau(\bar{L}_3)$. Then $\bar{L}_6 \cdot \bar{L}_6 = -1$ and $-K_{\bar{X}} \cdot \bar{L}_6 = E_6 \cdot \bar{L}_6 = 1$, which implies that

$$E_1 \cdot \bar{L}_6 = E_2 \cdot \bar{L}_6 = E_3 \cdot \bar{L}_6 = E_4 \cdot \bar{L}_6 = E_5 \cdot \bar{L}_6 = E_7 \cdot \bar{L}_6 = \bar{C} \cdot \bar{L}_6 = 0.$$

Put $L_3 = \pi(\bar{L}_3)$ and $L_6 = \pi(\bar{L}_6)$. Then $3L_3 \sim 3L_6 \sim -3K_X$. On the other hand, we have

$$\begin{aligned} \bar{L}_3 &\sim_{\mathbb{Q}} \pi^*(L_3) - \frac{2}{3}E_1 - \frac{4}{3}E_2 - 2E_3 - \frac{5}{3}E_4 - \frac{4}{3}E_5 - E_6 - \frac{2}{3}E_7 - \frac{1}{3}E_8, \\ \bar{L}_6 &\sim_{\mathbb{Q}} \pi^*(L_6) - \frac{1}{3}E_1 - \frac{2}{3}E_2 - E_3 - \frac{4}{3}E_4 - \frac{5}{3}E_5 - 2E_6 - \frac{4}{3}E_7 - \frac{2}{3}E_8, \end{aligned}$$

which implies $c(X, L_3) = c(X, L_6) = 1/2$. Then $\text{lct}_3(X) \leq 1/2$.

To complete the proof, it is enough to show that $\mu \geq 1/2$.

Suppose that $\mu < 1/2$. Let us derive a contradiction.

By Remark 2.1, we may assume that $\text{Supp}(\bar{D})$ does not contain \bar{L}_3 and \bar{L}_6 . Then

$$1 - a_3 = \bar{D} \cdot \bar{L}_3 \geq 0,$$

which implies that $a_3 \leq 1$. Similarly, we have $a_6 \leq 1$. Then it follows from (4) that

$$a_1 \leq \frac{8}{9}, \quad a_2 \leq \frac{7}{6}, \quad a_3 \leq 1, \quad a_4 \leq \frac{4}{3}, \quad a_5 \leq \frac{4}{3}, \quad a_6 \leq 1, \quad a_7 \leq \frac{7}{6}, \quad a_8 \leq \frac{8}{9}.$$

By Lemma 4.4, there exists $k \in \{2, 3, 4, 5, 6\}$ such that (3) is not Kawamata log terminal at the point $E_k \cap E_{k+1}$ and is Kawamata log terminal outside of the point $E_k \cap E_{k+1}$.

Put $Q = E_k \cap E_{k+1}$. Then it follows from Lemma 2.3 that

$$\begin{cases} 2a_k - a_{k-1} - a_{k+1} = \bar{D} \cdot E_k \geq \text{mult}_Q(\bar{D} \cdot E_k) > \frac{1}{\mu} - a_{k+1} > \frac{1}{2} - a_{k+1}, \\ 2a_{k+1} - a_k - a_{k+2} = \bar{D} \cdot E_{k+1} \geq \text{mult}_Q(\bar{D} \cdot E_{k+1}) > \frac{1}{\mu} - a_k \geq \frac{1}{2} - a_k, \end{cases}$$

which is impossible by (4), since $a_3 \leq 1$ and $a_6 \leq 1$. □

The assertion of Theorem 4.1 is proved.

5 One Non-Cyclic Singular Point

Let X be a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ with canonical singularities such that $|\text{Sing}(X)| = 1$, and $\text{Sing}(X)$ consists of a singular point of type $\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{D}_7, \mathbb{D}_8, \mathbb{E}_6, \mathbb{E}_7$, or \mathbb{E}_8 .

Theorem 5.1 *The following equality holds:*

$$\text{lct}(X) = \begin{cases} \text{lct}_2(X) = 1/3 & \text{if } P \text{ is a point of type } \mathbb{D}_8, \\ \text{lct}_2(X) = 2/5 & \text{if } P \text{ is a point of type } \mathbb{D}_7, \\ \text{lct}_1(X) & \text{in the remaining cases.} \end{cases}$$

Corollary 5.2 *The inequality $\text{lct}(X) \leq 1/2$ holds.*

In the rest of this section, we will prove Theorem 5.1.

Let D be an effective \mathbb{Q} -divisor on X such that $D \sim_{\mathbb{Q}} -K_X$. We must show that

$$c(X, D) \geq \begin{cases} \text{lct}_2(X) = 1/3 & \text{if } P \text{ is a point of type } \mathbb{D}_8, \\ \text{lct}_2(X) = 2/5 & \text{if } P \text{ is a point of type } \mathbb{D}_7, \\ \text{lct}_1(X) & \text{in the remaining cases.} \end{cases}$$

To prove Theorem 5.1, put $\mu = c(X, D)$.

Suppose that $\mu < \text{lct}_1(X)$. Then $\text{LCS}(X, \mu D) = \text{Sing}(X)$ by Lemma 2.6. Put $P = \text{Sing}(X)$.

Let $\pi : \bar{X} \rightarrow X$ be a minimal resolution, let E_1, E_2, \dots, E_m be irreducible π -exceptional curves, let C be the curve in $|-K_X|$ such that $P \in C$, and let \bar{C} be its proper transform on \bar{X} . Then

$$\bar{C} \sim_{\mathbb{Q}} \pi^*(C) - \sum_{i=1}^m n_i E_i,$$

where $n_i \in \mathbb{N}$. Without loss of generality, we may assume that $E_3 \cdot \sum_{i \neq 3} E_i = 3$. Then

$$\text{lct}_1(X) = c(X, C) = \frac{1}{n_3} = \begin{cases} 1/2 & \text{if } P \text{ is of type } \mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{D}_7, \text{ or } \mathbb{D}_8, \\ 1/3 & \text{if } P \text{ is of type } \mathbb{E}_6, \\ 1/4 & \text{if } P \text{ is of type } \mathbb{E}_7, \\ 1/6 & \text{if } P \text{ is of type } \mathbb{E}_8. \end{cases}$$

By Remark 2.1, we may assume that $C \not\subset \text{Supp}(D)$, since the curve C is irreducible.

Let \bar{D} be the proper transform of the divisor D on the surface \bar{X} . Then

$$\bar{D} \sim_{\mathbb{Q}} \pi^*(D) - \sum_{i=1}^m a_i E_i,$$

where a_i is a non-negative rational number. Then

$$K_{\bar{X}} + \mu \left(\bar{D} + \sum_{i=1}^m a_i E_i \right) \sim_{\mathbb{Q}} \pi^*(K_X + \mu D),$$

which implies that $(\bar{X}, \mu \bar{D} + \sum_{i=1}^m \mu a_i E_i)$ is not Kawamata log terminal (see Remark 2.4).

Lemma 5.3 *The equality $\mu a_3 = 1$ holds.*

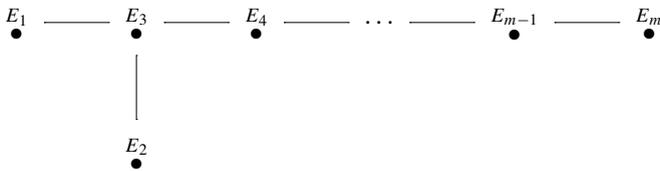
Proof The equality $\mu a_3 = 1$ follows from Lemma 2.5. □

Lemma 5.4 *Suppose that P is not a point of type $\mathbb{E}_6, \mathbb{E}_7,$ or \mathbb{E}_8 . Then*

$$\mu \geq \begin{cases} \text{lct}_2(X) = 1/3 & \text{if } P \text{ is a point of type } \mathbb{D}_8, \\ \text{lct}_2(X) = 2/5 & \text{if } P \text{ is a point of type } \mathbb{D}_7, \end{cases}$$

and P is either a point of type \mathbb{D}_7 or is a point of type \mathbb{D}_8 .

Proof Without loss of generality, we may assume that the diagram



shows how the π -exceptional curves intersect each other. Then

$$\bar{C} \sim_{\mathbb{Q}} \pi^*(C) - E_1 - E_2 - E_m - \sum_{i=3}^{m-1} 2E_i,$$

which implies that $\bar{C} \cdot E_{m-1} = 1$ and $\bar{C} \cdot E_i = 0 \iff i \neq m - 1$. Then

$$\left\{ \begin{array}{l} 1 - a_{m-1} = \bar{D} \cdot \bar{C} \geq 0, \\ 2a_1 - a_3 = \bar{D} \cdot E_1 \geq 0, \\ 2a_2 - a_3 = \bar{D} \cdot E_2 \geq 0, \\ 2a_3 - a_1 - a_2 - a_3 = \bar{D} \cdot E_3 \geq 0, \\ \vdots \\ 2a_{m-1} - a_{m-2} - a_m = \bar{D} \cdot E_{m-1} \geq 0, \\ 2a_m - a_{m-1} = \bar{D} \cdot E_m \geq 0, \end{array} \right. \tag{6}$$

which easily implies that $a_3 \leq 2$ if $m \leq 6$. But $\mu a_3 = 1$ and $\mu < \text{lct}_1(X) = 1/2$ by Lemma 5.3, which implies that either $m = 7$ or $m = 8$.

Arguing as in the proofs of Lemmas 4.7 and 4.8, we may assume that there is an irreducible smooth rational curve \bar{L}_1 on the surface \bar{X} such that $\bar{L}_1 \cdot \bar{L}_1 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_1 = E_1 \cdot \bar{L}_1 = 1,$$

which implies that $\bar{C} \cdot \bar{L}_1 = 0$ and $E_i \cdot \bar{L}_1 = 0 \iff i \neq 1$.

Let $\omega: X \rightarrow \mathbb{P}(1, 1, 2)$ be the natural double cover given by $|-2K_X|$, and let τ be a biregular involution of the surface \bar{X} that is induced by ω . Put $\bar{L}_2 = \tau(\bar{L}_1)$. If $m = 7$, then

$$-K_{\bar{X}} \cdot \bar{L}_2 = E_2 \cdot \bar{L}_2 = 1$$

and $\bar{L}_2 \cdot \bar{L}_2 = -1$, which implies that $\bar{C} \cdot \bar{L}_2 = 0$ and $E_i \cdot \bar{L}_2 = 0 \iff i \neq 2$.

Put $L_1 = \pi(\bar{L}_1)$ and $L_2 = \pi(\bar{L}_2)$. Then $L_1 + L_2 \sim -2K_X$. If $m = 7$, then

$$\begin{aligned} \bar{L}_1 &\sim_{\mathbb{Q}} \pi^*(L_1) - \frac{7}{4}E_1 - \frac{5}{4}E_2 - \frac{5}{2}E_3 - 2E_4 - \frac{3}{2}E_5 - E_6 - \frac{1}{2}E_7, \\ \bar{L}_2 &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{5}{4}E_1 - \frac{7}{4}E_2 - \frac{5}{2}E_3 - 2E_4 - \frac{3}{2}E_5 - E_6 - \frac{1}{2}E_7, \end{aligned}$$

which implies that $c(X, L_1 + L_2) = 1/5$ and $\text{lct}_2(X) \leq 2/5$. If $m = 7$, then

$$a_3 \leq \frac{5}{2}$$

by (6). But $\mu a_3 = 1$ by Lemma 5.3. Then $\mu \geq 2/5$ if $m = 7$, which is exactly what we need.

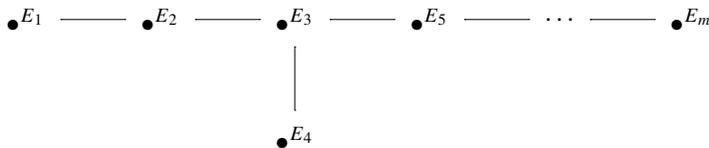
We may assume that $m = 8$. Then $\bar{L}_2 = \bar{L}_1$ and

$$\bar{L}_1 \sim_{\mathbb{Q}} \pi^*(L_1) - 2E_1 - \frac{3}{2}E_2 - 3E_3 - \frac{5}{2}E_4 - 2E_5 - \frac{3}{2}E_6 - E_7 - \frac{1}{2}E_8,$$

which implies that $\text{lct}_2(X) \leq c(X, L_1) = 1/3$. But $a_3 \leq 1/3$ by (6) and $\mu a_3 = 1$ by Lemma 5.3, which implies that $\mu \geq 1/3$, which completes the proof since $\text{lct}_2(X) \geq \text{lct}(X)$. □

To complete the proof of Theorem 5.1, we may assume that P is a point of type $\mathbb{E}_6, \mathbb{E}_7$, or \mathbb{E}_8 .

Without loss of generality, we may assume that the diagram



shows how the π -exceptional curves intersect each other. It is well known (cf. [29, 30]) that

- if $m = 6$, then $\bar{C} \cdot E_4 = 1$, which implies that $\bar{C} \cdot E_i = 0 \iff i \neq 4$,
- if $m = 7$, then $\bar{C} \cdot E_1 = 1$, which implies that $\bar{C} \cdot E_i = 0 \iff i \neq 1$,
- if $m = 8$, then $\bar{C} \cdot E_8 = 1$, which implies that $\bar{C} \cdot E_i = 0 \iff i \neq 8$.

Put $k = 4$ if $m = 6$, put $k = 1$ if $m = 7$, put $k = 8$ if $m = 8$. Then

$$\left\{ \begin{array}{l} 1 - a_k = \bar{D} \cdot \bar{C} \geq 0, \\ 2a_1 - a_3 = \bar{D} \cdot E_1 \geq 0, \\ 2a_2 - a_3 - a_1 = \bar{D} \cdot E_2 \geq 0, \\ 2a_3 - a_2 - a_4 - a_5 = \bar{D} \cdot E_3 \geq 0, \\ 2a_4 - a_3 = \bar{D} \cdot E_4 \geq 0, \\ 2a_5 - a_3 - a_6 = \bar{D} \cdot E_5 \geq 0, \\ \vdots \\ 2a_{m-1} - a_{m-2} - a_m = \bar{D} \cdot E_{m-1} \geq 0, \\ 2a_m - a_{m-1} = \bar{D} \cdot E_m \geq 0, \end{array} \right. \tag{7}$$

which implies that $a_3 < n_3$. But $n_3 = 1/\text{lct}_1(X)$ and $\mu a_3 = 1$ by Lemma 5.3. Then $\mu \geq \text{lct}_1(X)$.

The assertion of Theorem 5.1 is proved.

6 Many Singular Points

Let X be a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ with canonical singularities such that $|\text{Sing}(X)| \geq 2$.

Theorem 6.1 *The following equality holds:*

$\text{lct}(X)$

$$= \begin{cases} \text{lct}_2(X) = 1/2 & \text{if } \text{Sing}(X) \text{ consists of a point of type } \mathbb{A}_7 \text{ and a point of type } \mathbb{A}_1, \\ \text{lct}_2(X) = 2/3 & \text{if } X \text{ has a singular point of type } \mathbb{A}_6, \\ \text{lct}_2(X) = 2/3 & \text{if } X \text{ has a singular point of type } \mathbb{A}_5, \\ \text{lct}_2(X) = \min(\text{lct}_1(X), 4/5) & \text{if } X \text{ has a singular point of type } \mathbb{A}_4, \\ \text{lct}_1(X) & \text{in the remaining cases,} \end{cases}$$

and if there exists an effective \mathbb{Q} -divisor D on the surface X such that $D \sim_{\mathbb{Q}} -K_X$ and

$$c(X, D) = \text{lct}(X) = \frac{2}{3},$$

then either D is an irreducible curve in $|-K_X|$ with a cusp at a point in $\text{Sing}(X)$ of type \mathbb{A}_2 , or the divisor D is uniquely defined and it can be explicitly described.

Let D be an arbitrary effective \mathbb{Q} -divisor on the surface X such that

$$D \sim_{\mathbb{Q}} -K_X,$$

and put $\mu = c(X, D)$. To prove Theorem 6.1, it is enough to show that

$$\mu \geq \begin{cases} \text{lct}_2(X) = 1/2 & \text{if } \text{Sing}(X) \text{ consists of a point of type } \mathbb{A}_7 \text{ and a point of type } \mathbb{A}_1, \\ \text{lct}_2(X) = 2/3 & \text{if } X \text{ has a singular point of type } \mathbb{A}_6, \\ \text{lct}_2(X) = 2/3 & \text{if } X \text{ has a singular point of type } \mathbb{A}_5, \\ \text{lct}_2(X) = \min(\text{lct}_1(X), 4/5) & \text{if } X \text{ has a singular point of type } \mathbb{A}_4, \\ \text{lct}_1(X) & \text{in the remaining cases,} \end{cases}$$

and if $\mu = \text{lct}(X) = 2/3$, then we have the following two possibilities:

- either D is a curve in $|-K_X|$ with a cusp at a point in $\text{Sing}(X)$ of type \mathbb{A}_2 ,
- or the divisor D is uniquely defined and it can be explicitly described.

Lemma 6.2 *If $\text{Sing}(X)$ has a point of type $\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{E}_6, \mathbb{E}_7$, or \mathbb{E}_8 , then $\mu \geq \text{lct}_1(X)$.*

Proof Suppose that $\text{Sing}(X)$ has a point of type $\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{E}_6, \mathbb{E}_7$, or \mathbb{E}_8 , but $\mu < \text{lct}_1(X)$. Then

$$\text{LCS}(X, \mu D) \subsetneq \text{Sing}(X)$$

and $\text{LCS}(X, \mu D)$ consists of a point in $\text{Sing}(X)$ that is not of type \mathbb{A}_1 or \mathbb{A}_2 by Lemma 2.6.

If the locus $\text{LCS}(X, \mu D)$ is a singular point of the surface X of type $\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{E}_6, \mathbb{E}_7$, or \mathbb{E}_8 , then arguing as in the proof of Theorem 5.1, we immediately obtain a contradiction.

By Remark 1.21, the locus $\text{LCS}(X, \mu D)$ must be a singular point of the surface X of type \mathbb{A}_3 , and we can easily obtain a contradiction arguing as in the proof of Corollary 4.5. □

Lemma 6.3 *Suppose that $\text{Sing}(X)$ consists of points of type $\mathbb{A}_1, \mathbb{A}_2$, or \mathbb{A}_3 . Then $\mu \geq \text{lct}_1(X)$. If*

$$\mu = \text{lct}_1(X) = \frac{2}{3},$$

then D is a curve in $|-K_X|$ with a cusp at a point in $\text{Sing}(X)$ of type \mathbb{A}_2 .

Proof This follows from Lemma 2.6 and the proof of Corollary 4.5. □

By Remark 1.21 and Lemmas 6.2 and 6.2, we may assume that

$\text{Sing}(X)$

$$\in \left\{ \begin{array}{l} \mathbb{A}_7 + \mathbb{A}_1, \mathbb{A}_6 + \mathbb{A}_1, \mathbb{A}_5 + \mathbb{A}_1, \mathbb{A}_5 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{A}_5 + \mathbb{A}_2, \mathbb{A}_5 + \mathbb{A}_2 + \mathbb{A}_1, \\ \mathbb{A}_4 + \mathbb{A}_4, \mathbb{A}_4 + \mathbb{A}_3, \mathbb{A}_4 + \mathbb{A}_2 + \mathbb{A}_1, \mathbb{A}_4 + \mathbb{A}_2, \mathbb{A}_4 + \mathbb{A}_1 + \mathbb{A}_1, \mathbb{A}_4 + \mathbb{A}_1 \end{array} \right\},$$

which implies that there is a point $P \in \text{Sing}(X)$ that is a point of type \mathbb{A}_m for $m \in \{4, 5, 6, 7\}$.

Let $\pi: \bar{X} \rightarrow X$ be a minimal resolution, let E_1, E_2, \dots, E_m be π -exceptional curves such that

$$E_i \cdot E_j \neq 0 \iff |i - j| \leq 1$$

and $\pi(E_i) = P$ for every $i \in \{1, \dots, m\}$, let C be the unique curve in $|-K_X|$ such that $P \in C$, and let \bar{C} be the proper transform of the curve C on the surface \bar{X} . Then

$$\bar{C} \cdot E_1 = \bar{C} \cdot E_m = 1,$$

and $\bar{C} \cdot E_2 = \bar{C} \cdot E_3 = \dots = \bar{C} \cdot E_{m-1} = 0$. Note that $\bar{C} \cong \mathbb{P}^1$ and $\bar{C} \cdot \bar{C} = -1$.

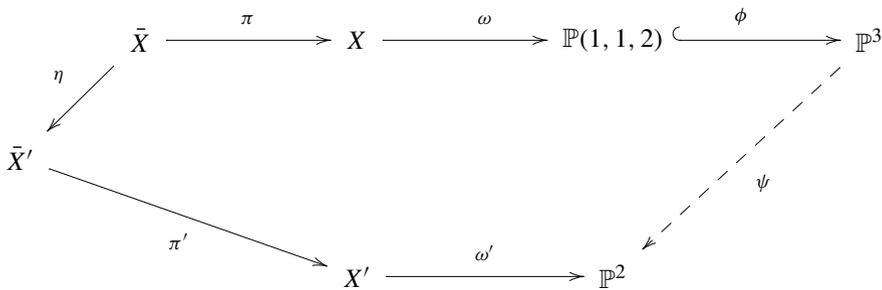
Let \bar{D} be the proper transform of D on the surface \bar{X} . Then

$$\bar{D} \sim_{\mathbb{Q}} \pi^*(D) - \sum_{i=1}^m a_i E_i,$$

where a_i is a non-negative rational number. Then

$$\left\{ \begin{array}{l} 1 - a_1 - a_m = \bar{D} \cdot \bar{C} \geq 0, \\ 2a_1 - a_2 = \bar{D} \cdot E_1 \geq 0, \\ \vdots \\ 2a_{m-1} - a_{m-2} - a_m = \bar{D} \cdot E_{m-1} \geq 0, \\ 2a_m - a_{m-1} = \bar{D} \cdot E_m \geq 0. \end{array} \right. \tag{8}$$

Let $\eta: \bar{X} \rightarrow \bar{X}'$ be a contraction of the curve \bar{C} . Then there is a commutative diagram



where ω and ω' are natural double covers π' is a minimal resolution, ϕ is an anti-canonical embedding, and ψ is a projection from $\phi \circ \omega(P)$. Put $P' = \eta(E_2)$. Then $P' \in \text{Sing}(X')$.

Remark 6.4 The birational morphism π' contracts the smooth curves $\eta(E_2), \eta(E_3), \dots, \eta(E_{m-1})$, and $\pi' \circ \eta$ contracts all π -exceptional curves that are different from the curves E_1, E_2, \dots, E_m .

Let R be the branch curve in $\mathbb{P}(1, 1, 2)$ of the double cover ω . Put $R' = \psi \circ \phi(R)$.

Lemma 6.5 *Suppose that $m = 7$. Then $\mu \geq \text{lct}_2(X) = 1/2$.*

Proof Let $\alpha: \bar{X} \rightarrow \check{X}$ be a contraction of the irreducible curves $\bar{C}, E_7, E_6, E_5, E_4, E_3$, and E_2 , and let F be the π -exceptional curve such that $\pi(F)$ is a point of type \mathbb{A}_1 . Then

$$\check{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)).$$

Let \check{L}_2 be the fiber of the projection $\check{X} \rightarrow \mathbb{P}^1$ such that $\alpha(\bar{C}) \in \check{L}_2$, and let \bar{L}_2 be the proper transform of the curve \check{L}_2 on the surface \bar{X} via α . Then $\bar{L}_2 \cdot \bar{L}_2 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_2 = E_2 \cdot \bar{L}_2 = F \cdot \bar{L}_2 = 1,$$

which implies that $E_1 \cdot \bar{L}_2 = E_3 \cdot \bar{L}_2 = E_4 \cdot \bar{L}_2 = E_5 \cdot \bar{L}_2 = E_6 \cdot \bar{L}_2 = E_7 \cdot \bar{L}_2 = \bar{C} \cdot \bar{L}_2 = 0$.

Let $\beta: \bar{X} \rightarrow \check{X}$ be a contraction of the curves $\bar{L}_2, E_2, \bar{C}, E_7, E_6, E_5, E_4$. Then

$$\beta(E_3) \cdot \beta(E_3) = \beta(F) \cdot \beta(F) = 0,$$

and \check{X} is a smooth del Pezzo surface such that $K_{\check{X}}^2 = 8$. Then $\check{X} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let \check{L}_4 be the curve in $|\beta(F)|$ such that $\beta(E_4) \in \check{L}_4$, and let \bar{L}_3 be its proper transform on the surface \bar{X} via β . Then one can easily check that $\bar{L}_4 \cdot \bar{L}_4 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_4 = E_4 \cdot \bar{L}_4 = 1,$$

which implies that $E_1 \cdot \bar{L}_4 = E_2 \cdot \bar{L}_4 = E_3 \cdot \bar{L}_4 = E_5 \cdot \bar{L}_4 = E_6 \cdot \bar{L}_4 = E_7 \cdot \bar{L}_4 = \bar{C} \cdot \bar{L}_4 = F \cdot \bar{L}_4 = 0$.

Put $L_4 = \pi(\bar{L}_4)$. Then one can easily check that

$$\bar{L}_4 \sim_{\mathbb{Q}} \pi^*(L_4) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - 2E_4 - \frac{3}{2}E_5 - E_6 - \frac{1}{2}E_7,$$

which implies that $c(X, L_4) = 1/2$. But $2L_4 \sim -2K_X$, which implies that $\text{lct}_2(X) \leq 1/2$.

Arguing as in the proof of Lemma 4.9, we see that $\omega(L_4) \subset \text{Supp}(R)$.

Arguing as in the proof of Lemma 4.11 and using (8), we see that $\mu \geq \text{lct}_2(X) = 1/2$. □

Lemma 6.6 *Suppose that $m = 6$. Then $\mu \geq \text{lct}_2(X) = 2/3$, and if $\mu = 2/3$, then*

- either D is a curve in $|-K_X|$ with a cusp at a point in $\text{Sing}(X)$ of type \mathbb{A}_2 ,
- or the divisor D is uniquely defined and can be explicitly described.

Proof Let $\alpha: \bar{X} \rightarrow \check{X}$ be a contraction of the curves $\bar{C}, E_6, E_5, E_4, E_3, E_2$. Then \check{X} is a smooth surface such that $K_{\check{X}}^2 = 7$, and $-K_X$ is nef. There is a birational morphism $\gamma: \check{X} \rightarrow \hat{X}$ such that

$$\hat{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)),$$

and γ is a blow-down of a smooth irreducible rational curve that does not contain the point $\alpha(\bar{C})$.

Let \tilde{L}_2 be the fiber of the projection $\hat{X} \rightarrow \mathbb{P}^1$ such that $\gamma \circ \alpha(\bar{C}) \in \hat{L}_2$, and let \bar{L}_2 be the proper transform of the curve \hat{L}_2 on the surface \bar{X} via $\gamma \circ \alpha$. Then $\bar{L}_2 \cdot \bar{L}_2 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_2 = E_2 \cdot \bar{L}_2 = 1,$$

which implies that $E_1 \cdot \bar{L}_2 = E_3 \cdot \bar{L}_2 = E_4 \cdot \bar{L}_2 = E_5 \cdot \bar{L}_2 = E_6 \cdot \bar{L}_2 = \bar{C} \cdot \bar{L}_2 = 0$.

Let $\beta: \check{X} \rightarrow \bar{X}$ be a contraction of the curves $\bar{L}_2, \bar{C}, E_6, E_5, E_4$, and let F be the π -exceptional curve such that $\pi(F)$ is a point of type \mathbb{A}_1 . Then

$$\beta(E_2) \cdot \beta(E_2) = \beta(E_3) \cdot \beta(E_3) = \beta(F) \cdot \beta(F) = -1,$$

and \check{X} is a smooth del Pezzo surface such that $K_{\check{X}}^2 = 6$. Thus, there exists an irreducible smooth rational curve \check{L}_3 on the surface \check{X} such that $\check{L}_3 \cdot \check{L}_3 = -1$, $\check{L}_3 \cdot \beta(E_3) = 1$, and $\check{L}_3 \cdot \beta(F) = 0$.

Let \bar{L}_3 be the proper transform of the curve \check{L}_3 on the surface \bar{X} . Then $\bar{L}_3 \cdot \bar{L}_3 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_3 = E_3 \cdot \bar{L}_3 = 1,$$

which implies that $E_1 \cdot \bar{L}_3 = E_2 \cdot \bar{L}_3 = E_4 \cdot \bar{L}_3 = E_5 \cdot \bar{L}_3 = E_6 \cdot \bar{L}_3 = \bar{C} \cdot \bar{L}_3 = F \cdot \bar{L}_3 = 0$.

Put $\bar{L}_4 = \tau(\bar{L}_3)$ and $\bar{L}_5 = \tau(\bar{L}_2)$. Then $\bar{C} \cdot \bar{L}_4 = \bar{C} \cdot \bar{L}_5 = 0$ and

$$-K_{\bar{X}} \cdot \bar{L}_4 = -K_{\bar{X}} \cdot \bar{L}_5 = E_4 \cdot \bar{L}_4 = E_5 \cdot \bar{L}_5 = 1,$$

which implies that $E_i \cdot \bar{L}_5 = E_j \cdot \bar{L}_4 = 0$ for every $i \neq 5$ and $j \neq 4$.

Put $L_3 = \pi(\bar{L}_3)$, $L_4 = \pi(\bar{L}_4)$, $L_2 = \pi(\bar{L}_2)$ and $L_5 = \pi(\bar{L}_5)$. Then

$$L_3 + L_4 \sim L_2 + L_5 \sim -2K_X,$$

which implies that $c(X, L_3 + L_4) = 1/3$ and $c(X, L_2 + L_5) = 1/2$. Then $\text{lct}_2(X) \leq 2/3$. But

$$\bar{L}_2 \sim_{\mathbb{Q}} \pi^*(L_2) - \frac{5}{7}E_1 - \frac{10}{7}E_2 - \frac{8}{7}E_3 - \frac{6}{7}E_4 - \frac{4}{7}E_5 - \frac{2}{7}E_6 - \frac{1}{2}F,$$

$$\bar{L}_3 \sim_{\mathbb{Q}} \pi^*(L_3) - \frac{4}{7}E_1 - \frac{8}{7}E_2 - \frac{12}{7}E_3 - \frac{9}{7}E_4 - \frac{6}{7}E_5 - \frac{3}{7}E_6,$$

which implies that $c(X, 2L_2 + L_3) = 1/4$. Then $2L_2 + L_3 \sim_{\mathbb{Q}} -3K_X$, since $\text{Pic}(X) \cong \mathbb{Z}^2$ and

$$L_2 \cdot L_2 = \frac{3}{7}, \quad L_3 \cdot L_3 = \frac{5}{7}, \quad L_2 \cdot L_3 = \frac{8}{7},$$

but $2L_2 + L_3$ is a Cartier divisor, which implies that $2L_2 + L_3 \sim -3K_X$.

If D is not a curve in $|-K_X|$ and $D \neq (L_3 + L_4)/2$, then arguing as in the proof of Lemma 4.8, we easily see that $\mu > 2/3$, since we can use (8). The lemma is proved (see Example 1.26). \square

Lemma 6.7 *Suppose that $m = 5$. Then $\mu \geq \text{lct}_2(X) = 2/3$, and if $\mu = 2/3$, then*

- *either D is a curve in $|-K_X|$ with a cusp at a point in $\text{Sing}(X)$ of type \mathbb{A}_2 ,*
- *or the divisor D is uniquely defined and can be explicitly described.*

Proof The curve R' has an ordinary tacnodal singularity at the point $\omega'(P')$, which implies that there exists a line $L' \subset \mathbb{P}^2$ such that either $L' \subset \text{Supp}(R')$ or $L' \not\subset \text{Supp}(R')$ and

$$\text{mult}_{\omega'(P')}(L' \cdot R') = 4.$$

There are irreducible smooth rational curves L'_3 and L'_4 on the surface X' such that

$$\omega'(L'_3) = \omega'(L'_4) = L'$$

and $L'_3 = L'_4 \iff L' \subset \text{Supp}(R')$. Note that neither L'_3 nor L'_4 contains a point in $\text{Sing}(X') \setminus R'$.

Let \bar{L}'_3 be the proper transform of the curve L'_3 on the surface \bar{X}' . Then

$$\bar{L}'_3 \cap \eta(E_1) = \bar{L}'_3 \cap \eta(E_2) = \bar{L}'_3 \cap \eta(E_4) = \bar{L}'_3 \cap \eta(E_5) = \emptyset,$$

and $\bar{L}'_3 \cdot \eta(E_3) = 1$. Let \bar{L}'_4 be the proper transform of the curve L'_4 on the surface \bar{X}' . Then

$$\bar{L}'_4 \cap \eta(E_1) = \bar{L}'_4 \cap \eta(E_2) = \bar{L}'_4 \cap \eta(E_4) = \bar{L}'_4 \cap \eta(E_5) = \emptyset,$$

and $\bar{L}'_4 \cdot \eta(E_3) = 1$. One can also check that $\bar{L}'_3 \cap \bar{L}'_4 = \emptyset$ if $\bar{L}'_3 \neq \bar{L}'_4$.

Let \bar{L}_3 and \bar{L}_4 be the proper transforms of the curves \bar{L}'_3 and \bar{L}'_4 on the surface \bar{X} , respectively, and let us put $L_3 = \pi(\bar{L}_3)$ and $L_4 = \pi(\bar{L}_4)$. Then

$$\bar{L}_3 + \bar{L}_4 \sim -2K_X$$

and $c(X, \bar{L}_3 + \bar{L}_4) = 1/3$, which implies that $\text{lct}_2(X) \leq 2/3$.

If $D \neq (\bar{L}_3 + \bar{L}_4)/2$, then (8), the proof of Lemma 4.7, and Lemma 2.6 imply that

$$\mu \geq \text{lct}_2(X) = \frac{2}{3},$$

and if $\mu = 2/3$, then D is a curve in $|-K_X|$ with a cusp at a point in $\text{Sing}(X)$ of type \mathbb{A}_2 . □

Lemma 6.8 *Suppose that $m = 4$. Then*

$$\mu \geq \text{lct}_2(X) = \min(\text{lct}_1(X), 4/5) \geq \frac{2}{3},$$

and if $\mu = 2/3$, then D is a curve in $|-K_X|$ with a cusp at a point in $\text{Sing}(X)$ of type \mathbb{A}_2 .

Proof The point $\omega'(P')$ is an ordinary cusp of the curve R' . Then there is a line $L' \subset \mathbb{P}^2$ such that

$$\text{mult}_{\omega'(P')}(L' \cdot R') = 3.$$

Let Z' be a curve in X' such that $\omega'(Z') = L'$ and $-K_{X'} \cdot Z' = 2$. Then

$$Z' \cap \text{Sing}(X') = \text{Sing}(Z') = R',$$

and Z' is irreducible curve that has an ordinary cusp at the point R' .

Let \bar{Z}' be the proper transform of the curve Z' on the surface \bar{X}' . Then Z' is smooth and

$$\eta(E_2) \cap \eta(E_3) \in \bar{Z}'.$$

Let \bar{Z} be the proper transform of the curve \bar{Z}' on the surface \bar{X} . Put $Z = \pi(\bar{Z})$. Then

$$\bar{Z} \sim \pi^*(Z) - E_1 - 2E_2 - 2E_3 - E_4$$

and $E_2 \cap E_3 \in Z$. Then $c(X, Z) = 2/5$, which implies that $\text{lct}_2(X) \leq 4/5$.

Arguing as in the proof of Lemma 4.6 and using Lemma 2.6 and (8), we see that

$$\mu \geq \text{lct}_2(X) = \min(\text{lct}_1(X), 4/5)$$

and if $\mu = 2/3$, then D is a curve in $|-K_X|$ with a cusp at a point in $\text{Sing}(X)$ of type \mathbb{A}_2 . □

The assertion of Theorem 6.1 is proved.

Acknowledgements The authors would like to thank an anonymous referee for many useful remarks.

The authors thank G. Brown, N. Budur, J. Kollár, M. Mustata, J. Park, Y. Prokhorov for valuable comments.

References

1. Alexeev, V., Nikulin, V.: Del Pezzo and K3 Surfaces. MSJ Memoirs, vol. 15. Mathematical Society of Japan, Tokyo (2006)
2. Cheltsov, I.: Log canonical thresholds on hypersurfaces. Sb. Math. **192**, 1241–1257 (2001)
3. Cheltsov, I.: Log canonical thresholds of del Pezzo surfaces. Geom. Funct. Anal. **18**, 1118–1144 (2008)
4. Cheltsov, I.: On singular cubic surfaces. Asian J. Math. **19**, 191–214 (2009)
5. Cheltsov, I., Park, J., Won, J.: Log canonical thresholds of certain Fano hypersurfaces (2007). [arXiv:0706.0751](https://arxiv.org/abs/0706.0751)
6. Cheltsov, I., Shramov, C.: Log canonical thresholds of smooth Fano threefolds. Russ. Math. Surv. **63**, 859–958 (2008)
7. Cheltsov, I., Shramov, C.: On exceptional quotient singularities. Geom. Topol. **15**, 1843–1882 (2011)
8. Chen, X., Wang, B.: Kähler–Ricci flow on Fano manifolds. [arXiv:0909.2391](https://arxiv.org/abs/0909.2391) (2009)
9. Cools, F., Coppens, M.: Star points on smooth hypersurfaces. J. Algebra **323**, 261–286 (2010)
10. Demailly, J.-P., Kollár, J.: Semi-continuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds. Ann. Sci. l’École Normale Sup. **34**, 525–556 (2001)

11. Ding, W., Tian, G.: Kähler-Einstein metrics and the generalized Futaki invariant. *Invent. Math.* **110**, 315–335 (1992)
12. Donaldson, S.: Stability, birational transformations and the Kähler-Einstein problem (2010). [arXiv:1007.4220](https://arxiv.org/abs/1007.4220)
13. de Fernex, T., Ein, L., Mustata, M.: Bounds for log canonical thresholds with applications to birational rigidity. *Math. Res. Lett.* **10**, 219–236 (2003)
14. Futaki, A.: An obstruction to the existence of Einstein–Kähler metrics. *Invent. Math.* **73**, 437–443 (1983)
15. Gauntlett, J., Martelli, D., Sparks, J., Yau, S.-T.: Obstructions to the existence of Sasaki-Einstein metrics. *Commun. Math. Phys.* **273**, 803–827 (2007)
16. Ghigi, A., Kollár, J.: Kähler-Einstein metrics on orbifolds and Einstein metrics on spheres. *Comment. Math. Helv.* **82**, 877–902 (2007)
17. Hacking, P., Prokhorov, Y.: Smoothable del Pezzo surfaces with quotient singularities. *Compos. Math.* **146**, 169–192 (2010)
18. Hidaka, F., Watanabe, K.: Normal Gorenstein surfaces with ample anti-canonical divisor. *Tokyo J. Math.* **4**, 319–330 (1981)
19. Jeffres, T.: Singular set of some kähler orbifolds. *Trans. Am. Math. Soc.* **349**, 1961–1971 (1997)
20. Keel, S., McKernan, J.: Rational Curves on Quasi-projective Surfaces. *Memoirs of the American Mathematical Society*, vol. 669. American Mathematical Society, Providence (1999)
21. Kollár, J.: Singularities of pairs. *Proc. Symp. Pure Math.* **62**, 221–287 (1997)
22. Kollár, J., et al.: In: Flips and abundance for algebraic threefolds. *Astérisque*, vol. 211 (1992)
23. Lyubeznik, G.: On Bernstein–Sato polynomials. *Proc. Am. Math. Soc.* **125**, 1941–1944 (1997)
24. Mabuchi, T., Mukai, S.: Stability and Einstein–Kähler metric of a quartic del Pezzo surface. *Lect. Notes Pure Appl. Math.* **145**, 133–160 (1993)
25. Matsushima, Y.: Sur la structure du groupe d’homéomorphismes analytiques d’une certaine variété kählérienne. *Nagoya Math. J.* **11**, 145–150 (1957)
26. Miyanishi, M., Zhang, D.-Q.: Gorenstein log del Pezzo surfaces of rank one. *J. Algebra* **118**, 63–84 (1988)
27. Mustata, M.: Singularities of pairs via jet schemes. *J. Am. Math. Soc.* **15**, 599–615 (2002)
28. Park, J.: Birational maps of del Pezzo fibrations. *J. Reine Angew. Math.* **538**, 213–221 (2001)
29. Park, J.: A note on del Pezzo fibrations of degree 1. *Commun. Algebra* **31**, 5755–5768 (2003)
30. Park, J., Won, J.: Log canonical thresholds on Gorenstein canonical del Pezzo surfaces. *Proc. Edinb. Math. Soc. (2)* **54**(1), 187–219 (2011)
31. Park, J., Won, J.: Log-canonical thresholds on del Pezzo surfaces of degrees ≥ 2 . *Nagoya Math. J.* **200**, 1–26 (2010)
32. Prokhorov, Y.G.: Blow-ups of canonical singularities. In: *Algebra*, p. 301–317. de Gruyter, Berlin (2000)
33. Pukhlikov, A.: Birational geometry of Fano direct products. *Izv. Math.* **69**, 1225–1255 (2005)
34. Ross, J., Thomas, R.: Weighted projective embeddings, stability of orbifolds and constant scalar curvature Kähler metrics (2009). [arXiv:0907.5214](https://arxiv.org/abs/0907.5214)
35. Shi, Y.: On the α -invariants of Cubic surfaces with Eckardt points (2009). [arXiv:0902.3203](https://arxiv.org/abs/0902.3203)
36. Tian, G.: On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$. *Invent. Math.* **89**, 225–246 (1987)
37. Tian, G.: On Calabi’s conjecture for complex surfaces with positive first Chern class. *Invent. Math.* **101**, 101–172 (1990)
38. Tian, G.: On a set of polarized Kähler metrics on algebraic manifolds. *J. Differ. Geom.* **32**, 99–130 (1990)
39. Tian, G.: Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.* **130**, 1–37 (1997)
40. Urabe, T.: On singularities on degenerate del Pezzo surfaces of degree 1, 2. *Proc. Symp. Pure Math.* **40**, 587–590 (1983)
41. Wang, B.: Ricci flow on orbifolds. [arXiv:1003.0151](https://arxiv.org/abs/1003.0151) (2010)