Modular metric spaces generated by F-modulars[1]

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Abstract. A *metric modular* on a set X is a function $w : (0, \infty) \times X \times X \to [0, \infty]$ such that, for all $x, y, z \in X$, one has [V. V. Chistyakov, Metric modulars and their application, Dokl. Math. 73 (2006) 32–35]: $x = y$ iff $w(\lambda, x, y) = 0$ for all $\lambda > 0$; $w(\lambda, x, y) = w(\lambda, y, x)$ for all $\lambda > 0$; $w(\lambda + \mu, x, y) \le w(\lambda, x, z) + w(\mu, y, z)$ for all $\lambda, \mu > 0$. Given $x_0 \in X$, the set $X_w = \{x \in X : \lim_{\lambda\to\infty} w(\lambda, x, x_0) = 0\}$ is a metric space with metric $d_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \le \lambda\}$, called *modular space*. The modular w is said to be *convex* if $(\lambda, x, y) \mapsto \lambda w(\lambda, x, y)$ is also a metric modular on X. In this case X_w coincides with the set of all $x \in X$ such that $w(\lambda, x, x_0) < \infty$ for some $\lambda = \lambda(x) > 0$ and is metrizable by $d^*_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \le 1\}$. Moreover, $(d_w(x, y))^2 \le d^*_w(x, y) \le d_w(x, y)$ if $d_w(x, y) < 1$ or $d^*_w(x, y) < 1$; otherwise, the reverse inequalities hold. In this paper the notion of a metric modular is extended with respect to a generalized addition operation (called F-operation) on the set of nonnegative reals, several metrics are defined and compared on the respective modular metric space and the assertions above are generalized to a more general setting.

Key Words and Phrases: F-operation, F-modular, modular metric space, modular convergence, φ-convexity, equivalent metrics.

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1. Introduction

A *modular* on a real linear space X is a functional $\rho : X \to [0,\infty]$ satisfying the conditions [14]: (A.1) $\rho(0)=0$; (A.2) if $x \in X$ and $\rho(\alpha x)=0$ for all $\alpha>0$, then $x=0$; (A.3) $\rho(-x)=\rho(x)$ for all $x\in X$; and (A.4) $\rho(\alpha x+\beta y)\le\rho(x)+\rho(y)$ for all α, $\beta\ge 0$ with $\alpha+\beta=1$ and x, $y\in X$; if the inequality $\rho(\alpha x+\beta y)\le\alpha\rho(x)+\beta\rho(y)$ holds in (A.4), the modular ρ is called *convex* [12]. It is known from [11] that $X_\rho=\{x\in X:\lim_{\alpha\to+0}\rho(\alpha x)=0\}$ is a linear space, called *modular space*, which can be endowed with an F-norm by setting $|x|_\rho=\inf\{\varepsilon>0:\rho(x/\varepsilon)\le\varepsilon\}$ for $x\in X_\rho$. In addition, if ρ is convex, the modular space X_ρ coincides with $X_\rho^*=\{x\in X:\exists\,\alpha=\alpha(x)>0 \text{ such that } \rho(\alpha x)<\infty\}$ and the functional $|x|_\rho^*=\inf\{\varepsilon>0:\rho(x/\varepsilon)\le 1\}$ is an ordinary norm on X_ρ^*, which is "equivalent" to $|x|_\rho$ in the following sense [12]: if $|x|_\rho<1$ or $|x|_\rho^*<1$, then $|x|_\rho^2\le|x|_\rho^*\le|x|_\rho$; otherwise, $|x|_\rho\le|x|_\rho^*\le|x|_\rho^2$.

By now the theory of modular *linear* (or close to linear) spaces is well known and well developed including several generalizations (e.g., [8], [13], [16]) and a number of textbooks is devoted to the theory and applications ([7], [9], [15], to mention only a few), which contain comprehensive bibliography on the subject and historical comments. However, for certain problems from set-valued analysis ([1]–[4]) the notion of a modular on a set X with an additional algebraic structure is too limited, and "linear" modular theory fails. In order to overcome this insufficiency, in [5] the following approach to the modular theory on an *arbitrary* set X is proposed. For the sake of clarity and comparison, in the next paragraph we describe the basic ideas in a fashion parallel to the first paragraph above.

A *metric modular* on a set X is a function $w:(0,\infty)\times X\times X\to[0,\infty]$ satisfying, for all x, y, $z\in X$, the following conditions: (i) $w(\lambda,x,y)=0$ for all $\lambda>0$ iff $x=y$; (ii) $w(\lambda,x,y)=w(\lambda,y,x)$ for all $\lambda>0$; and (iii) $w(\lambda+\mu,x,y)\le w(\lambda,x,z)+w(\mu,y,z)$ for all λ, $\mu>0$; if the function $(\lambda,x,y)\mapsto\lambda w(\lambda,x,y)$ is also a metric modular on X, the metric modular w is called *convex*. Given $x_0\in X$, the set $X_w=\{x\in X:\lim_{\lambda\to\infty}w(\lambda,x,x_0)=0\}$ is a metric space, called *modular space*, whose metric is given by $d_w(x,y)=\inf\{\lambda>0:w(\lambda,x,y)\le\lambda\}$ for x, $y\in X_w$. Moreover, if w is convex, the modular set X_w is equal to $X_w^*=\{x\in X:\exists\,\lambda=\lambda(x)>0 \text{ such that } w(\lambda,x,x_0)<\infty\}$ and metrizable by $d_w^*(x,y)=\inf\{\lambda>0:w(\lambda,x,y)\le 1\}$ for x, $y\in X_w^*$, and metrics d_w and d_w^* are "equivalent" in the sense that if $d_w(x,y)<1$ or $d_w^*(x,y)<1$ for x, $y\in X_w^*=X_w$, then $(d_w(x,y))^2\le d_w^*(x,y)\le d_w(x,y)$; otherwise, the last two inequalities should be reversed. The resulting theory agrees with the classical one in that if X is a real linear space, $\rho : X \to [0, \infty]$ and $w(\lambda, x, y) = \rho((x-y)/\lambda)$ for all $\lambda > 0$ and $x, y \in X$, then ρ is a (convex) modular on X in the sense of (A.1)–(A.4) iff w is a (convex) modular in the sense of (i)–(iii).

The aim of this paper is to extend the notion of a (convex) metric modular to that of a (convex) F-modular where F is an F-operation (i.e., a generalized addition) on the set of all nonnegative reals in the sense of [9, Section 3], [13]. The main results of the paper are concerned with metrizability of modular sets X_w and X_w^* by different metrics (Theorems 1, 4 and 5) and, further, their comparison (Theorems 4 and 6). Also, we establish specific inequalities between a metric F-modular w and the corresponding metric d_w (Theorem 2), which proved to be useful in the study of abstract superposition (Nemytskii) operators in [4] and in establishing the equivalence of d_w-metric convergence and w-modular convergence in Theorem 3. We make no attempt to present any applications aiming at the general theory, which is of interest in its own right (for applications of convex metric modulars see [4] and [5]).

The paper is organized as follows. In Section 2 we recall the notion of an F-operation on $\mathbb{R}^+$ and its main properties, which are needed for our results. Metric F-modulars are introduced in Section 3, which also contains their properties and examples. In Section 4 two different metrics d_w and d_w^1 are defined on modular sets and a relation between them is established. Finally, in Section 5 we present a generalization of the notion of convexity for metric F-modulars and study the relations between appropriate metrics in convex and non-convex settings.

2. F-operations on $\mathbb{R}^+$ revisited

Throughout the paper $\mathbb{R}^+ = [0, \infty)$ denotes the set of all nonnegative real numbers.

In this section we recall the notions and properties of an F-operation on $\mathbb{R}^+$, an equivalence relation between F-operations and an F-superadditive function needed below (for the classical exposition, which we generally follow here, see [9, Section 3, 3.1–3.10] and [13]).

2.1. A continuous function $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an *F-operation* (or *generalized addition* operation) on $\mathbb{R}^+$ if, for all $u, v, w \in \mathbb{R}^+$, the following conditions hold: (a) $F(u, 0) = u$; (b) $F(u, v) = F(v, u)$; (c) if $u \le v$, then $F(u, w) \le F(v, w)$; and (d) $F(u, F(v, w)) = F(F(u, v), w)$. If, instead of (c), $u < v$ implies $F(u, w) < F(v, w)$, the F-operation F is called *strict*. We denote by $\mathcal{F}(\mathbb{R}^+)$ the set of all F-operations on $\mathbb{R}^+$. Any $F \in \mathcal{F}(\mathbb{R}^+)$ may be formally extended to $\overline{\mathbb{R}}^+ = [0, \infty]$ by setting

$F(u,v)=\infty$ provided $u=\infty$ or $v=\infty$. Also, in extending an F to any finite number of terms it is convenient to set $u\oplus v\equiv u\oplus_F v=F(u,v)$ for $u,\,v\in\mathbb{R}^+$ and, given $u_1,\dots,u_n\in\mathbb{R}^+$ with $n\in\mathbb{N}$,

$$\oplus_{i=1}^{1}u_i=u_1,\ \ \oplus_{i=1}^{2}u_i=u_1\oplus u_2\ \text{ and }\ \oplus_{i=1}^{n}u_i=F\big(\oplus_{i=1}^{n-1}u_i,u_n\big),\ \ n\ge 3.$$

Clearly, the set $\mathcal{F}(\mathbb{R}^+)$ is closed under the uniform convergence. The following properties of an F-operation F are straightforward: $F(0,0)=0$, $F(u_1,v_1)\le F(u_2,v_2)$ if $0\le u_1\le u_2$ and $0\le v_1\le v_2$, and $F(u,v)\ge F_\infty(u,v)=\max\{u,v\}$, i.e., the F-operation F_∞ is the minimal element of $\mathcal{F}(\mathbb{R}^+)$. For more examples of F-operations see 2.4 below.

2.2. A function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is said to be a φ-function if it is nondecreasing, continuous, vanishes at zero only and $\varphi(u)\to\infty$ as $u\to\infty$; if, moreover, φ is strictly increasing (or convex), then it is called an *increasing* (or *convex*) φ-function.

2.3. Given an increasing φ-function φ, the following function $\varphi^*:\mathcal{F}(\mathbb{R}^+)\to\mathcal{F}(\mathbb{R}^+)$ is well defined:

$$(\varphi^*F)(u,v)=\varphi^{-1}\big(F(\varphi(u),\varphi(v))\big),\qquad F\in\mathcal{F}(\mathbb{R}^+),\quad u,\,v\in\mathbb{R}^+,$$

where $\varphi^*F=\varphi^*(F)$ and φ^{-1} is the inverse function of φ. Clearly, $(\mathrm{id}_{\mathbb{R}^+})^*=\mathrm{id}_{\mathcal{F}(\mathbb{R}^+)}$ where id_X is the identity function of the set X, i.e., $\mathrm{id}_X(x)=x$ for all $x\in X$. Given two φ-functions φ and ψ, we have $(\varphi\circ\psi)^*=\psi^*\circ\varphi^*$, where $\circ$ denotes the usual composition of functions. It follows that the relation $\overset{*}{\sim}$, defined for $F,G\in\mathcal{F}(\mathbb{R}^+)$ by: $G\overset{*}{\sim}F$ if and only if there exists an increasing φ-function φ such that $G=\varphi^*F$, is an equivalence relation on $\mathcal{F}(\mathbb{R}^+)$. The equivalence class $[F]$ of $F\in\mathcal{F}(\mathbb{R}^+)$ under $\overset{*}{\sim}$ is given by $[F]=\{\varphi^*F:\varphi \text{ is an increasing } \varphi\text{-function}\}\subset\mathcal{F}(\mathbb{R}^+)$.

2.4. Examples of F-operations. Given $u,\,v\in\mathbb{R}^+$ and $p>0$, we have:

(a) $F_\infty(u,v)=\max\{u,v\}$, and $\varphi^*F_\infty=F_\infty$ for any increasing φ-function φ;

(b) $F_1(u,v)=u+v$, the usual addition operation in $\mathbb{R}^+$;

(c) $F_p(u,v)=(u^p+v^p)^{1/p}$, and $F_p=\varphi^*F_1$ with $\varphi(u)=u^p$;

(d) $F_e(u,v)=\frac{1}{p}\log(e^{pu}+e^{pv}-1)$, and $F_e=\varphi^*F_1$ with $\varphi(u)=e^{pu}-1$;

(e) $F_{\log}(u,v)=u+v+puv$, and $F_{\log}=\varphi^*F_1$ with $\varphi(u)=\log(1+pu)$;

(f) $$F_\diamond(u,v)=\begin{cases}u+v & \text{if } \ u<1,\ v<1 \text{ and } u+v<1,\\ 1 & \text{if } \ u<1,\ v<1 \text{ and } u+v\ge 1,\\ \max\{u,v\} & \text{if } \ u\ge 1 \text{ or } v\ge 1.\end{cases}$$

The F-operations F_1, F_p, F_e and $F_{\log}$ are equivalent (under $\overset{*}{\sim}$) and strict, while $[F_\infty]=\{F_\infty\}$.

2.5. Given $F \in \mathcal{F}(\mathbb{R}^+)$, a function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *F-superadditive* if $\kappa(u) > 0$ for all $u > 0$ and

$$F(\kappa(u), \kappa(v)) \le \kappa(u+v) \quad \text{for all} \quad u,\, v \in \mathbb{R}^+. \tag{1}$$

The function κ is *nondecreasing* (*strictly increasing* if F is strict) on $\mathbb{R}^+$, for if $0 \le u < v$, then, by virtue of (1),

$$\kappa(u) = F(\kappa(u), 0) \le F(\kappa(u), \kappa(v-u)) \le \kappa(u + (v-u)) = \kappa(v).$$

2.6. Examples of F-superadditive functions.

(a) Any nondecreasing function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ with $\kappa(u) > 0$ for $u > 0$ is F_∞-superadditive.

(b) Any convex φ-function κ is F_1-superadditive (in particular, $\kappa(u) = u$): in fact, the convexity of κ implies $\kappa(\theta u) \le \theta\kappa(u)$ for all $0 \le \theta \le 1$ and $u \in \mathbb{R}^+$, and so, given $u > 0$ and $v > 0$, we have:

$$\kappa(u) = \kappa\Big(\frac{u}{u+v} \cdot (u+v)\Big) \le \frac{u}{u+v}\kappa(u+v) \quad \text{and} \quad \kappa(v) \le \frac{v}{u+v}\kappa(u+v),$$

which gives $F_1(\kappa(u), \kappa(v)) = \kappa(u) + \kappa(v) \le \kappa(u+v)$ for all $u,\, v \in \mathbb{R}^+$.

(c) The following assertion holds: given $F \in \mathcal{F}(\mathbb{R}^+)$ and an increasing φ-function φ, a function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is F-superadditive if and only if the function $\varphi^{-1} \circ \kappa$ is $\varphi^* F$-superadditive.

(d) Example (b) and assertion (c) provide a large number of examples of functions κ. According to (b), the convex function $\kappa(u) = e^{qu} - 1$ is F_1-superadditive for all $q > 0$. Employing examples 2.4(c)–(e), respectively, by virtue of 2.6(c), we find that $\kappa_1(u) = (e^{qu}-1)^{1/p}$ is F_p-superadditive, $\kappa_2(u) = (q/p)u$ is F_e-superadditive and $\kappa_3(u) = \frac{1}{p}(e^{\kappa(u)} - 1)$ is $F_{\log}$-superadditive.

3. Metric F-modulars

In what follows X is a nonempty set, $\lambda > 0$ is understood in the sense that $\lambda \in (0, \infty)$, and if the domain of a function w is $(0,\infty) \times X \times X$, we always write $w_\lambda(x, y) = w(\lambda, x, y)$ for $\lambda > 0$ and $x,\, y \in X$.

3.1. Definition. A function $w : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a *metric F-modular on X*, where F is a given F-operation, if the following three axioms are satisfied:

(F.i) for all $x, y \in X$ we have: $w_\lambda(x, y) = 0$ for all $\lambda > 0$ iff $x = y$;

(F.ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x,\, y \in X$;

(F.iii) $w_{\lambda+\mu}(x, y) \le F\big(w_\lambda(x, z), w_\mu(y, z)\big)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If instead of (F.i) we have only

(F.i′) $w_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$,

then the function w is called a *metric F-pseudomodular on X*.

3.2. Properties of F-(pseudo)modulars. Let w be a metric F-pseudomodular on a set X with $F \in \mathcal{F}(\mathbb{R}^+)$. We have:

(a) The function $0 < \lambda \mapsto w_\lambda(x, y) \in [0, \infty]$ is *nonincreasing* on $(0, \infty)$ for all x, $y \in X$: in fact, if $0 < \mu < \lambda$, then (F.iii), (F.i′), (F.ii) and 2.1 imply

$$w_\lambda(x, y) \le F\big(w_{\lambda-\mu}(x, x), w_\mu(y, x)\big) = F(0, w_\mu(x, y)) = w_\mu(x, y).$$

It follows that for any $\lambda > 0$ the following finite or infinite right and left limits exist:

$$w_{\lambda+0}(x, y) = \lim_{\varepsilon \to +0} w_{\lambda+\varepsilon}(x, y), \quad w_{\lambda-0}(x, y) = \lim_{\mu \to \lambda-0} w_\mu(x, y),$$

such that

$$w_{\lambda+0}(x, y) \le w_\lambda(x, y) \le w_{\lambda-0}(x, y), \qquad \lambda > 0, \quad x, y \in X. \tag{2}$$

(b) Given $\lambda_1 > 0, \ldots, \lambda_n > 0$ and $x_0, x_1, \ldots, x_n \in X$, the standard induction on $n \in \mathbb{N}$ gives:

$$w_{\lambda_1+\cdots+\lambda_n}(x_0, x_n) \le \oplus_{i=1}^n w_{\lambda_i}(x_{i-1}, x_i) \quad \text{with} \quad \oplus = \oplus_F.$$

(c) Given an increasing φ-function φ, we have: w is a metric F-(pseudo)-modular on X if and only if $\varphi^{-1} \circ w$ is a metric $\varphi^* F$-(pseudo)modular on X. In fact, the axioms (F.i) and (F.ii) are clear, and as for (F.iii), we have:

$$w_{\lambda+\mu}(x, y) \le F\big(w_\lambda(x, z), w_\mu(y, z)\big), \qquad x, y, z \in X, \quad \lambda > 0,$$

is equivalent to (cf. 2.3)

$$\varphi^{-1}\big(w_{\lambda+\mu}(x, y)\big) \le \varphi^{-1}\Big(F\Big[\varphi\big(\varphi^{-1}(w_\lambda(x, z))\big), \varphi\big(\varphi^{-1}(w_\mu(y, z))\big)\Big]\Big).$$

3.3. Examples of F-modulars. Here we present mainly examples of metric F_1-modulars, since this case is the basic one in the theory. Applying property 3.2(c) one obtains further examples of metric F-modulars. Let $\lambda > 0$ and x, $y \in X$ in (a), (b) and (c).

(a) If $w_\lambda(x, x) = 0$ and $w_\lambda(x, y) = \infty$ for $x \neq y$, then w is a metric F-modular on X for every $F \in \mathcal{F}(\mathbb{R}^+)$.

Now let (X, d) be a metric space (or a pseudometric space).

(b) Setting $w_\lambda(x, y) = \infty$ if $\lambda \le d(x, y)$ and $w_\lambda(x, y) = 0$ if $\lambda > d(x, y)$, we find that w is a metric F-modular on X for every $F \in \mathcal{F}(\mathbb{R}^+)$.

(c) Let $w_\lambda(x, y) = \lambda\varphi\big(d(x, y)/\lambda\big)$ where φ is a convex φ-function or $w_\lambda(x, y) = d(x, y)/(\varphi(\lambda) + d(x, y))$ where $\varphi : (0, \infty) \to (0, \infty)$ is a nondecreasing function. Then w is a metric F_1-modular on X.

(d) Let $Y = X^{\mathbb{N}}$ be the set of all sequences $x : \mathbb{N} \to X$. If

$$w_\lambda(x, y) = \sup_{n\in\mathbb{N}} \Big(\frac{d(x(n), y(n))}{\varphi(\lambda)}\Big)^{1/n} \quad \text{for} \quad x,\, y \in Y \quad \text{and} \quad \lambda > 0, \tag{3}$$

where $\varphi : (0, \infty) \to (0, \infty)$ is a nondecreasing function, then w is a metric F_1-modular on Y. Another example of an F_1-modular in this context is

$$w_\lambda(x, y) = \sum_{n=1}^{\infty} \Phi_n\Big(\frac{d(x(n), y(n))}{\varphi_n(\lambda)}\Big), \qquad \lambda > 0 \quad \text{and} \quad x,\, y \in Y, \tag{4}$$

where Φ_n is a φ-function and φ_n is a convex φ-function for each $n \in \mathbb{N}$ (in checking the axiom (F.iii) relations (6) and (7) below can be helpful).

(e) Let $(X, d, +)$ be a metric semigroup, i.e., (X, d) is a metric space, $(X, +)$ is an Abelian semigroup and $d(x, y) = d(x+z, y+z)$ for all $x, y, z \in X$. Given x, y, $\overline{x}$, $\overline{y} \in X$, the following inequalities hold:

$$d(x, y) \le d(x + \overline{x}, y + \overline{y}) + d(\overline{x}, \overline{y}), \qquad d(x + \overline{x}, y + \overline{y}) \le d(x, y) + d(\overline{x}, \overline{y}).$$

Let $I = [a, b]$ be a closed interval in $\mathbb{R}$ and $Y = X^I$ be the set of all functions $x : I \to X$. Given a φ-function Φ, a convex φ-function φ, $\lambda > 0$ and x, $y \in Y$, we set

$$w_\lambda(x, y) = \sup \sum_{i=1}^{m} \Phi\Big(\frac{d\big(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1})\big)}{\varphi(\lambda)}\Big), \tag{5}$$

where the supremum is taken over all $m \in \mathbb{N}$ and $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$. Then w is a metric F_1-pseudomodular on Y. Let us verify only axiom (F.iii). In order to do it, we first note that if α, $\beta \ge 0$, $\alpha+\beta \le 1$ and A, $B \ge 0$, then

$$\Phi(\alpha A + \beta B) \le \max\{\Phi(A), \Phi(B)\} \le \Phi(A) + \Phi(B). \tag{6}$$

Now if λ, $\mu > 0$, x, y, $z \in Y$, $m \in \mathbb{N}$, $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ and $i \in \{1, \ldots, m\}$, then by virtue of the inequalities for d above, we have:

$$\begin{aligned} C_i &\equiv d\big(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1})\big) \\ &\le d\big(x(t_i) + y(t_{i-1}) + y(t_i) + z(t_{i-1}), y(t_i) + x(t_{i-1}) + z(t_i) + y(t_{i-1})\big) \\ &\quad + d\big(y(t_i) + z(t_{i-1}), z(t_i) + y(t_{i-1})\big) = \\ &= d\big(x(t_i) + z(t_{i-1}), z(t_i) + x(t_{i-1})\big) + d\big(y(t_i) + z(t_{i-1}), z(t_i) + y(t_{i-1})\big) \\ &\equiv A_i + B_i, \end{aligned}$$

whence the monotonicity of Φ, inequality $\varphi(\lambda)+\varphi(\mu) \le \varphi(\lambda+\mu)$ (cf. 2.6(b)) and (6) imply

$$\Phi\Big(\frac{C_i}{\varphi(\lambda+\mu)}\Big) \le \Phi\Big(\frac{\varphi(\lambda)}{\varphi(\lambda+\mu)}\cdot\frac{A_i}{\varphi(\lambda)}+\frac{\varphi(\mu)}{\varphi(\lambda+\mu)}\cdot\frac{B_i}{\varphi(\mu)}\Big) \le \Phi\Big(\frac{A_i}{\varphi(\lambda)}\Big)+\Phi\Big(\frac{B_i}{\varphi(\mu)}\Big). \tag{7}$$

Summing over $i = 1, \dots, m$ and taking the supremum as in (5), we arrive at $w_{\lambda+\mu}(x, y) \le w_\lambda(x, z) + w_\mu(y, z) = F_1\big(w_\lambda(x, z), w_\mu(y, z)\big)$.

If we set $\overline{w}_\lambda(x, y) = d(x(a), y(a)) + w_\lambda(x, y)$ with w from (5), then $\overline{w}$ is an F_1-modular on Y. In fact, suppose that $\overline{w}_\lambda(x, y) = 0$ for all $\lambda > 0$. Then $d(x(a), y(a)) = 0$, $w_\lambda(x, y) = 0$ and (5) implies

$$\Phi\Big(\frac{d(x(t)+y(s), y(t)+x(s))}{\varphi(\lambda)}\Big) \le w_\lambda(x, y) = 0 \quad \text{for all} \quad t, s \in I,$$

and so, $d(x(t) + y(s), y(t) + x(s)) = 0$. Thus, for any $t \in I$ it follows that

$$d(x(t), y(t)) = |d(x(t), y(t))-d(x(a), y(a))| \le d(x(t)+y(a), y(t)+x(a)) = 0,$$

that is, $x = y$ as elements of $Y = X^I$.

Note that F-modulars from examples 3.3(c), (d), (e) are not allowed in the classical linear modular theory where usually $\varphi(\lambda) = \lambda$. The modular (5) generates the modular set (see the next subsection) related to the space of functions $x : I \to X$ of generalized Φ-variation in the sense of N. Wiener and L. C. Young (see also [4] and [10]).

3.4. Modular sets. Given an F-pseudomodular on a set X, we define a relation $\overset{w}{\sim}$ on X as follows: if x, $y \in X$, we set

$$x \overset{w}{\sim} y \quad \text{if and only if} \quad \lim_{\lambda\to\infty} w_\lambda(x, y) = 0. \tag{8}$$

Then $\overset{w}{\sim}$ is an *equivalence relation* on X: this is a consequence of definition 3.1; for instance, if $x \overset{w}{\sim} z$ and $z \overset{w}{\sim} y$ for x, y, $z \in X$, then, by virtue of (F.iii), continuity of F and (8), we have:

$$w_\lambda(x, y) \le F\big(w_{\lambda/2}(x, z), w_{\lambda/2}(y, z)\big) \to 0 \quad \text{as} \quad \lambda \to \infty, \tag{9}$$

and so, $x \overset{w}{\sim} y$.

Let $X/\overset{w}{\sim}$ be the quotient set of X under $\overset{w}{\sim}$. Given $x \in X$, the equivalence class of x in $X/\overset{w}{\sim}$ is given by

$$X_w(x) \equiv \widetilde{x} = \{y \in X : y \overset{w}{\sim} x\},$$

and it is called a *modular set*. In particular, $X/\overset{w}{\sim} = \{\widetilde{x} : x \in X\}$, and it is shown in (9) that $x \overset{w}{\sim} y$ iff x, $y \in X_w(z)$ iff $\widetilde{x} = \widetilde{y} = X_w(z)$ for some $z \in X$.

3.5. Space $(X/\overset{w}{\sim}, \widetilde{d})$. If w is an F-pseudomodular on X, we set

$$\widetilde{d}(\widetilde{x}, \widetilde{y}) = \lim_{\lambda\to\infty} w_\lambda(x, y) \quad \text{for} \quad \widetilde{x},\, \widetilde{y} \in X/\overset{w}{\sim}\,. \tag{10}$$

The function $\widetilde{d} : (X/\overset{w}{\sim}) \times (X/\overset{w}{\sim}) \to [0, \infty]$ has the following properties:

(a) $\widetilde{d}(\widetilde{x}, \widetilde{y}) = 0$ if and only if $\widetilde{x} = \widetilde{y}$ in $X/\overset{w}{\sim}$;

(b) $\widetilde{d}(\widetilde{x}, \widetilde{y}) = \widetilde{d}(\widetilde{y}, \widetilde{x})$;

(c) $\widetilde{d}(\widetilde{x}, \widetilde{y}) \le F\big(\widetilde{d}(\widetilde{x}, \widetilde{z}), \widetilde{d}(\widetilde{y}, \widetilde{z})\big)$.

According to 3.2(a), the limit (10) exists in $[0, \infty]$; moreover, $\widetilde{d}$ is well defined, that is, the value (10) does not depend on the choice of representatives: in fact, if $\widetilde{x_1} = \widetilde{x}$ and $\widetilde{y_1} = \widetilde{y}$, then $x_1 \overset{w}{\sim} x$ and $y_1 \overset{w}{\sim} y$, and so,

$$w_\lambda(x_1, y_1) \le F\big(w_{\lambda/3}(x_1, x), F(w_{\lambda/3}(x, y), w_{\lambda/3}(y, y_1))\big), \quad \lambda > 0.$$

By the continuity of F and 2.1(a), (b), we get

$$\lim_{\lambda\to\infty} w_\lambda(x_1, y_1) \le F\Big(0, F\big(\lim_{\lambda\to\infty} w_\lambda(x, y), 0\big)\Big) = \lim_{\lambda\to\infty} w_\lambda(x, y),$$

and, in a similar manner, we obtain the reverse inequality.

We remark that if $F(u, v) \le u + v$ for $u, v \in \mathbb{R}^+$, then $\widetilde{d}$ satisfies the axioms of a metric on $X/\overset{w}{\sim}$, but *may assume* the value ∞ (see 3.3(a)).

In some interesting and important cases $X/\overset{w}{\sim}$ and $\widetilde{d}$ may degenerate, i.e., $X/\overset{w}{\sim} = \{X\}$ and $\widetilde{d} = 0$: this is the case, for instance, for the metric F_1-modular $w_\lambda(x, y) = d(x, y)/\varphi(\lambda)$ on a metric space (X, d) where $\varphi : (0, \infty) \to (0, \infty)$ is a nondecreasing function such that $\varphi(\lambda) \to \infty$ as $\lambda \to \infty$. So in what follows the equivalence classes $\widetilde{x} \in X/\overset{w}{\sim}$ (modular sets) will be under consideration.

3.6. Convention. Throughout the rest of the paper we arbitrarily fix $x_0 \in X$ and define the modular set by $X_w = \widetilde{x_0} = X_w(x_0)$. If not specified otherwise, $\inf \varnothing = \infty$ and $F \in \mathcal{F}(\mathbb{R}^+)$ is an F-operation. Metric F-(pseudo)modulars will be termed simply as F-(pseudo)modulars. If a function κ (as in 2.5) appears in a context, it will be assumed to be F-superadditive. Additional assumptions on κ may involve the limits $\kappa(+0) = \lim_{u\to+0} \kappa(u) \in \mathbb{R}^+$ and $\kappa(\infty) = \lim_{u\to\infty} \kappa(u) \in [0, \infty]$.

4. Metrizability theorems

4.1. Now we are ready to define the basic metric d_w on the modular set X_w generalizing the corresponding F-norm from [13].

Theorem 1. *Suppose that w is an F-pseudomodular on X. Then the function $d_w : X \times X \to [0, \infty]$ defined by*

$$d_w(x, y) \equiv d_{w,F,\kappa}(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \le \kappa(\lambda)\}, \quad x, y \in X, \quad (11)$$

is a pseudometric on X assuming the value ∞, i.e., for all x, y, $z \in X$ we have: (a) $d_w(x, x) = 0$; (b) $d_w(x, y) = d_w(y, x)$; *and* (c) $d_w(x, y) \le d_w(x, z) + d_w(y, z)$.

Assume, moreover, that w is an F-modular on X and $\kappa(+0) = 0$. Then (d) *given x, $y \in X$, $d_w(x, y) = 0$ if and only if $x = y$;* (e) *d_w is a metric on each set $X' \subset X$ such that $d_w(x, y) < \infty$ for all x, $y \in X'$ and, in particular, one may always set $X' = X_w$.*

Proof. (a) Given $x \in X$, axiom (F.i$'$) and 2.5 imply $w_\lambda(x, x) = 0 < \kappa(\lambda)$ for all $\lambda > 0$, and so, $\{\lambda > 0 : w_\lambda(x, x) \le \kappa(\lambda)\} = (0, \infty)$ yielding $d_w(x, x) = \inf(0, \infty) = 0$.

Since (b) is clear by virtue of (F.ii), we prove (c). If $d_w(x, z) = \infty$ or $d_w(y, z) = \infty$, the inequality is clear. Suppose that the right hand side in (c) is finite. Then for any $\lambda > d_w(x, z)$ and $\mu > d_w(y, z)$ we have from (11): $w_\lambda(x, z) \le \kappa(\lambda)$ and $w_\mu(y, z) \le \kappa(\mu)$, and so, by (F.iii) and properties of F,

$$w_{\lambda+\mu}(x, y) \le F\big(w_\lambda(x, z), w_\mu(y, z)\big) \le F\big(\kappa(\lambda), \kappa(\mu)\big) \le \kappa(\lambda + \mu).$$

This gives $d_w(x, y) \le \lambda + \mu$, and it remains to pass to the limit as $\lambda \to d_w(x, z)$ and $\mu \to d_w(y, z)$.

Now let w be an F-modular on X and $\kappa(+0) = 0$.

(d) Let x, $y \in X$ and $d_w(x, y) = 0$. It follows from (11) that $w_\mu(x, y) \le \kappa(\mu)$ for all $\mu > 0$. Given $\lambda > 0$, for any $0 < \mu < \lambda$ we have, according to 3.2(a),

$$w_\lambda(x, y) \le w_\mu(x, y) \le \kappa(\mu) \to \kappa(+0) = 0 \quad \text{as} \quad \mu \to +0.$$

Thus, $w_\lambda(x, y) = 0$ for all $\lambda > 0$, and so, by (F.i), $x = y$.

(e) Let us show that $d_w(x, y) < \infty$ for all x, $y \in X_w$. Since $x \overset{w}{\sim} x_0$ and $y \overset{w}{\sim} x_0$, then $x \overset{w}{\sim} y$, and so, by (8) there exists $\lambda_0 > 0$ such that $w_\lambda(x, y) \le \kappa(1)$ for all $\lambda \ge \lambda_0$. Setting $\lambda_1 = \max\{1, \lambda_0\}$, we find $\lambda_1 \ge \lambda_0$, and so, $w_{\lambda_1}(x, y) \le \kappa(1)$. Also, since κ is nondecreasing (see 2.5) and $\lambda_1 \ge 1$, then $\kappa(\lambda_1) \ge \kappa(1)$, and it follows that $w_{\lambda_1}(x, y) \le \kappa(1) \le \kappa(\lambda_1)$, which gives $d_w(x, y) \le \lambda_1 < \infty$. □

4.2. Particular cases. Unlike the other parts of the paper, here we consider the ordinary addition operation $F_1(u, v) = u + v$ and the function $\kappa(u) = u$, which, of course, is F_1-superadditive.

(a) An F_1-(pseudo)modular w on a set X, called simply *(pseudo)modular*, satisfies conditions (F.i′) or (F.i) and (F.ii) of 3.1 and

$$w_{\lambda+\mu}(x,y) \le w_\lambda(x,z) + w_\mu(y,z), \qquad \lambda,\mu>0, \quad x,\,y,\,z\in X. \tag{12}$$

By Theorem 1, the (pseudo)metric d_w on the modular set X_w is given by (cf. [5])

$$d_w(x,y) = \inf\{\lambda>0 : w_\lambda(x,y)\le\lambda\} \in \mathbb{R}^+, \qquad x,\,y\in X_w. \tag{13}$$

(b) A (pseudo)modular w on X (in the sense of (a)) is said to be *convex* (cf. [5, Section 2]) if, instead of (12), it satisfies the condition:

$$w_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu}\,w_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\,w_\mu(y,z), \quad \lambda,\mu>0, \quad x,y,z\in X, \tag{14}$$

or, in other words, if the function $\widehat{w}_\lambda(x,y)=\lambda w_\lambda(x,y)$, $\lambda>0$, $x,\,y\in X$, is also a (pseudo)modular on X.

Everywhere in this example we assume that w is a convex (pseudo)modular on X.

If the set $X_w^*\subset X$ is given by

$$X_w^* \equiv X_w^*(x_0) = \{x\in X : \exists\,\lambda=\lambda(x)>0 \text{ such that } w_\lambda(x,x_0)<\infty\}, \tag{15}$$

then

$$X_w = X_w^*;$$

in fact, the inclusion $X_w\subset X_w^*$ is always true, and if $x\in X_w^*$, then $w_\mu(x,x_0)<\infty$ for some constant $\mu=\mu(x)>0$, and since by virtue of 3.2(a) the function $\lambda\mapsto\widehat{w}_\lambda(x,x_0)=\lambda w_\lambda(x,x_0)$ is nonincreasing on $(0,\infty)$, for any $\lambda>\mu$ we have

$$w_\lambda(x,x_0) \le (\mu/\lambda)w_\mu(x,x_0) \to 0 \quad \text{as} \quad \lambda\to\infty,$$

and so, $x\in X_w$.

Along with the (pseudo)metric d_w on X_w (cf. (13)), another (pseudo)-metric d_w^* on $X_w=X_w^*$ can be defined from the fact that $\widehat{w}$ is also a (pseudo)modular on X, i.e.,

$$\begin{aligned} d_w^*(x,y) &\equiv d_{\widehat{w}}(x,y) = \inf\{\lambda>0 : \widehat{w}_\lambda(x,y)=\lambda w_\lambda(x,y)\le\lambda\} = \\ &= \inf\{\lambda>0 : w_\lambda(x,y)\le 1\}, \qquad x,\,y\in X_w=X_w^*. \end{aligned} \tag{16}$$

That d_w^* is well defined will follow from Theorem 1 if we verify that $d_w^*(x,y)$ is finite for all $x,\,y\in X'=X_w^*$ (note that $X_w=X_w^*=X_{\widehat{w}}^*\supset X_{\widehat{w}}$ in general): indeed, if x, $y \in X_w^*$, then $w_\lambda(x, x_0) < \infty$ and $w_\mu(y, x_0) < \infty$ for some numbers $\lambda = \lambda(x) > 0$ and $\mu = \mu(y) > 0$, and so, since $\lambda \mapsto \lambda w_\lambda(x, y)$ in nonincreasing, applying (14), for any $\nu \geq \lambda + \mu$ we have:

$$w_\nu(x,y) \leq \frac{\lambda+\mu}{\nu}\, w_{\lambda+\mu}(x,y) \leq \frac{\lambda+\mu}{\nu}\Big(\frac{\lambda}{\lambda+\mu}\, w_\lambda(x,x_0) + \frac{\mu}{\lambda+\mu}\, w_\mu(y,x_0)\Big)$$
$$= \frac{1}{\nu}\big(\lambda w_\lambda(x,x_0) + \mu w_\mu(y,x_0)\big) \to 0 \quad \text{as} \quad \nu \to \infty.$$

Thus, there exists $\nu_0 = \nu_0(x, y) > 0$ such that $w_\nu(x, y) \leq 1$ for all $\nu \geq \nu_0$, and so, $d_w^*(x, y) \leq \nu_0 < \infty$.

(c) The modulars from examples 3.3(a), (b) and $w_\lambda(x, y) = d(x, y)/\lambda$ are convex; (4) is convex if Φ_n is convex and $\varphi_n(\lambda) = \lambda$ for all $n \in \mathbb{N}$; and (5) is convex if Φ is convex and $\varphi(\lambda) = \lambda$. On the other hand, the modulars from 3.3(c) are not convex (see Remark 5.5(c) below). In order to see that (3) is non-convex, let us show that $Y_w(\mathbf{x}_0) \neq Y_w^*(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in Y$ (cf. notation in 3.4). Choose $x, x_0 \in X$, $x \neq x_0$, and define the corresponding constant sequences $\mathbf{x}, \mathbf{x}_0 \in Y = X^{\mathbb{N}}$ by $\mathbf{x}(n) = x$ and $\mathbf{x}_0(n) = x_0$ for all $n \in \mathbb{N}$. Then for $\lambda > \varphi^{-1}(d(x, x_0))$ we have

$$w_\lambda(\mathbf{x},\mathbf{x}_0) = \sup_{n\in\mathbb{N}}\Big(\frac{d(\mathbf{x}(n),\mathbf{x}_0(n))}{\varphi(\lambda)}\Big)^{1/n} = \lim_{n\to\infty}\Big(\frac{d(x,x_0)}{\varphi(\lambda)}\Big)^{1/n} = 1,$$

and so, $\mathbf{x} \in Y_w^*(\mathbf{x}_0) \setminus Y_w(\mathbf{x}_0)$.

4.3. Now we present specific inequalities between an F-pseudomodular and the pseudometric, which the F-pseudomodular generates according to Theorem 1.

Theorem 2. *Let w be an F-pseudomodular on X, κ an F-superadditive function, $\lambda > 0$ and x, $y \in X_w$. We have:*

(a) *if $d_w(x, y) < \lambda$, then $w_\lambda(x, y) \leq \kappa\big(d_w(x, y) + 0\big) \leq \kappa(\lambda)$;*

(b) *if $d_w(x, y) = \lambda$, then $w_{\lambda+0}(x, y) \leq \kappa(\lambda+0)$ and $\kappa(\lambda-0) \leq w_{\lambda-0}(x, y)$;*

(c) *if $\mu \mapsto w_\mu(x, y)$ and κ are continuous from the right on $(0, \infty)$, then the inequalities $d_w(x, y) \leq \lambda$ and $w_\lambda(x, y) \leq \kappa(\lambda)$ are equivalent.*

Suppose also that F is strict or κ is increasing. Then we have:

(d) *if $d_w(x, y) < \lambda$, then $w_\lambda(x, y) < \kappa(\lambda)$, and $w_\lambda(x, y) = \kappa(\lambda)$ implies $d_w(x, y) = \lambda$;*

(e) *if $\mu \mapsto w_\mu(x, y)$ and κ are continuous from the left on $(0, \infty)$, then the inequalities $d_w(x, y) < \lambda$ and $w_\lambda(x, y) < \kappa(\lambda)$ are equivalent;*

(f) *if $\mu \mapsto w_\mu(x, y)$ and κ are continuous on $(0, \infty)$, then $d_w(x, y) = \lambda$ if and only if $w_\lambda(x, y) = \kappa(\lambda)$.*

Proof. (a) By 3.2(a), (11) and 2.5, for any $\mu > 0$ such that $d_w(x,y) < \mu < \lambda$ we have: $w_\lambda(x,y) \le w_\mu(x,y) \le \kappa(\mu) \le \kappa(\lambda)$, and it suffices to pass to the limit as $\mu \to d_w(x,y) + 0$.

(b) By the definition of d_w, for any $\mu > \lambda = d_w(x,y)$ we find $w_\mu(x,y) \le \kappa(\mu)$, and so, letting μ go to $\lambda + 0$, we get:

$$w_{\lambda+0}(x,y) = \lim_{\mu\to\lambda+0} w_\mu(x,y) \le \lim_{\mu\to\lambda+0} \kappa(\mu) = \kappa(\lambda+0).$$

For any $0 < \mu < \lambda$ we find $w_\mu(x,y) > \kappa(\mu)$ (for otherwise if $w_\mu(x,y) \le \kappa(\mu)$, the definition of d_w would give $\lambda = d_w(x,y) \le \mu$), and, as above, it remains to pass to the limit as $\mu \to \lambda - 0$.

(c) If $w_\lambda(x,y) \le \kappa(\lambda)$, then $d_w(x,y) \le \lambda$ follows from the definition of d_w. Suppose that $d_w(x,y) \le \lambda$. If $d_w(x,y) < \lambda$, then, by (a), $w_\lambda(x,y) \le \kappa(\lambda)$, and if $d_w(x,y) = \lambda$, then, by (b) and the continuity from the right of functions κ and $\mu \mapsto w_\mu(x,y)$, $w_\lambda(x,y) = w_{\lambda+0}(x,y) \le \kappa(\lambda+0) = \kappa(\lambda)$.

(d) If F is strict, then we know from 2.5 that κ is strictly increasing. By (a), if $d_w(x,y) < \mu < \lambda$, then $w_\lambda(x,y) \le w_\mu(x,y) \le \kappa(\mu) < \kappa(\lambda)$. Now suppose that $w_\lambda(x,y) = \kappa(\lambda)$. The definition of d_w implies $d_w(x,y) \le \lambda$. The inequality $d_w(x,y) < \lambda$ cannot hold, because by the just proved fact, $w_\lambda(x,y) < \kappa(\lambda)$. Therefore, $d_w(x,y) = \lambda$.

(e) The part "$d_w(x,y) < \lambda$ implies $w_\lambda(x,y) < \kappa(\lambda)$" follows from (d). Let $w_\lambda(x,y) < \kappa(\lambda)$. By definition (11), $d_w(x,y) \le \lambda$, and the equality here is impossible, for if $d_w(x,y) = \lambda$, then, by (b), $w_\lambda(x,y) = w_{\lambda-0}(x,y) \ge \kappa(\lambda-0) = \kappa(\lambda)$, which contradicts the assumption.

(f) Part "$\Leftarrow$" follows from (d), and part "$\Rightarrow$" follows from (b):

$$w_\lambda(x,y) = w_{\lambda+0}(x,y) \le \kappa(\lambda+0) = \kappa(\lambda) = \kappa(\lambda-0) \le w_{\lambda-0}(x,y) = w_\lambda(x,y).$$

This finishes the proof of Theorem 2. $\square$

4.4. Examples. (a) The one-sided limits used in the previous theorem are essential for its validity. Let (X,d) be a metric space, $\lambda > 0$, $x, y \in X$, and set $w_\lambda(x,y) = \infty$ if $\lambda < d(x,y)$ and $w_\lambda(x,y) = 0$ if $\lambda \ge d(x,y)$. Then w is an F_1-modular on X (actually, an F-modular for every $F \in \mathcal{F}(\mathbb{R}^+)$, cf. 3.3(b)), for which the function $\mu \mapsto w_\mu(x,y)$ is continuous from the right on $(0,\infty)$ and discontinuous from the left at $\lambda = d(x,y) > 0$. If $\kappa(u) = u$ and $x \ne y$ in Theorem 2(b), then for $\lambda = d_w(x,y) = d(x,y) > 0$ we find

$$w_{\lambda+0}(x,y) = w_\lambda(x,y) = 0 < \kappa(\lambda) = \lambda = d(x,y) < \infty = w_{\lambda-0}(x,y),$$

and so, $w_{\lambda-0}(x,y)$ cannot be replaced by $w_\lambda(x,y)$. Similarly, 3.3(b) shows that $w_{\lambda+0}(x,y)$ cannot be replaced by $w_\lambda(x,y)$ in Theorem 2(b). As a consequence, assertions (c), (e) and (f) in that theorem may not be valid without the assumptions of one-sided continuity.

(b) An example of an F-pseudomodular with property (c) from Theorem 2 is (5) from example 3.3(e). In order to see that $w_{\lambda+0}(x,y)=w_\lambda(x,y)$ for all $\lambda>0$ and $x,y\in Y$, by virtue of (2) it suffices to show that $w_\lambda(x,y)\le w_{\lambda+0}(x,y)$. If $a=t_0<t_1<\cdots<t_{m-1}<t_m=b$, definition (5) implies

$$\sum_{i=1}^{m}\Phi\Big(\frac{d\big(x(t_i)+y(t_{i-1}),y(t_i)+x(t_{i-1})\big)}{\varphi(\mu)}\Big)\le w_\mu(x,y)\ \text{for all}\ \mu>\lambda.$$

Taking into account the continuity of Φ and φ, we have, as $\mu\to\lambda+0$,

$$\sum_{i=1}^{m}\Phi\Big(\frac{d\big(x(t_i)+y(t_{i-1}),y(t_i)+x(t_{i-1})\big)}{\varphi(\lambda)}\Big)\le w_{\lambda+0}(x,y),$$

and it remains to take the supremum such as in (5). Similarly, for w from 3.3(d) one can show that the function $\mu\mapsto w_\mu(x,y)$ is continuous from the right on $(0,\infty)$ provided that the function φ from 3.3(d) is continuous from the right on $(0,\infty)$.

4.5. Now we study to what extent the modular convergence is related to the metric convergence:

Theorem 3. *Let w be an F-pseudomodular on X, κ an F-superadditive function, $x\in X_w$ and $\{x_n\}_{n=1}^\infty\subset X_w$ be a sequence.* (a) *If $w_\lambda(x_n,x)\to0$ as $n\to\infty$ for all $\lambda>0$, then $d_w(x_n,x)\to0$ as $n\to\infty$.* (b) *Conversely, if κ is continuous from the left on $(0,\infty)$, $\kappa(0)=0$ and $\kappa(\infty)=\infty$, then the condition $d_w(x_n,x)\to0$ as $n\to\infty$ implies $w_\lambda(x_n,x)\to0$ as $n\to\infty$ for all $\lambda>0$.* (c) *Assertions, similar to* (a) *and* (b), *hold for Cauchy sequences $\{x_n\}_{n=1}^\infty\subset X_w$.*

Proof. (a) Given $\varepsilon>0$, $w_\varepsilon(x_n,x)\to0$ as $n\to\infty$, and so, since $\kappa(\varepsilon)>0$, there exists $n_0(\varepsilon)\in\mathbb{N}$ such that $w_\varepsilon(x_n,x)\le\kappa(\varepsilon)$ for all $n\ge n_0(\varepsilon)$. By the definition of d_w, this means that $d_w(x_n,x)\le\varepsilon$ for all $n\ge n_0(\varepsilon)$, and the assertion follows.

(b) Let us fix $\lambda>0$ arbitrarily. Given $\varepsilon>0$, we set $\mu(\varepsilon)\equiv\kappa_+^{-1}(\varepsilon)=\sup\{u\in\mathbb{R}^+:\kappa(u)\le\varepsilon\}$ (the right inverse of κ) and note that $\kappa(\mu(\varepsilon))\le\varepsilon$: in fact, $\kappa(u)\le\varepsilon$ for all $u<\mu(\varepsilon)$, and it remains to pass to the limit as $u\to\mu(\varepsilon)-0$ and take into account the left continuity of κ. We consider two possibilities: (i) $0<\varepsilon<\kappa(\lambda)$, and ($ii$) $\varepsilon\ge\kappa(\lambda)$.

(i) Let $0<\varepsilon<\kappa(\lambda)$. Since $\mu(\varepsilon)>0$, by the assumption there exists $n_0(\varepsilon)\in\mathbb{N}$ such that $d_w(x_n,x)<\mu(\varepsilon)$ for all $n\ge n_0(\varepsilon)$, and so, by Theorem 2(a), $w_{\mu(\varepsilon)}(x_n,x)\le\kappa(\mu(\varepsilon))\le\varepsilon$. Since $\varepsilon<\kappa(\lambda)$, we have $\mu(\varepsilon)\le\lambda$ (otherwise, condition $\lambda<\mu(\varepsilon)$ and the definition of $\mu(\varepsilon)$ imply $\kappa(\lambda)\le\varepsilon$), and the property in 3.2(a) gives:

$$w_\lambda(x_n,x)\le w_{\mu(\varepsilon)}(x_n,x)\le\varepsilon\quad\text{for}\quad\text{all}\quad n\ge n_0(\varepsilon).$$

(ii) If $\varepsilon\ge\kappa(\lambda)$, we set $\varepsilon_1=\kappa(\lambda)/2<\kappa(\lambda)$ and apply the arguments in (i) for $\varepsilon=\varepsilon_1$: because, as above, $0<\mu(\varepsilon_1)\le\lambda$, we get

$$w_\lambda(x_n,x)\le w_{\mu(\varepsilon_1)}(x_n,x)\le\varepsilon_1=\frac{\kappa(\lambda)}{2}<\kappa(\lambda)\le\varepsilon$$

for all $n\ge n_0(\varepsilon_1)$. □

4.6. The pseudometric d_w in Theorem 1 is defined for any F-superadditive function κ. Under additional restrictions on κ (with respect to F) another pseudometric can be defined on X as Theorem 4 below shows. Also, in the next theorem the function d_w^1 corresponding to $\kappa(u)=u$ is a more general variant of the F-norm from [6].

Theorem 4. *Let w be an F-(pseudo)modular on X and κ be an increasing φ-function. Then the function $d_w^1:X\times X\to[0,\infty]$ defined by (with the convention that $\kappa^{-1}(\infty)=\infty$)*

$$d_w^1(x,y)=\inf_{\lambda>0}\big(\lambda+\kappa^{-1}(w_\lambda(x,y))\big)\quad\textit{for}\quad x,\,y\in X\tag{17}$$

is a (pseudo)metric on X assuming the value ∞ (cf. Theorem 1(a)–(d)) such that

$$d_w(x,y)\le d_w^1(x,y)\le 2d_w(x,y)\quad\textit{for all}\quad x,\,y\in X.\tag{18}$$

In particular, the function d_w^1 is finite on each set $X'\subset X$ such that $d_w(x,y)<\infty$ for all $x,\,y\in X'$, and one may always set $X'=X_w$.

Proof. Let w be an F-pseudomodular on X (recall that the function κ is F-superadditive according to convention 3.6).

1. To show that $d_w^1(x,x)=0$ for all $x\in X$, we note that $w_\lambda(x,x)=0$ for all $\lambda>0$ implying $\kappa^{-1}\big(w_\lambda(x,x)\big)=0$, and so, $d_w^1(x,x)=\inf_{\lambda>0}\lambda=0$.

2. Since the symmetry of d_w^1 is clear including the value ∞, we prove the triangle inequality $d_w^1(x,y)\le d_w^1(x,z)+d_w^1(y,z)$ for all $x,y,z\in X$. If $d_w^1(x,z)=\infty$ or $d_w^1(y,z)=\infty$, this inequality is obvious, and so, we suppose

that $d_w^1(x,z)$ and $d_w^1(y,z)$ are finite. By the definition of these quantities, for each $\varepsilon > 0$ there exist $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$\lambda + \kappa^{-1}\big(w_\lambda(x,z)\big) \le d_w^1(x,z) + \varepsilon \quad \text{and} \quad \mu + \kappa^{-1}\big(w_\mu(y,z)\big) \le d_w^1(y,z) + \varepsilon$$

or, equivalently,

$$w_\lambda(x,z) \le \kappa\big(d_w^1(x,z) + \varepsilon - \lambda\big) \quad \text{and} \quad w_\mu(y,z) \le \kappa\big(d_w^1(y,z) + \varepsilon - \mu\big). \tag{19}$$

Because κ is F-superadditive, inequality (1) can be equivalently rewritten in the form

$$\kappa^{-1}\big(F(\kappa(u),\kappa(v))\big) \le u + v \quad \text{for all} \quad u,\, v \in \mathbb{R}^+. \tag{20}$$

Then applying (17), 3.1(F.iii), (19), the monotonicity of F and κ^{-1} and (20), we get:

$$\begin{aligned} d_w^1(x,y) &\le \lambda + \mu + \kappa^{-1}\big(w_{\lambda+\mu}(x,y)\big) \\ &\le \lambda + \mu + \kappa^{-1}\Big(F\big(w_\lambda(x,z), w_\mu(y,z)\big)\Big) \\ &\le \lambda + \mu + \kappa^{-1}\Big(F\big(\kappa(d_w^1(x,z) + \varepsilon - \lambda), \kappa(d_w^1(y,z) + \varepsilon - \mu)\big)\Big) \\ &\le \lambda + \mu + \big(d_w^1(x,z) + \varepsilon - \lambda\big) + \big(d_w^1(y,z) + \varepsilon - \mu\big) \\ &= d_w^1(x,z) + d_w^1(y,z) + 2\varepsilon, \end{aligned}$$

and it remains to take into account the arbitrariness of $\varepsilon > 0$.

3. Let us prove the inequalities in (18). The right hand side inequality is clear if $d_w(x,y) = \infty$, so we suppose that it is finite. Then for any $\lambda > 0$ such that $d_w(x,y) < \lambda$ the definition of $d_w(x,y)$ implies $w_\lambda(x,y) \le \kappa(\lambda)$, whence

$$d_w^1(x,y) \le \lambda + \kappa^{-1}\big(w_\lambda(x,y)\big) \le \lambda + \kappa^{-1}(\kappa(\lambda)) = 2\lambda.$$

Passing to the limit as $\lambda \to d_w(x,y)$, we get $d_w^1(x,y) \le 2d_w(x,y)$, $x,y \in X$.

In order to prove the left hand side inequality, let $\lambda > 0$ be arbitrary and $x,y \in X$. If $w_\lambda(x,y) \le \kappa(\lambda)$, then, according to the definition of $d_w(x,y)$, $d_w(x,y) \le \lambda$. Let us show that if $w_\lambda(x,y) > \kappa(\lambda)$, then $d_w(x,y) \le \kappa^{-1}(w_\lambda(x,y))$. In fact, this inequality is clear if $w_\lambda(x,y) = \infty$, and if $w_\lambda(x,y) < \infty$, then by the continuity of κ and the fact that $\kappa(\mathbb{R}^+) = \mathbb{R}^+$ we find $\mu > \lambda$ such that $\kappa(\mu) = w_\lambda(x,y)$ (otherwise, if $\mu \le \lambda$, then $w_\lambda(x,y) = \kappa(\mu) \le \kappa(\lambda)$), and it follows from 3.2(a) that $w_\mu(x,y) \le w_\lambda(x,y) = \kappa(\mu)$. Then the definition of $d_w(x,y)$ implies $d_w(x,y) \le \mu = \kappa^{-1}(w_\lambda(x,y))$. Thus,

$$d_w(x,y) \le \max\{\lambda, \kappa^{-1}(w_\lambda(x,y))\} \le \lambda + \kappa^{-1}(w_\lambda(x,y)) \text{ for all } \lambda > 0.$$

Taking the infimum over all $\lambda > 0$, we arrive at $d_w(x,y) \le d_w^1(x,y)$.

Inequalities (18) mean that the quantities $d_w(x,y)$ and $d_w^1(x,y)$ are finite or infinite simultaneously, proving the last assertion of our theorem.

4. Now let w be an F-modular on X, and let us prove that, given $x, y \in X$, condition $d_w^1(x,y) = 0$ implies $x = y$ or, equivalently, that $w_\lambda(x,y) = 0$ for all $\lambda > 0$. On the contrary, suppose that there exists $\lambda_0 > 0$ such that $w_{\lambda_0}(x,y) > 0$. Then we have

$$\lambda + \kappa^{-1}(w_\lambda(x,y)) \ge \lambda_0 \quad \text{for all} \quad \lambda \ge \lambda_0,$$

and if $0 < \lambda < \lambda_0$, then by virtue of the monotonicity of functions $\lambda \mapsto w_\lambda(x,y)$ and κ^{-1}, we find $0 < w_{\lambda_0}(x,y) \le w_\lambda(x,y)$, and so,

$$0 < \kappa^{-1}(w_{\lambda_0}(x,y)) \le \kappa^{-1}(w_\lambda(x,y)) \le \lambda + \kappa^{-1}(w_\lambda(x,y)).$$

It follows that

$$\lambda + \kappa^{-1}(w_\lambda(x,y)) \ge \min\bigl\{\lambda_0, \kappa^{-1}(w_{\lambda_0}(x,y))\bigr\} \equiv \lambda_1 \quad \text{for all} \quad \lambda > 0,$$

and the definition of $d_w^1(x,y)$ implies $d_w^1(x,y) \ge \lambda_1 > 0$, a contradiction.

This completes the proof of Theorem 4. □

5. φ-convex metric F-modulars

Throughout this section φ is an increasing φ-function.

5.1. Definition. A function $w : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a *φ-convex (metric) F-(pseudo)modular on the set X* if it satisfies axioms (F.i′) or (F.i) and (F.ii) from 3.1 and, for all $\lambda, \mu > 0$ and $x, y, z \in X$,

(F.φ) $(\lambda + \mu) w_{\varphi(\lambda+\mu)}(x,y) \le F\bigl(\lambda w_{\varphi(\lambda)}(x,z), \mu w_{\varphi(\mu)}(y,z)\bigr)$.

In other words, w is a φ-convex F-(pseudo)modular on X if and only if the function $\widehat{w} : (0,\infty) \times X \times X \to [0,\infty]$ given by $\widehat{w}_\lambda(x,y) = \lambda w_{\varphi(\lambda)}(x,y)$ for all $\lambda > 0$ and $x, y \in X$ is an F-(pseudo)modular on X in the sense of 3.1. We recover the ordinary convex (pseudo)modular w from [5, Section 2] (see also 4.2(b)) by setting $\varphi(\lambda) = \lambda$ and $F(u,v) = u + v$ in the axiom (F.φ).

Given $w : (0,\infty) \times X \times X \to [0,\infty]$, we set $\mathbf{w}_\lambda(x,y) = w_{\varphi(\lambda)}(x,y)$ for $\lambda > 0$ and $x, y \in X$, so that $w_\lambda(x,y) = \mathbf{w}_{\varphi^{-1}(\lambda)}(x,y)$. We have: w is a φ-convex F-modular on X iff $\mathbf{w}$ satisfies (F.i), (F.ii) and

$$(\lambda + \mu)\mathbf{w}_{\lambda+\mu}(x,y) \le F\bigl(\lambda \mathbf{w}_\lambda(x,z), \mu \mathbf{w}_\mu(y,z)\bigr).$$

In particular, if ρ is a convex modular on a real linear space X, then the function $w_\lambda(x,y) = \rho\bigl((x-y)/\varphi^{-1}(\lambda)\bigr)$ is a φ-convex metric modular on X, which is not considered in the classical theory.

5.2. Some properties of w. (a) If the function w satisfies 5.1, then the functions $\lambda \mapsto w_{\varphi(\lambda)}(x,y)$, $\lambda \mapsto \lambda w_{\varphi(\lambda)}(x,y)$ and $\lambda \mapsto \varphi^{-1}(\lambda)w_\lambda(x,y)$ are nonincreasing on $(0,\infty)$ for all $x, y \in X$: in fact, if $0<\mu<\lambda$, then (F.φ) with $z=x$, (F.i′) and (F.ii) yield:

$$\begin{aligned}\lambda w_{\varphi(\lambda)}(x,y) &\le F\big((\lambda-\mu)w_{\varphi(\lambda-\mu)}(x,x), \mu w_{\varphi(\mu)}(y,x)\big)\\ &= F\big(0, \mu w_{\varphi(\mu)}(x,y)\big) = \mu w_{\varphi(\mu)}(x,y),\end{aligned}$$

whence

$$w_{\varphi(\lambda)}(x,y) \le \frac{\mu}{\lambda} w_{\varphi(\mu)}(x,y) \le w_{\varphi(\mu)}(x,y), \quad 0<\mu<\lambda, \quad x,y\in X, \tag{21}$$

or, equivalently, the first inequality can be rewritten as

$$w_\lambda(x,y) \le \frac{\varphi^{-1}(\mu)}{\varphi^{-1}(\lambda)} w_\mu(x,y) \quad \text{for all} \quad 0<\mu<\lambda \text{ and } x,y\in X. \tag{22}$$

(b) If w is a φ-convex F-(pseudo)modular on X, then $X_w = X_w^*$ (cf. (15)). It suffices to show that $X_w^* \subset X_w$. Let $x \in X_w^*$. Then there exists $\mu = \mu(x) > 0$ such that $w_\mu(x, x_0) < \infty$, and it follows from (22) that if $0<\mu<\lambda$, then

$$w_\lambda(x,x_0) \le \frac{\varphi^{-1}(\mu)}{\varphi^{-1}(\lambda)} w_\mu(x,x_0) \to 0 \quad \text{as} \quad \lambda\to\infty,$$

and so, $x \in X_w$.

5.3. In the next theorem we introduce a metric on the modular space corresponding to the case under consideration.

Theorem 5. *Let w be a φ-convex F-(pseudo)modular on X and the function κ be such that $\kappa(+0)=0$ and*

$$\limsup_{\lambda\to\infty} \frac{\lambda}{\kappa(\lambda)} < \infty. \tag{23}$$

Then the function d_w^φ defined by

$$d_w^\varphi(x,y) = \inf\{\lambda>0 : \lambda w_{\varphi(\lambda)}(x,y) \le \kappa(\lambda)\}, \qquad x,\, y \in X_w^*,$$

is a (pseudo)metric on the set X_w^*.

Proof. According to the assumption, the function $\widehat{w}_\lambda(x,y)$ from 5.1 is an F-(pseudo)modular on X, and so, Theorem 1 applies to $d_w^\varphi = d_{\widehat{w}}$.

All we are left to verify is that $d_w^\varphi(x,y)<\infty$ for all $x,\,y\in X_w^*$. There exist $\lambda=\lambda(x)>0$ and $\mu=\mu(y)>0$ such that $w_\lambda(x,x_0)<\infty$ and $w_\mu(y,x_0)<\infty$. Setting $\lambda'=\varphi^{-1}(\lambda)$ and $\mu'=\varphi^{-1}(\mu)$, by virtue of (21) and (F.φ), for $\nu\ge\lambda'+\mu'$ we find

$$\begin{aligned} w_{\varphi(\nu)}(x,y) &\le \frac{\lambda'+\mu'}{\nu}\,w_{\varphi(\lambda'+\mu')}(x,y)\\ &\le \frac{1}{\nu}\,F\big(\lambda' w_{\varphi(\lambda')}(x,x_0),\mu' w_{\varphi(\mu')}(y,x_0)\big)\\ &= \frac{1}{\nu}\,F\big(\varphi^{-1}(\lambda)w_\lambda(x,x_0),\varphi^{-1}(\mu)w_\mu(y,x_0)\big)\to 0 \end{aligned} \tag{24}$$

as $\nu\to\infty$. Condition (23) implies the existence of $c_1>0$ and $c_2>0$ such that $\kappa(\lambda)\ge c_2\lambda$ for all $\lambda\ge c_1$. By (24), there exists $\lambda_0>0$ such that $w_{\varphi(\nu)}(x,y)\le c_2$ for all $\nu\ge\lambda_0$. We set $\lambda_1=\max\{\lambda_0,c_1\}$. Since $\lambda_1\ge\lambda_0$, we have $w_{\varphi(\lambda_1)}(x,y)\le c_2$, and $\lambda_1\ge c_1$ implies $c_2\le\kappa(\lambda_1)/\lambda_1$, and so, $\lambda_1 w_{\varphi(\lambda_1)}(x,y)\le\kappa(\lambda_1)$, proving that $d_w^\varphi(x,y)\le\lambda_1<\infty$. ☐

5.4. In the special case when $\kappa(u)=u$ is F-superadditive, that is, $F(u,v)\le u+v$ (examples of such F are contained in 2.4(c) with $p\ge 1$ and 2.4(d) for any $p>0$), the functions d_w from Theorem 1 given by (13) and d_w^φ from Theorem 5 defined by (cf. (16))

$$d_w^\varphi(x,y)=\inf\{\lambda>0: w_{\varphi(\lambda)}(x,y)\le 1\},\qquad x,\,y\in X_w^*=X_w, \tag{25}$$

are *specifically equivalent* as can be seen from the following theorem.

Theorem 6. *Let w be a φ-convex F-(pseudo)modular on X, $\kappa(u)=u$ be F-superadditive and $x,\,y\in X_w^*=X_w$. Then we have:*

(a) *if $d_w(x,y)<1$, then $d_w^\varphi(x,y)<\varphi^{-1}(1)$, and vice versa, and the following two inequalities hold:*

$$d_w(x,y)\cdot\varphi^{-1}(d_w(x,y))\le d_w^\varphi(x,y)\le\varphi^{-1}(d_w(x,y));$$

(b) *if $d_w(x,y)\ge 1$, then $d_w^\varphi(x,y)\ge\varphi^{-1}(1)$, and vice versa, and the following two inequalities hold:*

$$\varphi^{-1}(d_w(x,y))\le d_w^\varphi(x,y)\le d_w(x,y)\cdot\varphi^{-1}(d_w(x,y)).$$

Proof. We divide the proof into four steps for clarity. The first two steps are devoted to the proof of (a).

1. First we show that $d_w(x,y)<1$ implies $d_w^\varphi(x,y)\le\varphi^{-1}(d_w(x,y))$. In fact, since $d_w(x,y)<1$, then $\varphi^{-1}(d_w(x,y))<\varphi^{-1}(1)$, and so, for any number $\lambda>0$ such that $\varphi^{-1}(d_w(x,y))<\lambda<\varphi^{-1}(1)$ we find $d_w(x,y)<\varphi(\lambda)<1$.

By (13), $w_{\varphi(\lambda)}(x,y) \le \varphi(\lambda)$, and so, $w_{\varphi(\lambda)}(x,y) < 1$, which, by virtue of (25), gives $d_w^\varphi(x,y) \le \lambda$. Passing to the limit as $\lambda \to \varphi^{-1}(d_w(x,y))$ we arrive at the desired inequality. This inequality also shows that if $d_w(x,y) < 1$, then $d_w^\varphi(x,y) < \varphi^{-1}(1)$.

2. Let us prove that if $d_w^\varphi(x,y) < \varphi^{-1}(1)$, then $d_w(x,y) \cdot \varphi^{-1}(d_w(x,y)) \le d_w^\varphi(x,y)$. For this, we define an auxiliary φ-function by $\widehat{\varphi}(\lambda) = \lambda\varphi(\lambda)$, $\lambda \in \mathbb{R}^+$, and note that

$$\widehat{\varphi}(\varphi^{-1}(\lambda)) = \varphi^{-1}(\lambda) \cdot \varphi(\varphi^{-1}(\lambda)) = \lambda\varphi^{-1}(\lambda), \qquad \lambda \in \mathbb{R}^+. \tag{26}$$

Since $d_w^\varphi(x,y) < \varphi^{-1}(1)$ or, equivalently, $\varphi(d_w^\varphi(x,y)) < 1$, we have $d_w^\varphi(x,y)\varphi(d_w^\varphi(x,y)) \le d_w^\varphi(x,y)$, and so, $\widehat{\varphi}(d_w^\varphi(x,y)) \le d_w^\varphi(x,y)$ implying $d_w^\varphi(x,y) \le \widehat{\varphi}^{-1}(d_w^\varphi(x,y))$. Again, since $d_w^\varphi(x,y) < \varphi^{-1}(1)$ and (26) implies $\varphi^{-1}(1) = \widehat{\varphi}(\varphi^{-1}(1))$, we get $\widehat{\varphi}^{-1}(d_w^\varphi(x,y)) < \varphi^{-1}(1)$. Thus, we have shown that

$$\text{if } \ d_w^\varphi(x,y) < \varphi^{-1}(1), \ \text{ then } \ d_w^\varphi(x,y) \le \widehat{\varphi}^{-1}(d_w^\varphi(x,y)) < \varphi^{-1}(1). \tag{27}$$

Now let $\lambda > 0$ be arbitrary such that

$$\widehat{\varphi}^{-1}(d_w^\varphi(x,y)) < \lambda < \varphi^{-1}(1). \tag{28}$$

Then the first inequality in (27) and the first inequality in (28) yield

$$d_w^\varphi(x,y) < \lambda, \tag{29}$$

and the first inequality in (28) implies also $d_w^\varphi(x,y) < \widehat{\varphi}(\lambda) = \lambda\varphi(\lambda)$, whence

$$d_w^\varphi(x,y)/\lambda < \varphi(\lambda). \tag{30}$$

Taking into account (29), for any $\mu > 0$ such that $d_w^\varphi(x,y) < \mu < \lambda$ we get, in view of (25), $w_{\varphi(\mu)}(x,y) \le 1$, and so, (21) gives

$$w_{\varphi(\lambda)}(x,y) \le (\mu/\lambda) w_{\varphi(\mu)}(x,y) \le \mu/\lambda.$$

Passing to the limit as $\mu \to d_w^\varphi(x,y)$ we find $w_{\varphi(\lambda)}(x,y) \le d_w^\varphi(x,y)/\lambda$. It follows from (30) now that $w_{\varphi(\lambda)}(x,y) < \varphi(\lambda)$, and so, (13) implies that $d_w(x,y) \le \varphi(\lambda)$ for all λ such as in (28). Letting λ tend to $\widehat{\varphi}^{-1}(d_w^\varphi(x,y))$, we get

$$d_w(x,y) \le \varphi\big(\widehat{\varphi}^{-1}(d_w^\varphi(x,y))\big). \tag{31}$$

Taking φ^{-1} and then $\widehat{\varphi}$ in (31) and applying (26), we obtain the desired inequality in step 2.

That $d_w(x,y)<1$ and $d_w^\varphi(x,y)<\varphi^{-1}(1)$ are equivalent can be exposed as follows: we know from step 1 that $d_w(x,y)<1$ implies $d_w^\varphi(x,y)<\varphi^{-1}(1)$; conversely, noting that, by (26), $\widehat{\varphi}(\varphi^{-1}(1))=\varphi^{-1}(1)$ or $\widehat{\varphi}^{-1}(\varphi^{-1}(1))=\varphi^{-1}(1)$, we find that if $d_w^\varphi(x,y)<\varphi^{-1}(1)$, then by virtue of (31),

$$d_w(x,y)\le\varphi\big(\widehat{\varphi}^{-1}(d_w^\varphi(x,y))\big)<\varphi\big(\widehat{\varphi}^{-1}(\varphi^{-1}(1))\big)=\varphi(\varphi^{-1}(1))=1.$$

It follows that the inequalities $d_w(x,y)\ge 1$ and $d_w^\varphi(x,y)\ge\varphi^{-1}(1)$ are equivalent, as well.

Now we turn to the proof of (b) in steps 3 and 4.

3. Let us show that if $d_w^\varphi(x,y)\ge\varphi^{-1}(1)$, then $\varphi^{-1}(d_w(x,y))\le d_w^\varphi(x,y)$. In fact, condition $d_w^\varphi(x,y)\ge\varphi^{-1}(1)$ implies $1\le\varphi(d_w^\varphi(x,y))$, and so, given $\lambda>d_w^\varphi(x,y)$, definition (25) gives

$$w_{\varphi(\lambda)}(x,y)\le 1\le\varphi(d_w^\varphi(x,y))<\varphi(\lambda),$$

whence (13) yields $d_w(x,y)\le\varphi(\lambda)$. Passing to the limit as $\lambda\to d_w^\varphi(x,y)$, we get $d_w(x,y)\le\varphi(d_w^\varphi(x,y))$, and the desired inequality follows.

4. Finally, we show that $d_w(x,y)\ge 1$ implies the right hand side inequality in (b). Let $d_w(x,y)\ge 1$. Then for any $\lambda>d_w(x,y)$ we have, by (13), $w_\lambda(x,y)\le\lambda$. Noting that $\lambda>1$, we get $\lambda\varphi^{-1}(\lambda)>\varphi^{-1}(\lambda)$, and so, setting $\lambda'=\varphi(\lambda\varphi^{-1}(\lambda))>\lambda$, by virtue of (22), we find

$$w_{\lambda'}(x,y)\le\frac{\varphi^{-1}(\lambda)}{\varphi^{-1}(\lambda')}\,w_\lambda(x,y)\le\frac{\varphi^{-1}(\lambda)}{\lambda\varphi^{-1}(\lambda)}\cdot\lambda=1$$

or $w_{\varphi(\lambda\varphi^{-1}(\lambda))}(x,y)\le 1$, which in accordance with (25) gives $d_w^\varphi(x,y)\le\lambda\varphi^{-1}(\lambda)$. So, letting λ go to $d_w(x,y)$ we arrive at the desired inequality.

This finishes the proof of Theorem 6. □

5.5. Remarks. (a) The inequalities in (a) and (b) in Theorem 6 may be rewritten equivalently as

(a′) $\varphi(d_w^\varphi(x,y))\le d_w(x,y)\le\varphi\big(\widehat{\varphi}^{-1}(d_w^\varphi(x,y))\big)$,

(b′) $\varphi\big(\widehat{\varphi}^{-1}(d_w^\varphi(x,y))\big)\le d_w(x,y)\le\varphi(d_w^\varphi(x,y))$,

respectively. In fact, (a′) follows from the right hand side inequality in (a) and (31). Since the inequality at the right in (b′) is equivalent to the inequality at the left in (b), we establish only the left hand side inequality in (b′): noting that $\widehat{\varphi}(\lambda)=\lambda\varphi(\lambda)$ iff $\lambda=\widehat{\varphi}^{-1}(\lambda\varphi(\lambda))$ and setting $\lambda=\varphi^{-1}(d_w(x,y))$ in the inequality $d_w^\varphi(x,y)\le d_w(x,y)\varphi^{-1}(d_w(x,y))$, we get

$$\widehat{\varphi}^{-1}(d_w^\varphi(x,y))\le\widehat{\varphi}^{-1}(\varphi(\lambda)\cdot\lambda)=\lambda=\varphi^{-1}(d_w(x,y)),$$

and so, $\varphi\big(\widehat{\varphi}^{-1}(d_w^\varphi(x,y))\big)\le d_w(x,y)$, as desired.

(b) In Theorem 6 we have assumed that w satisfies definition 5.1, but not definition 3.1 (i.e., axiom (F.iii) may fail). So, according to Theorem 5, function d_w^φ from (25) is a (pseudo)metric in Theorem 6, and function d_w from (13) is not, in general. Now if we assume in Theorem 6 that w is also an F-(pseudo)modular in the sense of definition 3.1, then, by virtue of Theorem 1, function d_w is a (pseudo)metric on any set $X' \subset X$, on which it is finite. The right hand sides in (a′) and (b′) clearly show that d_w is finite on $X' = X_w^*$ (which also follows from Theorem 1 and equality $X_w^* = X_w$).

(c) We also note that without the "φ-convexity" assumption in Theorem 6 the inequalities at the left in (a) and at the right in (b) may not be true, while the other two inequalities always hold (for $\varphi(\lambda) = \lambda$ see [5, Section 2.3]). In our more general situation this can be seen as follows. For a metric space (X, d) and a number $p > 0$ we set

$$w_\lambda(x, y) = \frac{d(x, y)}{\lambda^p + d(x, y)}, \qquad \lambda > 0, \quad x,\, y \in X.$$

According to 3.3(c), w is an F_1-modular on X, and $X_w = X_w^* = X$. For any increasing φ-function φ we have $w_{\varphi(\lambda)}(x, y) < 1$ for all $\lambda > 0$ and $x,\, y \in X$, and so, $d_w^\varphi \equiv 0$ on X identically. On the other hand, if $x \neq y$, then $0 < d_w(x, y) < 1$: this follows from (13), Theorem 2(f) and the fact that if $\lambda \geq 0$ is the solution of $\lambda^{p+1} = d - \lambda d$ with $d > 0$, then $0 < \lambda < 1$ (draw the picture of the graphs). Thus, the left hand side inequality in Theorem 6(a) does not hold. This example shows that the modular w above is not φ-convex for any increasing φ-function φ (which can be also verified by the definition).

(d) Definition 5.1 generalizes the notions of: 1) an s-convex modular from [9, Chapter I], [14] corresponding to $\varphi(\lambda) = \lambda^{1/s}$ with $0 < s \leq 1$; 2) an (F, φ)-modular from [8] where φ additionally satisfies $\varphi(uv) \geq \varphi(u)\varphi(v)$ for all $u, v \in \mathbb{R}^+$; 3) a φ-convex modular from [5] for $F(u, v) = u + v$ and general function φ. In the case where $\varphi(\lambda) = \lambda^{1/s}$ with $0 < s \leq 1$ the inequalities in Theorem 6 are of the form known from the classical theory of modulars on linear spaces [7, p. 7, Remark 3], [12] (see also Introduction):

$$\big(d_w(x, y)\big)^{s+1} \leq d_w^\varphi(x, y) \leq \big(d_w(x, y)\big)^s \ \text{ if } d_w(x, y) < 1 \text{ or } d_w^\varphi(x, y) < 1,$$

and these inequalities should be reversed if $d_w(x, y) \geq 1$ or $d_w^\varphi(x, y) \geq 1$. The inequalities in Theorem 6 are new if φ is a general increasing φ-function.

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