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BOUNDEDNESS OF THE VALUE FUNCTION OF THE WORST-CASE PORTFOLIO SELECTION PROBLEM WITH LINEAR CONSTRAINTS

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BOUNDEDNESS OF THE VALUE FUNCTION OF THE WORST-CASE PORTFOLIO SELECTION PROBLEM WITH LINEAR CONSTRAINTS [2]

We study the boundedness properties of the value function for a general worst-case scenario stochastic dynamic programming problem. For the portfolio selection problem,we present sufficient economically reasonable conditions for the finitness and uniform boundedness of the value function. The results can be used to decide if the problem is ill-posed and to correctly solve the Bellman-Isaacs equation with an appropriate numeric scheme.

JEL Classification: C61, C63, G11.

Key words: portfolio selection, Bellman-Isaacs equation, stochastic dynamic programming, value function, worst-case optimization.

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Introduction

A classic portfolio optimization problem with terminal utility maximization criterion is usually solved via the dynamic programming approach which reduces to solving a Bellman equation or a quasi-variational inequality (for recent developments see [Vath et al., 2007, Øksendal and Sulem, 2010, Ma et al., 2013]). We consider a generalization of the approach in discrete time when the underlying process driving the stochastic system is not explicit, rather two of its properties are known: the expected value and range at any given moment, plus the range is considered compact. For a detailed discussion of the approach, see [Andreev, 2015]. Ideologically similar research for a simplified one-step problem can be found in [Deng et al., 2005].

One of the main steps in solving a dynamic programming problem is establishing the necessary properties of the value function. In finance, the pioneering work of Merton [Merton, 1969] considers a simplified problem statement which allows for the analytic solution of the problem. A more elaborate framework requires proof of the correctness of the problem itself by proving that the value function exists and finite over a required domain. For this purpose, various approaches are implemented (usually a unique research is required for each problem). In [Davis and Norman, 1990] the value function properties are derived from the specific form of the price process and the utility function which make the function homogeneous. The paper [Vath et al., 2007] presents the boundaries of the value function based on the form of the corresponding Hamilton-Jakobi-Bellman quasi-variational inequality. [Ma et al., 2013] studies the problem in continuous time by approximating it with a sequence finite-step problems and researching properties of their value functions. The framework and assumptions of [Fruth et al., 2013] allow to represent the value function as a Riemann-Stieltjes integral and prove its existence and boundedness.

In this paper we provide practically useful sufficient conditions for the finiteness and boundedness of the value function for the worst-case optimization problem. For the particular case of the portfolio selection problem, we present economically reasonable conditions for the boundedness above and below and provide their interpretation. The paper is structured as follows: Section 1 provides a quick overview of the worst-case optimization framework and the Bellman-Isaacs equation; Section 2 presents results for the general optimization problem; Section 3 presents results for the portfolio selection problem; Section 4 concludes.

1 The worst-case optimization framework

Consider a general stochastic system at moments $t_0, \dots, t_N$ for a filtration $\{\mathcal{F}_n\}_{n=0}^N$. Let H_n denote the $\mathcal{F}_{n-1}$-measurable strategy at t_n and $\{H_k\}_{k=1}^N$ denote the whole strategy, let $\mathcal{S}_n$ be the $\mathcal{F}_n$-measurable system state at t_n. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In this section we consider the state space $\bar{\mathbb{R}}^l$ with the Euclidian metric. The following notation will be used for $n = \overline{1, N}$:

- $\mathcal{A}$ is a set of admissible strategies;
- $\mathbb{S}_n \subseteq \bar{\mathbb{R}}^l$ is a state space of the system at t_n, while $\mathcal{S}_n$ denotes the particular state at t_n;
- $\mathcal{A} \mid \mathcal{S}_{n-1}$ is a set of admissible strategies starting at moment t_{n-1} given the state $\mathcal{S}_{n-1} \in \mathbb{S}_{n-1}$, $n = \overline{1, N}$; an element of $\mathcal{A} \mid \mathcal{S}_{n-1}$ is denoted $H_{\geq n}$;
- $\mathbb{S}_n \mid H_{\leq n} \subseteq \bar{\mathbb{R}}^l$ is the range of $\mathcal{S}_n$ given the strategy $H_{\leq n} = \{H_k\}_{k=1}^n$;
- $\mathbb{S}_n \mid (H_n, \mathcal{S}_{n-1}) \subseteq \bar{\mathbb{R}}^l$ is range of $\mathcal{S}_n$ given that the strategy at t_n is H_n and the state at t_{n-1} is $\mathcal{S}_{n-1}$;
- $\mathbb{S}_N \mid (H_{\geq n}, \mathcal{S}_{n-1}) \subseteq \bar{\mathbb{R}}^l$ is the range of $\mathcal{S}_N$ given the state $\mathcal{S}_{n-1} \in \mathbb{S}_{n-1}$ and the strategy $H_{\geq n} \in \mathcal{A} \mid \mathcal{S}_{n-1}$.

$\mathcal{S}_0 \in \mathbb{R}^l$ is the given finite initial state. For further reference, we formally assume $\mathbb{S}_0 = \{\mathcal{S}_0\}$ and $H_{\leq 0} = H_0$.

The stochastic parameters of the system are driven by the $\mathcal{F}_n$-measurable process Θ_n with a distribution Q from the class of distributions $\mathbb{Q}^E$ with the expected value E_n and compact support K_n at t_n, $n = \overline{1, N}$. Indexing for $\mathbb{Q}$ will be analogous to H. Consider the following dynamics of the system:

$$S_n \mid (H_n, \mathcal{S}_{n-1}) = f_n\left(\Theta_n, H_n, \mathcal{S}_{n-1}\right). \tag{1}$$

Note that since the dynamics of the system (1) is Markov, $\mathbb{S}_n \mid (H_{\leq n}, \mathcal{S}_{n-1}) = \mathbb{S}_n \mid \mathcal{S}_{n-1}$. For ease of notation we will also write $\mathbb{E}_Q^{\mathcal{S}} f(S_n \mid H_n)$ instead of $\mathbb{E}_Q^{\mathcal{S}} f\left(S_n \mid (H_n, \mathcal{S})\right)$ for $\mathcal{S} \in \mathbb{S}_{n-1}$ since the conditioning on $\mathcal{S}$ is implied by the conditional expectation operator.

Definition 1.1. For a given collection of $\mathcal{F}_{n-1}$-measurable sets D_n, $D_n \subseteq \bar{\mathbb{R}}^{m+1}$, $D_n \neq \emptyset$, *admissible strategy* is a strategy H such that for all $n = \overline{1, N}$

1. $H_n \in D_n$;

2. H_n is $\mathcal{F}_{n-1}$-measurable;

3.

$$Q\left\{H_n \in A_n \mid \mathcal{F}_{n-1}\right\} \overset{\text{a.-s.}}{=} Q\left\{H_n \in A_n \mid \mathcal{S}_{n-1}\right\} \qquad \forall A_n \in \mathcal{B}(D_n),\ \forall Q \in \mathbb{Q}^E$$

(Markov control policy).

Given the terminal utility function J on $\mathbb{S}_N$, the *optimal strategy* is a strategy $H^* \in \mathcal{A}$ such that

$$\inf_{Q\in\mathbb{Q}^E} \mathbb{E}_Q^{\mathcal{S}_0} J(\mathcal{S}_N \mid H_N^*) = \sup_{H\in\mathcal{A}} \inf_{Q\in\mathbb{Q}^E} \mathbb{E}_Q^{\mathcal{S}_0} J(\mathcal{S}_N \mid H_N). \tag{2}$$

The Bellman-Isaacs equation of the problem is

$$V_{n-1}(\mathcal{S}) = \sup_{H_n\in D_n(\mathcal{S})} \inf_{Q_n\in\mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}} V_n(\mathcal{S}_n \mid H_n), \quad \mathcal{S} \in \mathbb{S}_{n-1}, \qquad n = \overline{1, N}, \tag{3}$$

$$V_N(\mathcal{S}) = J(\mathcal{S}), \quad \mathcal{S} \in \mathbb{S}_N, \tag{4}$$

where $V_n(\mathcal{S})$ is the value function at t_n.

Consider the strategies

$$H_n^* \in \operatorname*{Arg\,max}_{H_n\in D_n(\mathcal{S})} \inf_{Q_n\in\mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}} V_n(\mathcal{S}_n \mid H_n), \qquad \mathcal{S} \in \mathbb{S}_{n-1}, \tag{5}$$

which can be proven optimal via the corresponding Verification theorem. Let $\mathcal{A}^*$ be the set of admissible strategies which satisfy (5) for $n = \overline{0, N-1}$, and let $\mathcal{A}^* \mid \mathcal{S}_{k-1}$, $k = \overline{1, N}$, be the set of strategies $H_{\geq k}^* \in \mathcal{A} \mid \mathcal{S}$ which satisfy (5) for $n \geq k$, given the initial state $\mathcal{S} \in \mathbb{S}_{k-1}$ at t_{k-1}.

2 Results for a general problem

Below we provide sufficient conditions for the finiteness of the value function.

Lemma 2.1. *Let $J(\mathcal{S}) \in [\underline{J}; \overline{J}]$ on $\mathbb{S}_N$, where $\underline{J}, \overline{J} \in \bar{\mathbb{R}}$, $\underline{J} \leq \overline{J}$; let $V_{n-1}(\mathcal{S})$ be defined by (3)-(4). Then $\sup\limits_{\mathcal{S}\in\mathbb{S}_{n-1}} V_{n-1}(\mathcal{S}) \in [\underline{J}; \overline{J}]$ on $\mathbb{S}_{n-1}$ for all $n = \overline{1, N}$.*

Proof. For $n = N$, let $\mathcal{S} \in \mathbb{S}_{N-1}$. For any $H_N \in D_N(\mathcal{S})$, $\mathcal{S}_N \mid (H_N, \mathcal{S}) \in \mathbb{S}_N$ and we have

$$f(\mathcal{S}, H_N) = \inf_{Q_N\in\mathbb{Q}_N^E} \mathbb{E}_{Q_N}^{\mathcal{S}} J(\mathcal{S}_N \mid H_N) \in [\underline{J}; \overline{J}].$$

Then

$$V_{N-1}(\mathcal{S}) = \sup_{H_N \in D_N(\mathcal{S})} \inf_{Q_N \in \mathbb{Q}_N^E} \mathbb{E}_{Q_N}^{\mathcal{S}} J(\mathcal{S}_N \mid H_N) = \sup_{H_N \in D_N(\mathcal{S})} f(\mathcal{S}, H_N) \in [\underline{J}; \overline{J}]$$

and we can prove the statement for the remaining n by induction. □

The conditions of the Lemma can be useful for a bounded utility functions such as CARA utility, but, generally, in case of the unbounded $\mathbb{S}_{N-1}$ the payoff can be infinite, e. g. for the CRRA utility, which makes the value function unbounded as well. This might be avoided if the optimal strategies are certainly bounded. Let H_n^* be any strategy that satisfies (5) and consider

$$D_n^*(\mathcal{S}) = \operatorname*{Arg\,max}_{H_n \in D_n(\mathcal{S})} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_Q^{\mathcal{S}} V_n(\mathcal{S}_n \mid H_n)$$

so that $H_n^* \in D_n^*(\mathcal{S})$ for all $\mathcal{S} \in \mathbb{S}_{n-1}$, $n = \overline{1, N}$.

Statement 2.1. *For any $n = \overline{1, N}$ and any $\mathcal{S} \in \mathbb{S}_{n-1}$ let $J(\cdot) \in [\underline{J}_n(\mathcal{S}); \overline{J}_n(\mathcal{S})]$ on $\mathbb{S}_N \mid (H_{\geq n}^*, \mathcal{S})$ where $\underline{J}_n(\mathcal{S}), \overline{J}_n(\mathcal{S}) \in \bar{\mathbb{R}}$. If $V_{n-1}(\mathcal{S})$ is defined by* (3)-(4)*, then $V_{n-1}(\mathcal{S}) \in [\underline{J}_n(\mathcal{S}); \overline{J}_n(\mathcal{S})]$.*

Proof. By the Verification theorem,

$$V_{n-1}(\mathcal{S}) = \inf_{Q_{\geq n} \in \mathbb{Q}_{\geq n}} E_{Q_{\geq n}}^{\mathcal{S}} J(\mathcal{S}_N \mid H_{\geq n}^*) \in [\underline{J}_n(\mathcal{S}); \overline{J}_n(\mathcal{S})].$$

□

Graph of a map $F\colon A \to B$ is the set $\mathrm{Gr}(F) = \{(a, b) \in A \times B\colon b \in F(a)\}$. *Domain* of the map F is the set $\{a \in A\colon F(a) \neq \emptyset\}$.

Statement 2.2. *For $A \subset \mathbb{R}^l$ and $B \subset \mathbb{R}^p$, consider the set-valued map $F\colon A \to B$ with closed domain. If $F(a)$ is upper hemicontinuous (u. h. c.) on A, compact for every $a \in A$ and A is non-empty and compact then* $\mathrm{Gr}(F)$ *is compact.*

Proof. It is well-known that the conditions yield the closedness of $\mathrm{Gr}(F)$. Assume that $\mathrm{Gr}(F)$ is not bounded. Then there is an unbounded sequence $(a_n, b_n) \in \mathrm{Gr}(F)$. Since A is compact then $a_n \to a^* \in A$ hence $b_n \to \infty$. On the other hand, $\mathrm{Gr}(F)$ is closed, therefore each point of convergence of b_n belongs to $F(a^*)$ which contradicts the boundedness of $F(a^*)$. This means that $\mathrm{Gr}(F)$ is bounded and closed hence compact. □

Lemma 2.2. *For each* $n = \overline{1, N}$ *let*

- $D_n^*(\mathcal{S})$ *be compact for each* $\mathcal{S} \in \mathbb{S}_{n-1}$*;*
- $D_n^*(\mathcal{S})$ *be u. h. c. on* $\mathbb{S}_{n-1}$*;*
- $f(\Theta, H, \mathcal{S})$ *be continuous on* $\big\{(\Theta, H, \mathcal{S})\colon \Theta \in K_n,\ H \in D_n^*(\mathcal{S}),\ \mathcal{S} \in \mathbb{S}_{n-1}\big\}$.

Also let $J(\mathcal{S})$ *be uniformly bounded above (below) on any compact subset of* $\mathbb{S}_N$*. Then* $V_{n-1}(\mathcal{S})$ *is uniformly bounded above (below) on any compact set in* $\mathbb{S}_{n-1}$ *for all* $n = \overline{1, N}$.

Proof. Assume that V_n is uniformly bounded above/below on any compact subset of $\mathbb{S}_n$ and let M_{n-1} be a compact subset of $\mathbb{S}_{n-1}$. Consider the set-valued function $R_n(\mathcal{S})$ on M_{n-1}:

$$R_n(\mathcal{S}) = \big\{(\Theta, H, \mathcal{S})\colon \Theta \in K_n, H \in D_n^*(\mathcal{S})\big\}.$$

Since $D_n^*(\mathcal{S})$ is u. h. c. on $\mathbb{S}_{n-1}$, $R_n(S)$ is u. h. c. on $\mathbb{S}_{n-1}$. By Statement 2.2, $\mathrm{Gr}(R_n)$ is compact. Then the image M_n of $\mathrm{Gr}(R_n)$ under f is a compact subset of $\mathbb{S}_n$, therefore $\underline{V}_n \le V_n(\mathcal{S}) \le \overline{V}_n$ on M_n, $\underline{V}_n, \overline{V}_n \in \bar{\mathbb{R}}$, and we have[3]

$$\forall \mathcal{S} \in M_{n-1} \quad V_{n-1}(\mathcal{S}) = \inf_{Q_n \in \mathbb{Q}_n} \mathbb{E}_{Q_n}^{\mathcal{S}} V_n(\mathcal{S}_n \mid H_n^*) \in [\underline{V}_n; \overline{V}_n],$$

i. e. V_{n-1} is uniformly bounded above/below on M_{n-1} respectively. Since V_N obviously satisfies the required boundedness property due to $V_N(\mathcal{S}) \equiv J(\mathcal{S})$, we can prove the boundedness by induction for all $n = \overline{1, N}$. □

Lemma 2.2′**.** *For each* $n = \overline{1, N}$ *let*

- $D_n(\mathcal{S})$ *be compact for each* $\mathcal{S} \in \mathbb{S}_{n-1}$*;*
- $D_n(\mathcal{S})$ *be upper hemicontinuous (u.h.c.) on* $\mathbb{S}_{n-1}$*;*
- $f(\Theta, H, \mathcal{S})$ *be continuous on* $\big\{(\Theta, H, \mathcal{S})\colon \Theta \in K_n,\ H \in D_n(\mathcal{S}),\ \mathcal{S} \in \mathbb{S}_{n-1}\big\}$.

Also let $J(\mathcal{S})$ *be uniformly bounded above (below) on any compact set in* $\mathbb{S}_N$*. Then* $V_{n-1}(\mathcal{S})$ *is uniformly bounded above (below) on any compact set in* $\mathbb{S}_{n-1}$ *for all* $n = \overline{1, N}$.

Proof. The proof almost completely repeats Lemma 2.2 for $D_n(\mathcal{S})$ instead of $D_n^*(\mathcal{S})$ and is left to the reader. □

[3]For the upper boundedness, take $\underline{V}_n = -\infty$ and $\overline{V}_n \in \mathbb{R}$; for the lower boundedness, take $\overline{V}_n = \infty$ and $\underline{V}_n \in \mathbb{R}$.

3 Application to the optimal portfolio selection problem

On a market with m risky assets and one risk-free asset, let the risky price model be multiplicative as

$$\Delta X_n = \mu_n X_{n-1} \Delta t_n + \sigma_n X_{n-1} \sqrt{\Delta t_n},\ n = \overline{1, N}, \tag{6}$$

with $\sigma_n \sim Q_n \in \mathbb{Q}_n^E$. The risk-free price dynamics follows the standard process

$$\Delta Y_n = r_n Y_{n-1} \Delta t_n,\ n = \overline{1, N}, \tag{7}$$

where r_n is the risk-free rate.

Assume that H_n^X is a vector of volumes of the risky assets at t_n, while H_n^Y is a volume of the risk-free asset. Let W_n^X and W_n^Y be the value of risky and risk-free positions at t_n respectively, the total wealth be $W_n = W_n^X + W_n^Y$ and the transaction costs function at t_{n-1} be $C_{n-1}(\Delta H_n^X, \mathcal{S}_{n-1})$, where ΔH_n^X is the volume of the transaction. Then the budget equation gives us

$$\Delta H_n^{X^T} X_{n-1} + \Delta H_n^Y Y_{n-1} = -C_{n-1}(\Delta H_n^X, \mathcal{S}_{n-1}) \tag{8}$$

$$\Leftrightarrow$$

$$H_n^Y = Y_{n-1}^{-1} \left(W_{n-1} - H_n^{X^T} X_{n-1} - C_{n-1}(\Delta H_n^X, \mathcal{S}_{n-1}) \right), \tag{9}$$

which allows to treat only H^X as the unknown strategy.

In the context of the portfolio selection framework, the state of the system is

$$\mathcal{S} = \left(\Theta, X, H^X, W^Y\right) \in \mathbb{R}^p \times \mathbb{R}_+^m \times \bar{\mathbb{R}}^m \times \bar{\mathbb{R}},$$

where Θ is the vector of $p = l - 2m - 1$ parameters, X is the vector of prices of the m risky assets, H^X is the portfolio of the risky assets and W^Y is the value of the risk-free position. Note that infinite values of positions are formally allowed to research the finiteness of the optimal strategy, whereas an infinite solution in the portfolio selection problem usually means that the problem is ill-posed or the market parameter values are not economically reasonable.

The dynamics of the system (1) can be written out as

$$S_n \mid (H_n^X, \mathcal{S}_{n-1}) = \left(\Theta_n, X_n(\Theta_n, \mathcal{S}_{n-1}),\ H_n^X,\ W_n^Y(H_n^X, \mathcal{S}_{n-1})\right), \tag{10}$$

where W_n^Y is derived from the budget equation (9) and the dynamics of the risk-free price (7) as

$$W_n^Y(H_n^X, \mathcal{S}_{n-1}) = W_{n-1}^Y \tilde{r}_n - (H_n^X - H_{n-1}^X)^T X_{n-1} \tilde{r}_n - C_{n-1}(H_n^X - H_{n-1}^X, \mathcal{S}_{n-1}) \tilde{r}_n, \tag{11}$$

$$\tilde{r}_n = 1 + r_n \Delta t_n.$$

The Bellman-Isaacs equation in this case can be written as

$$V_{n-1}(X, H^X, W^Y) = \sup_{H_n^X \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n\Big((1 + \mu_n \Delta t_n + \Theta_n \sqrt{\Delta t_n}) X, H_n^X, \\ \big(W^Y - (H_n^X - H^X)^T X - C_{n-1}(H_n^X - H^X, S_{n-1})\big) \tilde{r}_n\Big),\ n = \overline{1, N}, \tag{12}$$

$$V_N(X, H^X, W^Y) = J(X, H^X, W^Y), \tag{13}$$

Then we can provide the analog of Lemma 2.2 for the portfolio selection problem:

Theorem 3.1. *Let $V_n(\mathcal{S})$ be defined by* (12)-(13). *For each $n = \overline{1, N}$ let*

- *$D_n^*(\mathcal{S})$ be compact for each $\mathcal{S} \in \mathbb{S}_{n-1}$;*
- *$D_n^*(\mathcal{S})$ be upper hemicontinuous (u.h.c.) on $\mathbb{S}_{n-1}$;*
- *$C_{n-1}(H^X, \mathcal{S})$ be continuous on $\big\{(H, \mathcal{S})\colon H \in D_n^*(\mathcal{S}),\ \mathcal{S} \in \mathbb{S}_{n-1}\big\}$;*
- *$X_n(\Theta, \mathcal{S})$ be continuous on $K_n \times \mathbb{S}_{n-1}$;*

and let $J(\mathcal{S})$ be uniformly bounded above (below) on any compact subset of $\mathbb{S}_N$. Then $V_{n-1}(\mathcal{S})$ is uniformly bounded above (below) on any compact subset of $\mathbb{S}_{n-1}$ for all $n = \overline{1, N}$.

Proof. Since (11) defines a continuous map, (10) implies that the map f_n in (1) is continuous on $\big\{(\Theta, H, \mathcal{S})\colon \Theta \in K_n,\ H \in D_n^*(\mathcal{S}),\ \mathcal{S} \in \mathbb{S}_{n-1}\big\}$ for every $n = \overline{1, N}$. Then the statement follows from Lemma 2.2. □

The first condition of Theorem 3.1 has a simple economic interpretation. If the phase constraints allow infinite values of the strategy then the problem basically allows obtaining an infinite payoff by investing an infinite volume in certain assets which usually means that the problem is ill-posed and its formulation is economically unreasonable. It is appropriate to assume that the infinite positions in a portfolio are either *a priori* suboptimal or forbidden by the trading limits. Note that $D_n^*(\mathcal{S})$ is bounded if $D_n(\mathcal{S})$ is bounded, so the first condition of the Theorem can be replaced by the boundedness of the risky positions in the portfolio. Moreover, upper hemicontinuity of $D_n^*(\mathcal{S})$ can also be replaced by upper hemicontinuity of $D_n(\mathcal{S})$ and we can prove the equivalent of Theorem 3.1 in terms of $D_n(\mathcal{S})$ by using Lemma 2.2′.

Uniform upper boundedness on compact sets holds for a wide class of the terminal utility functions, e. g. CARA and CRRA utilities. Therefore, under the conditions of Theorem 3.1, we can usually assume that $V_n < +\infty$ for all n. On the other hand, the value function can attain $-\infty$ even for the "ordinary" system trajectories. In case of the portfolio selection problem, $J(\mathcal{S}) = -\infty$ might mean that the selected strategy has led to undesirable results, e. g. the portfolio has negative wealth which might be interpreted as bankruptcy. Below, we provide sufficient conditions for the lower boundedness of the value function for a wide class of the utility functions which depend on the terminal liquidation value of the portfolio. By *liquidation value* W^L of the portfolio we mean the potential gains from liquidating all positions at the real market. In presence of transaction costs, the liquidation value is less than the market value by the amount of the costs carried:

$$W_n^L(\mathcal{S}) = W^Y + H^{X^T} X - C_n(H^{X^T}, \mathcal{S}), \quad \mathcal{S} = (\Theta, X, H^X, W^Y) \in \mathbb{S}_n.$$

For a given positive $x \in \mathbb{R}$ and $n = \overline{1, N}$, consider

$$\mathbb{S}_{n-1}(x) = \left\{\mathcal{S} \in \mathbb{S}_{n-1} \colon W_{n-1}^L(\mathcal{S}) \geq x\right\}.$$

Theorem 3.2. *Let* $V_n(\mathcal{S})$ *be defined by* (12)-(13),

$$J(\mathcal{S}) = \begin{cases} F\left(W_N^L(\mathcal{S})\right), & \text{if } W_N^L(\mathcal{S}) \geq w, \quad w \in \mathbb{R},\ w \geq 0, \\ -\infty, & \text{otherwise,} \end{cases}$$

be defined on $\mathbb{S}_N$*, where* $F(x) \geq c$ *on* $[w, +\infty)$*, and let* $\tilde{w}_{n-1} = \frac{w}{\prod_{i=n}^{N} \tilde{r}_i}$*. If for every* $n = \overline{1, N}$

- $0 \in D_n(\mathcal{S})$ *for any* $\mathcal{S} \in \mathbb{S}_{n-1}(\tilde{w}_{n-1})$,

- $C_{n-1}(H, \mathcal{S}) = C_{n-1}(-H, \mathcal{S})$ *for any* $\mathcal{S} \in \mathbb{S}_{n-1}(\tilde{w}_{n-1})$ *and* $H \in D_n(\mathcal{S})$,

then for any $n = \overline{1, N}$, $V_{n-1}(\mathcal{S}) \geq c$ *on* $\mathbb{S}_{n-1}(\tilde{w}_{n-1})$.

Proof. We prove the statement by induction. To shorten notation, formally assume that $\prod_{i=N+1}^{N} \tilde{r}_i = 1$. Then the form of $J(\mathcal{S})$ gives us that $V_N(\mathcal{S}) \geq c$ on

$$\left\{ \mathcal{S} \in \mathbb{S}_N : W_N^L(\mathcal{S}) \geq \frac{w}{\prod_{i=N+1}^{N} (1 + r_i \Delta t_i)} \right\} = \mathbb{S}_N(\tilde{w}_N).$$

Assume that the statement is true for V_n. Then for $\mathcal{S} = (\Theta, X, H^X, W^Y) \in \mathbb{S}_{n-1}(\tilde{w}_{n-1})$,

$$\begin{aligned} V_{n-1}(\mathcal{S}) = \sup_{H_n^X \in D_n(\mathcal{S})} \inf_{Q_n \in \mathbb{Q}_n} \mathbb{E}^{\mathcal{S}}_{Q_n} V_n \big(\Theta_n, X_n(\Theta_n, \mathcal{S}), H_n^X, \\ W^Y \tilde{r}_n - (H_n^X - H^X)^T \tilde{r}_n - C_{n-1}(H_n^X - H^X, \mathcal{S}) \tilde{r}_n \big) \overset{0 \in D_n(\mathcal{S})}{\geq} \end{aligned}$$

$$\geq \inf_{Q_n \in \mathbb{Q}_n} \mathbb{E}^{\mathcal{S}}_{Q_n} V_n \Big(\Theta_n, X_n(\Theta_n, \mathcal{S}), 0, W^Y \tilde{r}_n + H^{X^T} X \tilde{r}_n - C_{n-1}(-H^X, \mathcal{S}) \tilde{r}_n \Big) =$$

$$\overset{C_{n-1}(-H^X, \mathcal{S}) = C_{n-1}(H^X, \mathcal{S})}{=} \inf_{Q_n \in \mathbb{Q}_n} \mathbb{E}^{\mathcal{S}}_{Q_n} V_n \big(\Theta_n, X_n(\Theta_n, \mathcal{S}), 0, W^L_{n-1}(\mathcal{S}) \tilde{r}_n \big) = \inf_{Q_n \in \mathbb{Q}_n} \mathbb{E}^{\mathcal{S}}_{Q_n} V_n (\mathcal{S}_n)$$

where $\mathcal{S}_n = \big(\Theta_n, X_n(\Theta_n, \mathcal{S}), 0, W^L_{n-1}(\mathcal{S}) \tilde{r}_n \big) \in \mathbb{S}_n$. If $W^L_{n-1}(\mathcal{S}) \geq \frac{w}{\prod_{i=n}^{N} \tilde{r}_i}$, then

$$W_n^L(\mathcal{S}_n) = 0^T X_n(\Theta_n, \mathcal{S}) + W^L_{n-1}(\mathcal{S}) \tilde{r}_n = W^L_{n-1}(\mathcal{S}) \tilde{r}_n \geq \frac{w}{\prod_{i=n+1}^{N} \tilde{r}_i},$$

hence $V_n(\mathcal{S}_n) \geq c$ for any $\Theta_n \in K_n$, thus

$$V_{n-1}(\mathcal{S}) \geq \inf_{Q_n \in \mathbb{Q}_n} \mathbb{E}^{\mathcal{S}}_{Q_n} V_n(\mathcal{S}_n) \geq c.$$

□

The condition $0 \in D_n(\mathcal{S})$ means that at t_{n-1} it is allowed to invest all the capital in the risk-free asset for the next period. This is an economically reasonable and not a constraining condition unless the portfolio manager has some specific investment limitations. The second condition of transaction costs symmetry is rather common in academic literature. For some order-driven markets it can even be shown [Gatheral, 2010] that the symmetry is a necessary condition for the absence of the round-trip arbitrage on the market. The specific form of the utility function J means that one of the necessary conditions for the optimality is that the portfolio terminal liquidation value must not fall below the given threshold. The result of the Theorem states that in this case, the value function is uniformly bounded from below for such portfolios whose liquidation value does not fall below the present value of the threshold.

To summarize the results of the section, the following statement can be formulated for a general form of trading limits:

Theorem 3.3. *Let $V_n(\mathcal{S})$ be defined by* (12)-(13),

$$J(\mathcal{S}) = \begin{cases} F\left(W_N^L(\mathcal{S})\right), & \text{if } W_N^L(\mathcal{S}) \geq w, \quad w \in \mathbb{R},\ w > 0, \\ -\infty, & \text{otherwise}, \end{cases}$$

be defined on $\mathbb{S}_N$, where $F(x) \geq c$ on $[w, +\infty)$ and $F(W_N^L(\mathcal{S}))$ is uniformly bounded above on any compact subset of $\mathbb{S}_N$. Let $\tilde{w}_{n-1} = \frac{w}{\prod_{i=n}^{N} \tilde{r}_i}$. If for every $n = \overline{1, N}$

- *$D_n^*(\mathcal{S})$ is compact for each $\mathcal{S} \in \mathbb{S}_{n-1}$,*
- *$D_n^*(\mathcal{S})$ is u. h. c. on $\mathbb{S}_{n-1}$,*
- *$0 \in D_n(\mathcal{S})$ for any $\mathcal{S} \in \mathbb{S}_{n-1}(\tilde{w}_{n-1})$,*
- *$C_{n-1}(H^X, \mathcal{S})$ is continuous on $\left\{(H, \mathcal{S})\colon H \in D_n^*(\mathcal{S}),\ \mathcal{S} \in \mathbb{S}_{n-1}\right\}$,*
- *$C_{n-1}(H, \mathcal{S}) = C_{n-1}(-H, \mathcal{S})$ for any $\mathcal{S} \in \mathbb{S}_{n-1}(\tilde{w}_{n-1})$ and $H \in D_n(\mathcal{S})$,*
- *$X_n(\Theta, \mathcal{S})$ is continuous on $K_n \times \mathbb{S}_{n-1}$,*

then for any $n = \overline{1, N}$, $V_{n-1}(\mathcal{S}) < +\infty$ on $\mathbb{S}_{n-1}$ and $V_{n-1}(\mathcal{S}) \geq c$ on $\mathbb{S}_{n-1}(\tilde{w}_{n-1})$.

Proof. The statement trivially follows from Theorems 3.1 and 3.2 since $J(\mathcal{S})$ is uniformly bounded above on any compact subset of $\mathbb{S}_N$ thus making $V_{n-1}(\mathcal{S})$ above-bounded at any point of $\mathbb{S}_{n-1}$ (a single-point set is a compact subset of $\mathbb{S}_{n-1}$) for every $n = \overline{1, N}$. □

As before, conditions for $D_n^*(\mathcal{S})$ of the Theorem can be replaced with the analogous conditions for $D_n(\mathcal{S})$ based on Lemma 2.2′. For an example of $J(S)$ from the Theorem, consider the CRRA utility

$$J(\mathcal{S}) = \begin{cases} \frac{W_N^L(\mathcal{S})^\gamma}{\gamma}, & \text{if } W_n^L(\mathcal{S}) \geq 0, \\ -\infty, & \text{otherwise}, \end{cases}$$

for which the required uniform boundedness follows from continuity on the effective domain.

For an example of the upper hemicontinuous and compact constraint sets, consider for $n = \overline{1, N}$ matrices $U^k_{n-1} \in \mathbb{R}^{m \times m}$, $k = \overline{1, r}$, and $D_n(\mathcal{S})$ which, for every $\mathcal{S} \in \mathbb{S}_{n-1}$, consists of portfolios $Z \in \mathbb{R}^m$ such that

$$-\beta^{X,k}_{n-1} W_{n-1} \leq X^T_{n-1} U^k_{n-1} Z \leq \beta^{Y,k}_{n-1} W_{n-1}, \quad k = \overline{1, q}, \tag{14}$$

$$X^T_{n-1} U^k_{n-1} |Z| \leq \tilde{\beta}^{Y,k}_{n-1} W_{n-1}, \quad k = \overline{q+1, r}, \tag{15}$$

where $W_{n-1} = W^Y_{n-1} + {H^X_{n-1}}^T X_{n-1}$, $\beta^{X,k}_{n-1}, \beta^{Y,k}_{n-1}, \tilde{\beta}^{Y,k}_{n-1} \in \bar{\mathbb{R}}$ and non-negative, and the $|\cdot|$ operator is applied element-wise. Inequalities (14) represent trading limits for the positions in risky assets and their linear combinations, the limit value being a fraction of the portfolio market value. This type of constraints can also incorporate a limit on position in the risk-free asset, since it can be restated as a risky-asset limit according to the budget equation. Assume that $\{U^k_{n-1}\}_{i,j} = \begin{cases} 1, \ i = j = k, \\ 0, \ \text{otherwise} \end{cases}$ for $k = \overline{1, q}$. Then the k-th inequality represents the limits for long and short position in k-th risky asset. In the absence of the individual limits, constraints (14) might not guarantee boundedness of D_n since some position can be increased infinitely by short-selling other assets. In this case, another set of constraints can be imposed for absolute values of positions and their combinations (15). Note that when D_n is non-empty, it is a bounded polyhedron hence compact. Infinite values of $\beta^{X,k}_{n-1}, \beta^{Y,k}_{n-1}, \tilde{\beta}^{Y,k}_{n-1}$ mean the absence of the corresponding limit.

Lemma 3.1. *If $D_n(\mathcal{S})$ is bounded for each $\mathcal{S} \in \mathbb{S}_{n-1}(0)$, then the set-valued function $D_n(\mathcal{S})$ is upper hemicontinuous on $\mathbb{S}_{n-1}(0)$.*

Proof. To demonstrate the idea of the proof, we only consider the case when at least one of $\beta^{X,k}_{n-1}$ or $\beta^{Y,k}_{n-1}$ is finite and at least one $\tilde{\beta}^{Y,k}_{n-1}$ is finite. For other cases the proof can be easily conducted by analogy.

Let I be a set of m-dimensional vectors, which elements are either 1 or -1. The set, defined by the system of inequalities (14)-(15), can be represented as a union of sets of linear inequalities, each element representing linear constraints active on a specific orthant of m-dimensional Euclidean space. For each $\delta \in I$, consider $I^\delta = \operatorname{diag}(\delta)$ and let $D^\delta_n(\mathcal{S})$ consist of such $Z \in \mathbb{R}^m$ that

$$\begin{cases} X_{n-1}^T U_{n-1}^k Z \leq \beta_{n-1}^{Y,k} W_{n-1}, \quad k = \overline{1,p}, \\ X_{n-1}^T U_{n-1}^k Z \geq -\beta_{n-1}^{X,k} W_{n-1}, \quad k = \overline{1,p}, \\ X_{n-1}^T U_{n-1}^k I^\delta Z \leq \tilde{\beta}_{n-1}^{Y,k} W_{n-1}, \quad k = \overline{p+1,r}, \\ I^\delta Z \geq 0. \end{cases} \tag{16}$$

Then $D_n(\mathcal{S}) = \bigcup\limits_{\delta \in I} D_n^\delta(\mathcal{S})$.

If $W_{n-1}^L \geq 0$ then $W_{n-1} \geq 0$ and D_n contains 0. Therefore, the sets $D_n(\mathcal{S})$ are compact on $\mathbb{S}_{n-1}(0)$. If $D_n(\mathcal{S})$ is bounded then $D_n^\delta(\mathcal{S})$ is bounded for every $\delta \in I$ as well. Since the system (16) can be represented in the form $D_n^\delta(\mathcal{S}) = \left\{Z \in \mathbb{R}^m \colon A_{n-1}^\delta(\mathcal{S})Z \leq B_{n-1}^\delta(\mathcal{S})\right\}$, where $A_{n-1}^\delta(\mathcal{S})$ and $B_{n-1}^\delta(\mathcal{S})$ are obviously continuous in $\mathcal{S}$, we use Theorem 5.1 to prove that $D_n^\delta(\mathcal{S})$ is upper hemicontinuous on $\mathbb{S}_{n-1}(0)$ for every $\delta \in I$. Then $D_n(\mathcal{S})$ is upper hemicontinuous on $\mathbb{S}_{n-1}(0)$ as the union of a finite number of upper hemicontinuous set-valued functions. □

Statement 3.1. *For any $n = \overline{1,N}$ and any $c \in \mathbb{R}$, the set $\{\mathcal{S} \in \mathbb{S}_{n-1} \colon W_{n-1} \geq c\}$ is closed.*

Proof. The proof trivially follows from the definition of W_{n-1} and is left to the reader. □

For the particular structure of the trading limits, we can formulate the following result:

Theorem 3.4. *Let $V_n(\mathcal{S})$ be defined by (12)-(13), $D_n(\mathcal{S})$ be defined by (14)-(15) and*

$$J(\mathcal{S}) = \begin{cases} F\left(W_N^L(\mathcal{S})\right), & \text{if } W_N^L(\mathcal{S}) \geq w, \quad w \in \mathbb{R},\ w > 0, \\ -\infty, & \text{otherwise,} \end{cases}$$

be defined on $\mathbb{S}_N$, where $F(x) \geq c$ on $[w, +\infty)$ and $F(W_N^L(\mathcal{S}))$ is uniformly bounded above on any compact subset of $\mathbb{S}_N$. Let $\tilde{w}_{n-1} = \frac{w}{\prod\limits_{i=n}^{N} \tilde{r}_i}$. If for every $n = \overline{1,N}$

- *$D_n(\mathcal{S})$ is bounded for each $\mathcal{S} \in \mathbb{S}_{n-1}(0)$,*
- *$C_{n-1}(H^X, \mathcal{S})$ is continuous on $\left\{(H, \mathcal{S}) \colon H \in D_n(\mathcal{S}),\ \mathcal{S} \in \mathbb{S}_{n-1}\right\}$,*
- *$C_{n-1}(H, \mathcal{S}) = C_{n-1}(-H, \mathcal{S})$ for any $\mathcal{S} \in \mathbb{S}_{n-1}(\tilde{w}_{n-1})$ and $H \in D_n(\mathcal{S})$,*
- *$X_n(\Theta, \mathcal{S})$ is continuous on $K_n \times \mathbb{S}_{n-1}$,*

then for any $n = \overline{1,N}$, $V_{n-1}(\mathcal{S}) < +\infty$ on $\mathbb{S}_{n-1}$ and $V_{n-1}(\mathcal{S}) \geq c$ on $\mathbb{S}_{n-1}(\tilde{w}_{n-1})$.

Proof. 1) By definition, $W_n^L \geq c$ implies $W_n \geq c$, therefore lower boundedness of V_{n-1} follows from Theorem 3.2 since $0 \in D_n(\mathcal{S})$ on $\mathbb{S}_{n-1}(\tilde{w}_{n-1})$. The proof of the upper boundedness closely follows Lemma 2.2. We prove for V_{n-1} an even stronger property of uniform upper boundedness on compact subsets of $\mathbb{S}_{n-1}$ from which the upper boundedness trivially follows. The uniform upper boundedness holds for $V_N \equiv J$. For the remaining n the proof is made by induction: assume that the statement is true for V_n. Let M_{n-1} be a compact subset of $\mathbb{S}_{n-1}$. Consider the set $M'_{n-1} = M_{n-1} \cap \{\mathcal{S} \in \mathbb{S}_{n-1} \colon W_{n-1} \geq 0\}$ which is bounded and closed (by virtue of Statement 3.1) hence compact.

2) If $M'_{n-1} = \emptyset$, then $W_{n-1} < 0$ and $D_{n-1}(S) = \emptyset$ on M_{n-1}. By convention, $V_{n-1}(\mathcal{S}) = -\infty$ when the set of admissible strategies is empty therefore $V_{n-1}(\mathcal{S})$ is uniformly upper bounded on M_{n-1}.

3) If M'_{n-1} is not empty, consider the set-valued function $R_n(\mathcal{S})$ on M'_{n-1}:

$$R_n(\mathcal{S}) = \big\{(\Theta, H, \mathcal{S}) \colon \Theta \in K_n, H \in D_n(\mathcal{S})\big\}.$$

Since $M'_{n-1} \subseteq \mathbb{S}_{n-1}(0)$, Lemma 3.1 implies that $D_n(\mathcal{S})$ is u. h. c. on M'_{n-1}, hence $R_n(S)$ is u. h. c. on M'_{n-1}. Compactness of $D_n(\mathcal{S})$ and K_n implies compactness of $R_n(\mathcal{S})$ for every $\mathcal{S} \in M'_{n-1}$; domain of R_n is M'_{n-1} since $0 \in D_n(\mathcal{S})$ on M'_{n-1}. By Statement 2.2, $\mathrm{Gr}(R_n)$ is compact. Then the image M'_n of $\mathrm{Gr}(R_n)$ under under the map (10) is a compact subset of $\mathbb{S}_n$, therefore $V_n(\mathcal{S}) \leq \overline{V}_n$ on M'_n, $\overline{V}_n \in \mathbb{R}$, and we have

$$\forall \mathcal{S} \in M'_{n-1} \quad V_{n-1}(\mathcal{S}) = \inf_{Q_n \in \mathbb{Q}_n} \mathbb{E}^{\mathcal{S}}_{Q_n} V_n(\mathcal{S}_n \mid H^*_n) \leq \overline{V}_n.$$

By analogy to the case of empty M'_{n-1}, we can see that $V_{n-1}(\mathcal{S}) = -\infty$ on $M_{n-1} \setminus M'_{n-1}$, therefore V_{n-1} is uniformly bounded above on M_{n-1}, hence bounded above on $\mathbb{S}_{n-1}$. □

4 Conclusion

In this research we have presented sufficient conditions for the boundedness of the value function for the worst case optimization problem. Since many practical (and especially financial) cases assume that infinite strategy values are forbidden by an exogenous constraints, we focus our attention on the properties of the value function itself. For a general framework, sufficient conditions are presented for the bounded utility and for a general class of utilities which includes continuous functions on the effective domain. For the portfolio selection problem, we consider a market with general multiplicative price model and non-zero transaction costs. For a wide class of utility functions, a general class of transaction costs functions and a general form of trading limits, we present sufficient conditions for the upper and lower boundedness of the value function over time. The considered form of the utility incorporates most of the commonly used utility functions (CRRA, CARA, HARA etc.). For the specific form of linear trading limits we provide less restrictive sufficient conditions for the upper and lower boundedness of the value function. In fact, we prove an even stronger property of uniform upper boundedness over compact sets which can be easily seen in the proofs. All the obtained results are practice-oriented and can be used to verify correctness of the numeric scheme when solving the Bellman-Isaacs equation.

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5 Appendix

The following section uses classic results from convex geometry. For references, see e. g. [Artamonov and Latyshev, 2004].

Definition 5.1. For a set $D \subset \mathbb{R}^m$, the *ε-neighborhood* of D is $U(D, \varepsilon) = \big\{u \colon d(u, D) < \varepsilon\big\}$.

Statement 5.1. *If D is a compact subset of $\mathbb{R}^m$ and V is an open subset of $\mathbb{R}^m$ containing D then there is $\varepsilon > 0$ such that $U(D, \varepsilon) \subset V$.*

Proof. Let $R = \mathbb{R}^m \setminus V$. Consider the function $f(x) = d(x, R)$ which is positive for every $x \in D$ since $D \cap R = \emptyset$. Therefore f attains a minimum value $\varepsilon > 0$ on compact D. If $u \in U(D, \varepsilon)$ then $d(x, u) < \varepsilon = d(x, R)$ for any $x \in D$, therefore $u \notin R$ i. e. $u \in V$, hence $U(D, \varepsilon) \subset V$. □

Consider the set-valued function

$$D(\mathcal{S}) = \big\{x \in \mathbb{R}^m \colon A(\mathcal{S})x \leq B(\mathcal{S})\big\}, \qquad A(\mathcal{S}) \in \mathbb{R}^{r \times m},\ B(\mathcal{S}) \in \mathbb{R}^r \tag{17}$$

defined on $\mathbb{S} \subseteq \mathbb{R}^l$. By A_n^i and B_n^i we would mean the $i-th$ row of A_n and the i-th component of B_n respectively.

Statement 5.2. *For any given $\mathcal{S}_0 \in \mathbb{S}$, if $\operatorname{int} D(\mathcal{S}_0) \neq \emptyset$ and both $A(\mathcal{S})$ and $B(\mathcal{S})$ are continuous at $\mathcal{S}_0$, then there is such $\varepsilon > 0$ that $D(\mathcal{S}) \neq \emptyset$ for every $\mathcal{S} \in U_\varepsilon(\mathcal{S}_0)$.*

Proof. Consider $x_0 \in \operatorname{int} D(S_0)$. By definition, $A(\mathcal{S}_0)x_0 < B(\mathcal{S}_0)$. By contradiction, assume that there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\varepsilon_n \xrightarrow[n \to \infty]{} 0$ and for each n there is such $\mathcal{S}_n \in$

$U_{\varepsilon_n}(\mathcal{S}_0)$ that $D(\mathcal{S}_n) = \emptyset$, hence $\max_{1\le i\le r}(A^i(\mathcal{S}_n)x_0 - B^i(\mathcal{S}_n)) > 0$. Since $\mathcal{S}_n \xrightarrow[n\to\infty]{} \mathcal{S}_0$, we have $\max_{1\le i\le r}(A(\mathcal{S}_0)x_0 - B(\mathcal{S}_0)) \ge 0$ which leads to contradiction. □

Lemma 5.1. *Consider $D(\mathcal{S})$ according to* (17)*. For any $\mathcal{S}_0 \in \mathbb{S}$, if*

- *$D(\mathcal{S}_0)$ is bounded,*
- $\operatorname{int} D(\mathcal{S}_0) \neq \emptyset$*,*
- *$A(\mathcal{S})$ and $B(\mathcal{S})$ are continuous at $\mathcal{S}_0$*

then $D(\mathcal{S})$ is upper hemicontinuous at $\mathcal{S}_0$.

Proof. 1) Throughout the proof we denote $D_n = D(\mathcal{S}_n)$, $A_n = A(\mathcal{S}_n)$, $B_n = B(\mathcal{S}_n)$ for $n \ge 0$. Consider an open set $V \supset D_0$. To show the upper hemicontinuity of D_0 we need to prove that there is an open neighborhood $U(\mathcal{S}_0)$ of $\mathcal{S}_0$ such that $D(\mathcal{S}) \subset V$ for every $\mathcal{S} \in U(\mathcal{S}_0)$.

Consider a sequence $\{\mathcal{S}_n\}_{n=1}^{\infty}$ such that $\mathcal{S}_n \xrightarrow[n\to\infty]{} \mathcal{S}_0$ and $D'_n = D_n \setminus D_0 \neq \emptyset$. (If such a sequence does not exist, then the statement is proven.) By Statement 5.2, $D_n \neq \emptyset$ for large enough n, so we shall assume that D_n are not empty since only large n will be of further interest. Statement 5.1 implies that there is such $\varepsilon > 0$ that $U(D_0, \varepsilon) \in V$. Since $\mathcal{S}_n$ converges to $\mathcal{S}_0$, $\mathcal{S}_n \in U_\varepsilon(\mathcal{S}_0)$ for sufficiently large n. Therefore, to prove upper hemicontinuity, we only need to prove that $D_n \in U(D_0, \varepsilon)$ for sufficiently large n.

2) Consider a point $x_0 \in \operatorname{int} D_0$. For each $x_n \in D'_n$ the interval $[x_0; x_n]$ intersects ∂D_0, let $Y_n = [x_0; x_n] \cap \partial D_0$. Since Y_n is compact, let $y_n = \arg\max_{y\in Y_n} d(y, \partial D_0)$. By definition,

$$y_n = t_n x_0 + (1 - t_n)x_n, \quad t_n \in (0; 1).$$

$y_n \in \partial D_0$, therefore $t_n \to 0$ iif $x_n \to \partial D_0$; $t_n < 1$ since $x_0 \in \operatorname{int} D_0$. We can also write

$$x_n = p_n y_n + (1 - p_n)x_0, \quad p_n = \frac{1}{1 - t_n} > 1. \tag{18}$$

Note that $p_n \to 1$ iif $x_n \to \partial D_0$; also x_n is unbounded iif $p_n \to \infty$.

First, we prove that there is a bounded set M and $n_0 > 0$ such that $D_n \subseteq M$ for $n \ge n_0$. By contradiction, we could construct an unbounded sequence of points $x_n \in D'_n$[4] for which

[4]by taking a set sequence $M_n = \{x : \|x\| \le n\}$ and constructing a corresponding sequence of points outside M_n.

$p_n \to \infty$. Then

$$A_n x_n \le B_n \qquad \Longleftrightarrow \qquad p_n A_n y_n + (1-p_n) A_n x_0 \le B_n \qquad \Longleftrightarrow$$

$$\Longleftrightarrow \qquad A_n y_n - A_n x_0 \le \frac{1}{p_n}(B_n - A_n x_0) = \eta_n. \tag{19}$$

$B_n - A_n x_0 \to B_0 - A_0 x_0$ hence $\eta_n \to 0$. $y_n \in \partial D_0$ hence bounded and there is a subsequence $y_{n_k} \to y^i \in \partial D_0$ such that $A^i_{n_k} y_{n_k} \to B^i_0$ for some $i = \overline{1, r}$. $x_0 \in \operatorname{int} D_0$ hence there is $\delta^i > 0$ such that $A^i_{n_k} x_0 \to B^i_0 - \delta^i$. Therefore, $A^i_{n_k} y_{n_k} - A^i_{n_k} x_0 \to \delta^i > 0$. Thus from (19) we have $0 < \delta^i \le \eta^i_n \to 0$ which leads to contradiction.

3) Note that for every $n \ge 0$ D_n is a polyhedron hence a closed convex set. Then it is compact for $n \ge n_0$. Let $n \ge n_0$ and consider $x_n \in \operatorname{Arg}\max\limits_{x \in D_n} d(x, D_0)$ for which define p_n as before. Since $\{p_n\}$ is bounded, it has a limit point. Assume that some limit point $\bar{p} > 1$ and $\{p_{n_k}\}$ is the corresponding convergent subsequence. As before, we have

$$p_{n'_k} A_{n'_k} y_{n'_k} + (1 - p_{n'_k}) A_{n'_k} x_0 \le B_{n'_k}$$

and $A^i_{n'_k} y_{n'_k} \to B^i_0$ for some $i = \overline{1, r}$ and a subsequence $\{p_{n'_k}\}$ of $\{p_{n_k}\}$. By letting $n'_k \to \infty$, we obtain

$$\bar{p} B^i_0 + (1-\bar{p}) A^i_0 x_0 \le B^i_0 \quad \Longleftrightarrow \quad (1-\bar{p}) A^i_0 x_0 \le (1-\bar{p}) B^i_0 \quad \Longleftrightarrow \quad A^i_0 x_0 \ge B^i_0,$$

which contradicts the fact that $x_0 \in \operatorname{int} D_0$. This means that $p_n \to 1$ which implies that $d(x_n, D_0) \to 0$. By definition of x_n we have

$$\max_{x \in D_n} d(x, D_0) \xrightarrow[n \to \infty]{} 0.$$

Therefore, for any $\varepsilon > 0$ there is such N that for any $n \ge N$ and any $x \in D_n$, $x \in U(D_0, \varepsilon)$ hence $D_n \in U(D_0, \varepsilon)$ which concludes the proof. □

Statement 5.3 ([Panik, 2013, Corollary 2.48]). *Polyhedron* $\{x \in \mathbb{R}^m \colon Ax \le B\}$ *is bounded iif* $\{x \in \mathbb{R}^m \colon Ax \le 0\} = \{0\}$.

Theorem 5.1. *Consider* $D(\mathcal{S})$ *according to* (17). *For any* $\mathcal{S}_0 \in \mathbb{S}$, *if*

- $D(\mathcal{S}_0)$ *is bounded,*
- $D(\mathcal{S}_0)$ *is non-empty;*

- *$A(\mathcal{S})$ and $B(\mathcal{S})$ are continuous at $\mathcal{S}_0$*

then $D(\mathcal{S})$ is upper hemicontinuous at $\mathcal{S}_0$.

Proof. Below we use the notation from the proof of Lemma 5.1.

1) First, assume that D_0 consists of more than one point. If $\operatorname{int} D_0 \neq \emptyset$ then the statement is obviously true due to Lemma 5.1. Otherwise D_0 belongs to one of its facets which means that there are such $i_1 < \ldots < i_p = \overline{1,r}$ that $A^{i_k}(\mathcal{S}_0)x = B^{i_k}(\mathcal{S}_0)$ for each $x \in D_0$, $k = \overline{1,p}$. Then there is a linear transform $C(\mathcal{S})$, continuous at $\mathcal{S}_0$, with $\det C(\mathcal{S}_0) \neq 0$, such that $x = C(\mathcal{S}_0)y$ where y is a vector of coordinates of a point in $\operatorname{aff}(D_0)$. Since D_0 consists of more than one point, $\operatorname{ri} D_0 \neq \emptyset$, therefore we can apply Lemma 5.1 to D_0 in a linear space $\operatorname{aff}(D_0)$ to show upper hemicontinuity. Upper hemicontinuity in the original space then follows from the continuity of C at $\mathcal{S}_0$.

2) Now we prove the statement for the single-point set D_0. Let $D_0 = \{x_0\}$. For any $\varepsilon > 0$ consider the set

$$D(\mathcal{S}, \varepsilon) = \big\{x \in \mathbb{R}^m \colon A(\mathcal{S})x \leq B(\mathcal{S}) + \varepsilon \cdot \mathbf{1}\big\}.$$

By definition, $x_0 \in D(\mathcal{S}_0, \varepsilon)$ for any $\varepsilon > 0$. Now consider a monotonic sequence $\{\varepsilon_n\} \to 0$ where each $\varepsilon_n > 0$. Since $A_0 x_0 \leq B_0 + \varepsilon_n \cdot \mathbf{1}$ and A_0, B_0 are finite, there is a neighborhood $U_{\delta_n}(x_0)$ of points near x_0 for which the inequality holds. Therefore, for each $\varepsilon_n > 0$ there is $\delta_n > 0$ such that

$$U_{\delta_n}(x_0) \subset D(\mathcal{S}_0, \varepsilon_n).$$

Since D_0 is bounded, $D(\mathcal{S}_0, \varepsilon)$ is bounded for any $\varepsilon > 0$ by virtue of Statement 5.3. Then by Lemma 5.1 $D(S, \varepsilon_n)$ is u. h. c. at $\mathcal{S}_0$.

Consider a sequence $\{x_n\}$, $x_n \in \operatorname*{Arg\,max}_{x \in D(\mathcal{S}_0, \varepsilon_n)} d(x_0, x)$. $D(\mathcal{S}_0, \varepsilon_n) \subseteq D(\mathcal{S}_0, \varepsilon_0)$ for any $n \geq 0$, hence $\{x_n\}$ is bounded and there are $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$. Consider a convergent subsequence $x_{n_k} \to \hat{x}$.

$$A_0 x_{n_k} \leq B_0 + \varepsilon_{n_k} \quad \Longrightarrow \quad A_0 \hat{x} \leq B_0 \quad \Longrightarrow \quad \hat{x} = x_0.$$

Therefore $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x_0$ and $x_n \to x_0$, thus

$$d\left(x_0, D(\mathcal{S}_0, \varepsilon_n)\right) \to 0.$$

As in Lemma 5.1, to prove the upper hemicontinuity of D_0 it is sufficient to prove that for every $\varepsilon > 0$ there is an open neighborhood $U(\mathcal{S}_0)$ of $\mathcal{S}_0$ such that $D(\mathcal{S}) \subset U(D_0, \varepsilon)$ for every $\mathcal{S} \in U(\mathcal{S}_0)$. Fix some $\varepsilon > 0$. For a large enough n

$$d\left(x_0, D(\mathcal{S}_0, \varepsilon_n)\right) < \frac{\varepsilon}{2} \quad \Longrightarrow \quad D(\mathcal{S}_0, \varepsilon_n) \subset U\left(\{x_0\}, \frac{\varepsilon}{2}\right) = U\left(D_0, \frac{\varepsilon}{2}\right). \tag{20}$$

Since $D(\mathcal{S}, \varepsilon_n)$ is u. h. c. at $\mathcal{S}_0$, there is a neighborhood $U(\mathcal{S}_0)$ such that for every $\mathcal{S} \in U(\mathcal{S}_0)$

$$D(\mathcal{S}, \varepsilon_n) \subset U\left(D(\mathcal{S}_0, \varepsilon_n), \frac{\varepsilon}{2}\right). \tag{21}$$

By combining (20) and (21) we receive that

$$D(\mathcal{S}) = D(\mathcal{S}, 0) \subseteq D(\mathcal{S}, \varepsilon_n) \subset U\left(D_0, \varepsilon\right)$$

for every $\mathcal{S} \in U(\mathcal{S}_0)$[5], hence $D(\mathcal{S})$ is u. h. c. at $\mathcal{S}_0$. □

[5]Due to compactness, for any $x \in D(\mathcal{S}, \varepsilon_n)$ $d\left(x, D(\mathcal{S}_0, \varepsilon_n)\right) = d(x, y_0(x))$, $y_0(x) \in D(\mathcal{S}_0, \varepsilon_n)$, and $d(D_0, y_0(x)) = d(x_0, y_0(x))$. Then $d(x, D_0) = d(x, x_0) \leq d(x, y_0(x)) + d(x_0, y_0(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

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