

# On quantum matrix algebras satisfying the Cayley-Hamilton-Newton identities

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## Abstract

The Cayley-Hamilton-Newton identities which generalize both the characteristic identity and the Newton relations have been recently obtained for the algebras of the RTT-type. We extend this result to a wider class of algebras  $\mathcal{M}(\hat{R}, \hat{F})$  defined by a pair of compatible solutions of the Yang-Baxter equation. This class includes the RTT-algebras as well as the Reflection equation algebras.

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In last years the two basic statements of matrix algebra — the Cayley-Hamilton theorem and the Newton relations — were generalized for quantum matrix algebras of the "RTT-" and the "Reflection equation" (RE) types [1, 2, 3, 4, 5, 6]. In [7] a new family of matrix identities called the Cayley-Hamilton-Newton (CHN) identities have been constructed. The Cayley-Hamilton theorem and the Newton relations are particular cases and combinations of these identities. However the proof of the CHN identities given in [7] is adapted for the RTT-algebra case. The factorization map from the RTT-algebra to the RE algebra produces, in the quasitriangular case, the CHN identities for the RE algebra. In the present note we introduce a wider class of algebras and extend for them the proof of the CHN identities given in [7].

The key observation for such a generalization is that there are two R-matrices lying behind the construction of the CHN identities. First of them which we denote  $\hat{R}$  is an R-matrix of the Hecke type. It is responsible, roughly speaking, for the commutation relations of quantum matrix entries. The second one which will be referred as  $\hat{F}$  is a closed R-matrix and it performs transition between different matrix spaces. These two R-matrices are related by certain compatibility conditions (see below, equations (2)).

While the role of the first R-matrix is widely recognized, the importance of the second one is usually not noticed. In the case of the RTT-algebra the R-matrix  $\hat{F}$  coincides with the permutation matrix  $P$  whereas for the RE algebra one has  $\hat{F} = \hat{R}$ . Therefore  $\hat{F}$  is in some sense trivial for these standard examples of quantum matrix algebras. Revealing an independent role of the R-matrix  $\hat{F}$  allows to broaden the class of algebras under consideration and to give a universal proof of the CHN identities for this whole class.

**1. Notation.** Consider a pair of R-matrices  $\hat{R}, \hat{F} \in \text{Aut}(V \otimes V)$  where  $V$  is a finite-dimensional vector space. We call them *compatible* if, besides the Yang-Baxter equations

$$\hat{R}_1 \hat{R}_2 \hat{R}_1 = \hat{R}_2 \hat{R}_1 \hat{R}_2, \quad \hat{F}_1 \hat{F}_2 \hat{F}_1 = \hat{F}_2 \hat{F}_1 \hat{F}_2, \quad (1)$$

they satisfy the conditions

$$\hat{R}_1 \hat{F}_2 \hat{F}_1 = \hat{F}_2 \hat{F}_1 \hat{R}_2, \quad \hat{F}_1 \hat{F}_2 \hat{R}_1 = \hat{R}_2 \hat{F}_1 \hat{F}_2. \quad (2)$$

We use here the matrix conventions of [8]. In particular,  $\hat{R}_k$  and  $\hat{F}_k$  denote the R-matrices acting in  $V_k \otimes V_{k+1}$  — the  $k$ -th and the  $(k+1)$ -st copies of the space  $V$ .

In the sequel, we assume that  $\hat{R}$  and  $\hat{F}$  are compatible. Further on, we assume that  $\hat{R}$  is an *even Hecke* R-matrix of a *height*  $n$  and  $\hat{F}$  is a *closed* R-matrix. Below we remind briefly these notions (for more details on the notation see [9, 4]).

**Conditions on the matrix  $\hat{R}$ .** An R-matrix  $\hat{R}$  satisfying the condition

$$\hat{R}^2 = I + (q - q^{-1})\hat{R} \quad (3)$$

is called the Hecke R-matrix. Here  $I$  is the identity operator and  $q \neq 0$  is a number.

Given a Hecke R-matrix, one constructs two sequences of projectors,  $A^{(k)}$  and  $S^{(k)} \in \text{End}(V^{\otimes k})$ , called *q-antisymmetrizers* and *q-symmetrizers*, correspondingly. They are defined inductively,

$$A^{(1)} := I, \quad A^{(k)} := \frac{1}{k_q} A^{(k-1)} \left( q^{k-1} - (k-1)_q \hat{R}_{k-1} \right) A^{(k-1)}, \quad (4)$$

$$S^{(1)} := I, \quad S^{(k)} := \frac{1}{k_q} S^{(k-1)} \left( q^{1-k} + (k-1)_q \hat{R}_{k-1} \right) S^{(k-1)}, \quad (5)$$

where it is additionally supposed that  $k_q := (q^k - q^{-k})/(q - q^{-1}) \neq 0, \forall k = 1, 2, \dots$ .

The Hecke R-matrix  $\hat{R}$  is called even if its sequence of *q-antisymmetrizers* vanishes at the  $(n + 1)$ -st step and  $\text{rank } A^{(n)} = 1$ . The number  $n$  is called then the height of  $\hat{R}$ .

**Conditions on the matrix  $\hat{F}$ .** An R-matrix  $\hat{F} = \hat{F}_{cd}^{ab}$  is called the closed R-matrix provided it is invertible in indices  $(a, c)$  and nonsingular (i.e., invertible in indices  $(a, b)$ ). The first condition means that there exists a matrix  $\Psi_{cd}^{ab}$  satisfying  $\Psi_{cg}^{af} \hat{F}_{fd}^{gb} = \delta_d^a \delta_c^b$  (summation over repeated indices is always assumed). Denote  $(D)_b^a = \Psi_{bc}^{ac}$ . Using the matrix  $D$ , one introduces the notion of the *quantum trace* for an arbitrary (not necessarily with commuting entries) matrix  $X$ ,

$$\text{Tr}_{\hat{F}} X := \text{Tr } DX. \quad (6)$$

The following properties of the matrix  $D$  will be important for us

$$\text{Tr}_{\hat{F}(2)} \hat{F}_1 = I_1, \quad (7)$$

$$\hat{F}_1 D_1 D_2 = D_1 D_2 \hat{F}_1, \quad (8)$$

$$\text{Tr}_{\hat{F}(2)} \hat{F}_1^{\pm 1} X_1 \hat{F}_1^{\mp 1} = I_1 \text{Tr}_{\hat{F}} X. \quad (9)$$

Here and below we use notation  $\text{Tr}_{(i_1 \dots i_k)}$  to denote the operation of taking traces in spaces with the numbers  $i_1 \dots i_k$ .

**Properties of compatible R-matrices.** Due to the compatibility conditions (2), a matrix

$$\hat{R}^{\hat{F}} := \hat{F} \hat{R} \hat{F}^{-1} , \quad (10)$$

satisfies the Yang-Baxter equation and is again compatible with  $\hat{F}$ . This transformation was called *twisting* of R-matrices [10] and, in the case of compatible  $\hat{R}$  and  $\hat{F}$ , it has been considered in [11].

Since  $\hat{R}^{\hat{F}}$  and  $\hat{F}$  are compatible, one can consider the square of the twist,  $\hat{R}^{\hat{F}\hat{F}} := (\hat{R}^{\hat{F}})^{\hat{F}}$ . One has the following relation

$$\hat{R}_1^{\hat{F}\hat{F}} D_1 D_2 = D_1 D_2 \hat{R}_1 . \quad (11)$$

We give the proof of this relation since we could not find it in literature.

Let  $Y_{12}$  denote an arbitrary element of  $\text{End}(V \otimes V)$ . Consider the following chain of transformations

$$\begin{aligned} \text{Tr}_{\hat{F}(1,2)}(\hat{R}_1 \hat{F}_1^2 Y_{12}) &= \text{Tr}_{\hat{F}(1,2,3)}(\hat{R}_1 \hat{F}_1 \hat{F}_2 \hat{F}_1 Y_{12}) = \text{Tr}_{\hat{F}(1,2,3,4)}(\hat{R}_1 \hat{F}_2 \hat{F}_1 \hat{F}_3 \hat{F}_2 Y_{12}) \\ &= \text{Tr}_{\hat{F}(1,2,3,4)}(\hat{F}_2 \hat{F}_1 \hat{F}_3 \hat{F}_2 Y_{12} \hat{R}_3) = \text{Tr}_{\hat{F}(1,2,3,4)}(Y_{12} \hat{R}_3 \hat{F}_2 \hat{F}_3 \hat{F}_1 \hat{F}_2) \\ &= \text{Tr}_{\hat{F}(1,2,3,4)}(Y_{12} \hat{F}_2 \hat{F}_3 \hat{F}_1 \hat{F}_2 \hat{R}_1) = \text{Tr}_{\hat{F}(1,2)}(Y_{12} \hat{F}_1^2 \hat{R}_1) . \end{aligned} \quad (12)$$

Here we have subsequently used the equations (7) and (1) in the first line, (2) and the cyclic property of the trace together with (8) in the second line, and again (2), (7) and (1) in the last line of the calculation. Substituting the definition of the quantum trace, the result of (12) can be presented in a form

$$\text{Tr}_{(1,2)}(Y_{12} D_1 D_2 \hat{R}_1 \hat{F}_1^2) = \text{Tr}_{(1,2)}(Y_{12} \hat{F}_1^2 \hat{R}_1 D_1 D_2)$$

which reduces to (11) if one takes into account the arbitrariness of  $Y_{12}$  and applies once again the equation (8).

**2. Algebra  $\mathcal{M}(\hat{R}, \hat{F})$ .** Consider a matrix  $M$ . Usually one associates with  $M$  a series of its copies  $M_k$  acting on the corresponding vector space  $V_k$ ,  $k = 1, 2, \dots$ . We need the following generalization of this notion.

With a matrix  $M$ , we associate a series of matrices  $M_k$  defined inductively as

$$M_{\overline{1}} := M_1 , \quad M_{\overline{k+1}} := \hat{F}_k M_k \hat{F}_k^{-1} . \quad (13)$$

For  $\hat{F} = P$  the new notation coincides with the old one:  $M_{\bar{k}} \equiv M_k$ . In general, the operator  $M_{\bar{k}}$  acts nontrivially on the space  $V_1 \otimes \dots \otimes V_k$ , not necessarily on  $V_k$  alone.

Now we define the main object of this note, the algebra  $\mathcal{M}(\hat{R}, \hat{F})$ . It is a unital associative algebra, generated by the components of a matrix  $M$  subject to a relation

$$\hat{R}_1 M_{\bar{1}} M_{\bar{2}} = M_{\bar{1}} M_{\bar{2}} \hat{R}_1^{\hat{F}\hat{F}} , \quad (14)$$

or  $\hat{R}_1 M_{\bar{1}} \hat{F}_1 M_{\bar{1}} = M_{\bar{1}} \hat{F}_1 M_{\bar{1}} \hat{R}_1^{\hat{F}}$ , in old notation. Specializing to  $\hat{F} = P$  or  $\hat{F} = \hat{R}$  one reproduces the RTT- or RE algebras, respectively. The algebras  $\mathcal{M}(\hat{R}, \hat{F})$  form a subclass of more general algebras discussed in [12, 13].

In the Lemma below we collect several useful results.

**Lemma. a)** *For a matrix  $M$  with arbitrary entries, the following relations hold*

$$\hat{F}_i M_{\bar{k}} = M_{\bar{k}} \hat{F}_i , \quad \text{for } k \neq i, i+1, \quad (15)$$

$$\hat{R}_i M_{\bar{k}} = M_{\bar{k}} \hat{R}_i , \quad \text{for } k \neq i, i+1, \quad (16)$$

$$\hat{F}_{i \rightarrow k} M_{\bar{i}} M_{\bar{i+1}} \dots M_{\bar{k}} = M_{\bar{i+1}} M_{\bar{i+2}} \dots M_{\bar{k+1}} \hat{F}_{i \rightarrow k}, \quad \text{for } i \leq k. \quad (17)$$

Here  $\hat{F}_{i \rightarrow k} := \hat{F}_i \hat{F}_{i+1} \dots \hat{F}_k$ .

**b)** *Let  $Y^{(k)} \equiv Y^{(k)}(\hat{R}_1, \dots, \hat{R}_{k-1})$  be any polynomial in  $\hat{R}_1, \dots, \hat{R}_{k-1}$ , and let  $Y^{(i,k)} := Y^{(k)}(\hat{R}_i, \dots, \hat{R}_{i+k-2})$ . Denote*

$$\alpha(Y^{(k)}) := \text{Tr}_{\hat{F}(1, \dots, k)}(Y^{(k)} M_{\bar{1}} \dots M_{\bar{k}}) .$$

*For a matrix  $M$  with arbitrary entries one has*

$$\text{Tr}_{\hat{F}(i, \dots, i+k-1)}(Y^{(i,k)} M_{\bar{i}} \dots M_{\bar{i+k-1}}) = I_{1, \dots, i-1} \alpha(Y^{(k)}) , \quad (18)$$

where  $I_{1, \dots, i-1}$  is the identity in the spaces  $1, \dots, i-1$ .

**c)** *If, in addition,  $M$  is the matrix of generators of  $\mathcal{M}(\hat{R}, \hat{F})$ , one has*

$$\hat{R}_k M_{\bar{k}} M_{\bar{k+1}} = M_{\bar{k}} M_{\bar{k+1}} \hat{R}_k^{\hat{F}\hat{F}} . \quad (19)$$

**Proof. a)** The relations (15) and (16) are trivial for  $i > k$ . For  $i < k-1$ , the relations (15) and (16) follow immediately from our definition (13) of  $M_{\bar{i}}$  and the conditions (1) and (2).

The relation (17) can be proved by induction. For  $k = i$  the formula (17) is just the definition of  $M_{\overline{k+1}}$ . Suppose that (17) is valid for some  $k = j - 1 \geq i$ . Then for  $k = j$  we have

$$\begin{aligned}\hat{F}_{i \rightarrow j} M_{\overline{i}} M_{\overline{i+1}} \dots M_{\overline{j}} &= (\hat{F}_{i \rightarrow j-1} M_{\overline{i}} \dots M_{\overline{j-1}}) (\hat{F}_j M_{\overline{j}}) \\ &= M_{\overline{i+1}} \dots M_{\overline{j}} (\hat{F}_{i \rightarrow j-1} M_{\overline{j+1}}) \hat{F}_j = M_{\overline{i+1}} \dots M_{\overline{j}} M_{\overline{j+1}} \hat{F}_{i \rightarrow j},\end{aligned}$$

which completes the induction. Here we applied several times the relations (15), used the induction assumption and the definition of  $\hat{F}_{i \rightarrow j+1}$  and  $M_{\overline{j+1}}$ .

**b)** It suffices to check (18) for the case  $i = 2$ . The calculation proceeds as follows

$$\begin{aligned}\mathrm{Tr}_{\hat{F}(2, \dots, k+1)}(Y^{(2,k)} M_{\overline{2}} \dots M_{\overline{k+1}}) &= \mathrm{Tr}_{\hat{F}(2, \dots, k+1)}(Y^{(2,k)} \hat{F}_{1 \rightarrow k} M_{\overline{1}} \dots M_{\overline{k}} \hat{F}_{1 \rightarrow k}^{-1}) \\ &= \mathrm{Tr}_{\hat{F}(2, \dots, k+1)}(\hat{F}_{1 \rightarrow k} Y^{(k)} M_{\overline{1}} \dots M_{\overline{k}} \hat{F}_{1 \rightarrow k}^{-1}) \\ &= \mathrm{Tr}_{\hat{F}(2, \dots, k)}(\hat{F}_{1 \rightarrow k-1} [\mathrm{Tr}_{\hat{F}(k+1)}(\hat{F}_k Y^{(k)} M_{\overline{1}} \dots M_{\overline{k}} \hat{F}_k^{-1})] \hat{F}_{1 \rightarrow k-1}^{-1}) \\ &= \mathrm{Tr}_{\hat{F}(2, \dots, k)}(\hat{F}_{1 \rightarrow k-1} [I_k \mathrm{Tr}_{\hat{F}(k)}(Y^{(k)} M_{\overline{1}} \dots M_{\overline{k}})] \hat{F}_{1 \rightarrow k-1}^{-1}) = \dots = I_1 \alpha(Y^{(k)}).\end{aligned}$$

Here we used the equations (17), (2) and (9). One should not be confused with the appearance of two  $\mathrm{Tr}_{\hat{F}(k)}$  in the left part of the last line of calculation. The inner of these quantum traces acts on arguments in parentheses while the outer one respects only the identity operator  $I_k$  among the terms enclosed by the square brackets. Therefore the outer quantum trace  $\mathrm{Tr}_{\hat{F}(k)}$  can be calculated in the next step and transformed into an inner  $\mathrm{Tr}_{\hat{F}(k-1)}$ . The procedure repeats until all the outer quantum traces transform into inner ones.

**c)** Induction in  $k$ . The relation (19) with  $k = 1$  is just the definition of  $\mathcal{M}(\hat{R}, \hat{F})$ . Assume that (19) is true for some  $k = i - 1 \geq 1$  and consider the case  $k = i$ ,

$$\begin{aligned}\hat{R}_i M_{\overline{i}} M_{\overline{i+1}} &= \hat{R}_i \hat{F}_{i-1} M_{\overline{i-1}} M_{\overline{i+1}} \hat{F}_{i-1}^{-1} = \hat{R}_i \hat{F}_{i-1} \hat{F}_i M_{\overline{i-1}} M_{\overline{i}} (\hat{F}_{i-1} \hat{F}_i)^{-1} \\ &= \hat{F}_{i-1} \hat{F}_i M_{\overline{i-1}} M_{\overline{i}} (\hat{F}_{i-1} \hat{F}_i)^{-1} \hat{R}_i^{\hat{F} \hat{F}} = M_{\overline{i}} M_{\overline{i+1}} \hat{R}_i^{\hat{F} \hat{F}}.\end{aligned}\quad (20)$$

Here we applied, first, the definition of  $M_{\overline{i}}$ ,  $M_{\overline{i+1}}$  and the relations (15). Next, we used (2) and the induction assumption and, then, performed the transformations of the first line of (20) in the inverse order. ■

**3. Characteristic subalgebra.** Let us consider three sequences of elements of the algebra  $\mathcal{M}(\hat{R}, \hat{F})$ :

$$s_k(M) := \text{Tr}_{\hat{F}(1\dots k)}(\hat{R}_{1 \rightarrow k-1} M_{\overline{1}} M_{\overline{2}} \dots M_{\overline{k}}) , \quad (21)$$

$$\sigma_k(M) := \text{Tr}_{\hat{F}(1\dots k)}(A^{(k)} M_{\overline{1}} M_{\overline{2}} \dots M_{\overline{k}}) , \quad (22)$$

$$\tau_k(M) := \text{Tr}_{\hat{F}(1\dots k)}(S^{(k)} M_{\overline{1}} M_{\overline{2}} \dots M_{\overline{k}}) , \quad k = 1, 2, \dots . \quad (23)$$

Also we put  $s_0(M) = \sigma_0(M) = \tau_0(M) = 1$ .

These elements are interpreted as symmetric polynomials on the spectrum of the matrix  $M$  (see [6, 7]). Namely,  $s_k(M)$  are the *power sums*,  $\sigma_k(M)$  are the *elementary symmetric functions* and  $\tau_k(M)$  are the *complete symmetric functions*.

It follows from the Newton and Wronski relations (see below) that, given any pair of the sets  $\{s_k(M)\}$ ,  $\{\sigma_k(M)\}$  or  $\{\tau_k(M)\}$ , one can express the elements of the first one of them as polynomials in the elements of the second one. Therefore all these sets generate the same subalgebra in  $\mathcal{M}(\hat{R}, \hat{F})$  which we call the *characteristic subalgebra* of  $\mathcal{M}(\hat{R}, \hat{F})$ .

**Proposition.** *The characteristic subalgebra of  $\mathcal{M}(\hat{R}, \hat{F})$  is abelian.*

**Proof.** The commutativity of the characteristic subalgebra in the particular case of the RTT-algebra was observed by J.M.Maillet [14] who has checked the commutativity of power sums. We extend Maillet's method to treat the general case. The proof is based on the relation (18) which is trivial for the RTT-algebra case but crucial for the general algebra  $\mathcal{M}(\hat{R}, \hat{F})$ .

Consider a pair  $\alpha(Y^{(k)})$  and  $\beta(Z^{(i)})$  of elements of the characteristic subalgebra. Using relations (18) one can present the product of  $\alpha$  and  $\beta$  in a form

$$\alpha(Y^{(k)}) \beta(Z^{(i)}) = \text{Tr}_{\hat{F}(1, \dots, k+i)}(Y^{(k)} Z^{(k+1, i)} M_{\overline{1}} M_{\overline{2}} \dots M_{\overline{k+i}}). \quad (24)$$

Further, consider an operator  $U_{\hat{R}} := \hat{R}_{i \rightarrow i+k-1} \dots \hat{R}_{2 \rightarrow k+1} \hat{R}_{1 \rightarrow k}$ . By virtue of the Yang-Baxter equation, one has

$$Y^{(k)} = U_{\hat{R}}^{-1} Y^{(i+1, k)} U_{\hat{R}} , \quad Z^{(k+1, i)} = U_{\hat{R}}^{-1} Z^{(i)} U_{\hat{R}} . \quad (25)$$

Substituting (25) into (24), one continues the transformation

$$\begin{aligned} \alpha(Y^{(k)}) \beta(Z^{(i)}) &= \text{Tr}_{\hat{F}(1, \dots, k+i)}(U_{\hat{R}}^{-1} Z^{(i)} Y^{(i+1, k)} U_{\hat{R}} M_{\overline{1}} \dots M_{\overline{k+i}}) \\ &= \text{Tr}_{\hat{F}(1, \dots, k+i)}(U_{\hat{R}}^{-1} Z^{(i)} Y^{(i+1, k)} M_{\overline{1}} \dots M_{\overline{k+i}} U_{\hat{R}}) = \beta(Z^{(i)}) \alpha(Y^{(k)}) . \end{aligned}$$

Here the relations (16), (19), the cyclic property of the trace and the relation (11) have been applied subsequently.  $\blacksquare$

**4. Cayley-Hamilton-Newton identities.** Finally, we need a proper generalization of the notion of a matrix power for the case of  $\mathcal{M}(\hat{R}, \hat{F})$ . Taking off the first quantum trace in the definitions of symmetric polynomials (21), (22), (23) one gets the following matrix expressions

$$M^{\bar{k}} := \text{Tr}_{\hat{F}(2, \dots, k)}(\hat{R}_{1 \rightarrow k-1} M_{\bar{1}} M_{\bar{2}} \dots M_{\bar{k}}) , \quad (26)$$

$$M^{\wedge k} := \text{Tr}_{\hat{F}(2, \dots, k)}(A^{(k)} M_{\bar{1}} M_{\bar{2}} \dots M_{\bar{k}}) , \quad (27)$$

$$M^{Sk} := \text{Tr}_{\hat{F}(2, \dots, k)}(S^{(k)} M_{\bar{1}} M_{\bar{2}} \dots M_{\bar{k}}) . \quad (28)$$

We call the matrix  $M^{\bar{k}}$  the *k-th power of the matrix M*. Certainly, this definition coincides with the usual one in the classical situation,  $\hat{R} = \hat{F} = P$ . More generally,  $M^{\bar{k}} \equiv M^k$  in the case  $\hat{R} = \hat{F}$ , i.e., for the RE algebra.

The matrices  $M^{\wedge k}$  and  $M^{Sk}$  will be relevant for the formulation of the Cayley-Hamilton-Newton identities. It is natural to call them the *k-wedge* and the *k-symmetric* powers of the matrix  $M$ , respectively.

With these definitions we can formulate our main result

**Cayley-Hamilton-Newton theorem.** *Let  $M$  be the matrix generating the algebra  $\mathcal{M}(\hat{R}, \hat{F})$ . Then, the following matrix identities hold*

$$(-1)^{k-1} k_q M^{\wedge k} = \sum_{i=0}^{k-1} (-q)^i M^{\overline{k-i}} \sigma_i(M) , \quad (29)$$

$$k_q M^{Sk} = \sum_{i=0}^{k-1} q^{-i} M^{\overline{k-i}} \tau_i(M) . \quad (30)$$

**Proof.** Consider the reflection  $q \rightarrow -q^{-1}$ , which is a symmetry transformation of a parameter of the Hecke R-matrix  $\hat{R}$ . It results in the substitutions  $k_q \leftrightarrow (-1)^{k-1} k_q$ ,  $A^{(k)} \leftrightarrow S^{(k)}$  and, hence, the two equations (29) and (30) map into each other. So, it suffices to prove only one of the two series of equations (29) and (30), say, the first one.

For the case of the RTT-algebra, these identities were proved in [7]. With the notation which we introduced in the present note, the proof of these identities given in [7] can be applied practically without changes for the algebra  $\mathcal{M}(\hat{R}, \hat{F})$ . The only additional remark should be given for the very first step

of the proof. It concerns the presentation of the typical term  $M^{\overline{k-i}} \sigma_i(M)$  from the right hand side of the CHN identities in a form

$$M^{\overline{k-i}} \sigma_i(M) = \text{Tr}_{\hat{F}(2, \dots, k)}(\hat{R}_{1 \rightarrow k-i-1} A^{(k-i+1, i)} M_{\overline{1}} \dots M_{\overline{k}}).$$

This equality being tautological in the RTT-algebra follows by an application of (18) and (16) in the general case.

For the rest of the proof we refer the reader to the paper [7]. ■

In conclusion we present several corollaries of the Cayley-Hamilton-Newton theorem. Their proofs given in [7] for the case of the RTT-algebra remain valid for the general algebra  $\mathcal{M}(\hat{R}, \hat{F})$  as well.

**Newton relations.**

$$(-1)^{k-1} k_q \sigma_k(M) = \sum_{i=0}^{k-1} (-q)^i s_{k-i}(M) \sigma_i(M),$$

$$k_q \tau_k(M) = \sum_{i=0}^{k-1} q^{-i} s_{k-i}(M) \tau_i(M).$$

**Wronski relations.**

$$0 = \sum_{i=0}^k (-1)^i \tau_{k-i}(M) \sigma_i(M).$$

**Cayley-Hamilton theorem.**

$$0 = \sum_{i=0}^n (-q)^i M^{\overline{n-i}} \sigma_i(M), \quad \text{where } M^{\overline{0}} := q^{-n} n_q \text{Tr}_{(2, \dots, n)}(A^{(n)}) D^{-1}.$$

**Inverse Cayley-Hamilton-Newton identities.**

$$M^{\overline{k}} = \sum_{i=1}^k (-1)^{i+1} q^{k-i} i_q M^{\wedge i} \tau_{k-i}(M) = \sum_{i=1}^k (-1)^{k-i} q^{i-k} i_q M^{S i} \sigma_{k-i}(M).$$

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