

N.I. Zhukova[†], A.Yu. Dolgonosova[‡]

PSEUDO-RIEMANNIAN FOLIATIONS AND THEIR GRAPHS

†, ‡ NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS
BOLSHAJA PECHERSKAJA STR., 25/12, NIZHNY NOVGOROD, 603155, RUSSIA

E-mail address: nina.i.zhukova@gmail.com, annadolgonosova@gmail.com

ABSTRACT. We prove that a foliation (M, F) of codimension q on a n -dimensional pseudo-Riemannian manifold is pseudo-Riemannian if and only if any geodesic that is orthogonal at one point to a leaf is orthogonal to every leaf it intersects.

We show that on the graph $G = G(F)$ of a pseudo-Riemannian foliation there exists a unique pseudo-Riemannian metric such that canonical projections are pseudo-Riemannian submersions and the fibres of different projections are orthogonal at common points. Relatively this metric the induced foliation (G, \mathbb{F}) on the graph is pseudo-Riemannian and the structure of the leaves of (G, \mathbb{F}) is described. Special attention is given to the structure of graphs of transversally (geodesically) complete pseudo-Riemannian foliations and totally geodesic pseudo-Riemannian ones.

1. INTRODUCTION

Let (M, F) be a smooth foliation. Recall that pseudo-Riemannian metric g on the manifold M is transversally projectable if the Lie derivative $L_X g$ along X is zero for any vector field X tangent to this foliation.

Definition 1. *A foliation on a pseudo-Riemannian manifold (M, F) is referred to as a pseudo-Riemannian foliation if every leaf L with induced metric is a pseudo-Riemannian manifold and g is transversally projectable.*

A pseudo-Riemannian submersion (see [11]) is a smooth map $p : M \rightarrow B$ which is onto and satisfies the following three axioms:

2010 Mathematical Subject Classification. 46L51, 46L53, 46E30, 46H05.

Key words and phrases. Pseudo-Riemannian foliation, graph of a foliation, geodesically invariant distribution, Ehresmann connection of a foliation.

Partially supported by the Russian Foundation of Basic Research (grant no. 16-01-00132) and by the Program of Basic Research Program at the National Research University Higher School of Economics in 2016 (project no. 98).

- (a) the differential $p_{*x} : T_x M \rightarrow T_{p(x)} B$ is onto for all $x \in M$;
- (b) the fibres $p^{-1}(b)$, $b \in B$, are pseudo-Riemannian submanifolds of M ;
- (c) the differential p_* preserves scalar products of vectors normal to fibres.

Definition 1 is equivalent to the fact that foliation (M, F) is given locally by pseudo-Riemannian submersions, i.e. it is equivalent to the following definition.

Definition 2. A foliation (M, F) on a pseudo-Riemannian manifold (M, g) is said to be a pseudo-Riemannian foliation if at any point there exists an adapted neighborhood U and a Riemannian metric g^V on the leaf space $V = U/F_U$ such that the canonical projection $f : U \rightarrow V$ is a pseudo-Riemannian submersion $(U, g|_U)$ onto (V, g^V) .

Let an operator of covariant differentiation be given on a manifold M . It is equivalent to the existence of a linear connection on M and also to the existence of an affine connection on M [8]. It is said to be a manifold of linear connection.

Pseudo-Riemannian foliations include Lorenzian and Riemannian foliations and may be considered as foliations with transverse linear connection. In the foliation category isomorphisms of two foliations are diffeomorphisms transforming leaves of one to the leaves of the other. We proved [[24], Theorem 1.1] that the group of all automorphisms of a foliation with transverse linear connection admits a structure of an infinite dimensional Lie group modelled on LF -spaces. Therefore the same statement is valid also for every pseudo-Riemannian foliation.

Further the pseudo-Riemannian manifolds (M, g) are considered with the Levi-Civita connection ∇ .

Definition 3. Let (M, ∇) be a manifold M with a linear connection ∇ . A smooth distribution D on a manifold M is called geodesically invariant if for any point $x \in M$ and each vector $X \in D_x$ the geodesic $\gamma = \gamma(s)$ of (M, ∇) such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X$ has the property $\dot{\gamma}(s) \in D_{\gamma(s)}$ for each s of the domain of γ .

A foliation (M, F) on (M, ∇) is called geodesically invariant or totally geodesic if its tangent distribution TF is geodesically invariant.

First we prove the following criterion of a pseudo-Riemannian nature for a foliation of a pseudo-Riemannian manifold.

Theorem 1. Let (M, F) be a foliation of codimension q on an n -dimensional pseudo-Riemannian manifold (M, g) , $0 < q < n$. Then (M, F) is a pseudo-Riemannian foliation if and only if the q -dimensional distribution D , orthogonal to TF , is geodesically invariant and the metric on the leaves is non-degenerate.

Corollary 1. Let M and B be pseudo-Riemannian manifolds and $p : M \rightarrow B$ be surjective submersion. Then this submersion is pseudo-Riemannian iff there is induced a pseudo-Riemannian metric on the fibers and any geodesic orthogonal to the fiber at one point also is orthogonal to every fiber it intersects.

For Riemannian foliations a similar result was proposed by B. Reinhardt [13], and it was proven in detail by P. Molino [[10], Propositions 3.5 and 6.1]. We emphasize

that Molino's proof bear on the property of a geodesic to be a local extremum of the length functional that does not have an analogue in pseudo-Riemannian geometry.

In the proof of Theorem 1 we essentially use the result of A.D. Lewis [9] on geodesically invariance of distributions on manifolds with affine connection.

By a holonomy group $\Gamma(L)$ of a leaf L of a foliation (M, F) we mean a germ holonomy group of L usually used in the foliation theory [16]. If $\Gamma(L) = 0$ the leaf L is said to be a leaf without holonomy.

A construction of the holonomy groupoid of a foliation was presented by S. Ehresmann. Another equivalent construction was given by H. Winkelkemper [17] and named by him *the graph* of a foliation. The graph $G(F)$ contains all information about the foliation (M, F) and its holonomy groups. C^* -algebras of complex valued functions of foliations (M, F) are determined on $G(F)$ and are one of the fundamental concepts in K -theory of foliations.

In the general case the graph of a smooth foliation (M, F) of codimension q on an n -dimensional smooth manifold M is a non-Hausdorff smooth $(2n - q)$ -manifold (the precise definition is given in Subsection 3.1).

For the graph $G(F)$ of a pseudo-Riemannian foliation (M, F) we prove the following statement.

Theorem 2. *Let (M, F) be a smooth pseudo-Riemannian foliation of codimension q on the n -dimensional pseudo-Riemannian manifold (M, g) . Let $G(F)$ be its graph with the canonical projections $p_i : G(F) \rightarrow M$, $i = 1, 2$. Then:*

- (1) *The graph $G(F)$ of a foliation (M, F) is a Hausdorff $(2n - q)$ -dimensional manifold with the induced foliation $\mathbb{F} = \{\mathbb{L}_\alpha = p_i^{-1}(L_\alpha) | L_\alpha \in F\}$, $i = 1, 2$. Moreover the germ holonomy groups $\Gamma(L_\alpha)$ and $\Gamma(\mathbb{L}_\alpha)$ of the appropriate leaves L_α and \mathbb{L}_α are isomorphic.*
- (2) *On the graph $G(F)$ there exists a unique pseudo-Riemannian metric d with respect to which $(G(F), \mathbb{F})$ is a pseudo-Riemannian foliation and p_i are pseudo-Riemannian submersions. In this case fibres of p_1 are orthogonal to the fibres of p_2 at common points.*
- (3) *Every leaf $\mathbb{L}_\alpha = p_i^{-1}(L_\alpha) \in \mathbb{F}$ is a reducible pseudo-Riemannian manifold that is isometric to the quotient manifold $(\mathcal{L}_\alpha \times \mathcal{L}_\alpha) / \Psi_\alpha$ of the pseudo-Riemannian product $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ of the pseudo-Riemannian holonomy covering space \mathcal{L}_α of L_α by the isometry group Ψ_α , and $\Psi_\alpha \cong \Gamma(L_\alpha) \cong \Gamma(\mathbb{L}_\alpha)$.*

Definition 4. *The pseudo-Riemannian metric d on the graph $G(F)$ satisfying Theorem 2 is called the induced metric.*

Corollary 2. *There exists a dense saturated G_δ -subset of $G(F)$ any leaf (\mathbb{L}_α, d) of which is isometric to the direct product $L_\alpha \times L_\alpha$ of a pseudo-Riemannian manifolds (L_α, g) .*

Remark 1. *We prove property 3 in Theorem 2 without the application of the well known Wu's theorem [18]. This theorem is not applicable here because the completeness of the pseudo-Riemannian metric d is not assumed.*

By a geodesic we mean a piecewise geodesic.

Definition 5. A pseudo-Riemannian foliation (M, F) is called transversally complete if the canonical parameter on every maximal orthogonal geodesic is defined on the whole real line.

Under the additional assumption of the transversal completeness of a pseudo-Riemannian foliation we prove the following statement.

Theorem 3. Let (M, F) be a transversally complete pseudo-Riemannian foliation on a pseudo-Riemannian manifold (M, g) and d be the induced pseudo-Riemannian metric on its graph $G(F)$. Then I:

- (1) The induced foliation $(G(F), \mathbb{F})$ is a transversally complete pseudo-Riemannian foliation.
- (2) The orthogonal q -dimensional distributions \mathfrak{M} and \mathfrak{N} are Ehresmann connections for the foliations (M, F) and $(G(F), \mathbb{F})$ respectively, and, for any $L_\alpha \in F$, $\mathbb{L}_\alpha = p_i^{-1}(L_\alpha)$, the following holonomy groups $\Gamma(L_\alpha)$, $H_{\mathfrak{M}}(L_\alpha)$, $\Gamma(\mathbb{L}_\alpha)$ and $H_{\mathfrak{N}}(\mathbb{L}_\alpha)$ are isomorphic.
- (3) The canonical projections $p_i : G(F) \rightarrow M$, $i = 1, 2$, form locally trivial fibrations with the same standard fibre L_0 , and L_0 is diffeomorphic to any leaf without holonomy of (M, F) .
- (4) Every leaf $\mathbb{L}_\alpha = p_i^{-1}(L_\alpha) \in \mathbb{F}$ is a reducible pseudo-Riemannian manifold that is isometric to the quotient manifold $(\mathcal{L}_\alpha \times \mathcal{L}_\alpha) / \Psi_\alpha$ of the pseudo-Riemannian product $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ of the pseudo-Riemannian holonomy covering space \mathcal{L}_α for L_α by an isometry group Ψ_α , where $\Psi_\alpha \cong \Gamma(L_\alpha) \cong H_{\mathfrak{M}}(L_\alpha) \cong \Gamma(\mathbb{L}_\alpha) \cong H_{\mathfrak{N}}(\mathbb{L}_\alpha)$, and every \mathcal{L}_α is diffeomorphic to L_0 .

II. If moreover, the foliation (M, F) is also geodesically invariant, then:

- (i) Each foliation $\mathbb{F}, F^{(i)} := \{p_i^{-1}(x) | x \in M\}$, $i = 1, 2$, is geodesically invariant and pseudo-Riemannian.
- (ii) Any leaf without of holonomy of (M, F) is isometric to any fibres of submersions p_1 and p_2 with respect to corresponding induced metrics.
- (iii) Every leaf \mathbb{L} without holonomy of the foliation $(G(F), \mathbb{F})$ is isometric to the pseudo-Riemannian product $L_0 \times L_0$ and any other leaf \mathbb{L}_α is isometric to the pseudo-Riemannian quotient manifold $(L_0 \times L_0) / \Psi_\alpha$, where $\Psi_\alpha \cong \Gamma(\mathbb{L}_\alpha)$.

We emphasize that the proofs of Theorems 2 and 3 essentially use the notion of an Ehresmann connection of a foliation proposed by R.A. Blumenthal and J.J Hebda [1] and also the results of the first author on graphs of foliations with Ehresmann connection [21], [23] and some other statements (see Sections 4-5).

Notations We denote by $\mathfrak{X}(N)$ the set of all smooth vector fields on a manifold N . If \mathfrak{M} is a smooth distribution on M , then $\mathfrak{X}_{\mathfrak{M}}(M) := \{X \in \mathfrak{X}(M) | X_u \in \mathfrak{M}_u \ \forall u \in M\}$. For a foliation (M, F) we denote $\mathfrak{X}_{TF}(M)$ also by $\mathfrak{X}_F(M)$. Let us denote the leaf of foliation (M, F) passing through a point $x \in M$ by $L(x)$.

Let \mathfrak{Fol} be the category of foliations where morphisms are smooth maps transforming leaves into leaves.

Let \cong be the symbol of a group isomorphism and also of a manifold diffeomorphism.

Further smoothness is understood to mean C^∞ .

2. A CRITERION OF PSEUDO-RIEMANNIAN OF A FOLIATION

2.1. Foliate and transversal vector fields. Let (M, F) be a smooth foliation of codimension q of a smooth n -dimensional manifold M .

A function $f \in \mathfrak{F}(M)$ is called *basic* if it is constant on every leaf L of this foliation. Let $\Omega_b^0(M, F)$ be a set of basic functions. Remark that $f \in \Omega_b^0(M, F)$ if and only if f depends on only transverse coordinates in any adapted charts.

Definition 6. Let (M, F) be an arbitrary foliation. A vector field $X \in \mathfrak{X}(M)$ is called *foliate* if for any $Y \in \mathfrak{X}_F(M)$ the vector field $[X, Y]$ belongs to $\mathfrak{X}_F(M)$.

Following Molino [10] we denote by $L(M, F)$ the set of all foliate vector fields. In this case $L(M, F)$ is a normalizer in $\mathfrak{X}(M)$ of the Lie subalgebra $\mathfrak{X}_F(M)$. Therefore $L(M, F)$ is a Lie subalgebra of $\mathfrak{X}(M)$.

A q -dimensional smooth distribution \mathfrak{M} on the manifold M is called *transversal* to the foliation (M, F) if the equality $T_x M = T_x F \oplus \mathfrak{M}_x$ holds for any $x \in M$, where \oplus stands for the direct sum of vector spaces. Let us identify the quotient vector bundle TM/TF with a transversal distribution \mathfrak{M} .

Let K_a^m be an open cube in R^m with the center at the null $0_m \in R^m$ and with the side $2a$. A chart (U, φ) of the manifold M is said to be *adapted* with respect to the foliation (M, F) at $p_0 \in U$ if $\varphi(U) = K_a^{n-q} \times K_a^q$, and $\varphi(p_0) = (0_{n-q}, 0_q)$, $\varphi(p) = (x^1, \dots, x^{n-q}, y^1, \dots, y^q) = (x^\alpha, y^\alpha) \in K_a^{n-q} \times K_a^q$ and $\varphi^{-1}(K_a^{n-q} \times \{y^\alpha\})$, $\{y^\alpha\} \in K_a^q$, is a plaque of p in U , i.e. it is a connected component of p in $L \cap U$. We say that x^α are leaf coordinates and y^α are transverse coordinates. At every point $x \in U$ the derivations $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^q}$ generate a subspace of $T_x M$ that is transversal to $T_x F$. Vectors $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^q}$ or for short $\frac{\partial}{\partial y^\beta}$, $\beta = 1, \dots, q$, are the corresponding local sections of the distribution \mathfrak{M}_x at each point $x \in U$. The section \bar{X} of \mathfrak{M} determined by the vector field $X \in \mathfrak{X}(M)$ has in U the form $\bar{X} = \sum_{\beta=1}^q \bar{X}^\beta \frac{\partial}{\partial y^\beta}$, where \bar{X}^i are the last q components of X in the local basis $\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\beta}\}$.

In particular, if X is a foliate vector field, then \bar{X}^β depend only on the transverse variables y^1, \dots, y^q . In this case the vector field \bar{X} is called the *transverse vector field* associated to X . Let $l(M, F)$ be the set of transverse vector fields. Then the projection $L(M, F) \rightarrow l(M, F) : X \mapsto \bar{X}$ is well defined with kernel is equal to $\mathfrak{X}_F(M)$. Therefore, for the foliation (M, F) there exists an exact sequence of vector spaces

$$(1) \quad 0 \rightarrow \mathfrak{X}_F(M) \rightarrow L(M, F) \rightarrow l(M, F) \rightarrow 0$$

Further we consider coordinate charts (U, φ) adapted to (M, F) with the base vector fields $\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^\beta}\}$, where $\frac{\partial}{\partial y^\beta} \in l(U, F_U)$. We call $\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^\beta}\}$ an \mathfrak{M} -adapted foliate basis on U .

The following characteristic properties of foliate vector fields are well known (see, for example, [10]).

Proposition 1. *Let (M, F) be a smooth foliation of codimension q and \mathfrak{M} be the transverse q -dimensional distribution. Then the following conditions are equivalent:*

- (1) *The vector field X is foliate in the sense of Definition 6.*
- (2) *For any basic function $f \in \Omega_b^0(M, F)$ the function Xf is also basic.*
- (3) *For any adapted chart (U, φ) to (M, F) with coordinates (x^a, y^β) and \mathfrak{M} -adapted foliate basis $\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^\beta}\}$ on U the vector field X has a form $X = \sum_{a=1}^p X^a \frac{\partial}{\partial x^a} + \sum_{\delta=1}^q \bar{X}^\delta \frac{\partial}{\partial y^\delta}$, and $\bar{X}^\delta = \bar{X}^\delta(y^\beta)$.*

2.2. Pseudo-Riemannian foliations of the pseudo-Riemannian manifolds.

Proposition 2. *Let (M, F) be a foliation of codimension q on a pseudo-Riemannian manifold (M, g) and \mathfrak{M} be the orthogonal q -dimensional distribution. The following conditions are equivalent:*

- (1) *The pseudo-Riemannian metric g is transversally projectable.*
- (2) *$g(X, Y) \in \Omega_b^0(M, F)$ for any foliate vector fields $X, Y \in \mathfrak{X}(M)$ orthogonal to this foliation.*
- (3) *The metric g in any \mathfrak{M} -adapted foliate basis $\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^\beta}\}$ corresponding to an adapted chart has the form*

$$(g_{ij}) = \begin{pmatrix} g_{ab}(x^a, y^\beta) & 0 \\ 0 & g_{\delta\varepsilon}(y^\beta) \end{pmatrix},$$

where $a, b, d = \overline{1, p}$; $\beta, \delta, \varepsilon = \overline{1, q}$.

2.3. **Lewis's criterion.** *Lewis's criterion [9]. A smooth distribution \mathfrak{M} on a manifold of affine connection (M, ∇) is geodesically invariant if and only if*

$$\nabla_X Y + \nabla_Y X \in \mathfrak{X}_{\mathfrak{M}}(M)$$

for any vector fields X, Y belonging to $\mathfrak{X}_{\mathfrak{M}}(M)$.

2.4. **Proof of Theorem 1.** Let (M, F) be a smooth foliation of codimension q on a pseudo-Riemannian manifold (M, g) such that the restriction of this metric to the leaves is non-degenerate. Suppose now that the q -dimensional distribution \mathfrak{M} orthogonal to (M, F) is geodesically invariant.

Let us consider any foliate vector fields $X^{\mathfrak{M}}, Y^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$ and an arbitrary vector field $Z^F \in \mathfrak{X}_F(M)$. Observe that the metric g is transversally projectable with respect to the foliation (M, F) if and only if

$$(2) \quad Z^F \cdot g(X^{\mathfrak{M}}, Y^{\mathfrak{M}}) = 0.$$

Recall that the equality $\nabla_X g = 0$ is equivalent to

$$(3) \quad Z \cdot g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \quad \forall X, Y, Z \in \mathfrak{X}(M).$$

Putting in (3) $Z = Z^F \in \mathfrak{X}_F(M)$ and $X = X^{\mathfrak{M}}, Y = Y^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$ and using the identity $g(X^{\mathfrak{M}}, Z^F) = 0$ we have

$$(4) \quad Y^{\mathfrak{M}} \cdot g(X^{\mathfrak{M}}, Z^F) = g(\nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}}, Z^F) + g(X^{\mathfrak{M}}, \nabla_{Y^{\mathfrak{M}}} Z^F) = 0.$$

By analogy, changing $Y^{\mathfrak{M}}$ and $X^{\mathfrak{M}}$ we get

$$(5) \quad X^{\mathfrak{M}} \cdot g(Y^{\mathfrak{M}}, Z^F) = g(\nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}}, Z^F) + g(Y^{\mathfrak{M}}, \nabla_{X^{\mathfrak{M}}} Z^F) = 0.$$

Add (4) to (5), then apply the bi-linearity of the pseudo-Riemannian metric g and obtain

$$(6) \quad g(\nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}} + \nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}}, Z^F) + g(X^{\mathfrak{M}}, \nabla_{Y^{\mathfrak{M}}} Z^F) + g(Y^{\mathfrak{M}}, \nabla_{X^{\mathfrak{M}}} Z^F) = 0.$$

Due to the geodesically invariance of the distribution \mathfrak{M} the Lewis's criterion implies $\nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}} + \nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$. Therefore the first term in (6) was equal to zero. Since ∇ is the Levi-Civita connection, it is torsion free and

$$(7) \quad \nabla_{Y^{\mathfrak{M}}} Z^F = \nabla_{Z^F} Y^{\mathfrak{M}} + [Y^{\mathfrak{M}}, Z^F],$$

similarly

$$(8) \quad \nabla_{X^{\mathfrak{M}}} Z^F = \nabla_{Z^F} X^{\mathfrak{M}} + [X^{\mathfrak{M}}, Z^F].$$

Putting (7) and (8) into (6) we obtain

$$g(X^{\mathfrak{M}}, \nabla_{Z^F} Y^{\mathfrak{M}}) + g(Y^{\mathfrak{M}}, \nabla_{Z^F} X^{\mathfrak{M}}) + g(X^{\mathfrak{M}}, [Y^{\mathfrak{M}}, Z^F]) + g(Y^{\mathfrak{M}}, [X^{\mathfrak{M}}, Z^F]) = 0.$$

In concordance with the choice, the vector fields $X^{\mathfrak{M}}$ and $Y^{\mathfrak{M}}$ are foliate, so $[Z^{\mathfrak{M}}, Y^F]$ and $[X^{\mathfrak{M}}, Y^F]$ belong to $\mathfrak{X}_F(M)$. Hence the third and fourth terms in the previous equation vanish. Therefore

$$(9) \quad g(X^{\mathfrak{M}}, \nabla_{Z^F} Y^{\mathfrak{M}}) + g(Y^{\mathfrak{M}}, \nabla_{Z^F} X^{\mathfrak{M}}) = 0.$$

Let in (3) $Z = Z^F \in \mathfrak{X}_F(M)$, $X = X^{\mathfrak{M}}, Y = Y^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$ and using the relation (9) we have the following

$$(10) \quad Z^F \cdot g(X^{\mathfrak{M}}, Y^{\mathfrak{M}}) = g(\nabla_{Z^F} X^{\mathfrak{M}}, Y^{\mathfrak{M}}) + g(X^{\mathfrak{M}}, \nabla_{Z^F} Y^{\mathfrak{M}}) = 0.$$

The equality (10) implies the pseudo-Riemannian of the foliation (M, F) .

Converse. Let (M, F) be a pseudo-Riemannian foliation of codimension q on a pseudo-Riemannian manifold (M, g) , hence by Definition 1 the restriction of this metric on leaves is non-degenerate. Denote by \mathfrak{M} the orthogonal q -dimensional distribution to this foliation. The pseudo-Riemannian of the foliation (M, F) implies that equalities (10) and (9) hold for an arbitrary vector field $Z^F \in \mathfrak{X}_F(M)$ and any foliate vector fields $X^{\mathfrak{M}}, Y^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$. From (7) and (8) we find

$$(11) \quad \nabla_{Z^F} Y^{\mathfrak{M}} = \nabla_{Y^{\mathfrak{M}}} Z^F - [Y^{\mathfrak{M}}, Z^F],$$

and similarly

$$(12) \quad \nabla_{Z^F} X^{\mathfrak{M}} = \nabla_{X^{\mathfrak{M}}} Z^F - [X^{\mathfrak{M}}, Z^F].$$

Putting (11) and (12) in (9) and taking into account that $[Z^{\mathfrak{M}}, Y^F]$ and $[X^{\mathfrak{M}}, Y^F]$ belong to $\mathfrak{X}_F(M)$ we get

$$(13) \quad g(X^{\mathfrak{M}}, \nabla_{Y^{\mathfrak{M}}} Z^F) + g(Y^{\mathfrak{M}}, \nabla_{X^{\mathfrak{M}}} Z^F) = 0.$$

Recall that the relation (6) is obtained using only the condition $\nabla g = 0$ and the orthogonality of the distributions \mathfrak{M} and TF . Therefore we may apply (6). Thus we have

$$(14) \quad g(\nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}} + \nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}}, Z^F) = 0$$

for any vector field Z^F tangent to the foliation (M, F) . Nondegeneracy of the restriction of g to the leaves of (M, F) implies

$$(15) \quad \nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}} + \nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M).$$

According to the Lewis's theorem mentioned above the relation (15) guarantees that the distribution \mathfrak{M} is geodesically invariant.

3. GRAPHS OF PSEUDO-RIEMANNIAN FOLIATIONS

3.1. The graph of a smooth foliation. Let (M, F) be a smooth foliation of codimension q of the n -dimensional manifold M and \mathfrak{M} be transversal q -dimension distribution on M . Denote by $A(x, y)$ the set of all piecewise smooth paths from x to y on the same leaf L . Two paths $h_1, h_2 \in A(x, y)$ are called equivalent and are denoted by $h_1 \sim h_2$ if and only if they define the same germ at the point x of the holonomy diffeomorphisms $D_x^q \rightarrow D_y^q$ from D_x^q to D_y^q , where D_x^q and D_y^q are q -dimensional disks transversal to this foliation.

We denote by $\langle h \rangle$ the equivalence class containing h .

Definition 7. *The set*

$$G(F) := \{(x, \langle h \rangle, y) \mid x \in M, y \in L(x), h \in A(x, y)\}$$

is called the graph of the foliation (M, F) [17].

Suppose that h and g are paths such that $h(1) = g(0)$. Denote by $h \cdot g$ the product of the path h and g , i.e. $(h \cdot g)(t) = h(2t), t \in [0, 1/2]$ and $(h \cdot g)(t) = g(2t - 1), t \in [1/2, 1]$. The equality

$$(x, \langle h \rangle, v) \circ (v, \langle g \rangle, y) := (x, \langle h \cdot g \rangle, y)$$

where $h(1) = g(0)$, defines a partial multiplication \circ in the graph $G(F)$. In relation to this operation $G(F)$ is a groupoid named the *the holonomy groupoid* of the foliation (M, F) .

Consider any coordinate chart (U, φ) with the center $x \in M$ adapted to the foliation (M, F) where $\varphi(U) = K_a^{n-q} \times K_a^q$. Further we call the submanifold $D_x^q := \varphi^{-1}(\{0_{n-q}\} \times K_a^q)$ a q -dimensional transversal disk at x .

Let (U, φ) and (U', φ') be two adapted to (M, F) charts at x and $x' \in L(x)$ accordingly. The chart (U', φ') is referred to as a *subordinate* to (U, φ) if $\varphi(U) = K_a^{n-q} \times K_a^q$, $\varphi'(U') = K_{a'}^{n-q} \times K_{a'}^q$, where $0 < a' \leq a$, and for any $c \in K_a^q$ the local

leaves $\varphi^{-1}(K_a^{n-q} \times \{c\})$ and $\varphi'^{-1}(K_{a'}^{n-q} \times \{c\})$ belong to the same leaf of the foliation (M, F) . For every adapted chart (U, φ) at x and any $y \in L(x)$ there exists a subordinated chart (U', φ') at y [12]. It's not difficult to show that for any points x, y from a leaf L there exists a pair of adapted charts (U, φ) and (U', φ') such that each of them is subordinated to the other.

Let $z = (x, \langle h \rangle, y)$ be any point in the graph $G(F)$. Let (U, φ) and (U', φ') be a pair a pair mutually subordinated charts and D_x, D_y be the q -dimensional transversal disks at x and y accordingly. Take any point $\tilde{x} \in D_x$, and denote by \tilde{h} the holonomy lift of h into point \tilde{x} [17]. Consider any point x' in the local leaf $L_{\tilde{x}}$ through \tilde{x} any y' in the local leaf $L_{\tilde{y}}$ through $\tilde{y} := \tilde{h}(1)$. Let $t_{x'}$ be a path in $L_{\tilde{x}}$ connected x' with \tilde{x} and $t_{y'}$ be a path in $L_{\tilde{y}}$ connected y' with \tilde{y} . Denote by V_z the set of $z' = (x', \langle t_{x'} \cdot \tilde{h} \cdot t_{y'}^{-1} \rangle, y')$ defined by the indicated above way. The set $\Sigma := \{V_z | z \in G(F)\}$ forms a base of a topology Ω in $G(F)$, and $(G(F), \Omega)$ is non-Hausdorff in general.

Define a map $\chi_z : V_z \rightarrow K_a^{n-q} \times K_a^n$ by the following equality

$$\chi_z(z') := (\varphi(x'), pr_2 \circ \varphi'(y')) \quad \forall z' = (x', \langle t_{x'} \cdot \tilde{h} \cdot t_{y'}^{-1} \rangle, y'),$$

where $pr_2 : K_a^{2n-q} \cong K_a^n \times K_a^{n-q} \rightarrow K_a^{n-q}$ is the canonical projection onto K_a^{n-q} . The set $\mathcal{A} = \{(V_z, \chi_z) | z \in G(F)\}$ is a smooth atlas on $G(F)$.

Thus, the graph $G(F)$ becomes a $(2n - q)$ -dimensional smooth manifold which is non-Hausdorff, in general.

Definition 8. A pseudogroup \mathcal{H} of local holonomy diffeomorphisms of a manifold N is called quasi-analytic if the existence of an open connected subset V in N such that $h|_V = id_V$ for an element $h \in \mathcal{H}$ implies that $h = id_{D(h)}$ in the whole connected domain $D(h)$ of h that contains V .

According to ([23], Proposition 2), Winkelkemper's criterion of the property of the graph $G(F)$ to be Hausdorff can be reformulated in the following form:

Proposition 3. The topological space of the graph $G(F)$ of a foliation (M, F) is Hausdorff iff the holonomy pseudogroup of this foliation is quasi-analytic.

The mappings

$$p_1 : G(F) \rightarrow M : (x, \langle h \rangle, y) \mapsto x, \quad p_2 : G(F) \rightarrow M : (x, \langle h \rangle, y) \mapsto y$$

are referred to as canonical projections, and p_1 and p_2 are submersions onto M .

Definition 9. A foliation $\mathbb{F} = \{\mathbb{L}_\alpha = p_1^{-1}(L_\alpha) = p_2^{-1}(L_\alpha) | L_\alpha \in F\}$ is defined on the graph $G(F)$ and is called the induced foliation.

3.2. An Ehresmann connection for a smooth foliation. Recall the notion of an Ehresmann connection which was introduced by R.A. Blumenthal and J.J. Hebda [1]. We use the term *a vertical-horizontal homotopy* introduced previously by R. Hermann. All mappings are supposed to be piecewise smooth.

Let (M, F) be a foliation of an arbitrary codimension $q \geq 1$. Let \mathfrak{M} be a transversal distribution on the manifold M , then for any $x \in M$ the equality $T_x M = T_x F \oplus \mathfrak{M}_x$

holds. Vectors from \mathfrak{M}_x , $x \in M$, are called horizontal. A piecewise smooth curve σ is horizontal (or \mathfrak{M} -horizontal) if each of its smooth segments is an integral curve of the distribution \mathfrak{M} . The distribution TF tangent to the leaves of the foliation (M, F) is called vertical. One says that a curve h is vertical if h is contained in a leaf of the foliation (M, F) .

A *vertical-horizontal homotopy* (v.h.h. for short) is a piecewise smooth map $H : I_1 \times I_2 \rightarrow M$, where $I_1 = I_2 = [0, 1]$, such that for any $(s, t) \in I_1 \times I_2$ the curve $H|_{I_1 \times \{t\}}$ is horizontal and the curve $H|_{\{s\} \times I_2}$ is vertical. The pair of curves $(H|_{I_1 \times \{0\}}, H|_{\{0\} \times I_2})$ is called a *base of the v.h.h. H*. Two paths (σ, h) with common origin $\sigma(0) = h(0)$, where σ is a horizontal path and h is a vertical one, are called an *admissible pair of paths*.

A distribution \mathfrak{M} transversal to a foliation (M, F) is called an *Ehresmann connection for (M, F)* if for any admissible pair of paths (σ, h) there exists a v.h.h. with the base (σ, h) .

Let \mathfrak{M} be an Ehresmann connection for a foliation (M, F) . Then for any admissible pair of paths (σ, h) there exists a unique v.h.h. H with the base (σ, h) . We say that $\tilde{\sigma} := H|_{I_1 \times \{1\}}$ is the result of the *transfer of the path σ along h with respect to the Ehresmann connection \mathfrak{M}* .

By analogy we define v.h.h. and transfers in cases, when I_1 and I_2 are replaced with half-intervals.

3.3. The structure of leaves of the induced foliation. Now we consider any smooth foliation (M, F) and its graph $G(F)$.

Proposition 4. *Let $\mathbb{L}_\alpha = p_1^{-1}(L_\alpha)$, $L_\alpha = L_\alpha(x) \in F$, be any leaf of the induced foliation $(G(F), \mathbb{F})$. Let $\mathcal{L}_\alpha = p_2^{-1}(x)$. Then:*

(i) *The restriction $p_1|_{\mathcal{L}_\alpha}$ is a regular covering map of \mathcal{L}_α onto L_α , and its deck transformation group is isomorphic to the germ holonomy group $\Gamma(L_\alpha, x)$ of the leaf L_α at x .*

(ii) *The diagonal action of $\Psi_\alpha = \Gamma(L_\alpha, x)$ is defined on the product $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ of manifolds, and the quotient manifold $(\mathcal{L}_\alpha \times \mathcal{L}_\alpha)/\Psi_\alpha$ is diffeomorphic to the leaf \mathbb{L}_α and there exists a diffeomorphism $\Xi_\alpha : \mathbb{L}_\alpha \rightarrow (\mathcal{L}_\alpha \times \mathcal{L}_\alpha)/\Psi_\alpha$ such that Ξ_α is an isomorphism of the foliations $(\mathbb{L}_\alpha, F^{(i)})$ and $((\mathcal{L}_\alpha \times \mathcal{L}_\alpha)/\Psi_\alpha, F_i)$, $i = 1, 2$, where (F_1, F_2) is the pair of foliations of complementary dimensions covered by the product $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$.*

The definitions of the graph $G(F)$ of its smooth structure and of the canonical projections imply that the restriction $p_1|_{p_2^{-1}(x)} : p_2^{-1}(x) \rightarrow L_\alpha(x)$ is a regular covering map with the deck transformation group isomorphic to $\Gamma(L_\alpha, x)$. Hence (i) is true.

For the proof of (ii) we consider two simple foliations $F^{(i)} := \{p_i^{-1}(x) | x \in M\}$, $i = 1, 2$, on $G(F)$. They induce two simple foliations on the leaf \mathbb{L}_α which will be denoted also by $F^{(1)}$ and $F^{(2)}$. Let us show that $TF^{(2)}$ is an Ehresmann connection for the foliation $(\mathbb{L}_\alpha, F^{(1)})$.

Take an admissible pair (g, f) of piecewise smooth paths in the leaf \mathbb{L}_α with the common origin at $g(0) = f(0) = z_0 = (x_0, \langle h_0 \rangle, y_0)$, where $g : I_1 \rightarrow p_2^{-1}(y_0)$ and $f : I_2 \rightarrow p_1^{-1}(x_0)$, $I_1 = I_2 = [0, 1]$. Let $g_0 = p_1 \circ g$. Denote by g_t , $t \in I_2$, the path with the origin in $f(t)$ covering the path g relatively the covering map $p_1|_{p_2^{-1}(y_t)}$, where $y_t = p_2(f(t))$. Put $H(s, t) := g_t(s) \forall (s, t) \in I_1 \times I_2$. It is easy to check that H is a vertical-horizontal homotopy with the base (g, f) . Hence $TF^{(2)}$ is an integrable Ehresmann connection for $(\mathbb{L}_\alpha, F^{(1)})$.

Let $k : \widetilde{\mathbb{L}} \rightarrow \mathbb{L}_\alpha$ be a universal covering map for the leaf \mathbb{L}_α and $\widetilde{F}^{(i)} = k^*F^{(i)}$, $i = 1, 2$, be an induced foliations on $\widetilde{\mathbb{L}}$. It is easy to show that $TF^{(2)}$ is an integrable Ehresmann connection for the foliation $\widetilde{F}^{(1)}$. Due to simple connectivity of $\widetilde{\mathbb{L}}$ according to the decomposition theorem proved by S. Kashiwabara [6], we identify $\widetilde{\mathbb{L}}$ with the product manifold $\widetilde{N}_1 \times \widetilde{N}_2$. In this case $\widetilde{F}^{(1)} = \{\widetilde{N}_1 \times \{v\} \mid v \in \widetilde{N}_2\}$, $\widetilde{F}^{(2)} = \{\{u\} \times \widetilde{N}_1 \mid u \in \widetilde{N}_1\}$. Thus, $(\mathbb{L}_\alpha, F^{(1)}, F^{(2)})$ is a pair of two transverse simple foliations on \mathbb{L}_α covered by the product of manifolds.

In the work [15] that construction is said to be a *simple transversal bifibration* and is denoted by $(\mathbb{L}_\alpha, p_1, p_2, L_\alpha, L_\alpha)$.

Let us fix a point $z = (x, \langle 1_x \rangle, x) \in \mathbb{L}_\alpha$. According (i) the group Ψ_α acts on the covering manifold \mathcal{L}_α as a group of deck transformations and the diagonal action of Ψ_α on the product manifolds $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ is defined. This action is proper and totally discontinuous, and preserves the product. The quotient manifold $(\mathcal{L}_\alpha \times \mathcal{L}_\alpha)/\Psi_\alpha$ with two foliations F_1, F_2 whose leaves are covered by the respective leaves of the trivial foliations of the product $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ is defined.

Proposition 1 from [15] implies the existence of a diffeomorphism

$$\Xi_\alpha : \mathbb{L} \rightarrow (\mathcal{L}_\alpha \times \mathcal{L}_\alpha)/\Psi_\alpha$$

such that Ξ_α is an isomorphism in the foliation category \mathfrak{Fol} of both pairs of foliations $(\mathbb{L}_\alpha, F^{(i)})$ and $((\mathcal{L}_\alpha \times \mathcal{L}_\alpha)/\Psi_\alpha, F_i)$, $i = 1, 2$.

3.4. Proof of Theorem 2. Denote by (M, F) an arbitrary pseudo-Riemannian foliation on a pseudo-Riemannian manifold (M, g) .

Observe that a Riemannian structure may be considered as a G -structure of the first order [7]. As for as every pseudogroup of local isometries of a pseudo-Riemannian manifold is quasi-analytic we may apply Winkelkemper's criterion indicated above, according to which the graph $G(F)$ is Hausdorff.

Observe that the definition of the inducted foliation $(G(F), \mathbb{F})$ implies that both foliations (M, F) and $(G(F), \mathbb{F})$ are given by the same holonomy pseudogroup \mathcal{H} . Recall that the germ holonomy group of an arbitrary foliation is interpreted as a group of germs at the relevant point v of the local transformations φ from the holonomy pseudogroup \mathcal{H} such that $\varphi(v) = v$ (see, for example, [14]).

This implies that the holonomy groups $\Gamma(L, x)$ and $\Gamma(\mathbb{L}, z)$, where $\mathbb{L} = p_1^{-1}(L)$, $z = (x, \langle h \rangle, y) \in G(F)$, are isomorphic.

Thus the assertion 1 of Theorem 2 is valid.

Let \mathbb{F} be the induced foliation and $F^{(i)} := \{p_i^{-1}(x) \mid x \in M\}$, $i = 1, 2$, be two simple foliations on the graph $G(F)$. Define a special pseudo-Riemannian metric on the graph $G(F)$. Denote by \mathfrak{M} the q -dimensional distribution on M orthogonal to the pseudo-Riemannian foliation (M, F) . Since the pseudo-Riemannian metric g is non-degenerate on the leaves of this foliation, there exists a decomposition of the tangent space $T_x M$, $x \in M$, of M into the orthogonal sum of vector subspaces:

$$T_x M = T_x F \oplus \mathfrak{M}_x.$$

For any $z = (x, \langle h \rangle, y) \in G(F)$ put

$$\mathfrak{N}_z := \{X \in T_z G(F) \mid p_{1*} X \in \mathfrak{M}_x, p_{2*} X \in \mathfrak{M}_y\}.$$

Emphasize that there exists a bijective mapping of the intersection $p_1^{-1}(x) \cap p_2^{-1}(y)$ to the holonomy group $\Gamma(L, x)$. Therefore $\mathfrak{N} = \{\mathfrak{N}_z \mid z \in G(F)\}$ is a smooth q -dimensional distribution on the graph $G(F)$ and, for any $z \in G(F)$, the tangent vector space $T_z G(F)$, admits the following decomposition into a direct sum of vector subspaces

$$(16) \quad T_z(G(F)) = T_z(F^{(1)}) \oplus \mathfrak{N}_z \oplus T_z(F^{(2)}).$$

According to the decomposition (16), any vector field $X \in \mathfrak{X}(G(F))$ may be represented in the form

$$(17) \quad X = X^{(1)} + X^{\mathfrak{N}} + X^{(2)},$$

where $X^{(i)} \in \mathfrak{X}_{F^{(i)}} G$ and $X^{\mathfrak{N}} \in \mathfrak{X}_{\mathfrak{N}} G(F)$. Let us define the pseudo-Riemannian metric d on $G(F)$ by the equality

$$(18) \quad d(X, Y) := g(p_{1*} X, p_{1*} Y) + g(p_{2*} X^{(1)}, p_{2*} Y^{(1)}),$$

where X, Y are represented in the form (17).

Observe that

$$d(X, Y) := g(p_{1*} X^{(2)}, p_{1*} Y^{(2)}) + g(p_{1*} X^{\mathfrak{N}}, p_{1*} Y^{\mathfrak{N}}) + g(p_{2*} X^{(1)}, p_{2*} Y^{(1)}),$$

for any $X, Y \in \mathfrak{X}(G)$. Note that the foliations $F^{(1)}, F^{(2)}$ and distribution \mathfrak{N} are pairwise orthogonal in the pseudo-Riemannian manifold $(G(F), d)$. Moreover, the restriction of d onto any leaf of each foliation $\mathbb{F}, F^{(1)}$ and $F^{(2)}$ on the graph $G(F)$ is non-degenerated, and, for any $X, Y \in \mathfrak{X}_{T\mathbb{F}}(G)$:

$$d(X, Y) := g(p_{1*} X^{(2)}, p_{1*} Y^{(2)}) + g(p_{2*} X^{(1)}, p_{2*} Y^{(1)}).$$

This means that the induced pseudo-Riemannian metric $d|_{\mathbb{L}}$ on a leaf \mathbb{L} is locally a direct product of the pseudo-Riemannian metric induced on leaves of foliations $F^{(1)}|_{\mathbb{L}}$ and $F^{(2)}|_{\mathbb{L}}$. Therefore (see, for example [18] or [8] for the Riemannian case) the distributions $TF^{(1)}$ and $TF^{(2)}$ are orthogonal and parallel on $(\mathbb{L}, d|_{\mathbb{L}})$. Hence $(\mathbb{L}, d|_{\mathbb{L}})$ is a non-degenerate reducible pseudo-Riemannian manifold.

Further the restriction of d (respectively, g or p_{i*}) onto the vector subspaces of corresponding vector subspaces of $T_z G(F)$, $z \in G(F)$, will be also denoted also d (respectively, g or p_{i*}).

The canonical projection $p_1 : G(F) \rightarrow M$ is pseudo-Riemannian because $p_{1*z} : T_z F^{(2)} \oplus \mathfrak{N} \rightarrow T_x M$ is an isomorphism of the pseudo-Euclidean vector spaces $(T_z F^{(2)} \oplus \mathfrak{N}_z, d)$ and $(T_x F, g|_{T_x F})$ by definition of d .

Let us show that the canonical projection p_2 is also a pseudo-Riemannian submersion. Take any point $z = (x, \langle h \rangle, y)$ in $G(F)$. The pseudo-Riemannianity of (M, F) implies the existence of a linear isomorphism $\phi_{xy} : \mathfrak{M}_x \rightarrow \mathfrak{M}_y$ induced by the local holonomy isometry along h of the pseudo-Euclidean vector spaces (\mathfrak{M}_x, g) and (\mathfrak{M}_y, g) . According to the definition of d the restriction $p_{2*z} : T_z F^{(1)} \rightarrow T_y F$ is an isomorphism of pseudo-Euclidean spaces. Observe that the restriction $p_{2*z} : T_z F^{(1)} \oplus \mathfrak{N}_z \rightarrow T_y M = T_y F \oplus \mathfrak{M}_y$ is equal to $p_{2*z} = (\phi_{xy} \circ p_{1*z}|_{\mathfrak{N}_z}, p_{2*z}|_{T_z F^{(1)}})$. This implies that p_2 is also a pseudo-Riemannian submersion.

Take any vector $X \in \mathfrak{N}_z$. Let $\gamma = \gamma(s)$ be the geodesic passing through the point z in the direction of the vector X , i.e. $\gamma(0) = z$, $\dot{\gamma}(0) = X$. Therefore γ is the geodesic orthogonal to the leaves $L^{(1)}(z)$ and $L^{(2)}(z)$ of the foliations $F^{(1)}$ and $F^{(2)}$. As for as $L^{(1)}$ and $L^{(2)}$ are the fibers of the pseudo-Riemannian submersions p_1 and p_2 , according to Corollary 1 at the any point of $\gamma(s)$ the tangent vector $\dot{\gamma}(s)$ is orthogonal to the both fibers $L^{(1)}(\gamma(s))$ and $L^{(2)}(\gamma(s))$. Since $T_{\gamma(s)} \mathbb{L} = T_{\gamma(s)} L^{(1)} \oplus T_{\gamma(s)} L^{(2)}$, then the tangent vector $\dot{\gamma}(s)$ is orthogonal to $T_{\gamma(s)} \mathbb{L}$. Thus geodesic $\gamma = \gamma(s)$ of the pseudo-Riemannian manifold $(G(F), d)$ orthogonal to leaves of the foliation $(G(F), \mathbb{F})$ at the one its point is orthogonal to the foliation $(G(F), \mathbb{F})$ at each its point.

According to Theorem 1, due to non-degeneracy of a pseudo-Riemannian metric on leaves of this foliation, $(G(F), \mathbb{F})$ is a pseudo-Riemannian foliation.

Assume that there exists another pseudo-Riemannian metric \hat{d} on $G(F)$ satisfying the second statement of Theorem 2. Let $\hat{\mathfrak{N}}$ be the q -dimensional distribution that is orthogonal to the foliation \mathbb{F} in $(G(F), \hat{d})$. Therefore $p_{1*\hat{\mathfrak{N}}_z} = \mathfrak{M}_x$ and $p_{2*\hat{\mathfrak{N}}_z} = \mathfrak{M}_y$ for every point $z = (x, \langle h \rangle, y)$ in $G(F)$. Consequently $\hat{\mathfrak{N}} = \mathfrak{N}$ and $\hat{d}(X, Y) = d(X, Y) = g(p_{1*X}, p_{1*Y})$ for any $X, Y \in \mathfrak{X}_{\mathfrak{N}} G(F)$. According to our assumption, in relation to both metrics d and \hat{d} the foliations $F^{(1)}$, $F^{(2)}$ are orthogonal, with p_i , $i = 1, 2$, are pseudo-Riemannian submersions. Due to the decomposition (17) and bilinearity of d and \hat{d} it is necessary that $d = \hat{d}$.

Thus the statements 1 and 2 of Theorem 2 are proven. The statement 3 of Theorem 2 follows from Proposition 4.

4. TWO GRAPHS OF A FOLIATION WITH AN EHRESMANN CONNECTION

4.1. Holonomy groups of foliations with Ehresmann connections. Let (M, F) be a foliation with an Ehresmann connection \mathfrak{M} . Take any point $x \in M$. Denote by Ω_x the set of horizontal curves with the origin at x . An action of the fundamental group $\pi_1(L, x)$ of the leaf $L = L(x)$ on the set Ω_x is defined in the following way:

$$\Phi_x : \pi_1(L, x) \times \Omega_x \rightarrow \Omega_x : ([h], \sigma) \mapsto \tilde{\sigma},$$

where $[h] \in \pi_1(L, x)$ and $\tilde{\sigma}$ is the result of transfer of $\sigma \in \Omega_x$ along h in relation to \mathfrak{M} . Let $K_{\mathfrak{M}}(L, x)$ be the kernel of the action Φ_x , i.e.

$$K_{\mathfrak{M}}(L, x) = \{\alpha \in \pi_1(L, x) \mid \alpha(\sigma) = \sigma \ \forall \sigma \in \Omega_x\}.$$

The quotient group $H_{\mathfrak{M}}(L, x) = \pi_1(L, x)/K_{\mathfrak{M}}(L, x)$ is the \mathfrak{M} -holonomy group of the leaf L , see [1]. Due to the pathwise connectedness of the leaves, the \mathfrak{M} -holonomy groups at different points on the same leaf are isomorphic.

Let $\Gamma(L, x)$ be a germinal holonomy group of the leaf L . Then there exists a unique group epimorphism $\chi : H_{\mathfrak{M}}(L, x) \rightarrow \Gamma(L, x)$ satisfying the equality

$$\chi \circ \mu = \nu, \tag{1}$$

where $\mu : \pi_1(L, x) \rightarrow H_{\mathfrak{M}}(L, x) : [h] \mapsto [h] \cdot K_{\mathfrak{M}}(L, x)$ is the quotient map and $\nu : \pi_1(L, x) \rightarrow \Gamma(L, x) : [h] \mapsto \langle h \rangle$, where $\langle h \rangle$ is a germ of the holonomy diffeomorphism of a transverse q -dimensional disk along the loop h at the point x .

Let us emphasize that the \mathfrak{M} -holonomy group $H_{\mathfrak{M}}(L, x)$ has a *global character* unlike the germinal holonomy group $\Gamma(L, x)$ having a local-global character: global along the leaves and local along the transverse directions.

4.2. The graph $G_{\mathfrak{M}}(F)$. The graph $G_{\mathfrak{M}}(F)$ of a foliation (M, F) with an Ehresmann connection was introduced by the first author in [21] (see also [23]).

Let (M, F) be a foliation of arbitrary dimension k on a n -manifold M and $q = n - k$ be the codimension of this foliation. Suppose that the foliation (M, F) admits an Ehresmann connection \mathfrak{M} .

Consider the set Ω_x of \mathfrak{M} -horizontal curves with the origin at $x \in M$.

Take any points x and y in the leaf L of (M, F) . Introduce an equivalence relation ρ on the set $A(x, y)$ of vertical paths in L connecting x with y . Paths h and f in $A(x, y)$ are called ρ -equivalent if the loop $h \cdot f^{-1}$ generates the trivial element of the \mathfrak{M} -holonomy group $H_{\mathfrak{M}}(L, x)$. In other words, paths h and f are ρ -equivalent iff they define the same transfers of \mathfrak{M} -horizontal curves from Ω_x to Ω_y in relation to the Ehresmann connection \mathfrak{M} . The ρ -equivalence class containing h is denoted by $\{h\}$.

The set of ordered triplets $(x, \{h\}, y)$, where x and y are any points of an arbitrary leaf L of the foliation (M, F) and $\{h\}$ is a class of ρ -equivalent paths from x to y in L , is called *the graph of the foliation (M, F) with an Ehresmann connection \mathfrak{M}* and is denoted by $G_{\mathfrak{M}}(F)$.

The maps

$$p_1 : G_{\mathfrak{M}}(F) \rightarrow M : (x, \{h\}, y) \mapsto x, \quad p_2 : G_{\mathfrak{M}}(F) \rightarrow M : (x, \{h\}, y) \mapsto y$$

are called the *canonical projections*.

The graph $G_{\mathfrak{M}}(F)$ is equipped with a smooth structure and the binary operation $(y, \{h_1\}, z) * (x, \{h_2\}, y) := (x, \{h_1 \cdot h_2\}, z)$ becomes a smooth \mathfrak{M} -holonomy groupoid.

We summarize the first author's results ([21] and [23]) into the following theorem.

Theorem 4. *1. The graph $G_{\mathfrak{M}}(F)$ of a foliation (M, F) with an Ehresmann connection \mathfrak{M} admits a structure of a Hausdorff manifold and the canonical projections p_1 and p_2 determine locally trivial fibrations with common typical fibre Y , and for any*

point $x \in M$ there is a regular covering map $p_x : Y \rightarrow L(x)$ with the deck transformation group isomorphic to the \mathfrak{M} -holonomy group $H_{\mathfrak{M}}(L, x)$.

2. The distribution $\mathfrak{N} := \{\mathfrak{N}_z \mid z = (x, \{h\}, y) \in G_{\mathfrak{M}}(F)\}$ where

$$\mathfrak{N}_z = \{X \in T_z G_{\mathfrak{M}}(F) \mid p_{1*}X \in \mathfrak{M}_x, p_{2*}X \in \mathfrak{M}_y\}$$

is an Ehresmann connection for the induced foliation $\mathbb{F} := \{p_1^{-1}(L_\alpha) \mid L_\alpha \in F\}$.

3. The group epimorphism $\chi : H_{\mathfrak{M}}(L, x) \rightarrow \Gamma(L, x)$ satisfying the equality (1) is a group isomorphism and the map

$$G_{\mathfrak{M}}(F) \rightarrow G(F) : (x, \{h\}, y) \rightarrow (x, \langle h \rangle, y)$$

is a groupoid isomorphism if and only if the holonomy pseudogroup $\mathcal{H}(M, F)$ of the foliation (M, F) is quasi-analytic.

5. THE PROOF OF THEOREM 3

I. Let us prove the following lemma which will be useful in this proof.

Lemma 1. *Let (M, F) be any transversally complete pseudo-Riemannian foliation. Then the orthogonal q -dimension distribution \mathfrak{M} is an Ehresmann connection for (M, F) .*

Theorem 1 implies that for any point $x \in M$ there exists an adapted chart (U, φ) such that projection $p_U : U \rightarrow V = U/F_U$ is a pseudo-Riemannian submersion, and \mathfrak{M}_U is a geodesically invariant distribution. Hence there is a chart $(\tilde{U}, \tilde{\varphi})$ at $x \in \tilde{U} \subset U$, $\tilde{\varphi} = \varphi|_{\tilde{U}}$ with the following property: for any admissible pair of the paths (γ, f) , with origin $\gamma(0) = f(0) = x$ in \tilde{U} , there exists a vertical-horizontal homotopy H with the base (γ, f) where γ is a horizontal geodesic. The application of the results from [20] to transversally complete pseudo-Riemannian foliations implies that all statements for foliations with an Ehresmann connection remain true if we prove the existence of a vertical-horizontal homotopy only for a pair of the form (γ, f) , where γ is a horizontal geodesic.

Let us take any admissible pair of paths (γ, f) where γ is a horizontal geodesic. Cover the path $f(I_2)$ by a finite chain of adapted charts $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_k$ which have the property indicated above. Therefore there is a number $\delta > 0$ such that for the pair $(\gamma|_{[0, \delta]}, f)$ there exists the v.h.h. $H_\delta : [0, \delta] \times I_2 \rightarrow \tilde{U}_1 \cup \tilde{U}_2 \cup \dots \cup \tilde{U}_k$ with the base $(\gamma|_{[0, \delta]}, f)$. We continue this construction and get the prolongation H onto the $[0, \delta_0] \times I_2$, where $\delta_0 \in (\delta, 1)$, and H is the vertical-horizontal homotopy with the base $(\gamma|_{[0, \delta_0]}, f)$. Hence for every $t \in I_2$ we have the transfer $\gamma_{f(t)}(s) := H(s, t)$, $s \in [0, \delta_0]$. Due to transversal completeness of the foliation (M, F) , the geodesic $\gamma_{f(t)}(s)$ is defined for any $s \in I_1$. Therefore we may put $H(s, t) := \gamma_{f(t)}(s) \forall (s, t) \in I_1 \times I_2$. The verification shows that H is the vertical-horizontal homotopy with the base (γ, f) , that finishes the proof of Lemma 1.

Let (M, F) be a transversally complete pseudo-Riemannian foliation. According to Lemma 1, the distribution \mathfrak{M} is an Ehresmann connection for (M, F) . Therefore two graphs $G(F)$ and $G_{\mathfrak{M}}(F)$ are defined. Since the holonomy pseudogroup $\mathcal{H}(M, F)$

of this foliation is quasi-analytic, according to the statement 3 of Theorem 4 the holonomy group $\Gamma(L, x)$ and $H_{\mathfrak{M}}(L, x)$ are isomorphic for any $x \in M$ and we may identify the graph $G(F)$ with the graph $G_{\mathfrak{M}}(F)$.

In this case in accordance with the first statement of Theorem 4 the canonical projections $p_i : G(F) \rightarrow M$, $i = 1, 2$, define locally trivial bundles with the same typical fibre Y . It is easy to see that $Y = L_0$ is diffeomorphic to any leaf L_α without the holonomy of the foliation (M, F) .

By statement 2 of Theorem 4 the q -dimensional distribution \mathfrak{N} is an Ehresmann connection for the induced foliation $(G(F), \mathbb{F})$ on the manifold $(G(F), d)$. It implies that the distribution $\mathfrak{K} = \mathfrak{N} \oplus TF^{(2)}$ is an Ehresmann connection for the submersion $p_1 : G(F) \rightarrow M$. Moreover, a \mathfrak{K} -lift of any \mathfrak{M} -curve in M is a \mathfrak{N} -curve in $G(F)$. Since p_1 is a pseudo-Riemannian submersion, any \mathfrak{N} -lift of a \mathfrak{M} -geodesic γ in (M, g) to any point $z \in p_1^{-1}(\gamma(0))$ is a \mathfrak{N} -geodesic in $(G(F), d)$ and the projection $p_1 \circ \hat{\gamma}$ of every \mathfrak{N} -geodesic $\hat{\gamma}$ in $(G(F), d)$ is the \mathfrak{M} -geodesic in (M, g) . These facts and the transversal completeness of (M, F) imply the transversal completeness of $(G(F), \mathbb{F})$.

Since the holonomy pseudogroup $\mathcal{H}(G(F), \mathbb{F})$ is also quasi-analytical, according to statement 3 of Theorem 4 the holonomy groups $\Gamma(\mathbb{L}_\alpha, z)$ and $H_{\mathfrak{N}}(\mathbb{L}_\alpha, z)$ are isomorphic. In accordance with the first statement of Theorem 2 the holonomy groups $\Gamma(\mathbb{L}_\alpha, z)$ and $\Gamma(L_\alpha, x)$, $x = p_1(z)$, are isomorphic. Therefore $\Gamma(L_\alpha) \cong H_{\mathfrak{M}}(L_\alpha) \cong \Gamma(\mathbb{L}_\alpha) \cong H_{\mathfrak{N}}(\mathbb{L}_\alpha)$.

Thus, three statements of Theorem 3 are proved.

Remark that statement 4 follows from the proven statements 2 and 3 of Theorem 2.

II. The following characteristic property is well known for the geodesically invariant foliations (M, F) on Riemannian manifolds M and by analogy is proved for pseudo-Riemannian manifolds M (see, for example [19], Proposition 2.7). Denote by \mathcal{L}_X the Lie derivative along the vector field X .

Proposition 5. *Let (M, g) be pseudo-Riemannian manifold and F be a codimension q foliation of M . Then F is geodesically invariant if and only if $(\mathcal{L}_X g)(Y, Z) = 0$ for all $X \in \mathfrak{X}_{\mathfrak{M}}(M)$ and for all $Y, Z \in \mathfrak{X}_F(M)$, where \mathfrak{M} is the distribution which consists of all vectors orthogonal to TF .*

Furthermore if F is geodesically invariant and $X \in \mathfrak{X}_{\mathfrak{M}}(M)$ is a local foliated vector field which can define a local 1-parameter group, then a local 1-parameter group of local transformations generated by X preserves the metrics induced on plaques.

It is known that for any two leaves L_0 and L there exists a piecewise smooth horizontal geodesic $\sigma : [0, 1] \rightarrow M$ such that $a_0 = \sigma(0) \in L_0$ and $a = \sigma(1) \in L$. Let L_0 be a fixed leaf without holonomy and L be any other leaf of the foliation (M, F) .

Let x be an arbitrary point in L_0 . Connect a_0 with x by a vertical path $h : [0, 1] \rightarrow L_0$, $h(0) = a_0$, $h(1) = x$. As (M, F) is a transversally complete pseudo-Riemannian foliation according to Lemma 1 \mathfrak{M} is an Ehresmann connection for foliation (M, F) . Then for any admissible pair (σ, h) there exists a vertical-horizontal homotopy $H : I_1 \times I_2 \rightarrow M$ with the base (σ, h) . Let $\tilde{\sigma} := H|_{I_1 \times \{1\}}$. In this case [[23], Lemma 1] the

map $f_\sigma : L_0 \rightarrow L : x \mapsto \tilde{\sigma}(1)$ is well defined and is a regular covering with the deck transformation group isomorphic to $H_{\text{gr}}(L, x) \cong \Gamma(L, x)$.

According to our assumption, the foliation (M, F) is geodesically invariant, then Proposition 5 implies that the covering $f_\sigma : L_0 \rightarrow L$ is local isometry in relation to the induced pseudo-Riemannian metric $g|_{L_0}$ and $g|_L$ on L_0 and L .

Therefore for any leaf L without holonomy the map $f_\sigma : L_0 \rightarrow L$ is an isometry.

Using Proposition 5 and considering that p_1 and p_2 are pseudo-Riemannian submersions we get that \mathbb{F} , $F^{(1)}$ and $F^{(2)}$ are geodesically invariant and pseudo-Riemannian foliations on the pseudo-Riemannian manifold $(G(F), d)$, i.e. (3) is true.

Now it is easy to check the fulfilment of statements (ii) and (iii) in II.

6. LORENTZIAN FOLIATIONS OF CODIMENSION 2 ON CLOSED 3-MANIFOLDS

6.1. Theorem of C. Boubel, P. Mounoud, C. Tarquini [3].

Definition 10. *The algebraic Anosov flows, up to finite coverings and finite quotients, are the following:*

- (1) *The geodesic flows of the unit tangent bundle of hyperbolic compact surfaces.*
- (2) *The flows defined by the suspensions of linear hyperbolic diffeomorphisms of the 2-torus.*

An application of Molino's theory of Riemannian foliations on compact manifolds [10] and the classification of the Lorentzian Anosov flows given by E. Ghys in [5] allowed C. Boubel, P. Mounoud, C. Tarquini [[3], Theorem 4.1] to describe the structure of transversally complete Lorentzian foliations of codimension 2 on closed 3-manifolds in the following way:

Theorem 5. *Up to finite coverings, a 1-dimensional transversally complete Lorentzian foliation on a compact closed 3-manifold is either smoothly equivalent to a foliation generated by an algebraic Anosov flow or a Riemannian foliation.*

The structure of suspended algebraic Lorentzian foliations of codimension 2 on closed 3-manifolds and their graphs is described in the following subsection 6.2.

6.2. Example. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ be a matrix of integers such that $ad - bc = 1$ and $a + d > 2$. Such matrix A induces an Anosov automorphism f_A of a torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ conserving its orientation.

Let us consider the action of the group of integers \mathbb{Z} by the formula

$$(19) \quad \Phi_A := \mathbb{T}^2 \times \mathbb{R}^1 \times \mathbb{Z} \rightarrow \mathbb{T}^2 \times \mathbb{R}^1 : (u, t, n) \mapsto (f_A^n(u), t + n), n \in \mathbb{Z}.$$

Then the quotient manifold $M := \mathbb{T}^2 \times_{\mathbb{Z}} \mathbb{R}^1$ is defined, and M is a closed 3-manifold. Let $\varphi : \mathbb{T}^2 \times \mathbb{R} \rightarrow M$ be the quotient mapping. Since Φ_A keeps the trivial foliation $F_{tr} = \{\{u\} \times \mathbb{R}^1 | u \in \mathbb{T}^2\}$, the foliation (M, F) of codimension 2 is defined.

There exists the Lorentzian metric $g = \eta \begin{pmatrix} -2c & a-d \\ a-d & 2b \end{pmatrix}$, $\eta \in R \setminus \{0\}$, on the plane \mathbb{R}^2 which is invariant in relation to A ([22], Theorem 1). This metric induces the flat Lorentzian metric on \mathbb{T}^2 which is denoted also by g . Therefore $\tilde{g} := \begin{pmatrix} -2c & a-d & 0 \\ a-d & 2b & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a flat Lorentzian metric on the manifold M , and (M, \tilde{g}) is a locally pseudo-Euclidean manifold with the Lorentzian totally geodesic foliation (M, F) .

Let $k : \tilde{M} \rightarrow M$ be the universal covering map for M and $\tilde{F} := k^*F$ be the induced foliation \tilde{M} . Then $\tilde{M} = \mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}^1$ and $\tilde{F} = \{\{v\} \times \mathbb{R}^1 \mid v \in \mathbb{R}^2\}$. Let $pr : \tilde{M} = \mathbb{R}^2 \times \mathbb{R}^1 \rightarrow \mathbb{R}^2$ be the projection onto the first multiple.

The group $\Psi := \langle A \rangle \cong \mathbb{Z}$ is the global holonomy group of the foliation (M, F) that is covered by the trivial fibre bundle $pr : \tilde{M} = \mathbb{R}^2 \times \mathbb{R}^1 \rightarrow \mathbb{R}^2$. Emphasize that (M, F) is Riemannian foliation if and only if there exists a Riemannian metric k on \mathbb{R}^2 such that Ψ becomes an isometry group of (\mathbb{R}^2, k) . In this case every stationary subgroup of Ψ must be relatively compact. That is impossible because Ψ considered as a stationary subgroup of $O_2 \in \mathbb{R}^2$ is non-compact discrete subgroup in $Diff(\mathbb{R}^2)$.

The restriction $k|_{\tilde{L}} : \tilde{L} \rightarrow L$ onto an arbitrary leaf $\tilde{L} \cong \mathbb{R}^1$ of (\tilde{M}, \tilde{F}) is the holonomy covering map and a local isometry.

The graph $G(F)$ is a 4-dimensional manifold with the induced foliation \mathbb{F} of codimensional 2. The generic leaf L of (M, F) has a trivial holonomy group and $L \cong \mathbb{R}^1$. Hence the generic leaf $\mathbb{L} \cong L \times L \cong \mathbb{R}^2$. The holonomy group of any other leaf L_α of (M, F) is isomorphic to \mathbb{Z} and $L_\alpha \cong S^1$. In this case the leaf $\mathbb{L}_\alpha := p_1^{-1}(L_\alpha)$ is isometric to the Euclidean cylinder $\mathbb{R}^1 \times_{\mathbb{Z}} \mathbb{R}^1$.

Let $\tilde{k} : \tilde{G} \rightarrow G(F)$ be the universal covering map, in this case \tilde{G} is diffeomorphic to \mathbb{R}^4 and it is provided by the induced foliation $\tilde{\mathbb{F}} := \{\{u\} \times \mathbb{R}^2 \mid u \in \mathbb{R}^2\}$, where $\tilde{G} = \tilde{G}(\tilde{F})$ is the graph of the foliation (\tilde{M}, \tilde{F}) and the following diagram

$$\begin{array}{ccccc} G(F) & \xleftarrow{\tilde{k}} & \tilde{G}(\tilde{F}) \cong \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\tilde{pr}} & \mathbb{R}^2 \\ \downarrow p_1 & & \downarrow \tilde{p}_1 & & \downarrow id \\ M & \xleftarrow{k} & \tilde{M} \cong \mathbb{R}^2 \times \mathbb{R}^1 & \xrightarrow{pr} & \mathbb{R}^2 \end{array}$$

is commutative, where $p_1 : G(F) \rightarrow M$ and $\tilde{p}_1 : G(\tilde{F}) \rightarrow \tilde{M}$ are the canonical projections, $\tilde{pr} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the projection onto the first multiplier.

Denote by $f : \mathbb{R}^3 \rightarrow \mathbb{T}^2 \times \mathbb{R}^1$ the universal covering map. Let $y_0 := f(0)$ where 0 is zero in \mathbb{R}^3 and $x_0 := \varphi(y_0)$. Compute the fundamental group $\pi_1(M, x_0)$. Remark that the regular covering map $\varphi : \mathbb{T}^2 \times \mathbb{R}^1 \rightarrow M$ induces a group monomorphism $\hat{\varphi} : \pi_1(\mathbb{T}^2 \times \mathbb{R}^1, y_0) \rightarrow \pi_1(M, x_0)$ onto a normal subgroup $N \cong \mathbb{Z}^2$ of $\pi_1(M, x_0)$, and, for fixed point y_0 , the quotient group $\pi_1(M, x_0)/N$ is isomorphic to the deck transformation group $\hat{G} \cong \mathbb{Z}$ with a generator $\Phi_A|_{\mathbb{T}^2 \times \mathbb{R}^1 \times \{1\}}$. Let us consider the leaf $L_0 = L_0(x_0)$ which is diffeomorphic to the circle. Observe that the inclusion $j : L_0 \rightarrow M$ induces a group monomorphism $\hat{j} : \pi_1(L_0, x_0) \rightarrow \pi_1(M, x_0)$ onto $H := Im(\hat{j}) \cong \mathbb{Z}$, and the deck

transformation group induced by H is equal \widehat{G} . Therefore [[4], Proposition 1.3.1.] the fundamental group $\pi_1(M, x_0)$ is the semi-direct product $H \rtimes N \cong \mathbb{Z} \rtimes \mathbb{Z}^2$.

Emphasize that foliations (M, F) and $(G(F), \mathbb{F})$ are both totally geodesic and Lorentzian, with the manifolds M and $G(F)$ are the Eilenberg–MacLane spaces of the type $K(H \rtimes N, 1)$, i.e. $\pi_n(M) = \pi_n(G(M)) = 0 \quad \forall n \geq 2$, $\pi_1(M) = \pi_1(G(F)) \cong H \rtimes N$.

REFERENCES

- [1] R.A. Blumenthal, J.J. Hebda, Ehresmann connections for foliations, *Indiana Univ. Math. J.* **33** (4), 597–611 (1984).
- [2] R.A. Blumenthal, J.J. Hebda, De Rham decomposition theorems for foliated manifolds, *Ann. Inst. Fourier.* **33** (2), 183–198 (1983)
- [3] C. Boubel, P. Mounoud, C. Tarquini, Lorentzian foliations on 3-manifolds, *Ergodic Theory Dynam. System* **26** (5), 1339–1362 (2006)
- [4] D. Bump, *Group representation theory* (2010) Sporadis.stanford.edu/bump/group/gind1_3.html.
- [5] E. Ghys, Deformations de flots d’Anosov et de groupes fuchsien, *Annales de l’institut Fourier* **42** (1-2), 209–247 (1992).
- [6] S. Kashiwabara, The decomposition of differential manifolds and its applications, *Tohoku Math. J.* **11**, 43–53 (1959).
- [7] S. Kobayashi, *Transformation groups in differential geometry* Springer, Berlin-Heidelberg-New York (1972).
- [8] S. Kobayashi, K. Nomizu K., *Foundations of differential geometry 1*, Nauka Press, Moscow (1981).
- [9] A.D. Lewis, Affine connections and distributions, *Reports on Mathematical Physics*, 135–164 (1998)
- [10] P. Molino, *Riemannian foliations*, *Progress in math.* **73**, Birkhauser, Boston. (1988).
- [11] B. O’Neill, *Semi-Riemannian Geometry with Applications to Relativity* Academic Press, New York, London (1983).
- [12] R. Palais, A global formulation of the Lie theory of transformation groups, *Memoris of Amer.Math. Soc.* **22**, 10–129 (1957).
- [13] B. Reinhart, Foliated manifolds with bundle-like metrics, *Ann. of Math.* **69**, 119–132 (1958)
- [14] E. Salem, Riemannian foliations and pseudogroups of isometries, Application D in [10]
- [15] Y.L. Shapiro, N.I. Zhukova, Simple transversal bifibrations, *Iz. VUZ. Mat.* (4), 104–113 (1974).
- [16] I. Tamura, *Topology of foliations* Iwanami Shoten, Japan (1979).
- [17] H.E. Winkelkemper, The graph of a foliation, *Ann. Glob. Analysis and Geometry* **1** (3), 51–75 (1983).
- [18] H. Wu, On the de Rham decomposition theorem, *Illinois J. Math.* **8**, 291–311 (1964).
- [19] K. Yokumoto, Mutual exclusiveness along spacelike, timelike, and lightlike leaves in totally geodesic foliations of lightlike complete Lorentzian two-dimensional tori, *Hokkaido Math.J.* **31** (3), 643–663 (2000).
- [20] Zhukova N.I.: Foliations that are compatible with systems of paths. *Iz. VUZ. Mat.* **7**, 5–13 (1984)

- [21] N.I. Zhukova, The graph of a foliation with Ehresmann connection and stability of leaves, *Iz. VUZ. Mat.* (2), 76–81 (1994).
- [22] N.I. Zhukova, E.A. Rogozhina, Classification of compact Lorentzian 2-orbifolds with non-compact full isometry graphs, *Siberian Mathematical Journal* **53** (6), 1037–1050 (2012).
- [23] N.I. Zhukova, Local and global stability of compact leaves and foliations, *J. of Math. Phys., Analysis and Geometry* **9** (3), 400–420 (2013).
- [24] N.I. Zhukova, A.Y. Dolgonosova, The automorphism groups of foliations with transverse linear connection, *Cent. Eur. J. Math.* **11** (12), 2076–2088 (2013).